Initial time singularities in nonequilibrium evolution of condensates and their resolution in the linearized approximation

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The real time nonequilibrium evolution of condensates in field theory requires an initial value problem specifying an initial quantum state or density matrix. Arbitrary specifications of the initial quantum state (pure or mixed) results in initial time singularities. These initial time singularities are of a different nature and independent of the ultraviolet divergences which are removed by the usual renormalization counterterms. The removal of the initial time singularities requires a specific choice of initial states. We study the initial time singularities in the linearized equation of motion for the scalar condensate in a renormalizable Yukawa theory in $3+1$ dimensions. In this renormalizable theory the initial time singularities are enhanced. We present a consistent method for removing these initial time singularities by specifying initial states where the distribution of high energy quanta is determined by the initial conditions and the interaction effects. This is done through a Bogoliubov transformation which is consistently obtained in a perturbative expansion. The usual renormalization counterterms and the proper choice of the Bogoliubov coefficients lead to a singularity free evolution equation. We establish the relationship between the evolution equations in the linearized approximation and linear response theory. It is found that only a very specific form of the external source for linear response leads to a real time evolution equation which is singularity free. We focus on the evolution of spatially inhomogeneous scalar condensates by implementing the initial state preparation via a Bogoliubov transformation up to one loop. As a concrete application, the evolution equation for an inhomogeneous condensate is solved analytically and the results are carefully analyzed. Symmetry breaking by initial quantum states is discussed.

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I. INTRODUCTION

The study of the real time dynamics and the evolution of nonequilibrium quantum states has now become ubiquitous in cosmology and high or intermediate energy physics. In cosmology the real time evolution of expectation values of quantum fields is a necessary component of a microscopic description of the inflationary dynamics and the subsequent hot Friedmann-Robertson-Walker (FRW) stage (big bang) seeking to give a realistic description of the early universe and the physical processes originated there. In the physics of heavy ion collisions a very active program seeks to establish potential experimental signatures from possible nonequilibrium stages of the evolution of the quark-gluon and chiral phase transitions [1]. In cosmology a program that incorporates consistently the nonequilibrium evolution of initial quantum states or density matrices of thermal or nonthermal origin including renormalization and back-reaction effects had been pursued vigorously during the last few years [4,6–8]. In high or intermediate energy the possibility of studying the quark-gluon plasma and chiral phase transition at the forthcoming ultrarelativistic heavy ion colliders [BNL Relativistic Heavy Ion Collider (RHIC) and CERN Large Hadron Collider (LHC)] has motivated a substantial effort to study out of equilibrium dynamics during phase transitions. In particular the formation of coherent pion domains [9], the evolution of nonequilibrium initial density matrices and states of high energy density [10], and isospin condensates [11]. Nonperturbative techniques had been developed to study consistently nonequilibrium dynamics of quantum field theories [3,4] and current computational facilities allow the possibility of studying the nonequilibrium dynamics of nonlinear, inhomogeneous configurations in quantum field theories [12] including gauge theories [13,14], for which recent lattice simulations of nonequilibrium gauge field theories with topological excitations had recently been reported [15].

The real time evolution of either density matrices or pure states, or alternatively of matrix elements must be set up as an initial value problem, either by specifying the initial state or by providing the Cauchy data (expectation values of the field and its time derivative) typically on spacelike hypersurfaces. Once this initial value problem has been set up at some initial time, the real-time evolution of the expectation values or other matrix elements can be studied either analytically in the case of small amplitudes [16,17] or numerically in the case of large amplitude configurations [3,10,12,14,15] (although analytic expressions are available in some extent).

An important but largely unnoticed subtlety arises in these situations in that besides the usual ultraviolet divergences associated with masses, couplings, and wave-function renormalizations there appear initial time singularities [18,19]. The physical reason for these initial time singularities can be understood as follows: the initial state (either pure or mixed) is typically chosen to reflect some physical description but generally is either some initial pure excited state with free field quanta or a thermal density matrix for free field theory. The choice of the initial state (including the field expectation value and its time derivative) has been essentially arbitrary and in particular independent of the field Hamiltonian. The time evolution with the interacting Hamiltonian suddenly couples at the initial time the infinite number of degrees of freedom of the theory, redistributing the spectral densities. In the case in which the underlying theory is renormalizable this redistribution of the spectral densities results in a divergent response. Such effect is also present on systems with a finite number of degrees of freedom but it is then finite.

The consideration of singularities associated with setting up initial conditions in a quantum field theory has been addressed originally by Stueckelberg [2], the similarity with sharp boundary conditions in a Euclidean formulation has been studied by Symanzik [20] and has since found different possible solutions [2,6,18,19,21].

It is important to emphasize that these initial time singularities are different from the usual ultraviolet divergences common in quantum field theories and are not cured by the renormalization counterterms associated with the removal of the ultraviolet divergences. These initial time singularities require a very different treatment for their resolution that hinges upon a judicious choice of initial states that includes the effects of the interactions.

A very appealing method to prepare initial states that lead to evolution equations without initial time singularities has been recently advocated [18,19] for self-consistent real-time evolution. This method consists in defining an initial state as a Bogoliubov transformation of the initial states in a free field theory. The Bogoliubov transformation is chosen to cancel the initial time singularities. The advantage of this method is that it is physically transparent and can be implemented both for small amplitude, i.e., the linearized problem, as well as for the large amplitude case which must be necessarily studied numerically.

In this article we focus on studying these initial time singularities and their resolution via the method of a Bogoliubov transformed initial state in a renormalizable theory in the case of small amplitudes of the scalar condensate. This case allows to obtain the evolution equations in a linearized approximation with an analytical solution to the evolution. Furthermore, we establish the correspondence between this method and linear response theory for the case of linearized equations of motion of condensates. An important corollary of this correspondence is that only very specific choices of the external source in the linear response approach lead to a singularity free initial value problem. We choose to study the initial value problem for the evolution of a scalar field condensate in a Yukawa theory in $3+1$ dimensions both for homogeneous as well as for inhomogeneous condensates. Whereas in [18,19] the homogeneous case has been studied in a self-consistent manner and the linearized approximation has been extracted from it, the *inhomogeneous* case has not been studied, and hence we devote our attention mainly to this important case. In a renormalizable theory the initial time singularities are enhanced and new infinities associated with the preparation of the initial state emerge, this situation is highlighted in the renormalizable Yukawa theory which is the focus of our study.

Main results. (i) The main results of our study can be summarized as follows: we show that the initial value problem in renormalizable quantum field theories is well defined and free of initial time singularities provided we apply an appropriate Bogoliubov transformation to the initial state. The initial data specifies the expectation value of the order parameter, i.e., the condensate and its time derivative at the initial time and the Bogoliubov coefficients required to specify the initial state. In order to eliminate the initial time singularities, the Bogoliubov coefficients are constrained to behave in a precise manner for high momentum modes. More precisely, the $1/p$ and $1/p^3$ contributions to the Bogoliubov coefficients for a mode of momentum p are uniquely fixed by the initial data, the coupling and the mass (Sec. IV). Choices of Bogoliubov coefficients that differ by contributions in $1/p$ of order higher than $1/p^3$ define different initial states, all of them free of initial time singularities. Thus, the time evolution of an initial state in quantum field theory is free of initial time singularities provided that the high energy distribution of quanta of the initial state is specified in a very precise manner.

This method is implemented consistently in the perturbative expansion and in combination with the usual renormalization of mass, wave-function and coupling leads to a real-time evolution free of ultraviolet and initial time singularities. (ii) As an example we study the real-time evolution of an inhomogeneous scalar condensate in the Yukawa theory, both in the case in which the scalar is heavy and can decay into fermion-antifermion pairs, and in the case in which the scalar is light and cannot decay into fermion pairs. Here we provide a detailed analysis of the real time evolution of an inhomogeneous scalar condensate corresponding to a spherical wave.

The article is organized as follows. In Sec. II we obtain the equations of motion for a scalar condensate. In Sec. III we analyze the ultraviolet and initial time singularities. For simplicity we present first the case of a homogeneous scalar condensate. Section IV introduces the Bogoliubov transformed initial state in the case of a homogeneous condensate, discusses in detail the choice of the Bogoliubov coefficients that lead to an evolution free of initial time singularities and presents the singularity free real-time equations of motion for the homogeneous case. In Sec. V we establish a relation between the initial value problem in the linearized approximation and linear response and discuss the constraints on the external sources that lead to a well defined initial value problem free of singularities. In Sec. VI we extend the treatment to the case of inhomogeneous scalar condensates, obtain the corresponding inhomogeneous Bogoliubov transformation consistently in perturbation theory and the equations of motion free of singularities to one-loop order. In Sec. VII we obtain an analytic solution of the real-time equations of evolution for an inhomogeneous scalar condensate. We also provide a numerical analysis of the solution and discuss its main features. The conclusions summarize our work and discusses the potential applications of the methods presented.

The Bogoliubov transformation of the tadpole and self-energy diagrams is presented in the Appendixes A–C. Furthermore, Appendix D establishes several sum rules on the spectral densities

II. LINEARIZED EQUATIONS OF MOTION FOR CONDENSATES

Although the initial time singularities that will be discussed in this article are generic features of initial value problems in field theory, they are highlighted in renormalizable theories. Therefore we choose to discuss these singularities and their resolution in a Yukawa theory in $3+1$ dimensions. The focus of this article is to understand the problems of setting up an initial value problem to describe the nonequilibrium evolution of condensates or field expectation values in the linearized (small amplitude) approximation. Furthermore, we compare with an alternative formulation based on linear response.

We consider a massive scalar field $\Phi(x)$ coupled to a massive Dirac field $\psi(x)$ in a Yukawa model specified by the Lagrangian density

$$\mathcal{L}(\Phi,\psi,\overline{\psi})=\frac{1}{2}\partial_\mu\Phi(x)\partial^\mu\Phi(x)+\frac{1}{2}M^2\Phi^2(x) + \overline{\psi}(x)[i\partial\!\!\!/+m+g\Phi(x)]\psi(x). \tag{2.1}$$

We study the time evolution of the expectation value of the scalar field via the real time generating functional in terms of a path integral defined on a contour in complex time (CTP) [22,23]. The effective Lagrangian that enters in the contour path integral is

$$\mathcal{L}_{\rm eff}=\mathcal{L}(\Phi^+,\psi^+,\overline{\psi}^+)-\mathcal{L}(\Phi^-,\psi^-,\overline{\psi}^-),$$

where the $\pm$ labels on the fields refer to the forward $(+)$ and backward $(-)$ branches corresponding to the forward and backward time evolution of the initially prepared density matrix. We now follow the procedure of references [16,17] and use the tadpole method to obtain the equation of motion for the expectation value of the scalar field

$$\phi(x)\equiv\langle\Phi(x)\rangle$$

and write

$$\Phi^\pm(x)=\chi^\pm(x)+\phi(x);\quad \langle\chi^\pm(x)\rangle=0.$$

We specify the initial data at the time $t_0=0$ by giving the initial condition,

$$\phi(\mathbf{x},0)\equiv\phi(\mathbf{x})\ \text{ and }\ \dot{\phi}(\mathbf{x},0)\equiv\dot{\phi}(\mathbf{x}).$$

Let us consider that the initial density matrix at time $t_0=0$ is given by

$$\rho(0)=|0\rangle\langle 0|$$

with $|0\rangle$ the free field Fock vacuum for the scalar and fermion fields. For $t>0$, $\rho(t)=e^{-iHt}\rho(0)e^{iHt}$ where H is the full Hamiltonian. This case is tantamount to considering an initial free field vacuum state and switching-on the interaction suddenly at $t=0$.

The equation of motion for $\phi(\mathbf{x},t)$ is obtained in a systematic perturbative expansion by imposing that $\langle\chi^\pm(\mathbf{x},t)\rangle=0$ to all orders in perturbation theory. We will restrict our study to the case of small amplitudes of the condensate and will obtain the equations of motion *linearized* in $\phi(\mathbf{x},t)$. In this linear approximation the self-energy kernel is obtained to any desired order in a perturbative expansion in the Yukawa coupling but in the state with *vanishing condensate*. Thus assuming that the state with vanishing condensate is spatially translational invariant it is convenient to perform a spatial Fourier transform for the condensate and the self-energy kernel [16,17]. Anticipating renormalization effects we introduce the renormalized field and mass in the Lagrangian before shifting the field by the condensate

$$\Phi(x)=\sqrt{Z_\phi}\,\Phi_R(x);\quad Z_\phi M^2=M_R^2+\delta M^2.$$

Since the fermionic fields will be integrated out to obtain the equation of motion for the expectation value of the scalar field, we do not introduce the renormalizations associated with the fermionic fields. We now drop the subscript R from the renormalized quantities to avoid cluttering of notation, with the understanding that the scalar field and its mass are the renormalized ones.

The equation of motion for the spatial Fourier transform of the condensate

$$\phi_{\mathbf{q}}(t)\equiv\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\phi(\mathbf{x},t),$$

to one loop order is given by (see Ref. [17] for details)

$$(1+\delta Z)[\ddot{\phi}_{\mathbf{q}}(t)+q^2\phi_{\mathbf{q}}(t)]+(M^2+\delta M^2)\,\phi_{\mathbf{q}}(t) + \int_0^t dt'\,\Sigma_{\mathbf{q}}(t-t')\,\phi_{\mathbf{q}}(t')-J=0, \tag{2.2}$$

with $\delta Z=Z_\phi-1$ and

$$J=-ig\,\delta_{\mathbf{q},\mathbf{0}}\int\frac{d^3k}{(2\pi)^3}\,\mathrm{tr}\,S^>_{\mathbf{k}}(t,t), \tag{2.3}$$

$$\Sigma_{\mathbf{q}}(t-t')=-ig^2\int\frac{d^3p}{(2\pi)^3}\,\mathrm{tr}[S^>_{\mathbf{p}}(t-t')S^<_{\mathbf{p}-\mathbf{q}}(t'-t) - S^<_{\mathbf{p}}(t-t')S^>_{\mathbf{p}-\mathbf{q}}(t'-t)]. \tag{2.4}$$

The contribution J is due to the fermion tadpole, it is usually absorbed in a constant shift of the field $\phi(x)$. The fermionic Green's functions in the vacuum state are given by [17]

$$S^{>}_{\mathbf{p}}(t,t')=-i\int d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}}\langle\psi(\mathbf{x},t)\overline{\psi}(\mathbf{0},t')\rangle$$

$$=-\frac{i}{2E_p}[e^{-iE_p(t-t')}(\not p+m)$$

$$+e^{iE_p(t-t')}\gamma_0(\not p-m)\gamma_0],$$

$$S^{>}_{\mathbf{p}}(t,t')=i\int d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}}\langle\overline{\psi}(\mathbf{0},t')\psi(\mathbf{x},t)\rangle$$

$$=\frac{i}{2E_p}[e^{-iE_p(t-t')}(\not p+m)$$

$$+e^{-iE_p(t-t')}\gamma_0(\not p-m)\gamma_0],$$

$$E_p=\sqrt{\mathbf{p}^2+m^2}.$$

III. ULTRAVIOLET RENORMALIZATION AND INITIAL TIME SINGULARITIES

The evolution equation of the type (2.2) contains two types of divergences: (i) ultraviolet divergences which are removed by the mass and wave function renormalizations; (ii) initial time singularities.

To illustrate these singularities in a more clear manner we now focus on the case of homogeneous condensate, i.e., $\mathbf{q}=0$. We find

$$\Sigma_{\mathbf{0}}(t-t')=-2g^2\int\frac{d^3p}{(2\pi)^3 2E_p}\frac{8p^2}{2E_p}\sin[2E_p(t-t')],$$

$$J=-4mg\int\frac{d^3p}{(2\pi)^3 2E_p}. \tag{3.1}$$

Whereas J acts as a constant source term and can be absorbed in a shift of the expectation value $\phi(x)$, the self-energy kernel leads to ultraviolet divergences as can be seen upon integrating by parts the nonlocal term in Eq. (2.2) three times

$$\int_0^t dt'\,\Sigma_{\mathbf{0}}(t-t')\phi_{\mathbf{0}}(t')=-g^2\int\frac{d^3p}{(2\pi)^3 2E_p}\frac{8p^2}{E_p}\int_0^t dt'\,\sin[2E_p(t-t')]\phi_{\mathbf{0}}(t')$$

$$=-g^2\int\frac{d^3p}{(2\pi)^3 2E_p}\frac{8p^2}{E_p}\Bigg\{\frac{1}{2E_p}\phi_{\mathbf{0}}(t)-\frac{1}{2E_p}\phi_{\mathbf{0}}(0)\cos(2E_pt)-\frac{1}{(2E_p)^2}\dot\phi_{\mathbf{0}}(0)\sin 2E_pt$$

$$-\frac{1}{(2E_p)^3}\ddot\phi_{\mathbf{0}}(t)+\frac{1}{(2E_p)^3}\ddot\phi_{\mathbf{0}}(0)\cos 2E_pt+\frac{1}{(2E_p)^3}\int_0^t dt'\,\cos[2E_p(t\text{-}t')]\dddot\phi_{\mathbf{0}}(t')\Bigg\}.$$

Using dimensional regularization the coefficient of $\phi_{\mathbf{0}}(t)$ becomes

$$-\delta M^2=-g^2\int\frac{d^3p}{(2\pi)^3 2E_p}\frac{4p^2}{E_p^2}=\frac{3g^2m^2}{4p^2}\left[\frac{2}{\epsilon}-\gamma+\frac{1}{3}+\ln\frac{4\pi\mu^2}{m^2}\right].$$

This agrees with the expression obtained by evaluation the corresponding Feynman graph in $4-\epsilon$ dimensions. The renormalization is performed here at $q^2=0$.

The coefficient proportional to $\ddot\phi_{\mathbf{0}}(t)$ is the wave function renormalization which again in dimensional regularization is given by

$$-\delta Z=g^2\int\frac{d^3p}{(2\pi)^3 2E_p}\frac{p^2}{E_p^4}=\frac{g^2}{8\pi^2}\left[\frac{2}{\epsilon}-\gamma-\frac{2}{3}+\ln\frac{4\pi\mu^2}{m^2}\right].$$

These ultraviolet renormalizations are cancelled by the mass and wave function counterterms but there still remain singularities arising from the terms that are evaluated at the initial time $t=0$, these are given by

$$\left[\int_0^t dt'\Sigma_{\mathbf{0}}(t-t')\phi_{\mathbf{0}}(t')\right]\Bigg|_{\text{sing}}=-g^2\int\frac{d^3p}{(2\pi)^3 2E_p}\frac{8p^2}{E_p}\Bigg\{-\frac{1}{2E_p}\phi_{\mathbf{0}}(0)\cos 2E_pt-\frac{1}{(2E_p)^2}\dot\phi_{\mathbf{0}}(0)\sin 2E_pt$$

$$+\frac{1}{(2E_p)^3}\ddot\phi_{\mathbf{0}}(0)\cos 2E_pt\Bigg\}.$$

Obviously this expression is singular as $t \to 0$. Simple power counting shows that the coefficients of $\phi_{\mathbf{0}}(0)$, $\dot{\phi}_{\mathbf{0}}(0)$, and $\ddot{\phi}_{\mathbf{0}}(0)$ diverge as $1/t^2$, $1/t$, and $\log t$, respectively. At finite t they are finite due to the oscillatory behavior of the integrand.

The physical reason for these singularities is the following: having prepared the initial state to be the free field Fock vacuum and switching-on the interaction suddenly at the initial time $t=0$, the interaction redistributes the spectral density of fields. The scalar field states overlap with the fermionic continuum of states and the particles become dressed by the interaction. This dressing effect which is responsible for mass, wave function and coupling renormalizations occurs suddenly when the interaction is switched on and is reflected as an initial time singularity. Obviously these short time singularities will be present at finite temperature or density, and are a consequence of the fact that the the underlying field theory possesses an infinite number of degrees of freedom.

Our main point in this article is that these initial time singularities can be removed and an initial value problem can be defined consistently and free of singularities by considering an appropriately chosen initial state that is dressed by the interactions. We thus propose to initialize the real time evolution by providing an initial state that includes the dressing as a Bogoliubov transformation from the free field Fock states. Furthermore we will argue that this construction leads to a consistent initial value problem for real-time dynamics and can be implemented systematically order by order in perturbation theory.

We now discuss in detail this procedure in the example under consideration to lowest order in the Yukawa coupling.

IV. EQUATIONS OF MOTION WITH BOGOLIUBOV TRANSFORMED STATES: HOMOGENEOUS CASE

In this section we introduce the Bogoliubov transformed states and show explicitly how the introduction of these dressed states provides a solution to the problem of initial time singularities. For the sake of clarity we study first the homogeneous case $\mathbf{q}=0$ and postpone to a later section the generalization to inhomogeneous condensates.

From the free field Fock vacuum state $|0\rangle$ a Bogoliubov transformed state is obtained after a unitary transformation

$$|0_b\rangle = e^{-Q}|0\rangle,$$

with Q an anti-Hermitian operator. Bogoliubov transformed operators are defined via

$$\mathcal{O}_b = \exp(-Q)\,\mathcal{O}\,\exp(Q),$$

therefore if the vacuum state $|0\rangle$ is annihilated by the destruction operators, the Bogoliubov transformed annihilation operators annihilate the state $|0_b\rangle$.

To lowest order in the Yukawa coupling the Bogoliubov transformation that required to cancel the initial time singularities only involve fermionic fields. Only when scalar contributions in higher order corrections arise there will be a need to introduce the Bogoliubov transformation for scalar fields.

Therefore we introduce the antihermitian operator Q that generates the unitary Bogoliubov transformation

$$Q = \sum_s \int \frac{d^3p}{(2\pi)^3 2E_p}\,\beta_{ps}[d^\dagger(-\mathbf{p},s)b^\dagger(\mathbf{p},s)e^{i\delta_{ps}} - b(\mathbf{p},s)d(-\mathbf{p},s)e^{-i\delta_{ps}}],$$

where $b(\mathbf{p},s)$ and $d(\mathbf{p},s)$ are annihilation operators for fermions and antifermions, respectively and $b^\dagger(\mathbf{p},s)$ and $d^\dagger(\mathbf{p},s)$ are creation operators for fermions and antifermions, respectively. The c-number function β_{ps} and δ_{ps} will be chosen such that the initial time singularities are removed.

This generator of Bogoliubov transformations illuminates at once the nature of the Bogoliubov transformed initial states. Acting on the free field Fock vacuum state, the Bogoliubov transformation leads to a state that is a linear combination of particle-antiparticle pairs of total zero momentum. The fact that the total momentum of these pairs is zero is of course a result of the fact that the scalar condensate is homogeneous.

The commutators of Q with the fermionic creation and annihilation operators are given by

$$[Q,b(\mathbf{p},s)] = \beta_{ps}\, e^{i\delta_{ps}} d^\dagger(-\mathbf{p},s),$$

$$[Q,d^\dagger(-\mathbf{p},s)] = -\beta_{ps}\, e^{-i\delta_{ps}}\, b(\mathbf{p},s),$$

which leads to the following relation between the transformed and the original operators:

$$b(\mathbf{p},s) = \cos\beta_{\mathbf{p},s}\, b_b(\mathbf{p},s) + \sin\beta_{\mathbf{p},s}\, e^{i\delta_{\mathbf{p},s}} d_b^\dagger(-\mathbf{p},s),$$

$$d^\dagger(-\mathbf{p},s) = -\sin\beta_{\mathbf{p},s}\, e^{-i\delta_{\mathbf{p},s}} b_b(\mathbf{p},s) + \cos\beta_{\mathbf{p},s} d_b^\dagger(-\mathbf{p},s). \tag{4.1}$$

The operators $b(\mathbf{p},s)$, $d(\mathbf{p},s)$ as well as $b_b(\mathbf{p},s)$, $d_b(\mathbf{p},s)$ obey the usual canonical anticommutation relations.

The initial density matrix is now given by

$$\rho_b(0) = e^{-Q}\rho(0)e^{Q} = |0_b\rangle\langle 0_b|. \tag{4.2}$$

Although we have focused on a pure (vacuum) state, obviously this can be easily generalized to thermal or nonthermal mixed states.

In Appendix A we provide the details that lead to the following Green's functions in the Bogoliubov transformed states.

With this restriction the transformed Green function becomes

$$iS_b^{>}(t,\mathbf{x};t',\mathbf{x}')=\langle 0_b|\psi(t,\mathbf{x})\overline{\psi}(t',\mathbf{x}')|0_b\rangle$$
$$=\int\frac{d^3p}{(2\pi)^3 2E_p}e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')}[\cos^2\beta_p(\not p+m)e^{-iE_p(t-t')}-\sin\beta_p\cos\beta_p e^{i\delta_p}\Sigma_{\hat{\mathbf{p}}}(\not p+m)\gamma_5\gamma_0 e^{-iE_p(t+t')}$$
$$-\sin\beta_p\cos\beta_p e^{-i\delta_p}\Sigma_{\hat{\mathbf{p}}}\gamma_5\gamma_0(\not p+m)e^{iE_p(t+t')}+\sin^2\beta_p\gamma_5\gamma_0(\not p+m)\gamma_5\gamma_0 e^{iE_p(t-t')}],$$

and

$$-iS_b^{<}(t,\mathbf{x};t',\mathbf{x}')=\langle 0_b|\overline{\psi}(t',\mathbf{x}')\psi(t,\mathbf{x})|0_b\rangle$$
$$=\int\frac{d^3p}{(2\pi)^3 2E_p}e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')}[\sin^2\beta_p(\not p+m)e^{-iE_p(t-t')}+\sin\beta_p\cos\beta_p e^{i\delta_p}\Sigma_{\hat{\mathbf{p}}}(\not p+m)\gamma_5\gamma_0 e^{-iE_p(t+t')}$$
$$+\sin\beta_p\cos\beta_p e^{-i\delta_p}\Sigma_{\hat{\mathbf{p}}}\gamma_5\gamma_0(\not p+m)e^{iE_p(t+t')}+\cos^2\beta_p\gamma_5\gamma_0(\not p+m)\gamma_5\gamma_0 e^{iE_p(t-t')}]. \tag{4.3}$$

Perhaps the most striking feature of these Green's functions is their lack of time translational invariance, the main reason is that the Bogoliubov transformed states are not eigenstates of the bare particle number. We note that in the terms with $t+t'$ a translation of the time variables can be compensated by a change in the phase δ_p, i.e., a gauge transformation of the fermionic fields.

It is this lack of time translational invariance that will allow to cancel the initial time singularities as shown explicitly below.

The evolution equation is obtained in the same manner as in the previous section and is exactly of the same form as Eq. (2.2) with $\mathbf{q}=0$ as befits the homogeneous case, with the self-energy and tadpole kernels now given by the expressions (2.3), (2.4) but in terms of the Bogoliubov transformed Green's functions

$$J_b(t)=-ig\,\mathrm{tr}\,S_b^{>}(t,\mathbf{x};t,\mathbf{x})$$
$$=-g\int\frac{d^3p}{(2\pi)^3 2E_p}[4m\cos 2\beta_p$$
$$-(-4p)\sin 2\beta_p\cos(2E_pt-\delta_p)] \tag{4.4}$$

and

$$\Sigma_{b,\mathbf{0}}(t,t')=-2g^2\int\frac{d^3p}{(2\pi)^3 2E_p}\frac{1}{2E_p}\{8p^2\cos(2\beta_p)$$
$$\times\sin[2E_p(t-t')]-8pm\sin(2\beta_p)$$
$$\times[\sin(2E_pt-\delta_p)-\sin(2E_pt'-\delta_p)]\}. \tag{4.5}$$

The first, time independent contribution to $J_b(t)$ in Eq. (4.4) can be absorbed into a constant shift of the condensate much in the same way as in the untransformed case. The second, time dependent term will be used to cancel the initial time singularities.

The self-energy kernel (4.5) can be written as $\Sigma_{\mathbf{0}}(t-t')+\Delta\Sigma_{b,\mathbf{0}}(t,t')$ with $\Sigma_{\mathbf{0}}(t-t')$ given by Eq. (3.1) and

$$\Delta\Sigma_{b,\mathbf{0}}(t,t')=-2g^2\int\frac{d^3p}{(2\pi)^3 2E_p}\frac{1}{2E_p}\{8p^2(\cos 2\beta_p-1)$$
$$\times\sin[2E_p(t-t')]-8pm\sin 2\beta_p$$
$$\times[\sin(2E_pt-\delta_p)-\sin(2E_pt'-\delta_p)]\}.$$

We will assume in the following that β_p decreases sufficiently fast with p so that terms proportional to $\sin 2\beta_p$ and $\cos(2\beta_p)-1$ lead to convergent integrals. This in fact will be checked *a posteriori* when we find the required expression for β_p below. Then the ultraviolet divergences in Eq. (4.5) are the same as in the perturbative vacuum.

Integrating by parts now three times in t' in order to single out the ultraviolet and equal-time singularities from $\Sigma_{b,\mathbf{0}}(t,t')$ in the equation of motion, we find that just as in the untransformed case the ultraviolet divergences proportional to $\phi_{\mathbf{0}}(t)$ and $\ddot{\phi}_{\mathbf{0}}(t)$ are cancelled by the mass and wave function renormalization counterterms (up to finite parts depending on the renormalization prescription).

The terms that result in initial-time singularities arise from

$$\left[\left.\int_0^t dt'\Sigma_{b,\mathbf{0}}(t-t')\phi_{\mathbf{0}}(t')\right]\right|_{\mathrm{sing}}$$
$$=-g^2\int\frac{d^3p}{(2\pi)^3 2E_p}\frac{8p^2}{E_p}\left\{-\frac{1}{2E_p}\phi_{\mathbf{0}}(0)\right.$$
$$\times\cos 2E_pt-\frac{1}{(2E_p)^2}\dot{\phi}_{\mathbf{0}}(0)\sin 2E_pt$$
$$\left.+\frac{1}{(2E_p)^3}\ddot{\phi}_{\mathbf{0}}(0)\cos 2E_pt\right\}\cos 2\beta_p.$$

We now require that these terms be cancelled by the time dependent terms of $J_b(t)$, Eq. (4.4). This requirement leads to the equation

$$\tan 2\beta_p \cos(2E_p t-\delta_p) = g\frac{2p}{E_p}\left\{-\frac{1}{2E_p}\phi_{\mathbf{0}}(0)\cos 2E_p t-\frac{1}{(2E_p)^2}\dot{\phi}_{\mathbf{0}}(0) \times \sin 2E_p t+\frac{1}{(2E_p)^3}\ddot{\phi}_{\mathbf{0}}(0)\cos 2E_p t\right\}.$$

Comparing the terms proportional to the sine and cosine we find two equations that determine δ_p,β_p. These equations can be solved perturbatively with

$$\beta_p=b_{1,p}g+b_{2,p}g^2+\mathcal{O}(g^3).$$

To one loop order we only need to keep the linear term in g leading to

$$\beta_p \cos\delta_p = g\frac{p}{E_p}\left[-\frac{1}{2E_p}\phi_{\mathbf{0}}(0)+\frac{1}{(2E_p)^3}\ddot{\phi}_{\mathbf{0}}(0)\right],$$

$$\beta_p \sin\delta_p = g\frac{p}{E_p}\left[-\frac{1}{(2E_p)^2}\dot{\phi}_{\mathbf{0}}(0)\right]. \tag{4.6}$$

At this stage we recognize that there is freedom in choosing the Bogoliubov parameters to cancel the initial time singularities. The initial value problem will be free of singularities by choosing the coefficients $\beta_p;\delta_p$ so as to cancel the terms proportional to $1/p, 1/p^3$ and so that β_p vanishes as $p\to\infty$.

Different choices of the Bogoliubov parameters that differ only in higher inverse powers of the momenta lead to different initial quantum states, but the initial value problem is free of initial singularities. This freedom is similar to choosing renormalization counterterms including finite parts, i.e., different renormalization prescriptions.

Now the Bogoliubov correction and the mass and wave-function renormalization counter terms remove exactly all ultraviolet and initial time divergences from the equation of motion. To lowest order we can set $\beta_p=0$ in the self-energy in the equation of motion and having removed all ultraviolet and initial time singularities and absorbing the time independent contribution from the tadpole term in a constant shift of the condensate, we finally obtain the evolution equation in the case of the homogeneous condensate

$$\ddot{\phi}_{\mathbf{0}}(t)+M^2\phi_{\mathbf{0}}(t)+\int_0^t dt'\,\Sigma_s(t-t')\dddot{\phi}_{\mathbf{0}}(t')=0, \tag{4.7}$$

where $\Sigma_s(t-t')$ is the subtracted self-energy kernel

$$\Sigma_s(t-t')=-g^2\int\frac{d^3p}{(2\pi)^3 2E_p}\frac{p^2}{E_p^4}\cos[2E_p(t-t')]. \tag{4.8}$$

The equation of motion above should be compared to Eqs. (7.21) and (7.22) in [19]. The equation of motion obtained there differs from Eq. (4.7) by a *finite* renormalization ΔZ that corresponds to a different renormalization scheme but otherwise the equations are the same up to one loop order.

V. LINEAR RESPONSE THEORY

The initial value problem can be obtained by establishing direct contact with linear response theory as presented in Ref. [24] in the case of scalar condensates and in Ref. [25] for fermionic coherent states. This is achieved by coupling an *external source* term to the scalar field in the Lagrangian density (2.1)

$$\mathcal{L}[\Phi,\psi,\bar{\psi}]\to\mathcal{L}[\Phi,\psi,\bar{\psi}]+J_{\rm ext}(x)\Phi(x).$$

The expectation value of the scalar field induced by this source term is given by

$$\langle\Phi(x)\rangle=i\int dx'\,J_{\rm ext}(x')[\langle\Phi^+(x)\Phi^+(x')\rangle -\langle\Phi^+(x)\Phi^-(x')\rangle], \tag{5.1}$$

where the superscripts $\pm$ refer to the forward and backward time branches in the real time generating functional. The bracket in Eq. (5.1) is the retarded commutator. As discussed in detail in Ref. [24] the inversion of Eq. (5.1) gives rise to the equation of motion for the scalar field with an inhomogeneity given by the external source term. Following the same steps detailed in the first section, we find the equation of motion for an homogenous condensate to be given by

$$(1+\delta Z)\ddot{\phi}_{\mathbf{0}}(t)+(M^2+\delta M^2)\phi_{\mathbf{0}}(t) +\int_{-\infty}^t dt'\,\Sigma_{\mathbf{0}}(t-t')\phi_{\mathbf{0}}(t')-J=J_{\mathrm{ext},\mathbf{0}}(t),$$

which differs from Eq. (2.2) in the lower limit in the nonlocal term with the self-energy. Assuming adiabatic switching-on of the interaction from $t=-\infty$ we can now rewrite this equation of motion in a form that is closer to Eq. (2.2) by integrating by parts the nonlocal term. Defining,

$$\Sigma_{\mathbf{0}}(t-t')=\frac{d}{dt'}\Sigma_{1,\mathbf{0}}(t-t') =\frac{d^2}{dt'^2}\Sigma_{2,\mathbf{0}}(t-t') =\frac{d^3}{dt'^3}\Sigma_{3,\mathbf{0}}(t-t')$$

and integrating by parts we obtain

$$(1+\delta Z)\ddot{\phi}_{\mathbf{0}}(t)+(M^2+\delta M^2)\phi_{\mathbf{0}}(t)+\Sigma_{1,\mathbf{0}}(0)\phi_{\mathbf{0}}(t) -\Sigma_{2,\mathbf{0}}(0)\dot{\phi}_{\mathbf{0}}(t)+\Sigma_{3,\mathbf{0}}(0)\ddot{\phi}_{\mathbf{0}}(t) +\int_{-\infty}^t dt'\,\Sigma_{s,\mathbf{0}}(t-t')\dddot{\phi}_{\mathbf{0}}(t')-J=J_{\mathrm{ext},\mathbf{0}}(t),$$

with $\Sigma_{s,\mathbf{0}}(t-t')$ given by Eq. (4.8). Using the explicit form of $\Sigma_{\mathbf{0}}(t)$ given by Eq. (3.1) we find

$$\Sigma_{1,\mathbf{0}}(0) = -g^2 \int \frac{d^3p}{(2\pi)^3 2E_p} \frac{4p^2}{E_p^2} = -\delta M^2, \tag{5.2}$$

$$\Sigma_{2,\mathbf{0}}(0) = 0,$$

$$\Sigma_{3,\mathbf{0}}(0) = g^2 \int \frac{d^3p}{(2\pi)^3 2E_p} \frac{4p^2}{E_p^4} = -\delta Z. \tag{5.3}$$

Therefore the specific choice of external current

$$J_{\text{ext}}(t) = \int_{-\infty}^{0} dt' \, \Sigma_s(t-t') \dddot{\phi}_{\mathbf{0}}(t') \tag{5.4}$$

leads to the initial value problem described by Eq. (4.7).

This current depends on the past history of the condensate, and in general *does not vanish for* $t>0$ as it would be desirable from the point of view of linear response. In linear response the initial value problem is envisaged to be prepared by switching-on an external source and the interaction adiabatically from $t=-\infty$. The external source acts as a Lagrange multiplier, slowly displacing the condensate to the value to be determined at $t=0$, and switching-off the source suddenly at this time. This allows the condensate to be formed and dressed over a very long period of time. The dressed condensate is then released when the external current is switched-off. However, in an interacting renormalizable theory this instantaneous switching-off of the external current results in singularities. To set up a consistent, singularity free and renormalized initial value problem as is the goal of this article, the choice of the current (5.4) is the one that establishes contact with a linear response formulation, in this case the current depends on the past history of the condensate, which obviously need not be specified for an initial value problem. On the other hand, such a choice of current, depending on the past history is rather artificial from the linear response point of view.

We note that the current can be made to vanish at all times with the particular choice

$$\phi_{\mathbf{0}}(t) = \phi_{\mathbf{0}}(0) + \dot{\phi}_{\mathbf{0}}(0)t + \frac{1}{2}\ddot{\phi}_{\mathbf{0}}(0)t^2$$

for $t<0$. Obviously this behavior manifests the problem of the initial time singularities as singularities in the behavior of the field at a remote past.

An alternative would be to assume that the external source and therefore the condensate is adiabatically switched-on with a damping factor $e^{\epsilon t}$ for $t<0$ but this results in *discontinuities* in the first or second derivatives at $t=0$ and that would produce additional contributions from the integration by parts from these discontinuities.

There are two main conclusions of this discussion on the relationship with linear response.

We have established a direct relationship between the evolution equations in the linearized approximation and linear response theory. The initial value problem free of UV and initial time singularities is shown to be obtained in the context of linear response through a particular choice of the external current. Such external current depends on the past history of the condensate and only a very specific form for it leads to a singularity free initial value problem. From the perspective of an initial value problem in which Cauchy data are specified at some given initial time on a spacelike hypersurface this is a consistent choice. However, from the point of view of linear response this choice is somewhat artificial. The resulting external current *does not vanish* for $t>0$. If we instead require an instantaneous switching-off of the external current at $t=0$, initial time singularities become unavoidable.

The method of preparing a dressed initial state via a Bogoliubov transformation leads to a satisfactory description of the initial value problem. The usual mass, wave function, and coupling constant renormalization counterterms cancel the ultraviolet divergences, and the Bogoliubov coefficients are judiciously chosen to cancel the initial time divergences consistently in perturbation theory. To lowest order in the Yukawa coupling and for a homogenous condensate such a choice is given by Eq. (4.6).

Having studied in detail the simpler case of the homogeneous condensate, we now move on to our main point, the study of the evolution of inhomogeneous condensates.

VI. EQUATIONS OF MOTION FOR INHOMOGENEOUS CONDENSATES

The equation of motion for nonhomogeneous condensates in the amplitude approximation and in terms of spatial Fourier transforms reads

$$(1+\delta Z)[\ddot{\phi}_{\mathbf{q}}(t) + \mathbf{q}^2\phi_{\mathbf{q}}(t)] + (M^2+\delta M^2)\phi_{\mathbf{q}}(t) + \int_0^t dt' \, \Sigma_{\mathbf{q}}(t,t')\phi_{\mathbf{q}}(t') + J_{b,\mathbf{q}}(t) = 0.$$

From the discussions of the previous sections we have learned that the Bogoliubov coefficients that define the dressed states at the initial time can be found consistently in perturbation theory. Since the self-energy is already of second order in the Yukawa coupling we will not need to consider the Bogoliubov corrections to the self-energy, but only to the tadpole term $J_{b\mathbf{q}}(t)$.

The one-loop self-energy is therefore the usual one and given by

$$i\Sigma_{\mathbf{q}}(t-t') = 4g^2 \int \frac{d^3p}{(2\pi)^3 2E_p 2E_{\mathbf{p}-\mathbf{q}}} \times [E_p + E_{\mathbf{p}-\mathbf{q}} + \mathbf{p}(\mathbf{p}-\mathbf{q}) - m^2](-2i) \times \sin[(E_{\mathbf{p}} + E_{\mathbf{p}-\mathbf{q}})(t-t')]. \tag{6.1}$$

The derivation of the tadpole diagram that determines $J_{b,\mathbf{q}}(t)$ using the Bogoliubov-transformed Green functions, is given in Appendix B. We find

$$J_{b,\mathbf{q}}(t) = 4mg \int \frac{d^3p}{(2\pi)^3 2E_p} - 2\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}} 2E_{\mathbf{p}-\mathbf{q}}} [\gamma^*(\mathbf{q},\mathbf{p}) e^{-i(E_{\mathbf{p}}+E_{\mathbf{p}-\mathbf{q}})t} + \gamma(\mathbf{q},\mathbf{p}') e^{i(E_{\mathbf{p}}+E_{\mathbf{p}-\mathbf{q}})t}].$$

The function $\gamma(\mathbf{q},\mathbf{p})$ is a function related to the angles of the Bogoliubov transformation, that will be specified below such as to remove the initial time singularities.

The analysis of the singular and divergent contributions in the equation of motion proceeds as in the homogeneous case, performing three integrations by parts with respect to the time in the nonlocal term, we find

$$\int_0^t dt' \sin[(E_{\mathbf{p}}+E_{\mathbf{p-q}})(t-t')]\phi_{\mathbf{q}}(t') = \frac{\phi_{\mathbf{q}}(t)}{E_{\mathbf{p}}+E_{\mathbf{p-q}}} - \frac{\phi_{\mathbf{q}}(0)}{E_{\mathbf{p}}+E_{\mathbf{p-q}}}\cos[(E_{\mathbf{p}}+E_{\mathbf{p-q}})t] - \frac{\dot{\phi}_{\mathbf{q}}(0)}{(E_{\mathbf{p}}+E_{\mathbf{p-q}})^2}\sin[(E_{\mathbf{p}}+E_{\mathbf{p-q}})t]$$
$$-\frac{\ddot{\phi}q(t)}{(E_{\mathbf{p}}+E_{\mathbf{p-q}})^3} + \frac{\ddot{\phi}_{\mathbf{q}}(0)}{(E_{\mathbf{p}}+E_{\mathbf{p-q}})^3}\cos[(E_{\mathbf{p}}+E_{\mathbf{p-q}})t]$$
$$+\frac{1}{(E_{\mathbf{p}}+E_{\mathbf{p-q}})^3}\int_0^t dt' \cos[(E_{\mathbf{p}}+E_{\mathbf{p-q}})(t-t')]\dddot{\phi}_{\mathbf{q}}(t').$$

The parts containing $\phi_{\mathbf{q}}(0)$ and its derivatives lead to initial time singularities in the equation of motion, these can be isolated by writing

$$\left[\int_0^t dt' \, \Sigma_{\mathbf{q}}(t-t')\phi_{\mathbf{q}}(t')\right]_{\text{sing}} = -8g^2\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}2E_{\mathbf{p-q}}}[E_{\mathbf{p}}E_{\mathbf{p-q}}+\mathbf{p}(\mathbf{p}-\mathbf{q})-m^2][\tau(\mathbf{q},\mathbf{p})e^{-i(E_{\mathbf{p}}+E_{\mathbf{p-q}})t}$$
$$+\tau^*(\mathbf{q},\mathbf{p})e^{i(E_{\mathbf{p}}+E_{\mathbf{p-q}})t}]$$

with

$$\tau(\mathbf{q},\mathbf{p}) = \frac{1}{2}\left[-\frac{\phi_{\mathbf{q}}(0)}{E_{\mathbf{p}}+E_{\mathbf{p-q}}} + i\frac{\dot{\phi}_{\mathbf{q}}(0)}{(E_{\mathbf{p}}+E_{\mathbf{p-q}})^2} + \frac{\ddot{\phi}_{\mathbf{q}}(0)}{(E_{\mathbf{p}}+E_{\mathbf{p-q}})^3}\right];$$

note that $\phi_{\mathbf{q}}(0) = \phi^*_{-\mathbf{q}}(0)$. With the choice

$$\gamma(\mathbf{q},\mathbf{p}) = -4g[E_{\mathbf{p}}E_{\mathbf{p-q}}+\mathbf{p}(\mathbf{p}-\mathbf{q})-m^2]\tau^*(\mathbf{q},\mathbf{p})$$

the current $J_{b,\mathbf{q}}(t)$ exactly cancels the initial time singularities in the nonlocal term with the self-energy. As in the homogeneous case, the current $J_{b,\mathbf{q}}(t)$, which is a Fourier transform of $\langle\psi(\mathbf{x})\psi(\mathbf{x}')\rangle$ is nonvanishing.

The ultraviolet divergent contributions of the self-energy to the equation of motion are given by

$$\left[\int_0^t dt' \, \Sigma_{\mathbf{q}}(t-t')\phi_{\mathbf{q}}(t')\right]_{UV\,\text{div}} = -8g^2\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}2E_{\mathbf{p-q}}}[E_{\mathbf{p}}E_{\mathbf{p-q}}+\mathbf{p}(\mathbf{p}-\mathbf{q})-m^2]\left[\frac{\phi_{\mathbf{q}}(t)}{E_{\mathbf{p}}+E_{\mathbf{p-q}}} - \frac{\ddot{\phi}_{\mathbf{q}}(t)}{(E_{\mathbf{p}}+E_{\mathbf{p-q}})^3}\right]$$
$$= \Sigma_1(\mathbf{q}^2)\phi_{\mathbf{q}}(t) + \tilde{\Sigma}_3(\mathbf{q}^2)\ddot{\phi}_{\mathbf{q}}(t).$$

Here we define the UV divergent parts of the self energy kernel in dimensional regularization as

$$\tilde{\Sigma}_1(\mathbf{q}^2) = -8g^2\int \frac{d^{3-\epsilon}p}{(2\pi)^{3-\epsilon}}\frac{E_{\mathbf{p}}E_{\mathbf{p-q}}+\mathbf{p}(\mathbf{p}-\mathbf{q})-m^2}{4E_{\mathbf{p}}E_{\mathbf{p-q}}(E_{\mathbf{p}}+E_{\mathbf{p-q}})},$$

$$\tilde{\Sigma}_3(\mathbf{q}^2) = 8g^2\int \frac{d^{3-\epsilon}p}{(2\pi)^{3-\epsilon}}\frac{E_{\mathbf{p}}E_{\mathbf{p-q}}+\mathbf{p}(\mathbf{p}-\mathbf{q})-m^2}{4E_{\mathbf{p}}E_{\mathbf{p-q}}(E_{\mathbf{p}}+E_{\mathbf{p-q}})^3}.$$

These expressions have to be regularized to obtain the renormalized equation of motion. This is discussed in detail in Appendix C. The singular and ultraviolet divergent parts are cancelled by the appropriate choice of the mass and wave function renormalization and the Bogoliubov coefficient in the tadpole. The final form of the subtracted self-energy kernel is given by

$$\Sigma_{s,\mathbf{q}}(t-t') = -8g^2\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}2E_{\mathbf{p-q}}}\frac{E_{\mathbf{p}}E_{\mathbf{p-q}}+\mathbf{p}(\mathbf{p}-\mathbf{q})-m^2}{(E_{\mathbf{p}}+E_{\mathbf{p-q}})^3}\cos[(E_{\mathbf{p}}+E_{\mathbf{p-q}})(t-t')].$$

It follows from rotation invariance that the kernel $\Sigma_{s,\mathbf{q}}(t)$ only depends on $\mathbf{q}^2$.

The equation of motion in momentum space becomes

$$[1+\delta Z+\tilde{\Sigma}_3(\mathbf{q}^2)]\ddot{\phi}_{\mathbf{q}}(t)+[q^2(1+\delta Z)+M^2+\delta M^2+\tilde{\Sigma}_1(\mathbf{q}^2)]\phi_{\mathbf{q}}(t)+\int_0^t dt'\,\tilde{\Sigma}_{s,\mathbf{q}}(t-t')\,\ddot{\phi}_{\mathbf{q}}(t')=0.$$

We show in Appendix C that we can decompose $\tilde{\Sigma}_1(\mathbf{q}^2)$ and $\tilde{\Sigma}_3(\mathbf{q}^2)$ as

$$\tilde{\Sigma}_1(\mathbf{q}^2)=-\delta M^2-\mathbf{q}^2\,\delta Z+\Delta\tilde{\Sigma}_1(\mathbf{q}^2),\quad \tilde{\Sigma}_3(\mathbf{q}^2)=-\delta Z+\Delta\tilde{\Sigma}^3(\mathbf{q}^2).$$

The divergent and finite parts are explicitly given in Appendix C. We finally obtain the renormalized equation of motion which is free from ultraviolet and initial time singularities

$$[1+\Delta\tilde{\Sigma}_3(\mathbf{q}^2)]\ddot{\phi}_{\mathbf{q}}(t)+[\mathbf{q}^2+M^2+\Delta\tilde{\Sigma}_1(\mathbf{q}^2)]\phi_{\mathbf{q}}(t)\int_0^t dt'\,\tilde{\Sigma}_{s,\mathbf{q}}(t-t')\,\ddot{\phi}_{\mathbf{q}}(t')=0, \tag{6.2}$$

where

$$\tilde{\Sigma}_{s,\mathbf{q}}(t-t')=-8g^2\int\frac{d^3p}{(2\pi)^3 4E_{\mathbf{p}}E_{\mathbf{p-q}}}\,\frac{E_{\mathbf{p}}E_{\mathbf{p-q}}+\mathbf{p}(\mathbf{p-q})-m^2}{(E_{\mathbf{p}}+E_{\mathbf{p-q}})^3}\cos[(E_p+E_{p'})\tau]. \tag{6.3}$$

VII. SOLUTION OF THE EQUATION OF MOTION: NUMERICAL ANALYSIS

We have derived in the previous section the renormalized equation of motion. It can be solved in a standard way via Laplace transform. We introduce

$$\tilde{\psi}_{\mathbf{q}}(s)=\int_0^\infty dt\,e^{-st}\phi_{\mathbf{q}}(t)$$

for the condensate, and

$$\tilde{\sigma}_s(s^2,q^2)=s\int_0^\infty dt\,e^{-st}\tilde{\Sigma}_{s,q}(t)$$

for the self-energy [26]. We find

$$\tilde{\sigma}_s(s^2,\mathbf{q}^2)=-8g^2 s\int_0^\infty d\tau\,e^{-s\tau}\int\frac{d^3p}{(2\pi)^3 4E_{\mathbf{p}}E_{\mathbf{p-q}}}\,\frac{E_{\mathbf{p}}E_{\mathbf{p-q}}+\mathbf{p}(\mathbf{p-q})-m^2}{(E_{\mathbf{p}}+E_{\mathbf{p-q}})^3}\cos[(E_{\mathbf{p}}+E_{\mathbf{p-q}})\tau]$$
$$=-8g^2s^2\int\frac{d^3p}{(2\pi)^3 4E_{\mathbf{p}}E_{\mathbf{p-q}}}\,\frac{E_{\mathbf{p}}E_{\mathbf{p-q}}+\mathbf{p}(\mathbf{p-q})-m_2}{(E_{\mathbf{p}}+E_{\mathbf{p-q}})^3[(E_{\mathbf{p}}+E_{\mathbf{p-q}})^2+s^2]}. \tag{7.1}$$

The Laplace transformed renormalized equation of motion becomes

$$\{s^2[1+\Delta\tilde{\Sigma}_3(\mathbf{q}^2)+\tilde{\sigma}_s(s^2,\mathbf{q}^2)]+\mathbf{q}^2+M^2+\Delta\tilde{\Sigma}_1(\mathbf{q}^2)\}\tilde{\psi}_{\mathbf{q}}(s)$$
$$=[\dot{\phi}_{\mathbf{q}}(0)+s\,\phi_{\mathbf{q}}(0)][1+\Delta\tilde{\Sigma}_3(\mathbf{q}^2)+\tilde{\sigma}_s(s^2,\mathbf{q}^2)]+\frac{\ddot{\phi}_{\mathbf{q}}(0)}{s}\tilde{\sigma}_s(s^2,\mathbf{q}^2),$$

so that

$$\tilde{\psi}_{\mathbf{q}}(s)=\frac{[\dot{\phi}_{\mathbf{q}}(0)+s\,\phi_{\mathbf{q}}(0)][1+\Delta\tilde{\Sigma}_3(\mathbf{q}^2)+\tilde{\sigma}_s(s^2,\mathbf{q})]+\ddot{\tilde{\phi}}(0,\mathbf{q})\,\tilde{\sigma}_s(s^2,\mathbf{q}^2)/s}{s^2[1+\Delta\tilde{\Sigma}_3(\mathbf{q}^2)+\tilde{\sigma}_s(s^2,\mathbf{q}^2)]+\mathbf{q}^2+M^2+\Delta\tilde{\Sigma}_1(\mathbf{q}^2)}.$$

The solution $\phi_{\mathbf{q}}(t)$ then is obtained by the inverse transformation

$$\phi_{\mathbf{q}}(t)=\int_{-i\infty+c}^{i\infty+c}\frac{ds}{2\pi i}\,e^{st}\tilde{\psi}_{\mathbf{q}}(s).$$

This solution is discussed in detail in Appendix D. The integral above is along the Bromwich contour with c a positive real constant to the right of all the singularities of the Laplace transform. The result can be written as (see Appendix D for details)

$$\phi_{\mathbf{q}}(t)=\frac{2}{\pi}\int_{0+}^{\infty}d\omega\Bigg[\cos(\omega t)\phi_{\mathbf{q}}(0)\ \omega\,\mathrm{Im}\,F_1(-\omega^2+io,\mathbf{q}^2)+\sin(\omega t)\dot{\phi}_{\mathbf{q}}(0)\mathrm{Im}\,F_1(-\omega^2+io,\mathbf{q}^2)$$
$$-\frac{\ddot{\phi}_{\mathbf{q}}(0)}{\omega}\cos(\omega t)\mathrm{Im}\,F_2(-\omega^2+io,\mathbf{q}^2)\Bigg],$$

where

$$F_1(s^2,\mathbf{q}^2)=\frac{1+\Delta\tilde{\tilde{\Sigma}}_3(\mathbf{q}^2)+\tilde{\sigma}_s(s^2,\mathbf{q}^2)}{s^2[1+\Delta\tilde{\tilde{\Sigma}}_3(\mathbf{q}^2)+\tilde{\sigma}_s(s^2,\mathbf{q}^2)]+\mathbf{q}^2+M^2+\Delta\tilde{\tilde{\Sigma}}_1(s^2,\mathbf{q}^2)},$$

$$F_2(s^2,\mathbf{q}^2)=\frac{\tilde{\sigma}_s(s^2,\mathbf{q}^2)}{s^2[1+\Delta\tilde{\tilde{\Sigma}}_3(\mathbf{q}^2)+\tilde{\sigma}_s(s^2,\mathbf{q}^2)]+\mathbf{q}^2+M^2+\Delta\tilde{\tilde{\Sigma}}_1(\mathbf{q}^2)}.$$

$\ddot{\phi}_{\mathbf{q}}(0)$ is not an independent initial value, it is determined by setting $t=0$ in the equation of motion (6.2) with the result

$$\ddot{\phi}_{\mathbf{q}}(0)=-\phi_{\mathbf{q}}(0)\frac{\mathbf{q}^2+M^2+\Delta\tilde{\tilde{\Sigma}}_1(\mathbf{q}^2)}{1+\Delta\tilde{\tilde{\Sigma}}_3(\mathbf{q})^2}.$$

It is then convenient to define a kernel F_3 which combines F_1 and F_2 with prefactors uniquely determined by $\phi_{\mathbf{q}}(0)$. With the definition

$$F_3(\omega,\mathbf{q}^2)=-\frac{1}{\omega}+\frac{\mathbf{q}^2+M^2+\Delta\Sigma_1}{(1+\Delta\Sigma_3)\omega}\times\frac{1}{-\omega^2+\dfrac{\mathbf{q}^2+M^2+\Delta\Sigma_1}{1+\Delta\Sigma_3+\tilde{\sigma}_s(-\omega^2+io,\mathbf{q}^2)}}$$

the solution can be written as

$$\phi_{\mathbf{q}}(t)=\frac{2}{\pi}\int_{\omega_c}^{\infty}d\omega[\phi_{\mathbf{q}}(0)\cos(\omega t)\mathrm{Im}\,F_3(\omega-io,\mathbf{q}^2)+\dot{\phi}_{\mathbf{q}}(0)\sin(\omega t)\mathrm{Im}\,F_1(-\omega^2+io,\mathbf{q}^2)].$$

The function F_3 which is, up to prefactors, the Laplace transform of the solution, exhibits a pole at

$$\omega_R=\omega_{\mathbf{q}}+\delta\omega,\quad \omega_{\mathbf{q}}=\sqrt{M^2+\mathbf{q}^2},$$

with $\delta\omega=\mathcal{O}(g^2)$. If $M>2m$ the scalar field can decay into a fermion-antifermion pair, the pole actually describes a resonance. In perturbation theory the width of this resonance is perturbatively small and near the resonance we can approximate the function F_3 by a Breit-Wigner resonance

$$F_3(\omega,\mathbf{q}^2)\simeq\frac{1}{2}\frac{Z_R}{\omega-\omega_R-i\Gamma_R}.$$

The resonance position is determined by

$$\omega_R=\mathrm{Re}\frac{\omega_{\mathbf{q}}^2+\Delta\Sigma_1}{1+\Delta\Sigma_3+\tilde{\sigma}_s(-\omega_{\mathbf{q}}^2+io,\mathbf{q}^2)},$$

the residue Z_R is given by

$$Z_R=\frac{\omega_{\mathbf{q}}^2+\Delta\Sigma_1}{(1+\Delta\Sigma_3)\omega_{\mathbf{q}}}\frac{1}{-2\omega_{\mathbf{q}}+\dfrac{d}{d\omega}\left[\mathrm{Re}\dfrac{\omega_{\mathbf{q}}^2+\Delta\Sigma_1}{1+\Delta\Sigma_3+\tilde{\sigma}_s(-\omega^2+io,\mathbf{q}^2)}\right]_{\omega=\omega_{\mathbf{q}}}},$$

and the width by

$$\Gamma_R = \frac{1}{2\omega_R}\,\mathrm{Im}\left[\frac{\omega_{\mathbf{q}}^2 + \Delta\Sigma_1}{1 + \Delta\Sigma_3 + \tilde{\sigma}_s(-\omega^2 + io, \mathbf{q}^2)}\right]_{\omega=\omega_{\mathbf{q}}}.$$

Numerical analysis

We are now in condition to study the evolution of an initial scalar condensate numerically by performing the inverse Laplace and Fourier transforms since all the quantities are given by the subtracted one-loop self-energy.

We consider two separate cases: $M>2m$ in which case the scalar can decay into fermion-antifermion pairs, and $M<2m$ in which case the scalar particle is stable. In both cases we studied the evolution for an initial spherical wave of Gaussian profile

$$\phi(0,\mathbf{x}) = N_0 \exp(-\mathbf{x}^2/2R_0^2),$$
$$\dot{\phi}(0,\mathbf{x}) = 0, \quad \text{with} \int d^3x\, \phi(0,\mathbf{x}) = 1. \tag{7.2}$$

Since this Gaussian wave packet has zero center of mass momentum, the peak of the wave packet will not displace under time evolution, but the wave packet will disperse and spread out in space-time.

$M>2m$. We have chosen $M=3m$ but there is a rather smooth variation of the wave function renormalization constant, position and width of the resonance for reasonable values of the ratio $2<M/m\leqslant 10$. As a first step we calculated the value of the wave function renormalization numerically and find that Z_R differs from unity by less than 3% and that the ratio of the width to the position of the pole $\Gamma/\omega_R \approx 0.02$ for $g=1$ or smaller and $2<M/m\leqslant 10$. Therefore if the coupling is of this order or smaller, the approximation of the spectral density (discontinuity across the cut) by the imaginary part of the Breit-Wigner form given above is excellent. The agreement obtained from the full numerical evolution and that obtained from the approximate Breit-Wigner form is excellent. The evolution in this case is depicted in Fig. 1 that displays the profile of the condensate as a function of t and $|\vec{x}|$ for $M=3; m=1; g=1$. We see that the propagation is inside the light cone and damped in time. The peak at the origin is the center of mass of the wave packet, it oscillates in time with a frequency $\approx\omega_R$ and decays on a time scale Γ^{-1}. The evolution at very early times is smooth.

$M<2m$. In this case the scalar field is stable, the spectral density features poles at $\omega=\pm\omega_R$ below the fermion-antifermion cut. We find again that $Z_P\approx 0.97$, implying, that the contribution from the fermion-antifermion continuum is negligible except for small t. Figure 2 displays the time evolution of the Gaussian wave packet in this case. The wave packet spreads in space-time, the evolution is always below the light cone as clearly illustrated in the figure. The amplitude of the wave packet decreases as a result of spreading. Its spatial integral, which is equal to $\phi_{\mathbf{0}}(t)$, asymptotically oscillates with the pole frequency and amplitude Z_P.

VIII. SCALAR THEORIES

The connection between the preparation of the initial state via Bogoliubov transformations and the formulation in terms of linear response allows to generalize the study presented above to scalar theories. In particular let us consider the case of a scalar self-coupled $\lambda\Phi^4$ model using the linear response analysis. Focusing on the evolution equation for a homogeneous condensate to order λ^2 we find

$$(1+\delta Z)\,\ddot{\phi}_{\mathbf{0}}(t) + (M^2+\delta M^2)\,\phi_{\mathbf{0}}(t) + \Sigma_{1,\mathbf{0}}(0)\,\phi_{\mathbf{0}}(t) + \Sigma_{3,\mathbf{0}}(0)\,\ddot{\phi}_{\mathbf{0}}(t) + \int_{-\infty}^{t} dt'\,\Sigma_{s,\mathbf{0}}(t-t')\,\ddot{\ddot{\phi}}_{\mathbf{0}}(t') = J_{\mathrm{ext},\mathbf{0}}(t). \tag{8.1}$$

The order λ tadpole has been absorbed into δM^2 as a mass renormalization and the self-energy to order λ^2 is given by

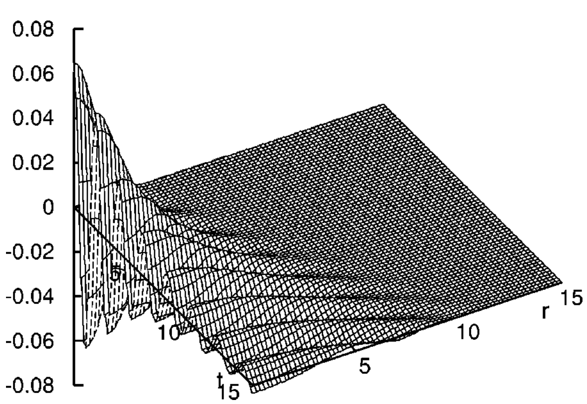

FIG. 1. Unstable case: $M=3$; $m=1$; $g=1$ with a Gaussian profile for the condensate at $t=0$, given by Eq. (7.2) with $R_0=1$ and normalized so that $\int d^3x\, \phi(0,\mathbf{x})=1$.

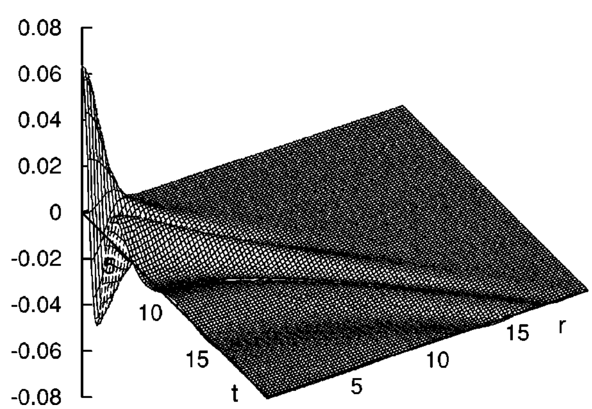

FIG. 2. Stable case: $M=1$; $m=1$; $g=1$ with a Gaussian profile for the condensate at $t=0$, given by Eq. (7.2) with $R_0=1$ and normalized so that $\int d^3x\, \phi(0,\mathbf{x})=1$.

the sunset diagram. The choice of counterterms $\delta Z, \delta M^2$ are as given in Eqs. (5.2), (5.3) and the external source introduced for linear response is specifically chosen as in Eq. (5.4). In this manner we obtain the initial value problem free of ultraviolet and initial time singularities. An alternative interpretation of the external source associated with the linear response formulation can be provided by introducing a Hartree factorization in the Lagrangian with a term of the form $4\lambda\Phi\langle\Phi^3\rangle$. This term acts now as an explicit source in the linearized equation of motion which is chosen so as to lead to a singularity free initial value problem, thus requiring a nontrivial Bogoliubov vacuum. The linear response analysis leads very simply to a well defined initial value problem for a proper choice of the external source. The equivalence between the preparation of the state via a Bogoliubov transformation of the free field Fock vacuum (or density matrix) and the initial value problem obtained from linear response for the proper choice of external source allows now to generalize the results obtained above for the linearized approximation.

IX. CONCLUSIONS

The nonequilibrium evolution of condensates in real time requires to provide Cauchy data at some initial time, hence an initial value problem which requires the specification of an initially prepared quantum state or density matrix. An initial pure or mixed state of free field Fock quanta leads to initial time singularities. These have a simple interpretation: the evolution of expectation values or matrix elements in the interacting theory implies that the interaction is switched-on suddenly. The interaction rearranges the spectral densities of the fields and the response to the sudden switching-on of the interaction results in initial time singularities which are enhanced in a renormalizable theory. For systems with a finite number of degrees of freedom such effects are also present but no singularities arise.

The Bogoliubov transformation (4.1) mix creation and annihilation operators. It mixes the fermion creation operator with the anti-fermion creation operator and similarly for the annihilation operators. The scalar products between transformed and untransformed states vanish in the infinite volume limit. As a consequence of that the transformed and untransformed states belong to unitarily inequivalent representations of the CCR. The physical space of states is the transformed one and the physical quantity must be computed there. Unlike the case of spontaneous breaking of chiral symmetry [5], the Bogoliubov transformation does not change the physical properties of the system like the symmetries or any other basic property. It just describes the reaction of the quantum modes of the vacuum to the external field, and leads, for the localized states considered later, to a vacuum polarization induced by the localized classical field.

In summary, the time evolution of arbitrary quantum states or density matrices in (interacting) field theory leads to short time divergences. Only for appropriately prepared pure or mixed initial states, as those considered in this paper, the time evolution is well defined. By appropriately prepared we mean states where the filling for high energy quanta follows a precise law determined by the initial data, couplings, and masses.

We have chosen to study these initial time singularities and provide a consistent resolution in a Yukawa theory in $3+1$ dimensions, this theory being renormalizable allows to identify all of the divergences and singularities: ultraviolet divergences associated with mass, coupling, and wave-function renormalizations and initial time singularities that cannot be cancelled by the usual counterterms.

After recognizing the initial time singularities and their physical significance in the case of homogeneous condensates, we have proposed a rather simple approach to provide a singularity free initial value problem. We introduced Bogoliubov transformed initial states that incorporate the effects of dressing of states by the interaction. The Bogoliubov coefficients can be obtained in a systematic series expansion in the Yukawa coupling and we have obtained them to one-loop order in this theory. The usual renormalization counterterms cancel the ultraviolet divergences associated with mass, coupling, and wave function, and the Bogoliubov coefficients are chosen consistently to cancel the initial time singularities. That is, their high energy behavior is fixed according to the initial data (Sec. IV).

We have established contact with linear response theory by obtaining the evolution equations for the scalar condensate and the initial value problem in the linearized approximation as the linear response to an external source coupled to the scalar condensate. This equivalent formulation clarifies at once the relationship between the linearized approximation for the evolution equations of the condensate and linear response. The corresponding initial value problem, i.e., providing Cauchy data for the field and its first derivative on a spatial hypersurface requires that the external source that couples to the scalar field *does not vanish after the initial time*. A very specific source term that depends on a given past history of the condensate furnished a singularity free initial value problem.

After presenting the method in the simpler homogeneous case and establishing the relationship to linear response theory we focused on the important case of inhomogeneous condensates. Following on the steps for the homogeneous case we have constructed the proper Bogoliubov transformation to lowest order in the Yukawa coupling and shown explicitly how a judicious choice of the Bogoliubov coefficients in combination with the usual renormalization counterterms leads to an initial value problem free of ultraviolet and initial time singularities. As an example of this consistent procedure we have provided a numerical study of the space-time evolution of an inhomogeneous scalar condensate both in the case in which the scalar can decay into fermion-antifermion pairs and in the case in which the scalar is light and stable.

We have also generalized the results of the fermionic case to scalar field theories by exploiting the relation to linear response, thus providing a generalized and consistent manner of describing nonequilibrium evolution of condensates in terms of an initial value problem free of ultraviolet divergences and initial time singularities.

Applications. We foresee several applications of these methods. (i) We can now study consistently the evolution of inhomogeneous pion condensates after a chiral phase transition by setting up a physically reasonable initial value problem that incorporates the important features of the transition in the *fully interacting* initial state (or density matrix). This approach is complementary to that advocated in Ref. [12]. (ii) In cosmology we can now study the nonequilibrium dynamics of inhomogeneous configurations by providing the initial field profile and the first derivative on a spacelike hypersurface and following the space-time evolution of this configuration. In particular a very relevant setting for cosmology is that of supersymmetric theories during for example the stages of rolling of the scalar field component. Treating the dynamics as an initial value problem, the initial conditions on the scalar field, displaced from the equilibrium position breaks supersymmetry. This breakdown of supersymmetry is not explicit at the level of the Lagrangian, but by the quantum state. Our formulation allows us to follow the dynamics consistently and study the consequences of this supersymmetry breaking. Work on these issues is in progress.

The next step in our program is to extend the results obtained in this article, valid in the linearized approximation, to a full nonlinear inhomogeneous problem. We expect to report on progress on these and other issues in the near future.

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APPENDIX A: FERMIONIC GREEN'S FUNCTIONS IN THE BOGOLIUBOV STATE FOR THE HOMOGENEOUS CASE

The Green function $S_b^>(t,\mathbf{x};t',\mathbf{x}')$ in the Bogoliubov-transformed state $|0_b\rangle$ is defined via

$$iS_b^>(t,\mathbf{x};t',\mathbf{x}')=\langle 0_b|\psi(t,\mathbf{x})\overline{\psi}(t',\mathbf{x}')|0_b\rangle \tag{A1}$$

and the (free) field operators are given, in terms of the creation and annihilation operators, by

$$\psi(t,\mathbf{x})=\sum_s \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i\mathbf{px}}[b(\mathbf{p},s)U(\mathbf{p},s)e^{-iE_pt}+d^\dagger(-\mathbf{p},s)V(-\mathbf{p},s)e^{iE_pt}],$$

$$\overline{\psi}(t',\mathbf{x}')=\sum_{s'} \int \frac{d^3p'}{(2\pi)^3 2E_{p'}} e^{-i\mathbf{p}'\mathbf{x}'}[b^\dagger(\mathbf{p}',s')\overline{U}(\mathbf{p}',s')e^{-iE_{p'}t'}+d(-\mathbf{p}',s')\overline{V}(-\mathbf{p}',s')e^{iE_{p'}t'}]. \tag{A2}$$

We use the normalization for the spinors $\overline{U}(\mathbf{p},s)U(\mathbf{p},s)=2m$ so that $U^\dagger(\mathbf{p},s)U(\mathbf{p},s)=1$.

Using Eqs. (4.1), (4.2), (A1), and (A2) we obtain the following expression for the transformed Green function $S_b^>(t,\mathbf{x};t',\mathbf{x}')$:

$$\begin{aligned} iS_b^>(t,\mathbf{x};t',\mathbf{x}')&=\langle 0_b|\psi(t,\mathbf{x})\overline{\psi}(t',\mathbf{x}')|0_b\rangle \\ &=\sum_s \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')}[\cos^2(\beta_{ps})U(\mathbf{p},s)\overline{U}(\mathbf{p},s)e^{-iE_p(t-t')}-\sin\beta_{ps}\cos\beta_{ps}e^{i\delta_{ps}}U(\mathbf{p},s)\overline{V} \\ &\quad\times(-\mathbf{p},s)e^{-iE_p(t+t')}-\sin\beta_{ps}\cos\beta_{ps}e^{-i\delta_{ps}}V(-\mathbf{p},s)\overline{U}(\mathbf{p},s)e^{iE_p(t+t')}+\sin^2\beta_{ps}V(-\mathbf{p},s)\overline{V}(-\mathbf{p},s)e^{iE_p(t-t')}]. \end{aligned}$$

As long as we consider homogeneous condensates, the angles β_{ps} and δ_{ps} can be chosen to depend only on the modulus $|\mathbf{p}|$, and on the helicities. The weight of the two possible helicities is still arbitrary and we consider these angles to be functions of the helicity matrix

$$\mathbf{\Sigma}\hat{\mathbf{p}}=\begin{pmatrix} \boldsymbol{\sigma}\hat{\mathbf{p}} & 0 \\ 0 & \boldsymbol{\sigma}\hat{\mathbf{p}} \end{pmatrix},$$

where $\hat{\mathbf{p}}=\mathbf{p}/|\mathbf{p}|$. We have, e.g.,

$$\begin{aligned} f(\mathbf{\Sigma}\hat{\mathbf{p}})\sum_s U(\mathbf{p},s)\overline{V}(-\mathbf{p},s)&=\sum_s U(\mathbf{p},s)\overline{V}(-\mathbf{p},s)f(\mathbf{\Sigma}\hat{\mathbf{p}}) \\ &=\sum_s f(s)U(\mathbf{p},s)\overline{V}(-\mathbf{p},s). \end{aligned}$$

$\mathbf{\Sigma}\hat{\mathbf{p}}$ commutes with all other matrices that are relevant to this discussion, γ_0, γ_5, and $\mathbf{p}\boldsymbol{\gamma}$, and can therefore be treated as a c number.

Specifying the vector p^μ to be on shell, $p^\mu=(E_p,\mathbf{p})$, and using $\gamma_0\boldsymbol{\gamma}\gamma_0=-\boldsymbol{\gamma}$ it is straightforward to find

$$\sum_s U(\mathbf{p},s)\overline{U}(\mathbf{p},s)=\not{p}+m,$$

$$\sum_s V(-\mathbf{p},s)\overline{V}(-\mathbf{p},s)=\gamma_0(\not{p}-m)\gamma_0$$
$$=\gamma_5\gamma_0(\not{p}+m)\gamma_5\gamma_0.$$

The mixed product can be found by resorting to the representation

$$U(\mathbf{p},s)=\frac{\not{p}+m}{\sqrt{E_p+m}}\begin{pmatrix}\chi_s\\0\end{pmatrix},$$

$$V(-\mathbf{p},s)=-\frac{\gamma_0(\not{p}-m)\gamma_0}{\sqrt{E_p+m}}\begin{pmatrix}0\\\chi_s\end{pmatrix},$$

where χ_s is an eigenspinor of $\boldsymbol{\sigma}\hat{\mathbf{p}}$ with eigenvalue s from which we find

$$\sum_s V(-\mathbf{p},s)\overline{U}(\mathbf{p},s)=\gamma_5\gamma_0(\not{p}+m),$$

$$\sum_s U(\mathbf{p},s)\overline{V}(-\mathbf{p},s)=(\not{p}+m)\gamma_5\gamma_0.$$

We note that the phase δ_{ps} appears in a combination such that a shift in this phase can be compensated by a shift in the origin of time, i.e., a time translation $t\to t+t_o\delta_{ps}\to\delta_{ps}+2E_pt_0$. Since the square of the helicity matrix is the identity and the only odd function of β_{ps} multiplies the mixed terms, we found that the simplest Bogoliubov transformation that is required to cancel the initial time singularities is such that the phase δ_{ps} is independent of s and that $\beta_{ps}=\boldsymbol{\Sigma}\hat{\mathbf{p}}\beta_p$ so that

$$\cos\beta_{ps}=\cos\beta_p,$$

$$\sin\beta_{ps}=\boldsymbol{\Sigma}\hat{\mathbf{p}}\sin\beta_p.$$

Such a choice has proven to be appropriate for removing the initial singularity for the full one-loop equations.

After some straightforward algebra, we find

$$\begin{aligned}iS_b^>(t,\mathbf{x};t',\mathbf{x}')&=\langle 0_b|\psi(t,\mathbf{x})\overline{\psi}(t',\mathbf{x}')|0_b\rangle\\&=\int\frac{d^3p}{(2\pi)^3 2E_p}e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')}[\cos\beta_p e^{iE_pt}\\&\quad-\sin\beta_p e^{-i\delta_p}\boldsymbol{\Sigma}\hat{\mathbf{p}}\gamma_5\gamma_0 e^{iE_pt}](\not{p}+m)\\&\quad\times[\cos\beta_p e^{iE_pt'}\\&\quad-\sin\beta_p e^{i\delta_p}\gamma_5\gamma_0\boldsymbol{\Sigma}\hat{\mathbf{p}}e^{-iE_pt'}],\end{aligned}\tag{A3}$$

Upon reordering of the terms we find the Green's function quoted in Eq. (4.3).

A similar calculation leads to the transformed Green function $S_b^<(t,\mathbf{x};t',\mathbf{x}')$ which is defined as

$$-iS_b^<(t,\mathbf{x};t',\mathbf{x}')=\langle 0_b|\overline{\psi}(t',\mathbf{x}')\psi(t,\mathbf{x})|0_b\rangle.$$

Following the same steps leading to Eq. (A3) we find

$$\begin{aligned}-iS_b^<(t,\mathbf{x};t',\mathbf{x}')&=\langle 0_b|\overline{\psi}(t',\mathbf{x}')\psi(t,\mathbf{x})|0_b\rangle\\&=\int\frac{d^3p}{(2\pi)^3 2E_p}e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')}\\&\quad\times[\sin\beta_p e^{i\delta_{ps}}e^{-iE_pt}\boldsymbol{\Sigma}\hat{\mathbf{p}}\\&\quad+\cos\beta_p\gamma_5\gamma_0 e^{iE_pt}](\not{p}+m)\\&\quad\times[\sin\beta_p e^{-i\delta_p}e^{iE_pt'}\boldsymbol{\Sigma}\hat{\mathbf{p}}\\&\quad+\cos\beta_p\gamma_5\gamma_0 e^{-iE_pt'}],\end{aligned}$$

which upon reordering of terms gives the form quoted in expression (4.4).

APPENDIX B: BOGOLIUBOV TRANSFORMATION AND TADPOLE DIAGRAM FOR INHOMOGENEOUS SYSTEMS

In this appendix we generalize the Bogoliubov transformations described in the homogeneous case to the case of inhomogeneous condensates. Unlike the homogeneous case in which the generator of the Bogoliubov transformation creates particle-antiparticle pairs of zero total momentum, in the inhomogeneous case the total momentum of the pair is nonzero.

Consistent with perturbation theory we now find the corresponding Bogoliubov transformation to lowest order in the Yukawa coupling, thus $\cos\beta_{ps}\approx 1$, $\sin\beta_{ps}\approx\beta_{ps}=\mathcal{O}(g)$. Since the self-energy is already of $\mathcal{O}(g^2)$, to lowest order we only need to focus on the tadpole term $J_b(\mathbf{x},t)$.

The Bogoliubov transformation in lowest order reads

$$\begin{aligned}b(\mathbf{p},s)&=b_b(\mathbf{p},s)+\int\frac{d^3p'}{(2\pi)^3 2E_{p'}}\rho_{ss'}(\mathbf{p},\mathbf{p}')d_b^\dagger\\&\quad\times(-\mathbf{p}',s'),\end{aligned}$$

$$\begin{aligned}d^\dagger(-\mathbf{p},s)&=-\int\frac{d^3p'}{(2\pi)^3 2E_{p'}}\rho^*_{s's}(\mathbf{p}',\mathbf{p})b_b(\mathbf{p}',\mathbf{s}')\\&\quad+d_b^\dagger(-\mathbf{p},s),\end{aligned}$$

with $\rho_{ss'}=\mathcal{O}(g)$. With this choice the transformation leaves the canonical anticommutation relations unchanged up to terms of order $\rho^2_{ss'}$. In order to compute the transformed Green functions we need the expectation values of bilinear combinations of creation and annihilation operator. We find the following expectation values that are necessary to compute the Green's functions:

$$\langle 0_b|b(\mathbf{p},s)d(-\mathbf{p}',s')|0_b\rangle = \int \frac{d^3p''}{(2\pi)^3 2E_{p''}}[-\rho_{s''s'}(\mathbf{p}'',\mathbf{p}')]\langle 0_b|b_b(\mathbf{p},s)b_b^\dagger(\mathbf{p}'',s'')|0_b\rangle\delta_{ss'}(2\pi)^3 2E_p\delta^3(\mathbf{p}-\mathbf{p}'')$$
$$= -\rho_{ss'}(\mathbf{p},\mathbf{p}'),$$

$$\langle 0_b|b(\mathbf{p},s)b^\dagger(\mathbf{p}',s')|0_b\rangle = (2\pi)^3 2E_{\mathbf{p}}\delta_{ss'}\delta^3(\mathbf{p}-\mathbf{p}'),$$

$$\langle 0_b|d^\dagger(-\mathbf{p},s)d(-\mathbf{p}',s')|0_b\rangle = 0,$$

$$\langle 0_b|d^\dagger(-\mathbf{p},s)b^\dagger(\mathbf{p}',s')|0_b\rangle = -\rho^*_{s's}(\mathbf{p}',\mathbf{p}).$$

$$\langle 0_b|d(\mathbf{p}',s')d^\dagger(\mathbf{p},s)|0_b\rangle = (2\pi)^3 2E_0\delta_{ss'}\delta^3(\mathbf{p}-\mathbf{p}'),$$

$$\langle 0_b|b^\dagger(-\mathbf{p}',s')b(-\mathbf{p},s)|0_b\rangle = 0,$$

$$\langle 0_b|d(-\mathbf{p}',s')b(\mathbf{p},s)|0_b\rangle = \rho_{ss'}(\mathbf{p},\mathbf{p}'),$$

$$\langle 0_b|b^\dagger(\mathbf{p}',s')d^\dagger(-\mathbf{p},s)|0_b\rangle = \rho^*_{s's}(\mathbf{p}',\mathbf{p}).$$

This yields the transformed Green's functions,

$$iS_b^>(t,\mathbf{x};t',\mathbf{x}') = \langle 0_b|\psi(t,\mathbf{x})\bar{\psi}(t',\mathbf{x}')|0_b\rangle$$
$$= \sum_s \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')}U(\mathbf{p},s)\bar{U}(\mathbf{p},s)e^{-iE_p(t-t')} - \sum_{ss'}\int \frac{d^3p}{(2\pi)^3 2E_p}\int \frac{d^3p'}{(2\pi)^3 2E_{p'}}$$
$$\times e^{i(\mathbf{px}-\mathbf{p}'\mathbf{x}')}\{\rho^*_{s's}(\mathbf{p}',\mathbf{p})V(-\mathbf{p},s)\bar{U}(\mathbf{p}',s')e^{-i(E_pt+E_{p'}t')} + \rho_{ss'}(\mathbf{p},\mathbf{p}')U(\mathbf{p},s)\bar{V}(-\mathbf{p}',s')e^{i(E_pt+E_{p'}t')}\}$$

and

$$-iS_b^<(t,\mathbf{x};t',\mathbf{x}') = \langle 0_b|\bar{\psi}(t',\mathbf{x}')\psi(t,\mathbf{x})|0_b\rangle$$
$$= \sum_s \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')}V(-\mathbf{p},s)\bar{V}(-\mathbf{p},s)e^{-iE_p(t-t')} + \sum_{ss'}\int \frac{d^3p}{(2\pi)^3 2E_p}\int \frac{d^3p'}{(2\pi)^3 2E_{p'}}$$
$$\times e^{i(\mathbf{px}-\mathbf{p}'\mathbf{s}')}\{\rho^*_{s's}(\mathbf{p}',\mathbf{p})V(-\mathbf{p},s)\bar{U}(\mathbf{p}',s')e^{-i(E_pt+E_{p'}t')} + \rho_{ss'}(\mathbf{p},\mathbf{p}')U(\mathbf{p},s)\bar{V}(-\mathbf{p}',s')e^{i(E_pt+E_{p'}t')}\},$$

to first order in $\rho_{ss'}(\mathbf{p},\mathbf{p}')$.

Using these results we now can evaluate the tadpole graph in the inhomogeneous condensate. That is, the expectation value of $\langle\bar{\psi}\psi\rangle$ which plays the rôle of an external current in the equation of motion. Inserting the above explicit expressions we find

$$J_b(t,\mathbf{x}) = ig\,\mathrm{tr}\,S_b^>(t,\mathbf{x};t,\mathbf{x})$$
$$= g\sum_s \int \frac{d^3p}{(2\pi)^3 2E_p}\mathrm{tr}[U(\mathbf{p},s)\bar{U}(\mathbf{p},s)] - \sum_{ss'}\int \frac{d^3p}{(2\pi)^3 2E_p}\int \frac{d^3p'}{(2\pi)^3 2E_{p'}}$$
$$\times e^{i(\mathbf{p}-\mathbf{p}')\mathbf{x}}\{\rho^*_{s's}(\mathbf{p}',\mathbf{p})\mathrm{tr}[V(-\mathbf{p},s)\bar{U}(\mathbf{p}',s')]e^{i(E_p+E_{p'})t}\rho_{ss'}(\mathbf{p},\mathbf{p}')\mathrm{tr}[U(\mathbf{p},s)\bar{V}(-\mathbf{p}',s')]e^{-i(E_p+E_{p'})t}\}.$$

The traces over the spinors yield

$$\mathrm{tr}\,V(-\mathbf{p},s)\bar{U}(\mathbf{p}',s') = \bar{U}(\mathbf{p}',s')V(-\mathbf{p},s)$$
$$= -\delta_{ss'}s[\sqrt{(E_p-m)(E_{p'}+m)} + \sqrt{(E_p+m)(E_{p'}-m)}], \tag{B1}$$

$$\mathrm{tr}\, U(\mathbf{p},s)\overline{V}(-\mathbf{p}',s') = \overline{V}(-\mathbf{p}',s')U(\mathbf{p},s)$$

$$= -\delta_{ss'}s[\sqrt{(E_p-m)(E_{p'}+m)} + \sqrt{(E_p+m)(E_{p'}-m)}].$$

We see that the relevant part of $\rho_{ss'}(\mathbf{p},\mathbf{p}')$ contributing to Eq. (B1) is odd in s and diagonal in ss'. We therefore choose

$$\rho_{ss'}(\mathbf{p},\mathbf{p}') = \gamma(\mathbf{p},\mathbf{p}')\frac{s\delta_{ss'}}{\sqrt{(E_p-m)(E_{p'}+m)} + \sqrt{(E_p+m)(E_{p'}-m)}}.$$

Then,

$$J_b(t,\mathbf{x}) = 4mg\int\frac{d^3p}{(2\pi)^3 2E_p} - 2\int\frac{d^3p}{(2\pi)^3 2E_p}\int\frac{d^3p'}{(2\pi)^3 2E_{p'}}e^{i(\mathbf{p}-\mathbf{p}')\mathbf{x}}\{\gamma^*(\mathbf{p}',\mathbf{p})e^{-i(E_p+E_{p'})t} + \gamma(\mathbf{p},\mathbf{p}')e^{i(E_p+E_{p'})t}\}.$$

The first term is again space and time independent and is absorbed into a shift of the condensate. The second term displays space and time dependence in a factored form. It will be used to compensate for the initial time singularities of the self-energy.

APPENDIX C: ANALYSIS OF THE SELF-ENERGY KERNEL

In this Appendix we provide the details for the various contributions to the self-energy. The integrals that enter in the expression for the self-energy kernel can be related to the following one defined in dimensional regularization:

$$I(q_0^2,\mathbf{q}^2) \equiv i\int\frac{d^{3-\epsilon}p}{(2\pi)^{3-\epsilon}}\frac{1}{2E_+E_-}\frac{E_+ + E_-}{(E_+ + E_-)^2 - q_0^2}$$

$$= \int\frac{d^{4-\epsilon}p}{(2\pi)^{4-\epsilon}}\frac{1}{[(p-q/2)^2 - m^2 + io][(p+q/2)^2 - m^2 + io]}$$

$$= \frac{1}{16\pi^2}\left[L_\epsilon + \int_0^1 d\alpha\,\ln\frac{m^2}{m^2 + \alpha(1-\alpha)(\mathbf{q}^2 - q_0^2)}\right],$$

where we have introduced the shifted momenta $\mathbf{p}_\pm = \mathbf{p} \pm \mathbf{q}/2$ and energies $E_\pm = \sqrt{\mathbf{p}_\pm^2 + m^2}$. L_ϵ is defined as

$$L_\epsilon = \frac{2}{\epsilon} - \gamma + \ln\frac{4\pi\mu^2}{m^2}.$$

We now consider the various integrals defined in Sec. VIII. In doing so we will shift the integration variable $\mathbf{p}$ so that $\mathbf{p}\to\mathbf{p}_+ = \mathbf{p} + \mathbf{q}/2$ and $\mathbf{p}' = \mathbf{p} - \mathbf{q} \to \mathbf{p}_- = \mathbf{p} - \mathbf{q}/2$. Then the numerator arising from the Dirac trace takes the form

$$E_+E_- + \mathbf{p}_+\mathbf{p}_- - m^2 = \frac{1}{2}(E_+ + E_-)^2 - 2\left(m^2 + \frac{\mathbf{q}^2}{4}\right).$$

We then have

$$\tilde{\Sigma}_1(\mathbf{q}^2) = -8g^2 I_1(\mathbf{q}^2),$$

with

$$I_1(\mathbf{q}^2) = \int\frac{d^{3-\epsilon}p}{(2\pi)^{3-\epsilon}}\frac{E_+E_- + \mathbf{p}_+\mathbf{p}_- - m^2}{4E_+E_-(E_+ + E_-)}$$

$$= \frac{1}{8}\int\frac{d^{3-\epsilon}p}{(2\pi)^{3-\epsilon}}\left(\frac{1}{E_+} + \frac{1}{E_-}\right) - \left(m^2 + \frac{\mathbf{q}^2}{4}\right)\int\frac{d^{3-\epsilon}p}{(2\pi)^{3-\epsilon}}\frac{1}{2E_+E_-(E_+ + E_-)}.$$

The first integral, including the prefactor, is equal to

$$\int\frac{d^{3-\epsilon}p}{(2\pi)^{3-\epsilon}}\frac{1}{4E} = -\frac{m^2}{32\pi^2}(L_\epsilon + 1).$$

The second integral is the basic integral $I(q_0^2,\mathbf{q}^2)$ at $q_0^2 = 0$. Altogether we obtain

$$\tilde{\Sigma}_1(\mathbf{q}^2) = \tilde{\Sigma}_1(0) + \mathbf{q}^2\tilde{\Sigma}_1'(0) + \Delta\tilde{\Sigma}_1(\mathbf{q}^2),$$

with

$$\tilde{\Sigma}_1(0) = -\delta M^2 = \frac{3m^2 g^2}{4\pi^2}\left(L_\epsilon + \frac{1}{3}\right),$$

$$\tilde{\Sigma}'_1(0) = -\delta Z = \frac{g^2}{8\pi^2} L_\epsilon ,$$

$$\Delta\tilde{\Sigma}_1(\mathbf{q}^2) = \left(m^2 + \frac{\mathbf{q}^2}{4}\right)\frac{g^2}{2\pi^2}\int_0^1 d\alpha \ln\frac{m^2}{m^2+\alpha(1-\alpha)\mathbf{q}^2}. \tag{C1}$$

Here we have introduced the renormalization constants corresponding to a renormalization at $q^2=0$.

For $\tilde{\Sigma}_3(\mathbf{q}^2)$ we have

$$\tilde{\Sigma}_3(\mathbf{q}^2) = 8g^2 I_3(\mathbf{q}^2),$$

with

$$I_3(\mathbf{q}^2) = \int \frac{d^{3-\epsilon}p}{(2\pi)^{3-\epsilon}}\frac{1}{4E_+E_-}\frac{(E_+ + E_-)^2/2 - 2(m^2+\mathbf{q}^2/4)}{(E_+ + E_-)^3}.$$

This integral can be related to the integral $I(q_0^2,\mathbf{q}^2)$ and its derivative with respect to q_0^2, at $q_0=0$. We find

$$\tilde{\Sigma}_3(\mathbf{q}^2) = -\delta Z + \Delta\tilde{\Sigma}_3(\mathbf{q}^2),$$

where δZ has been defined in Eq. (C1). The finite part is

$$\Delta\tilde{\Sigma}_3(\mathbf{q}^2) = -\frac{g^2}{8\pi^2}\int_0^1 d\alpha \ln\left[1+\alpha(1-\alpha)\frac{\mathbf{q}^2}{m^2}\right] + \frac{g^2}{2\pi^2}\left(m^2+\frac{\mathbf{q}^2}{4}\right)\int_0^1 d\alpha \frac{\alpha(1-\alpha)}{m^2+\alpha(1-\alpha)\mathbf{q}^2}.$$

From the way in which we have introduced δZ in $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_3$ it is apparent that the covariant counterterms $\delta Z(\ddot{\tilde{\phi}} + \mathbf{q}^2\tilde{\phi})$ in the equation of motion will absorb these divergences.

We finally consider the Laplace transform of the subtracted self-energy kernel introduced in Eq. (7.1)

$$\tilde{\sigma}_s(s^2,\mathbf{q}^2) = -8g^2 s^2\int\frac{d^3p}{(2\pi)^3}\frac{1}{4E_+E_-} \times\frac{(E_+ + E_-)^2/2 - 2(m^2+\mathbf{q}^2/4)}{(E_+ + E_-)^3[(E_+ + E_-)^2 + s^2]}.$$

Comparing with the standard integral $I(q_0^2,\mathbf{q}^2)$ we see that besides the continuation to the Euclidean region, $q_0^2\rightarrow -s^2$ we have additional denominators. These can be obtained via subtraction. We have

$$\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_+E_-}\frac{1}{(E_+ + E_-)[(E_+ + E_-)^2+s^2]} = -\frac{1}{s^2}[I(-s^2,\mathbf{q}^2) - I(0,\mathbf{q}^2)] = -\frac{1}{s^2}\frac{1}{16\pi^2}\int_0^1 d\alpha \ln\frac{m^2+\alpha(1-\alpha)\mathbf{q}^2}{m^2+\alpha(1-\alpha)(\mathbf{q}^2+s^2)}.$$

We proceed analogously for the second integral and finally obtain

$$\tilde{\sigma}_s(s^2,\mathbf{q}^2) = \frac{g^2}{2\pi^2}\int_0^1 d\alpha\left\{\frac{1}{s^2}\left[m^2+\frac{1}{4}(\mathbf{q}^2+s^2)\right] \times \ln\frac{m^2+\alpha(1-\alpha)\mathbf{q}^2}{m^2+\alpha(1-\alpha)(\mathbf{q}^2+s^2)} + \left(m^2+\frac{\mathbf{q}^2}{4}\right)\frac{\alpha(1-\alpha)}{m^2+\alpha(1-\alpha)\mathbf{q}^2}\right\}. \tag{C2}$$

The α integrations can be performed analytically.

APPENDIX D: DETAILS OF THE ANALYTIC SOLUTION

We have obtained in section the solution of the equation of motion and its solution via Laplace transform. We consider at first the unrenormalized equation. The solution reads

$$\psi_{\mathbf{q}}(s) = \frac{s\phi_{\mathbf{q}}(0) + \dot{\phi}_{\mathbf{q}}(0)}{s^2+M^2+\mathbf{q}^2+\tilde{\sigma}(s^2,\mathbf{q}^2)},$$

so that

$$\phi_{\mathbf{q}}(t) = \frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c} ds\, e^{st}\frac{s\phi_{\mathbf{q}}(0)+\dot{\phi}_{\mathbf{q}}(0)}{s^2+M^2+\mathbf{q}^2+\tilde{\sigma}(s^2,\mathbf{q}^2)}.$$

As usual [16,17] we shift the contour to the left so that finally it includes the cuts, and eventually poles, on the imaginary s axis and a circle at $|s|\rightarrow\infty$ around the left half, which does not contribute for positive t as the exponential $\exp st$ tends to zero there. In doing so we make use of the causality condition that there are no zeros in the left half of the complex s plane, as required by causality. Along the cut at $s=i\omega$ with $2\sqrt{m^2+\mathbf{q}^2}<\omega<\infty$ we define the real and imaginary parts of the kernel $\tilde{\sigma}$ by the convention

$$\tilde{\sigma}[(i\omega\pm\epsilon)^2,\mathbf{q}^2] = \tilde{\sigma}_R(-\omega^2,\mathbf{q}^2) \pm i\tilde{\sigma}_I(-\omega^2,\mathbf{q}^2).$$

As σ only depends on s^2 this also fixes the relative signs of the imaginary parts of $\tilde{\sigma}$ on the lower cut for which $-\infty<\omega<-2\sqrt{m^2+\mathbf{q}^2}$. We then obtain

$$\phi_{\mathbf{q}}(t)=\frac{1}{2\pi i}\int_{\omega_c}^{\infty} i\,d\omega\, e^{i\omega t}\mathrm{disc}\frac{i\omega\phi_{\mathbf{q}}(0)+\dot{\phi}_{\mathbf{q}}(0)}{-\omega^2+M^2+\mathbf{p}^2+\tilde{\sigma}(-\omega^2\pm i\epsilon,\mathbf{q}^2)}+\frac{1}{2\pi i}\int_{\omega_c}^{\infty}(-i)d\omega\, e^{-i\omega t}\mathrm{disc}\frac{-i\omega\phi_{\mathbf{q}}(0)+\dot{\phi}_{\mathbf{q}}(0)}{-\omega^2+M^2+\mathbf{p}^2+\tilde{\sigma}(-\omega^2\mp i\epsilon,\mathbf{q}^2)},$$

with $\omega_c=2\sqrt{\mathbf{q}^2+m^2}$ for the two fermion cut. The spectral density is obtained from the discontinuity across the cut

$$S(\omega,\mathbf{q})=i\,\mathrm{disc}\frac{1}{-\omega^2+M^2+\mathbf{p}^2+\tilde{\sigma}(-\omega^2+i\epsilon,\mathbf{q}^2)}=\frac{2\tilde{\sigma}_I(-\omega^2,\mathbf{q}^2)}{[-\omega^2+M^2+\mathbf{p}^2+\tilde{\sigma}_R(-\omega^2+i\epsilon,\mathbf{q}^2)]^2+\tilde{\sigma}_I^2(-\omega^2+i\epsilon,\mathbf{q}^2)}. \tag{D1}$$

In the case in which the scalar particle is unstable, i.e., $M>2m$ there is a resonance above the fermion-antifermion threshold and no support for the spectral density below threshold [16,17]. However in the case $M<2m$ the scalar is stable and cannot decay, now the spectral density has support above and *below* threshold. Below threshold the spectral density is a delta function at the position of the renormalized pole, to include the stable pole below the two particle threshold in the description we now define $\omega_c=0^+$ to distinguish that the origin is excluded from the integration region. The pole in the stable case is obtained from the identity

$$S(\omega,\mathbf{q})\overset{\sigma_1\to 0}{\to}\pi\delta[-\omega^2+M^2+\mathbf{p}^2+\tilde{\sigma}_R(-\omega^2+i\epsilon,\mathbf{q}^2)],$$

so that

$$\phi_{\mathbf{q}}(t)=\frac{2}{\pi}\int_{0^+}^{\infty}d\omega\cos\omega t\frac{\omega\phi_{\mathbf{q}}(0)\tilde{\sigma}_I(-\omega^2,\mathbf{q}^2)}{[-\omega^2+M^2+\mathbf{p}^2+\tilde{\sigma}_R(-\omega^2+i\epsilon,\mathbf{q}^2)]^2+\tilde{\sigma}_I^2(-\omega^2+i\epsilon,\mathbf{q}^2)}+\frac{2}{\pi}\int_{0+}^{\infty}d\omega\sin\omega t\frac{\dot{\phi}_{\mathbf{q}}(0)\tilde{\sigma}_I(-\omega^2,\mathbf{q}^2)}{[-\omega^2+M^2+\mathbf{p}^2+\tilde{\sigma}_R(-\omega^2+i\epsilon,\mathbf{q}^2)]^2+\tilde{\sigma}_I^2(-\omega^2+i\epsilon,\mathbf{q}^2)}. \tag{D2}$$

In order that this equation and its time derivative be consistent at $t=0$ we have to require the sum rule

$$1=\frac{2}{\pi}\int_{0^+}^{\infty}d\omega\frac{\omega\tilde{\sigma}_I(-\omega^2,\mathbf{q}^2)}{[-\omega^2+M^2+\mathbf{p}^2+\tilde{\sigma}_R(-\omega^2+i\epsilon,\mathbf{q}^2)]^2+\tilde{\sigma}_I^2(-\omega^2+i\epsilon,\mathbf{q}^2)}.$$

In order to derive this sume rule we require, as already mentioned above, that the denominator $s^2+\mathbf{q}^2+M^2+\sigma(s^2,\mathbf{q}^2)$ has no zeros in the left half of the complex plane. We have to assume furthermore that $\tilde{\sigma}(s^2,\mathbf{q}^2)$ increases less strongly as s^2 as $|s|\to\infty$ in the left half of the complex plane. Under these assumptions we have the identity

$$\frac{1}{i\pi}\oint ds\frac{s}{s^2+\mathbf{q}^2+M^2+\tilde{\sigma}(s^2,\mathbf{q}^2)}=0$$

if the integral is carried out along the contour enclosing the left half of the complex plane. The contour consists of an integral along the left of the imaginary s axis and a semicircle at $|s|=\infty$. The latter one contributes

$$\frac{1}{i\pi}\int_C\frac{ds}{s}=-1.$$

The integral along the imaginary axis is given by

$$\frac{1}{\pi}\int_0^{\infty}d\omega\left[\frac{i\omega}{s^2+\mathbf{q}^2+M^2+\tilde{\sigma}(-\omega^2-io)}+\frac{-i\omega}{s^2+\mathbf{q}^2+M^2+\tilde{\sigma}(-\omega^2+io)}\right]=\frac{2}{\pi}\int_{\omega_c}^{\infty}d\omega\frac{\omega\tilde{\sigma}_I(-\omega^2,\mathbf{q}^2)}{|-\omega^2+M^2+\mathbf{p}^2+\tilde{\sigma}(-\omega^2+i\epsilon,\mathbf{q}^2)|^2}. \tag{D3}$$

The two parts of the contour integral have to add up to zero, which yields the sum rule.

For the renormalized equation of motion we rewrite the result obtained in Sec. VII as

$$\phi_{\mathbf{q}}(t)=\frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c}ds\left\{[s\phi_{\mathbf{q}}(0)+\dot{\phi}_{\mathbf{q}}(0)]F_1(s^2,\mathbf{q}^2)+\ddot{\phi}_{\mathbf{q}}(0)\frac{1}{s}F_2(s^2,\mathbf{q}^2)\right\},$$

with

$$F_1(s^2,\mathbf{q}^2)=\frac{1+\Delta\tilde{\Sigma}_3(\mathbf{q}^2)+s\tilde{\sigma}(s^2,\mathbf{q}^2)}{s^2[1+\Delta\tilde{\Sigma}_3(\mathbf{q}^2)+\tilde{\sigma}_s(s^2,\mathbf{q}^2)]+\mathbf{q}^2+M^2+\Delta\tilde{\Sigma}_1(s^2,\mathbf{q}^2)},$$

$$F_2(s^2,\mathbf{q}^2)=\frac{\tilde{\sigma}_s(s^2,\mathbf{q}^2)}{s^2[1+\Delta\tilde{\Sigma}_3(\mathbf{q}^3)+\tilde{\sigma}_s(s^2,\mathbf{q}^2)]+\mathbf{q}^2+M^2+\Delta\tilde{\Sigma}_1(\mathbf{q}^2)}.$$

The function $F_1(s^2,\mathbf{q}^2)$ has analyticity properties analogous to the fraction $1/(s^2+\mathbf{q}^2+M^2+\tilde{\sigma})$ considered above. In particular the discontinuities along the positive and negative imaginary axis have the same relative signs, it has no singularities in the left half of the complex plane, and the limiting behavior as $|s|\to\infty$ is $1/s^2$. For the first property it is essential to note that $\tilde{\sigma}_s$ only depends on the square of the variable s; see Eq. (C2). For the last property is sufficient to notice that $\tilde{\sigma}_s(s^2,\mathbf{q}^2)$ behaves as $\ln s^2$ as $|s|\to\infty$, so the terms proportional to $\tilde{\sigma}_s$ dominate in numerator and denominator. The function $F_2(s^2,\mathbf{q}^2)$ has analyticity properties analogous to those of $F_1(s^2,\mathbf{q}^2)$, and decreases asymptotically as $1/s^2$. The relative signs of the imaginary parts along the cuts are the same as for $F_1(s^2,\mathbf{q}^2)$. Collecting all terms we find

$$\phi_{\mathbf{q}}(t)=\frac{2}{\pi}\int_{\omega_c}^{\infty}d\omega\bigg[\cos\omega t\,\phi_{\mathbf{q}}(0)\,\omega\,\mathrm{Im}\,F_1(-\omega^2+io,\mathbf{q}^2)+\sin\omega t\,\dot{\phi}_{\mathbf{q}}(0)\mathrm{Im}\,F_1(-\omega^2+io,\mathbf{q}^2)-\frac{\ddot{\phi}_{\mathbf{q}}(0)}{\omega}\cos\omega t\,\mathrm{Im}\,F_2(-\omega^2+io,\mathbf{q}^2)\bigg]. \quad \text{(D4)}$$

For the consistency of the left and right hand sides and their first derivatives with respect to t the sum rule

$$1=\frac{2}{\pi}\int_{\omega_c}^{\infty}d\omega\;\omega\;\mathrm{Im}\,F_1(-\omega^2+io,\mathbf{q}^2) \quad \text{(D5)}$$

has to be satisfied. It follows again by considering the integral

$$\frac{1}{i\pi}\oint ds\,sF_1(s^2,\mathbf{q}^2)$$

along a closed contour around the left complex half plane. One needs, furthermore,

$$0=\frac{2}{\pi}\int_{\omega_c}^{\infty}d\omega\,\frac{1}{\omega}\,\mathrm{Im}\,F_2(-\omega^2+io,\mathbf{q}^2)$$

which follows from analogous considerations, using that in this case the infinite semicircle does not contribute as the integrand behaves as $1/s^3$ there.

We next have to consider the term proportional to $\ddot{\phi}_{\mathbf{q}}(0)$ and the second time derivative of Eq. (D4). From the renormalized equation of motion (6.2) at $t=0$ we derive immediately

$$\ddot{\phi}_{\mathbf{q}}(0)=-\phi_{\mathbf{q}}(0)\,\frac{\mathbf{q}^2+M^2+\Delta\tilde{\Sigma}_1(\mathbf{q}^2)}{1+\Delta\tilde{\Sigma}_3(\mathbf{q})^2}. \quad \text{(D6)}$$

The second derivative of Eq. (D4) at $t=0$ reads

$$\ddot{\phi}_{\mathbf{q}}(0)=-\frac{2}{\pi}\int_{\omega_c}^{\infty}d\omega[\phi_{\mathbf{q}}(0)\omega^3\,\mathrm{Im}\,F_1(-\omega^2+io,\mathbf{q}^2)+\omega^2\ddot{\phi}_{\mathbf{q}}(0)\mathrm{Im}\,F_2(-\omega^2+io,\mathbf{q}^2)].$$

We express, on the right hand side, $\phi_{\mathbf{q}}(0)$ by $\ddot{\phi}_{\mathbf{q}}(0)$, using Eq. (D6). Then we can write the sum rule as

$$1=\frac{2}{\pi}\int_{\omega_c}^{\infty}d\omega\;\omega\;\mathrm{Im}\bigg[\omega^2F_1(-\omega^2+io,\mathbf{q}^2)\times\frac{1+\Delta\tilde{\Sigma}_3(\mathbf{q}^2)}{q^2+M^2+\Delta\tilde{\Sigma}_1(\mathbf{q}^2)}+F_2(-\omega^2+io,\mathbf{q}^2)\bigg]. \quad \text{(D7)}$$

The expression in the square brackets can be written explicitly as

$$\frac{\omega^2(1+\Delta\tilde{\Sigma}_3+\tilde{\sigma}_s)}{-\omega^2(1+\Delta\tilde{\Sigma}_3+\tilde{\sigma}_s)+\mathbf{q}^2+M^2+\Delta\tilde{\Sigma}_1}\,\frac{1+\Delta\tilde{\Sigma}_1}{\mathbf{q}^2+M^2+\Delta\tilde{\Sigma}_1}+\frac{\tilde{\sigma}_s}{-\omega^2(1+\Delta\tilde{\Sigma}_3+\tilde{\sigma}_s)+\mathbf{q}^2+M^2+\Delta\tilde{\Sigma}_1}$$

$$=-\frac{1+\Delta\tilde{\Sigma}_1}{\mathbf{q}^2+M^2+\Delta\tilde{\Sigma}_1}+\frac{1+\Delta\tilde{\Sigma}_3+\tilde{\sigma}_s}{-\omega^2(1+\Delta\tilde{\Sigma}_3+\tilde{\sigma}_s)+\mathbf{q}^2+M^2+\Delta\tilde{\Sigma}_1}.$$

Only the imaginary part of this expression occurs in the integrand. So the first term on the right-hand side does not contribute, and the second term is just F_1. The sum rule (D7) reduces therefore to the first one (D5).

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