Renormalization of periodic potentials

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The renormalization of the periodic potential is investigated in the framework of the Euclidean one-component scalar field theory by means of the differential RG approach. Some known results about the sine-Gordon model are recovered in an extremely simple manner. There are two phases: an ordered one with asymptotical freedom and a disordered one where the model is nonrenormalizable and trivial. The order parameter of the periodicity, the winding number, indicates spontaneous symmetry breaking in the ordered phase where the fundamental group symmetry is broken and the solitons acquire dynamical stability. It is argued that the periodicity and the convexity are such strong constraints on the effective potential that it always becomes flat. This flattening is reproduced by integrating out the RG equation.

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I. INTRODUCTION

The dynamics generated by a periodic potential represents a challenge in quantum field theory where the usual strategies to obtain the solution are based on the Taylor expansion which violate the essential symmetry of the problem, the periodicity. One expects two kinds of problems. A local one: the perturbation expansion for the small fluctuations around a minima of the potential should deal with infinitely many vertices in order to preserve the periodicity. A global problem appears in the construction of the effective potential as an instability, a conflict between the periodicity and the convexity which can be resolved in a trivial manner only, being the constant the only function which is periodic and convex at the same time. Another global problem, made rather puzzling by the flatness of the effective potential, is to find an order parameter distinguishing the phase with periodicity from the one where the periodicity is broken spontaneously.

The goal of this paper is to give a brief presentation of these issues in the case of a two-dimensional scalar model. The pertinent features of the sine-Gordon model, the simplest realization of a periodicity, and the more detailed goals of this paper are presented in Sec. II. Section III contains a very brief introduction into the Wegner-Houghton equation, applied for the periodic potential in Sec. IV. The linearized renormalization group flow is given around the two-dimensional Gaussian fixed point and the periodic operators of the potential are classified in Sec. V. The nonlinear flow is discussed in Sec. VI by means of numerical integration of the renormalization group equation. Section VII contains a few remarks about the signatures of the spontaneous breakdown of the fundamental group symmetry. Finally, Sec. VIII is for the summary of our findings.

II. THE SINE-GORDON, *X-Y* AND FERMIONIC MODELS

The simplest example for periodic potential is the two-dimensional sine-Gordon model which is described by the Lagrangian

$$L=\tfrac{1}{2}(\partial_\mu\phi)^2+V(\phi), \tag{1}$$

with

$$V(\phi)=u\cos\beta\phi. \tag{2}$$

The dynamics and the renormalization have been discussed by means of the straight perturbation expansion and the mappings between the sine-Gordon, the Thirring, and the *X-Y* planar models [1–8] The duality transformation connecting the sine-Gordon and the *X-Y* models provides the renormalization group flow in the complete coupling constant space [9,10].

The mappings between the different models are not exact. In fact, the bosonization of the Thirring model [1] and the Coulomb-gas representation of a periodic potential [7] are obtained perturbatively. The duality is established up to irrelevant terms in the action [10]. An exact equivalence exists between the *X-Y* and the compactified sine-Gordon model which is obtained by expressing (1) in terms of the compact variable [11]

$$z(x)=e^{\beta\phi(x)}. \tag{3}$$

Such a parametrization makes the kinetic energy periodic,

$$L=\frac{1}{2\beta^2}\partial_\mu z^*\partial_\mu z+\frac{u}{2}(z+z^*) \tag{4}$$

TABLE I. Comparison of the XY and the sine-Gordon model.

XY model with external field	Compactified sine-Gordon model
External field h	Fourier amplitude $u=h/\beta^2$
Temperature T	Coupling constant β^2
Molecular phase of the vortex gas $T<T_c$	Weak coupling phase $\beta^2<\beta_c^2$
Ionized phase of the vortex gas $T>T_c$	Strong coupling phase $\beta^2>\beta_c^2$
Vortex, antivortex	Soliton creation and annihilation

and introduces vortices in the dynamics. The models (1) and (4) are equivalent in any order of the perturbation expansion in continuous space-time. The modification of the regulator amounts to the introduction of nonrenormalizable terms in the action which are irrelevant in the UV scaling regime. Thus the equivalence is reached asymptotically only when the cutoff is removed. In lattice regularization the theory (4) is in the same universality class as the X-Y model which is described by the action

$$S=\frac{1}{T}\sum_{\langle x,x'\rangle}\cos(\theta_x-\theta_{x'})+\sum_x h\cos(\theta_x), \tag{5}$$

with $T=\beta^2$. The renormalization of the X-Y model induces a nontrivial value for the vortex fugacity y which appears as an additional evolving coupling constant in the compactified sine-Gordon model [11]. Therefore the complete coupling constant space contains three coupling constants, the external field h, the temperature T, and the vortex fugacity y.

As it is well known, there are two phases in the $y-T$ plane connected by the Kosterlitz-Thouless transition [9]. In the low temperature, molecular phase $T<T_{KT}$ the vortices and antivortices form closely bound pairs while above the transition temperature $T>T_{KT}$, in the ionized phase they dissociate into a plasma. Due to the duality transformation, the corresponding two phases appear in the $h-T$ plane as well as those of the dual electric Coulomb gas (DCG). In the $h-T$ plane the ionized phase of DCG is realized at low temperature, i.e., at weak coupling $\beta^2<\beta_c^2$ whereas the molecular phase of DCG is positioned at high temperature, i.e., at strong coupling $\beta^2>\beta_c^2$. Instead of this duality transformation connecting the X-Y and the sine-Gordon models we shall rely in this paper on the more direct equivalence of Eqs. (4) and (5) as noted above [11], which identifies the weakly coupled (small β) and the strongly coupled (large β) phase of the sine-Gordon model with the molecular and the ionized phase of the X-Y model. The vortices of Eqs. (4) and (5) correspond to each other in this scheme.

The transition between the phases can be characterized either perturbatively or by means of the vortex dynamics. As far as the perturbation expansion is concerned, the normal ordering is sufficient to remove the UV divergences for weak couplings. Though the divergence structure of the individual graphs is the same in either phase, the partial resummation of the perturbation expansion in u produces new UV divergences for $\beta^2>8\pi$ [1,4,5]. A further UV divergence was found at $\beta^2=4\pi$ [5]. The double expansion in u and $\beta^2/8\pi-1$ [8] indicates no special singularities at $\beta^2=4\pi$ and shows that the adjustment of u and the introduction of a wave function renormalization constant for the field ϕ in the strong coupling phase is sufficient to remove the UV divergences.

The inspection of the vortex dynamics allows us to follow the transition line for larger values of u. One way the vortices arise is that the vertices appearing in the perturbation expansion in Eq. (2) form a gas of vortices described by the X-Y model [7]. Another way to identify the vortices is to use the equivalence of Eqs. (4) and (5) in lattice regularization when the continuum limit is approached. Both ways indicate that the weak and the strong coupling phases are the molecular or the ionized phases, respectively, from the point of view of the vortex gas and that the phases are separated by the Kosterlitz-Thouless transition. It is worthwhile noting that this transition is rather peculiar because (i) its driving force, the vortex dynamics is generated by the UV modes rather than the IR ones as in the case of spontaneous symmetry breaking, and (ii) it is a higher than second order since the correlation length is infinite in the molecular phase and diverges faster than any power of the reduced temperature (exponentially), therefore one cannot introduce the critical exponent ν in the usual manner.

The sine-Gordon model possesses a topological current

$$j_\mu(x)=\frac{\beta}{2\pi}\epsilon_{\mu\nu}\partial_\nu\phi(x), \tag{6}$$

which is conserved in the semiclassical expansion, when the path integral is saturated by field configurations with analytic space-time dependence. The flux defined by j_μ is the vortex number, called the vorticity and the soliton number in the same time. In this manner the world lines of the sine-Gordon solitons end at the X-Y model vortices, making the soliton unstable and the topological current anomalous in the ionized phase where field configurations with singular space-time dependence survive the removal of the cutoff [11]. This destroys or at least modifies the bosonization transformation in the high temperature, ionized phase of the vortex gas. In fact, the nonconservation of the topological current requires fermion number nonconserving terms in the fermionic representation, a fundamental violation of the rules inferred from the weak coupling expansion.

The relation between the sine-Gordon and the X-Y model is summarized in Table I.

III. DIFFERENTIAL RG APPROACH IN MOMENTUM SPACE

The challenge in developing a renormalization group (RG) method for the sine-Gordon model is that it should follow the mixing of all the operators which become relevant in either phase. Since there is an infinite amount of relevant operators in two dimensions, one needs the functional form of the evolution equations for the blocked Wilsonian action [12–18]. We shall use the leading order gradient expansion in the Wegner-Houghton equation [12] to study the renormalization group flow of a generalized model with an arbitrary periodic potential. Such a drastic truncation leaves some doubts due to the supposed role of the wave function renormalization constant in the ionized phase. But the higher order contribution of the gradient expansion can only be treated consistently by the use of the effective action instead of the bare one. We shall find that the instability arising from the periodicity of the effective potential makes the Legendre transformation highly nontrivial and prevents us to use the effective action in the infrared regime of the molecular phase.

The differential RG transformations are realized by integrating out the high-frequency Fourier components of the field variable, in infinitesimal steps in momentum space successively from the UV cutoff k to $k-\delta k$,

$$e^{-S_{k-\delta k}[\phi]}=\int D[\phi']e^{-S_k[\phi+\phi']}, \tag{7}$$

where $S_k[\phi]$ stands for the blocked action with cutoff k and the field variables ϕ and ϕ' contain Fourier components with momenta $p<k-\delta k$, and $k-\delta k<p<k$, respectively. In every infinitesimal step, the path integration in Eq. (7) is evaluated by the help of the saddle point approximation. If the saddle point is at $\phi'=0$, one can find an integro-differential equation for the blocked action, called the Wegner-Houghton equation [12]. In the local potential approximation one uses the leading order expression for the action in the gradient expansion

$$S_k=\int d^dx[\tfrac{1}{2}(\partial_\mu\phi(x))^2+V_k(\phi(x))], \tag{8}$$

and the Wegner-Houghton equation reduces to a differential equation for the scale dependent potential $V_k(\phi)$ [16]

$$k\partial_k V_k(\phi)=-k^d\alpha\ln(k^2+\partial_\phi^2 V_k(\phi)), \tag{9}$$

with $\alpha=\frac{1}{2}\Omega_d(2\pi)^{-d}$, and the solid angle Ω_d in dimension d.

Notice that the argument of the logarithm in Eq. (9) must be non-negative for the expansion made around a stable saddle point. If the argument becomes negative at a critical value $k_{cr}>0$, given by $k_{cr}^2=-\partial_\phi^2 V_{k_{cr}}(\phi)$ then the Wegner-Houghton equation looses its validity for $k<k_{cr}$ and the saddle point becomes nonzero and the tree-level blocking relation must be used. We simplify the saddle point structure of the blocking (7) by retaining the plane waves only and we find the evolution equation

$$V_{k-\delta k}(\phi)=\min_\rho\left[k^2\rho^2+\frac{1}{2}\int_{-1}^{1}du V_k[\phi+2\rho\cos(\pi u)]\right], \tag{10}$$

where ρ is the amplitude of the plane wave [19].

IV. PERIODICITY OF THE POTENTIAL

The symmetry of the action under the transformation

$$\phi(x)\rightarrow\phi(x)+\Delta \tag{11}$$

is to be preserved by the blocking and the potential $V_k(\Phi)$ must be periodic with period length Δ. It is actually obvious that the blocking, the transformation

$$kV_{k-\delta k}(\phi)=kV_k(\phi)+[k^d\alpha\ln(k^2+\partial_\phi^2 V_k(\phi))]\delta k \tag{12}$$

preserves that periodicity of the potential. The impact of the periodicity of the potential on the dynamics can be understood by recalling that $V_k(\Phi)$ tends to the effective potential, $V_{eff}(\Phi)$ in the IR limit, $k\rightarrow 0$. This is because both potential give the action density within the functional space $\Phi=\langle\phi(x)\rangle$. The Legendre transform imposes the condition of the convexity on the effective potential [20] and thereby on $V_{k=0}(\Phi)$. Since the only periodic and convex function is the constant there is no way to have nontrivial effective potential when the transformation (11) is a formal symmetry of the action. Notice that this statement holds for any dimensions. The detailed RG study of the sine-Gordon model is pursued here to check such a general conjecture in a simpler case.

One could object by pointing out that it is enough to implement the horizontal Maxwell cut of the effective potential between the inflexion points $V''_{eff}(\Phi)=0$, where the function is concave. But the problem left by this construction is just at the inflection point where the potential obtained in this manner has singular higher derivatives. This is unacceptable, as one can see by placing the system in a large but finite box. On one hand, the effective potential should be close enough to the one obtained in the thermodynamical limit, i.e., the Maxwell cut should be present up to finite size corrections. On the other hand, the effective potential should be regular in the absence of IR divergences. It is easy to see that the singularity at the inflexion point gives rise to smooth but concave region for finite systems after being rounded off and the Maxwell cut must be extended further. In the case of the ϕ^4 model where the symmetry which is broken spontaneously is $\phi\rightarrow-\phi$ such an extension of the Maxwell cut leads to the degeneracy of the effective potential between the minima [19]. When the symmetry (11) is broken then infinitely many minima are expected in the effective potential. The only Maxwell cut which is stable in the thermodynamical limit is thus the one which renders $V_{eff}(\Phi)$ constant.

In order to allow a convenient truncation of the potential for the numerical solution of the evolution equation and to preserve the periodicity we write the $V_k(\phi)$ as a Fourier series,

$$V_k(\phi)=\sum_{n=0}^{\infty} u_n(k)\cos(n\beta\phi). \tag{13}$$

For the sake of simplicity we consider only potentials with Z(2) symmetry, $V_k(\phi)=V_k(-\phi)$. The whole scale dependence occurs in the Fourier amplitudes $u_n(k)$, the "coupling constants" of the scale dependent potential. In the case of a nontrivial saddle point, Eq. (10) can be rewritten as

$$V_{k-\delta k}(\phi)=\min_{\rho}\left[k^2\rho^2+\sum_{n=1}^{\infty} u_n(k)\cos(n\beta\phi)J_0(2n\beta\rho)\right], \tag{14}$$

where J_0 stands for the Bessel function and $\beta=2\pi/\Delta$. Let $\rho_k(\phi)$ denote the position of the minimum of the bracket on the right-hand-side of Eq. (14). Then $\rho_k(\phi)$ is periodic $\rho_k(\phi+\Delta)=\rho_k(\phi)$, since one has to minimize the same expression of ρ for ϕ and $\phi+\Delta$. Note that Eq. (14) preserves the periodicity.

The retaining of the higher order contributions in the gradient expansion, among them the leading order being the inclusion of a wavefunction renormalization constant $Z_k[\phi(x)]$ into the kinetic energy of the action (8) changes the situation. The action keeps its period length in the bare field under any circumstance. But the period length in terms of the renormalized field $\phi_{R,k}(x)=Z_k^{1/2}(\phi_0)\phi(x)$, where ϕ_0 minimizes the potential $V_k(\phi)$, is $\Delta Z_k^{1/2}(\phi_0)$. Such subleading contributions of the gradient expansion are neglected in the present work. We believe that their contribution will not change our results qualitatively.

It is easier to use the derivative of Eq. (9) with respect to ϕ rather than the original equation itself. For dimension d the general form of the evolution equation reads as follows:

$$\alpha\beta^2k^{d-2}n^2v_n(k)=(d+k\partial_k)v_n(k) -\tfrac{1}{2}\beta k^{d-2}\sum_{p=1}^{N} A_{np}(k)(d+k\partial_k)v_p(k), \tag{15}$$

where N is the truncation in the Fourier series, $v_n(k)=n\beta u_n(k)$ and

$$\begin{aligned}A_{np}(k)=&(n-p)v_{n-p}\Theta(n\geqslant p)\\ &+(p-n)v_{p-n}\Theta(p\geqslant n)\\ &-(n+p)v_{p+n}\Theta(N\geqslant n+p),\end{aligned} \tag{16}$$

with $\Theta(n\geqslant n')=\{1 \text{ if } n\geqslant n', 0 \text{ if } n<n'\}$.

Reaching the critical value k_{cr} one has to change automatically from the system of Eqs. (15) to Eq. (14). In every step δk in the momentum space, the potential at the scale $k-\delta k$ is then found by minimizing the expression on the right-hand side (r.h.s.) of Eq. (14). After the minimization, the potential $V_{k-\delta k}(\phi)$ is expanded in Fourier-series to define the new Fourier amplitudes at the scale $k-\delta k$. One repeats this algorithm step by step until $k=0$.

V. LINEARIZED SOLUTION

According to the power counting, theories with polynomial interactions are super-renormalizable in dimension $d=2$. Furthermore, the super-renormalizable interactions correspond to relevant operators in the UV scaling regime. How can we have new UV divergences in the ionized phase when the set of the renormalizable operators is fixed? The source of the complication is that in the usual perturbative proof of the renormalizability each monomial vertex is treated independently. This strategy is sufficient for polynomial interactions but is not necessarily applicable for periodic potentials where the symmetry is destroyed by any truncation of the Taylor expansion.

The treatment of an infinite series of operators instead of a single monomial may cause complications, an impression of having renormalized a manifestly nonrenormalizable model. In fact, we may find an infinite series of irrelevant operators in a renormalizable model, showing the possibility of the removal of the cutoff in the presence of nonrenormalizable operators. An obvious example is when a regulator, represented as an interaction vertex, yields irrelevant operators. We find this situation by introducing the finite difference operator appearing in the lattice regularized theories in the continuum. The finite difference operator generates an infinite power series of the gradient whose monomials are nonrenormalizable. The basic question in the removal of the cutoff is whether the series of the irrelevant operators is chosen in such a manner that the divergences can be removed by the fine-tuning of a finite number of parameters in the action. The infinite series of irrelevant operators is usually required by some symmetry of the theory, such as the periodicity in momentum space on the lattice [21], the global $O(N)$ symmetry of the nonlinear sigma model [22], or gauge theories on the lattice [23]. The symmetry imposes such constraints on the radiative corrections that the divergences can in fact be removed within the given functional family of the action and the apparently nonrenormalizable model becomes renormalizable.

One touches upon here a fundamental difference between the ways renormalization group is used in statistical mechanics and particle physics. In statistical mechanics the UV cutoff is physical and we may not ignore the effects taking place at that scale. In particle physics we insist that the cutoff is sufficiently far from the scale of the phenomenon we are interested even in the effective theories. The gain coming from this constraining of the set of observables is that universality arguments apply and it is enough to consider renormalizable models.[1] We may need irrelevant operators in statistical mechanics which lead to complicated nonrenormalizable models. The error caused by the omission of the irrelevant pieces in the particle physical applications is negligible. We can turn this insensitivity into a freedom. The suppression mechanism responsible of this simplification gives us the possibility to include irrelevant pieces in the

[1]The universality might need certain generalization in case of several scaling regimes or instabilities [19,24].

theories without specific fine tuning of their coupling constants in particle physics so long as the UV divergences can be removed.

In the case of the periodic potential we have the opposite effect, a restriction on the renormalizability due to the presence of infinitely many vertices in the model. This is because the new UV divergences of the ionized phase arise from the summation of infinitely many, individually finite graphs [4,5,8]. We show that this highly nontrivial effect can be reproduced in a very simple manner in the framework of the functional form of the renormalization group method. We shall use the Fourier amplitudes $u_n(k)$ as coupling constants and we linearize the $d=2$ dimensional renormalization group flow (15) around the fixed point $u_n=0$ by assuming $|\partial_\phi^2 V_k(\phi)| \ll k^2$. Notice that the UV Gaussian fixed point is well defined in any dimensions because modes with nonvanishing momentum are considered in the blocking. The singularity building up at or below the lower critical dimension can be found in the flow generated by such a blocking procedure as an instability and as an inconsistency of the massless IR fixed point without influencing the UV scaling laws.

It is more natural to express the flow in terms of the dimensionless coupling constants, $\tilde{u}(k)=k^{-d}u(k)$. The solution of the linearized renormalization group equation satisfying the initial conditions $\tilde{u}_n(k=\Lambda)=\tilde{u}_{n\Lambda}$ is

$$\tilde{u}_n(k)=\tilde{u}_{n\Lambda}\left(\frac{k}{\Lambda}\right)^{(\alpha\beta^2 n^2-2)}. \tag{17}$$

Thus, the coupling constants $\tilde{u}_n(k)$ are relevant, marginal, or irrelevant for $\beta^2<8\pi/n^2$, $\beta^2=8\pi/n^2$, or $\beta^2>8\pi/n^2$, respectively.

For $\beta^2>8\pi$ all dimensionless coupling constants are irrelevant and Eq. (17) is consistent and keeps the trivial saddle point of the blocking stable. This result indicates the inaccessibility of the Gaussian fixed point in the UV limit, the nonrenormalizability of the model. The infrared fixed point is a trivial, noninteracting massless theory.

It is instructive to compare this result with the $O(u^3)$ prediction of the double expansion in u and $\beta^2/8\pi-1$ of the sine-Gordon model which shows the possibility of absorbing the UV divergences into u and a wave function renormalization constant for the field [8]. The doubt that the higher order contributions of the perturbation expansion may render the theory nonrenormalizable is resolved in Ref. [8] by relying on the bosonization method. Unfortunately there are problems with the bosonization in the ionized phase which is not surprising since this mapping is based either on the saddle point or the weak coupling expansion. The problem is the anomaly of the topological current which requires the presence of fermionic operators with odd power in the fermion field [11]. We believe that the theory is nonrenormalizable in this phase but the real proof requires further efforts.

The potential becomes flat under RG transformation in the limit $k\to 0$ and the flat effective potential suggests the presence of a massless particle which is in contradiction with the dimensionality of the system [25]. The resolution of this apparent contradiction is that according to Eq. (17), the non-Gaussian vertices of the model tend to zero sufficiently fast in the infrared limit to suppress the IR divergences of the perturbation expansion when each mode in the loop integral is coupled to the effective coupling constants at the appropriate scale.

The critical point at $\beta^2=8\pi$ is a well-known result for the sine-Gordon model. Furthermore it was known that the higher harmonics, corresponding to the vortices with higher vorticity are irrelevant around the critical point [8]. What is interesting in our solution is that one sees a change in the scaling laws for the higher harmonics at a finite distance away from this critical point. In fact, the nth harmonic is found to be relevant for $\beta^2 \leqslant 8\pi/n^2$. The apparency of a new relevant operator implies not only a new renormalization condition but the possibility of new ultraviolet divergences. Though we find no evidence for the singularity at $\beta^2=4\pi$ [5], a series of new singularities is expected at $\beta^2=8\pi/n^2$, due to the vortices with higher vorticity.

VI. NUMERICAL SOLUTION

For $\beta^2<8\pi$ the first few Fourier-amplitudes are relevant, that is they increase for decreasing value of k and consequently the linearization ceases to be reliable. The solution can only be found numerically in this case. Monitoring the coupling constants numerically we compare the solution of Eq. (15) with the result obtained from the analogous relation for the polynomial potential [19],

$$V_k(\phi)=\sum \frac{1}{n!} g_n \phi^n. \tag{18}$$

The initial conditions for the polynomial potential were chosen $g_2=-0.001$, $g_4=0.01$, and $g_n=0$, if $n>4$. Then the saddle point remains trivial for any k and the singularity of Eq. (9) at $k_{cr}^2(\phi)=-\partial_\phi^2 V_k(\phi)$ is avoided. In order to compare the periodic and the polynomial cases we choose the initial conditions for the Fourier-amplitudes $u_n(k=\Lambda)$ for $\beta^2=0.1\beta_c^2$ such that after Taylor expansion the initial conditions for the polynomial case are recovered. Therefore, the initial conditions for various truncations N of the Fourier series are different. For the increasing values of N the coupling constants $g_n(k)$ determined from the periodic potential by the Taylor expansion approach the running coupling constants of the polynomial potential in the UV regime. This is understandable since the quantum field ϕ does not feel the global properties of the potential, i.e., the periodicity due to its small fluctuations in the UV regime. Just the opposite holds in the IR regime, where the field fluctuations become larger and they make the global features of the potential manifest. Therefore, it is expected that the solutions for the periodic and the polynomial potentials become different in the IR regime.

We integrated numerically Eq. (15) starting from the UV cutoff $\Lambda=1$ down to the critical value $k_{cr}^2(\phi)=-\partial_\phi^2 V_{k_{cr}}(\phi)$ by using the fourth-order Runge-Kutta method and $\delta k=10^{-p}k$ with $p=3$ or $p=4$. There were no changes in the numerical results by increasing p further. In Figs. 1 and 2 we show the scaling of the dimensionful coupling

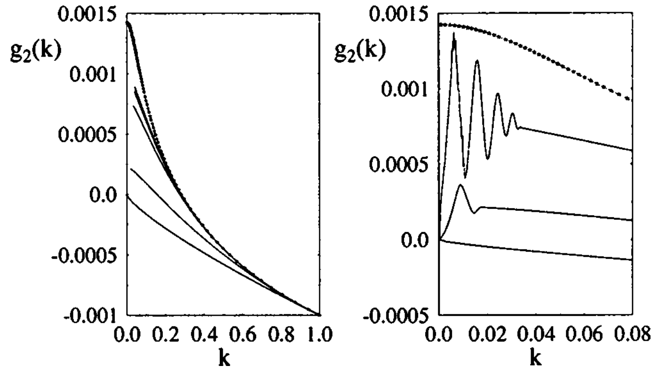

FIG. 1. Comparison of the scaling of the dimensionful coupling constant g_2 obtained for the polynomial (dashed line with stars) and for the periodic potential for various values of the truncation $N = 2,3,10,20,30$ in the Fourier series. The figures on the left (right) show the scaling of the coupling above (below) k_{cr}. Below k_{cr} we show the scaling of g_2 for the cases $N = 2,3,10$. The increasing order of the full-line curves corresponds to increasing N.

constants g_2 and g_4 for different truncations N. Increasing the value of N the differences in the results obtained for the periodic potential decrease.

There are relevant coupling constants for $\beta^2 < 8\pi$ which become large enough to destabilize the trivial saddle point of the blocking and we reach a nonvanishing critical value k_{cr} where the saddle point becomes nontrivial and the tree-level blocking Eq. (10) must be used. By following the solution of this equation all dimensionful Fourier-amplitudes are found to approach zero as $k \to 0$. The typical behavior is depicted in Figs. 1 and 2.

There is a remarkable difference in the behavior of the theory with a periodic potential and that with the corresponding polynomial potential [26]. Namely, that all the dimensionful coupling constants $g_n(k)$ obtained for the periodic potential tend to zero in contrary to those of the polynomial potential which remain finite as $k \to 0$. In Fig. 3 we show this flattening starting from the value k_{cr}.

Integrating numerically Eq. (14) we have shown that under the successive infinitesimal RG transformations the periodic potential becomes a constant potential up to the accuracy 10^{-5}. We compared our results with the analytic formula for the saddle point amplitude $\rho_k(\phi)$ obtained for the polynomial potential [19]:

$$\rho_k(\phi) = \tfrac{1}{2}[\phi_{vac}(k) - |\phi|]. \tag{19}$$

For periodic potentials $V(\Phi + \Delta) = V(\Phi)$, the amplitude $\rho_k(\phi)$ should be periodic in the field variable with the length of period Δ for any scale k [see Eq. (14)]. Thus, for periodic potentials, an expression similar to Eq. (19) is valid in the period $\phi \in [-\Delta/2, \Delta/2]$ and then the same pattern of the function $\rho_k(\phi)$ is repeated in all other periods. In the particular case investigated by us, the minus sign in expression (19) should be changed to a plus sign and the field independent term is $\phi_{vac}^{per}(k_{cr}) = -\Delta/2$ and $\phi_{vac}^{per}(0) = 0$. Therefore we can compare the result $\rho_{k=0}^{per}(\phi) = \frac{1}{2}|\phi|$ with that obtained by numerical integration of Eq. (14), see Fig. 4. We have established that with the increasing number of Fourier-modes taken into account the computed curves get closer to the dashed line defined by $\rho_{k=0}^{per}(\phi) = \frac{1}{2}|\phi|$.

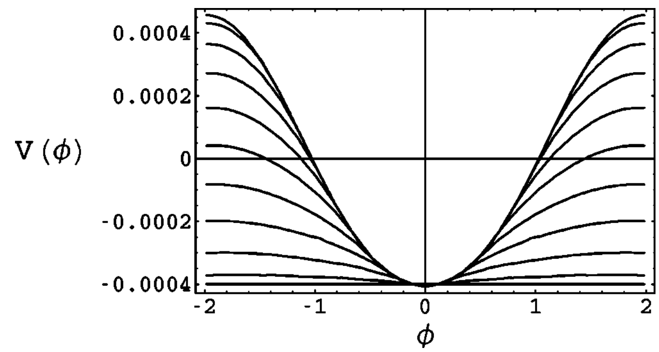

FIG. 3. Flattening of the periodic potential below k_{cr}. The decreasing order of the curves corresponds to decreasing values of the scale k with the step $\delta k = k_{cr}/10$.

VII. SPONTANEOUS SYMMETRY BREAKING

We found two different phases of the model, $\beta^2 < 8\pi$ and $\beta^2 > 8\pi$, for small u with different scaling laws. The effective potential flattens out in either case. How can we reconcile this result with the usual perturbation expansion of the

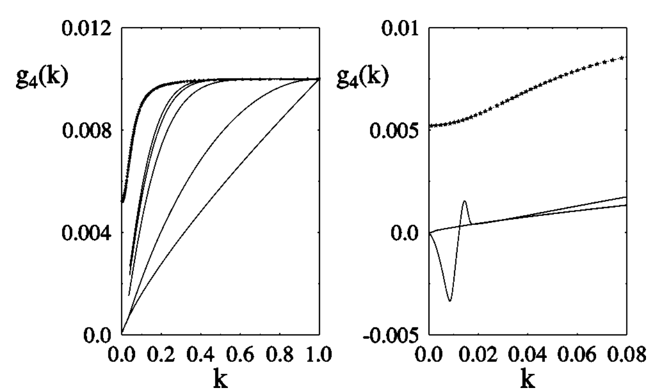

FIG. 2. The scaling of the dimensionful coupling constant g_4 in the same cases as for g_2 in Fig. 1. The figures on the left (right) show the scaling of the coupling above (below) k_{cr}.

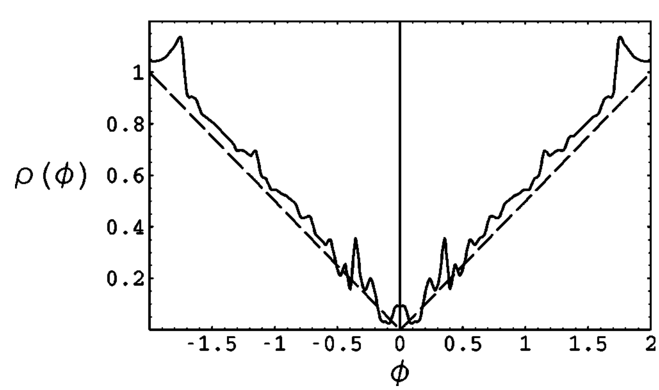

FIG. 4. Comparison of the function $\rho_{k=0}(\phi)$ obtained by numerical integration of Eq. (14) with the analytic expression $\rho_{k=0}^{per}(\phi) = \frac{1}{2}|\phi|$ (dashed line).

sine-Gordon model [1] where one expects small fluctuations around one of the minima of the periodic potential for $\beta^2 < 8\pi$? What does happen with the symmetry $\phi \to \phi + 2\pi$? The flattening of the effective potential makes the naive order parameter to detect the realization of the periodicity, $\int dx \phi(x)$, useless. What is the corresponding order parameter?

The common difficulty behind these questions is that the physical configuration space is actually multiply connected. In fact, the sine-Gordon Hamiltonian describes a family of coupled pendulums and the configurations $\phi(x)$ and $\phi(x) + \Delta$ are physically indistinguishable. We may use either the multiply connected description in terms of the complex variable $e^{2\pi i \phi(x)/\Delta} \in \mathrm{U}(1)$ or the unconstrained $\phi(x) \in \mathcal{R}$ variable on the covering space for the description of the system, and the transformation (11) belongs to the fundamental group $\pi_1[\mathrm{U}(1)] = Z$.

The special difficulty in detecting the spontaneous breakdown of the fundamental group symmetry is the degeneracy of the vacuum energy density, the only resolution of the conflict between the periodicity and the convexity of the effective potential in either phase. The flattening of the effective action in the region of the Maxwell-cut indicates the weakness of the restoring force acting on the fluctuations around the equilibrium position and raises the possibility of the collapse of the topological stability of the solitons. This, in turn, suggests the winding number as an order parameter to distinguish between the phases with the usual and the unusual realization of the fundamental group symmetry. This circumstance reflects a further similarity with gauge models at finite temperature [27].

To understand these questions better we start by distinguishing the local potential of the effective theory $V_k(\phi)$ with a low cutoff k from the effective potential, $V_{eff}(\phi) = V_{k=0}(\phi)$. In theories with infrared stable dynamics the limit $k \to 0$ is safe and this difference is negligible. But the infrared instability inducing the mixed phase and the Maxwell cut in the effective potential makes the limit $k \to 0$ more involved even for theories with massive particles only [19]. By turning this complication into an advantage, we suggest to consider the infrared instabilities as the signature of the spontaneous symmetry breaking.

Let us consider the dynamics of the modes with momentum $p < k \neq 0$. The local potential $V_k(\phi)$ of the corresponding effective theory is nontrivial and periodic and the perturbation expansion might be justified around one of the minima for $k > 0$. The real question is the relative speed the different coupling constants approach 0 as $k \to 0$, and whether the saddle point and the perturbation expansion remain consistent in this limit. As mentioned above, we adopt the infrared instability, i.e., the stability of the trivial saddle point of the blocking as a signature of the absence of the change of the symmetry pattern of the model.

The saddle point of the blocking remains trivial in the ionic phase and we expect no spontaneous symmetry breaking there. The coupling constants decrease as $k \to 0$, the theory becomes trivial as it happens with other nonrenormalizable models. The infinitesimal fluctuations gradually fill up the valleys of the local potential and the naive order parameter for the periodicity, $\int dx \phi(x)$, decouples.

The large amplitude, tree level fluctuations which lie beyond the realm of the perturbation expansion but are picked up by the tree-level renormalization start to fill up the valleys of the potential in the molecular phase already at a finite scale, $k < k_{cr}$. We take this instability as the indication of spontaneous symmetry breaking. This seems to be in agreement with the perturbative approach which is meaningful only when the potential has a nontrivial structure.

In order to identify the order parameter for the periodicity we elucidate the topological differences between the two phases. Let us introduce periodic boundary conditions in the Euclidean space-time and consider the following quantities. One of them is the winding number,

$$Q(x^0) = \int dx^1 j_0(x^0, x^1), \tag{20}$$

the space integral of the topological current density at a given time, x^0. We shall argue that this is the order parameter for the fundamental group symmetry. One can construct another independent topological invariant by exchanging the time and the space axes. But this does not modify the discussion what follows. The other quantities are the vorticities of the space-time regions before and after the time x^0,

$$V_\pm = \int_{\pm(x^0 - y^0) > 0} d^2y \, \partial_\mu j_\mu(y). \tag{21}$$

The flux of the topological current agrees with the vorticity of the enclosed vortices $Q = V_- = -V_+$ [11].

In the molecular phase the distance between the vortex-antivortex pairs is shrinking with the increasing UV cutoff and the quantum fluctuations cannot change the value of Q. This makes the path integration consistent when constrained within a homotopy class, characterized by a fixed value of Q. In fact, the consistency of the path integration is the requirement that the path integral as the function of the end point of the trajectories should satisfy the (functional) Schrödinger equation. The Schrödinger equation can be derived for the path integral by performing infinitesimal variations on the trajectories at the final time. Thus the path integral is consistent if the functional space over which it is evaluated is closed with respect to infinitesimal deformation of the trajectories. Since the discontinuous field configurations are suppressed in the path integral the quantization process is well defined in a given homotopy class [27]. Note that the consistent constraining the functional integration into a given homotopy class removes the fundamental group symmetry. This is obvious in the semiclassical quantization of a soliton where the dynamical stability of the whole construction comes from the spontaneous breakdown, the suppression of the winding number changing processes. Thus the condition for the stability of the solitons, the sufficient smoothness of the configurations dominating the path integral, is in the same time the signature of the breakdown of the fundamental group symmetry.

There are vortex-antivortex pairs with cutoff independent separation in the ionized phase and Q fluctuates in an uncontrollable manner. The path integration cannot be constrained into a given homotopy sector, the fundamental group symmetry is realized but the semiclassical structure based on smoothness is destroyed, the solitons become unstable. In the same time the bosonization relations are either lost or fundamentally modified because the soliton (fermion number) nonconserving processes require the introduction of operators with odd powers of the fermion field in the action.

The susceptibility of the topological charge,

$$\chi = \langle Q^2 \rangle - \langle Q \rangle^2 \tag{22}$$

may serve as a disorder parameter to distinguish the different realizations of the periodicity. In fact, the vortex fugacity tends to zero or stays finite in the molecular phase or the ionized phase when the cutoff is removed according to the Kosterlitz-Thouless RG flow. Since the topological susceptibility is vanishing whenever the fugacity is zero, the former may serve as an order parameter.

VIII. SUMMARY

A simple two dimensional scalar model with periodic potential was investigated in this paper. The first question considered was the renormalization of the potential. The study of this problem requires the handling of infinitely many operators which was achieved by the Wegner-Houghton equation. We found a disordered phase where the model is nonrenormalizable and trivial. In another, ordered phase the relevant operators compatible with the symmetry were identified.

The periodicity and the convexity impose triviality on the effective potential, a phenomenon verified in detail in both phases. The coupling constants approach zero regularly in the disordered phase as the theory becomes trivial, leading to the flattening of the effective potential. Instabilities were found in the ordered phase where the effective potential is flattened out by the Maxwell construction only, indicating a nontrivial dynamics behind the trivial final result.

It was pointed out that the configuration space of the model is multiply connected and that the winding number is suggested as a nonlocal order parameter distinguishing the explicit realization and the spontaneous breakdown of the fundamental group, the shift of the field variable by the period length of the potential. The relation between the applicability of the semiclassical arguments and the spontaneous breakdown of the fundamental group symmetry is shown in the context of our model.

We have started to extend this work to include the wave function renormalization constant for the field, and to investigate the phase structure in higher dimensions. The scalar model with periodic potential can be considered as a nonlinear $O(2)$ model with a symmetry breaking term, and one expects similarities with non-Abelian gauge theories due to the compact nature of the dynamical variable. In fact, the molecular phase is asymptotically free and the vacuum is filled up with vortices in its large distance structure. Furthermore, the spontaneous breakdown of the fundamental group symmetry is reminiscent of the breakdown of the center symmetry in finite temperature gauge theories. Thus, our results offer interesting lessons to be learned in constructing the quark confinement mechanism.

We ignored the wave function renormalization constant in this work. In order to go beyond this approximation and to take into account some higher order contributions in the gradient expansion one has to turn to the evolution equations for the effective action, instead of the bare action [15,17,18]. But the problem is just that the effective action, appearing in the renormalization group treatment, hides a large part of the dynamics due to the Maxwell cut. This is because the effective action, obtained by a smooth cutoff, is after all the Legendre transform of the logarithm of a path integral which does contain the IR modes. These modes, though they appear with small amplitude, generate the Maxwell cut. It is not clear to us how to improve the renormalization group method to make it applicable for models with spontaneous symmetry breaking or with compact variables (e.g., gauge models with gauge fixing which are based on compact gauge group) beyond the local potential approximation.

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