Gauge invariant derivative expansion of the effective action at finite temperature and density and the scalar field in 2+1 dimensions

C. García-Recio and L. L. Salcedo
Departamento de Física Moderna, Universidad de Granada, E-18071 Granada, Spain
(Received 25 July 2000; published 26 January 2001)

A method is presented for the computation of the one-loop effective action at finite temperature and density. The method is based on an expansion in the number of spatial covariant derivatives. It applies to general background field configurations with an arbitrary internal symmetry group and space-time dependence. Full invariance under small and large gauge transformations is preserved without assuming stationary or Abelian fields or fixing the gauge. The method is applied to the computation of the effective action of spin zero particles in $2+1$ dimensions at finite temperature and density and in the presence of background gauge fields. The calculation is carried out through second order in the number of spatial covariant derivatives. Some limiting cases are worked out.

DOI: 10.1103/PhysRevD.63.045016 PACS number(s): 11.10.Wx, 05.30.Jp, 11.10.Kk

I. INTRODUCTION

In their pioneering work, Deser, Jackiw, and Templeton [1] noted that gauge theories in odd-dimensional spaces naturally admit a local term of topological nature, known as the Chern-Simons term [2] (see Ref. [3] for a recent review). One of the interesting properties of the non-Abelian Chern-Simons term is that under gauge transformations it changes proportionally to the winding number of the transformation. Thus when the action contains a Chern-Simons term, the partition functional of the system is only well defined if the coupling constant of the Chern-Simons term is properly quantized.

It was later realized [4–6] that the Chern-Simons term is induced by quantum fluctuations when gauge fields are coupled to odd-dimensional fermions. Such a term comes out with the correctly quantized coupling constant and so full gauge invariance is preserved (although possibly at the price of spoiling parity invariance [4]). Because the Chern-Simons term is a polynomial in the gauge fields and their derivatives, it can be obtained through a combination of perturbative and derivative expansions.

In the so-called imaginary time formalism for field theory at finite temperature [7,8], the space time has a nontrivial topology, since the time is effectively compactified to a circle. This allows the existence of topologically large gauge transformations even in the Abelian case. When the problem of the induced Chern-Simons term is studied using the method just mentioned of retaining a low number of fields and of derivatives, a puzzling situation appears, namely, the coefficient of the Chern-Simons term turns out to be a smooth function of the temperature, and hence it violates the quantization condition [9]. The situation has recently been clarified by considering a simple $0+1$-dimensional model [10] which can be computed in closed form. There it is seen that full gauge invariance holds for the exact result but it is broken by perturbation theory. This is not difficult to understand, since in simple cases gauge invariance under large gauge transformations is equivalent to periodicity of the effective action as a function of the gauge field, whereas perturbation theory corresponds to a Taylor expansion of that function. Clearly, the property of being periodic is not maintained in general by a truncated Taylor expansion. In Ref. [11] it was noted that full gauge invariance is always a property of the exact result, since it follows straightforwardly from using a ζ-function regularization. In Ref. [12] the exact effective action of fermions in $2+1$ dimensions was obtained for the case of Abelian and stationary background gauge fields.

The problem of preserving full gauge invariance at finite temperature is not tied to odd-dimensional theories [13–17] nor to fermions [18,19]. It appears whenever perturbation theory is involved. This is unfortunate, since, as noted in Ref. [3], ''at finite temperature, perturbation theory is one of the few tools we have.'' In this work we show that it is possible to carry out detailed calculations of the effective action fully preserving gauge invariance, without restricting oneself to particular configurations such as Abelian or stationary ones, and without choosing a particular gauge. The study of simple cases [10–12] shows that the problem with gauge invariance comes through the scalar potential $A_0(x)$. The finite temperature effective action is nonlocal in time but it is local in the space variables. This suggests to consider an expansion in the number of spatial covariant derivatives only. The time component is treated nonperturbatively in order to avoid destroying gauge invariance. (See Ref. [20] for another discussion of derivative expansions at finite temperature.)

It should be emphasized that the expansion in the number of spatial covariant derivatives is not tied to a particular method of computation, since it can be obtained from the exact result (namely, by considering an appropriate spatial dilatation of the background fields) and is fully gauge invariant. What is shown here is that it is also amenable to explicit computation order by order, through a combination of the method of symbols and a ζ function, for instance. This combination works very well and has been applied at zero temperature for fermions with local [21] and nonlocal actions [22]. At finite temperature it has been applied to fermions in odd dimensions [23] as well as in even dimensions [17],

however, for technical reasons, this has been done choosing a particular gauge. In the present work we remove the necessity of any choice of gauge. It turns out that previous formulas in Refs. [23] and [17] can be reinterpreted and rewritten in a manifest gauge invariant form.

Although a naive perturbative expansion in A_0 breaks gauge invariance, the very method of calculation suggests an expansion in powers of the temporal covariant derivative in the adjoint representation which preserves gauge invariance. This yields the remarkable result that, even at finite temperature, the theory is local if expressed in the appropriate variables.

The method is explicitly applied to the case of relativistic scalar particles in $2+1$ dimensions. The computation is carried out through second order in the number of spatial covariant derivatives. Contact is made with the relativistic Bose gas.

II. GAUGE INVARIANT DERIVATIVE EXPANSION AT FINITE TEMPERATURE

A. The mathematical problem

The aim of this section is to present a scheme to address the computation of the one-loop effective action at finite temperature preserving gauge invariance at every step. (We note that gauge invariance in this work refers always to vector gauge transformations.) To fix ideas consider the case of scalar particles in $d+1$ dimensions in presence of background gauge fields. This case will be worked out later for $d=2$. The Euclidean action of the system is

$$S=\int d^{d+1}x[(D_\mu\phi)^\dagger(D_\mu\phi)+m^2\phi^\dagger\phi], \tag{1}$$

where $D_\mu=\partial_\mu+A_\mu$ is the covariant derivative. The finite temperature condition can be implemented by using the imaginary time formalism, that is, by compactification of the Euclidean time to a circle so that the fields ϕ and A_μ are periodic functions of x_0 with period $\beta=1/T$ (T being the temperature). After functional integration over $\phi(x)$, the Euclidean effective action is formally given by

$$W_s[m,A]=\mathrm{Tr}_b\log(-D_\mu^2+m^2). \tag{2}$$

The subindex b recalls that the functional trace is to be taken in the Hilbert space of bosonic wave functions, i.e., with periodic boundary conditions.

Presently the mathematical problem to be addressed is the computation of quantities of the form

$$\Gamma[M,A]=\mathrm{Tr}(f(M,D)), \tag{3}$$

where D_μ is the covariant derivative and $M(x)$ collectively denote one or more matrix valued functions of x_μ representing other external fields in addition to the gauge fields. The trace refers to the Hilbert space $\mathcal{H}$ of wave functions with space time and internal degrees of freedom, the space-time manifold has topology $\mathcal{M}_{d+1}=\mathrm{S}^1\times\mathcal{M}_d$ and the wave functions are periodic for bosons and antiperiodic for fermions.

Although slightly pedantic, it will occasionally be convenient to regard M not as functions but rather as multiplicative operators in $\mathcal{H}$, i.e., operators commuting with the operators x_μ and otherwise with arbitrary structure in internal space. Likewise D_μ are differential operators of the form $\partial_\mu+A_\mu$ with A_μ multiplicative. The quantity $f(M,D)$ denotes an operator constructed out of M and D_μ in the algebraic sense, that is, $f(M,D)$ is a linear combination (or series) of products of M and D_μ multiplied in any order with constant c-number coefficients. In order for M and D_μ to be well-defined operators in $\mathcal{H}$, $M(x)$, and $A_\mu(x)$ are required to be periodic functions of x_0. In addition we will assume that the fields are sufficiently convergent at infinity and the function f is well behaved. This means, in particular, that f is one-valued and sufficiently convergent at infinity as a function of D_μ to ensure the existence of the trace (by avoiding ultraviolet divergences).

A gauge transformed configuration (M^U,A^U) is one of the form

$$\begin{aligned} M^U(x)&=U^{-1}(x)M(x)U(x),\\ A_\mu^U(x)&=U^{-1}(x)\partial_\mu U(x)+U^{-1}(x)A_\mu(x)U(x), \end{aligned} \tag{4}$$

where the gauge transformation $U(x)$ is a periodic function of x_0 which takes values on matrices in internal space. This corresponds to a similarity transformation of D_μ, namely, $D_\mu^U=\partial_\mu+A_\mu^U=U^{-1}D_\mu U$ where U is to be regarded as a multiplicative operator in $\mathcal{H}$. Because $f(M,D)$ is constructed with M, D and c numbers, it follows that $f(M,D)$ also transforms under a similarity transformation

$$f(M^U,D^U)=U^{-1}f(M,D)U \tag{5}$$

and so

$$\Gamma[M^U,A^U]=\Gamma[M,A] \tag{6}$$

using the cyclic property of the trace, which holds due to our regularity assumptions for $f(M,D)$.[1]

B. The method of symbols

Assuming that the operator $\hat{f}=f(M,D)$ admits a complete set of eigenfunctions, $\hat{f}|n\rangle=\lambda_n|n\rangle$, the functional trace is simply $\Gamma[M,A]=\Sigma_n\lambda_n$. In this form gauge invariance is obvious since $\hat{f}$ and $\hat{f}^U$ are related by a similarity transformation and hence they have the same spectrum.

The gauge invariance of $\Gamma[M,A]$ is also manifest computing the trace in the basis $|x\rangle$ of eigenfunctions of x_μ,

[1]In practice, $\Gamma[M,A]$ is only computed for a subset of configurations (M,A) and only the subgroup of gauge transformations which leave invariant such a subset are relevant. For fermions, the internal space includes Dirac space as well as flavor degrees of freedom, and the γ_μ matrices are included in M (they are not c numbers). In this case only gauge transformations in flavor space are relevant since they are the ones that preserve the form of γ_μ.

normalized as $\langle x|x'\rangle=\delta(x-x')$ (a periodic delta function in the temporal direction),

$$\Gamma[M,A]=\int d^{d+1}x\ \mathrm{tr}\langle x|f(M,D)|x\rangle \tag{7}$$

(where tr refers to internal space) because U is a multiplicative operator and so $\mathrm{tr}\langle x|f(M,D)|x\rangle$ is gauge invariant without integration over x. However, computationally it is more convenient to use a basis in momentum space $|p\rangle$,

$$\langle x|p\rangle=e^{px},\quad \langle p|p'\rangle=\beta\delta_{p_0p_0'}(2\pi)^d\delta(\boldsymbol{p}-\boldsymbol{p}') \tag{8}$$

[to avoid unessential factors of i we take the convention of using purely imaginary momenta p_μ but $\int d^d\boldsymbol{p}$ below denotes the usual integral in $\mathbb{R}^d$ and $\delta(\boldsymbol{p}-\boldsymbol{p}')$ denotes the corresponding delta function]. The frequency takes the Matsubara values $p_0=2\pi in/\beta$ for bosons and $p_0=2\pi i(n+\frac{1}{2})/\beta$ for fermions. Note that we have assumed that the space manifold $\mathcal{M}_d$ has a topology $\mathbb{R}^d$. In this basis

$$\Gamma[M,A]=\frac{1}{\beta}\sum_{p_0}\int\frac{d^d\boldsymbol{p}}{(2\pi)^d}\mathrm{tr}\langle p|f(M,D)|p\rangle. \tag{9}$$

At this point the symbols method can be used (see, e.g., Refs. [21] and [24]): let $|0\rangle$ denote the state with $p=0$, then using the identities $|p\rangle=e^{xp}|0\rangle$ (where e^{xp} acts as a multiplicative operator and the quantities p_μ are constant c numbers) as well as $e^{-xp}D_\mu e^{xp}=D_\mu+p_\mu$, $e^{-xp}Me^{xp}=M$, one obtains

$$\begin{aligned}\langle p|f(M,D)|p\rangle&=\langle 0|e^{-xp}f(M,D)e^{xp}|0\rangle\\&=\langle 0|f(M,D+p)|0\rangle,\end{aligned} \tag{10}$$

and so the functional trace can be cast in the form

$$\Gamma[M,A]=\frac{1}{\beta}\sum_{p_0}\int\frac{d^d\boldsymbol{p}}{(2\pi)^d}\mathrm{tr}\langle 0|f(M,D+p)|0\rangle. \tag{11}$$

In this expression it is clear the requirement of regularity on f: the functional trace comes after integration over momenta and sum over frequencies and this requires f to be sufficiently convergent for large p_μ. Let us remark that $|0\rangle$ is periodic rather than antiperiodic in the temporal direction. The information on whether we are dealing with bosons or fermions is now contained solely in the values taken by p_0. The state $|0\rangle$ satisfies

$$\langle x|0\rangle=1,\quad \partial_\mu|0\rangle=\langle 0|\partial_\mu=0,\quad \langle 0|0\rangle=\int d^{d+1}x. \tag{12}$$

In addition, when $\hat{h}$ is a multiplicative operator, $\hat{h}|x\rangle=h(x)|x\rangle$,

$$\langle 0|\hat{h}|0\rangle=\int d^{d+1}x\,h(x). \tag{13}$$

It follows that ∂_μ appearing inside $f(M,D+p)$ in Eq. (11) acts derivating everything to its right (or its left, by parts) and then vanishes after it reaches $|0\rangle$ (or $\langle 0|$). This is a well-defined working rule and from it stems the usefulness of the symbols method.

Unfortunately, gauge invariance is no longer manifest when using the momentum basis. In fact, $\mathrm{tr}\langle 0|f(M,D+p)|0\rangle$ is not gauge invariant because $|0\rangle$ (or more generally $|p\rangle$) is not covariant under local transformations. For instance, according to the rules given in Eq. (12),

$$\mathrm{tr}\langle 0|D_\mu^2|0\rangle=\mathrm{tr}\langle 0|A_\mu^2|0\rangle=\int d^{d+1}x\ \mathrm{tr}[A_\mu^2(x)] \tag{14}$$

breaks gauge invariance. However,

$$\mathrm{tr}\langle 0|[D_\mu,D_\nu]^2|0\rangle=\int d^{d+1}x\ \mathrm{tr}[F_{\mu\nu}^2(x)] \tag{15}$$

does not. Note that $[D_\mu,D_\nu]$ is a multiplicative operator whereas D_μ^2 is not. As a rule, when an operator $g(M,D)$ (a gauge covariant operator) is multiplicative, Eq. (13) applies and $\mathrm{tr}\langle 0|g(M,D)|0\rangle$ is gauge invariant [21]. In Eq. (11) gauge invariance is only recovered after integration over momenta and sum over frequencies.[2] This will be further discussed subsequently.

C. The derivative expansion at finite temperature

By computing the functional trace we essentially mean to end up with purely multiplicative operators, since this implies that the functional is expressed as the integral of a function over space time. At zero temperature this is usually equivalent to saying that all derivative operators D_μ appear inside commutators. In addition it means to carry out as many implied sums and integrations (over frequencies and momenta or other parameters) as possible.

In general it is not possible to compute $\Gamma[M,A]$ in closed form and one must resort to approximations. The standard approach is to make power expansions in one or more operators appearing in $f(M,D)$ while the remaining operators are treated nonperturbatively. As will be clear below, a naive expansion in powers of D_0 would break gauge invariance, therefore, because it is in general difficult to work with two or more nonperturbative operators unless they are commuting, and our present emphasis is in the preservation of manifest gauge invariance rather than in a particular computation, we will keep D_0 as the only operator to be treated nonperturbatively, and expand in all other operators M and $\boldsymbol{D}$.

Before proceeding, let us be more precise about the meaning of expanding in powers of M and $\boldsymbol{D}$. A convenient way to define the expansion is by introducing constant c-number

[2] An elegant method has been presented in Ref. [24] which yields gauge invariant expressions prior to momentum integration, at the price of introducing derivatives with respect to p_μ. The method has not yet been extended to include discrete momenta, as required at finite temperature but it can be applied to the integration over $\boldsymbol{p}$.

bookkeeping parameters, $f(M,D_0,\boldsymbol{D})\to f(\lambda_1 M,D_0,\lambda_2\boldsymbol{D})$, so that counting powers of those operators is equivalent to counting powers of $\lambda_{1,2}$. This procedure preserves gauge invariance since it amounts to a modification of the function f. After applying the symbols method [through $D_\mu\to D_\mu+p_\mu$, cf. Eq. (11)] the factor λ_2 will affect $\boldsymbol{D}$ and $\boldsymbol{p}$, however, this can be brought to the form $f(\lambda_1 M,D_0+p_0,\lambda_2\boldsymbol{D}+\boldsymbol{p})$ by a redefinition of $\boldsymbol{p}$. Therefore the expansion can be formulated as an expansion in powers of M and $\boldsymbol{D}$ in $f(M,D+p)$.[3] Equivalently, the expansion in $\boldsymbol{D}$ can be obtained directly from the functional $\Gamma[M,A]$ by means of a covariant spatial dilatation, namely, $M(x_0,\boldsymbol{x})\to M(x_0,\lambda_2\boldsymbol{x})$, $A_0(x_0,\boldsymbol{x})\to A_0(x_0,\lambda_2\boldsymbol{x})$, and $\boldsymbol{A}(x_0,\boldsymbol{x})\to\lambda_2\boldsymbol{A}(x_0,\lambda_2\boldsymbol{x})$. This guarantees that the expansion is well defined, i.e., it depends on the functional itself and not on how it is written or computed.

The situation is completely different for an expansion in powers of D_0. A bookkeeping parameter $D_0\to\lambda_3 D_0$ can be introduced in $f(M,D)$ and this defines a new λ_3-dependent gauge invariant functional. Nevertheless this functional is not useful since it presents an essential singularity at $\lambda_3=0$, as can be seen in the simple case of fermions in $0+1$ dimensions [10]. (The dependence on $\lambda_{1,2}$ is analytic or at least asymptotic under suitable regularity conditions on the fields and on the function f.) After applying the symbols method, $f(M,\lambda_3 D_0+\lambda_3 p_0,\boldsymbol{D})$ is obtained. However, because p_0 is a discrete variable this is not equivalent to $f(M,\lambda_3 D_0+p_0,\boldsymbol{D})$. Therefore expanding in the explicit D_0 in $f(M,D+p)$ does not correspond to a modification of f and in fact violates gauge invariance. Also, it is not possible to introduce λ_3 by means of a rescaling of type $x_0\to\lambda_3 x_0$ of the field configuration (M,A) since this transformation violates the periodicity condition on the wave functions of $\mathcal{H}$.

After expansion there will be all kind of terms which will be products of single factors of M and $\boldsymbol{D}$ as well as operators depending nonperturbatively on D_0 (by this we merely mean that all orders of D_0 are retained). It is always possible to bring all $\boldsymbol{D}$ operators to the right producing commutators, so that we end up with two kind of terms: (i) terms in which all operators $\boldsymbol{D}$ appear only in commutators (more precisely, in the form $[\boldsymbol{D},]$) and (ii) terms with unsaturated factors $\boldsymbol{D}$ at the right (i.e., $\boldsymbol{D}$ not inside a commutator). The terms of the first type are multiplicative operators regarding $\boldsymbol{x}$ space, although they are still differential (or pseudodifferential) operators with respect to x_0 space. The terms of the second type are nonmultiplicative in $\boldsymbol{x}$ space. As we have argued above, these latter terms break gauge invariance and in fact they will cancel after integration over $\boldsymbol{p}$. This can be seen as follows: let us replace $\boldsymbol{D}$ by $\boldsymbol{D}+\boldsymbol{a}$, where $\boldsymbol{a}$ is a constant c number. This replacement has no effect on the terms where $\boldsymbol{D}$ is in commutators, but counts the contribution from the terms with unsaturated $\boldsymbol{D}$. However, it is clear that there is no such contribution after integration over momenta since $\boldsymbol{a}$ can be compensated by a similar shift in the integration variable $\boldsymbol{p}$. Thus at the end, all operators $\boldsymbol{D}$ appear in commutators only. (The same result is obtained directly using the method of Pletnev and Banin [24].) A similar argument would break down for D_0: a shift to D_0+a_0 cannot, in general, be compensated by a shift in p_0 since at finite temperature the frequency is a discrete variable.

From the previous discussion it follows that we only have to retain those terms where all operators $\boldsymbol{D}$ are in commutators. Because all operators are now multiplicative in $\boldsymbol{x}$ space, $\boldsymbol{x}$ becomes just a parameter in what follows. Let us consider a typical term:

$$\mathrm{TT}=\frac{1}{\beta}\sum_{p_0}\int\frac{d^d\boldsymbol{p}}{(2\pi)^d}\,\mathrm{tr}\langle 0|\alpha_1(D_0+p_0,\boldsymbol{p})\times X\alpha_2(D_0+p_0,\boldsymbol{p})Y\alpha_3(D_0+p_0,\boldsymbol{p})|0\rangle. \tag{16}$$

X and Y are multiplicative and gauge covariant operators constructed with D_μ and M, and the $\alpha_i(x,y)$ are some functions. At this point the integration over p_μ is nontrivial [even for the simplest forms of the functions $\alpha_i(x,y)$] because p_μ appears in different and noncommuting operators. A possible approach is to express the operators in terms of their matrix elements using as basis a complete set of eigenstates of D_0 (in the Hilbert space of time and internal degrees of freedom). These matrix elements are then ordinary functions of p_μ. Instead of that, we will use the equivalent prescription of labelling the operators D_0 according to their position with respect to X and Y: the symbols D_{01}, D_{02}, and D_{03} will be used to denote the operator D_0 in positions 1 (before X), 2 (between X and Y), and 3 (after Y), respectively. In this notation,

$$\mathrm{TT}=\frac{1}{\beta}\sum_{p_0}\int\frac{d^d\boldsymbol{p}}{(2\pi)^d}\,\mathrm{tr}\langle 0|\alpha_1(D_{01}+p_0,\boldsymbol{p})\times\alpha_2(D_{02}+p_0,\boldsymbol{p})\alpha_3(D_{03}+p_0,\boldsymbol{p})XY|0\rangle. \tag{17}$$

An immediate consequence is that the labeled operators are effectively commuting and the momentum integration and frequency summation can be carried out as for ordinary functions. The result can be written as

$$\mathrm{TT}=\mathrm{tr}\langle 0|g(D_{01},D_{02},D_{03})XY|0\rangle, \tag{18}$$

where the function g is defined by

$$g(x,y,z)=\frac{1}{\beta}\sum_{p_0}\int\frac{d^d\boldsymbol{p}}{(2\pi)^d}\,\alpha_1(x+p_0,\boldsymbol{p})\alpha_2(y+p_0,\boldsymbol{p})\times\alpha_3(z+p_0,\boldsymbol{p}). \tag{19}$$

(Note that there will be two versions of g, the bosonic one and the fermionic one, which are related by a shift of $i\pi/\beta$ in their arguments.) By construction the function g is periodic:

[3]Of course, the actual expansion in powers of $\boldsymbol{D}$ must be done after arriving at Eq. (11) (or other similar formulas in other approaches, such as Schwinger proper time method), i.e., after D_μ has become $D_\mu+p_\mu$, since otherwise powers of p_μ would be generated as well and that would destroy the ultraviolet convergence of the formula.

$$g(x,y,z)=g\left(x+\frac{2\pi i}{\beta},y+\frac{2\pi i}{\beta},z+\frac{2\pi i}{\beta}\right). \tag{20}$$

This is an immediate consequence of the sum over Matsubara frequencies and ultraviolet convergence of the expressions.

This periodicity property is essential to codify the gauge invariance of the original expression. To see this, let us introduce the operation $\mathcal{D}_\mu$ which is defined as $\mathcal{D}_\mu X=[D_\mu,X]$ for any operator X. Consistently with our previous notation, we will denote by $\mathcal{D}_{01}$ the action of $\mathcal{D}_0$ in position 1 (i.e., on X) and by $\mathcal{D}_{02}$ the action of $\mathcal{D}_0$ in position 2 (i.e., on Y). Then clearly

$$\mathcal{D}_{01}=D_{01}-D_{02},\quad \mathcal{D}_{02}=D_{02}-D_{03}. \tag{21}$$

The interesting point is that these formulas hold for arbitrary functions of $\mathcal{D}_{01}$ and $\mathcal{D}_{02}$ as well. This follows from the well-known identity $e^A B e^{-A}=e^{[A,]}B$: for any c number λ,

$$e^{\lambda(D_{01}-D_{02})}XY=e^{\lambda D_0}Xe^{-\lambda D_0}Y=(e^{\lambda\mathcal{D}_0}X)Y=e^{\lambda\mathcal{D}_{01}}XY, \tag{22}$$

and this identity immediately extends to arbitrary functions of $D_{01}-D_{02}$, and analogously for $\mathcal{D}_{02}$. This allows us to make everywhere the replacements

$$D_{02}=D_{01}-\mathcal{D}_{01},\quad D_{03}=D_{01}-\mathcal{D}_{01}-\mathcal{D}_{02} \tag{23}$$

and use D_{01}, $\mathcal{D}_{01}$, and $\mathcal{D}_{02}$ as the independent variables to work with. The advantage of doing this is that the action of $\mathcal{D}_{01}$ and $\mathcal{D}_{02}$ on X and Y produces multiplicative and gauge covariant operators. On the other hand, the presence of the operator D_{01}, which is outside commutators, combined with the gauge noncovariant operation $\langle 0|\ |0\rangle$, still can introduce gauge noninvariant contributions. This is avoided thanks to the periodicity property of $g(x,y,z)$ as we will show now. Indeed, the periodicity property allows to write the term as

$$\mathrm{TT}=\mathrm{tr}\langle 0|\varphi(e^{-\beta D_{01}},\mathcal{D}_{01},\mathcal{D}_{02})XY|0\rangle, \tag{24}$$

where the function φ is defined by

$$\varphi(e^{-\beta x},y,z)=g(x,x-y,x-y-z). \tag{25}$$

The periodicity condition of g ensures that the function $\varphi(\omega,y,z)$ is one-valued [it depends on ω and not just on $\log(\omega)$]. In order to bring the expression into a manifestly gauge invariant form, we will use the following property:

$$e^{-tD_0}=e^{-t\partial_0}Te^{-\int_{x_0}^{x_0+t}A_0(x_0',\mathbf{x})dx_0'},\quad t\geqslant 0. \tag{26}$$

Here t is just a parameter and T denotes time ordered product. (The quantities D_0, ∂_0, and x_μ represent operators and the product refers to a product of operators so that ∂_0 is not directly derivating x_0.) This equation can be easily proved by the standard procedure of showing that the two expressions satisfy the same first order differential equation in t and coincide at $t=0$. The left-hand side is a manifestly gauge covariant operator. It is interesting to see how gauge covariance is realized in the right-hand side: the time-ordered product from x_0 to x_0+t (with $\mathbf{x}$ fixed) transforms with $U(x_0,\mathbf{x})$ at the right and $U^{-1}(x_0+t,\mathbf{x})$ at the left, and this latter factor is transformed into $U^{-1}(x_0,\mathbf{x})$ after commutation with $e^{-t\partial_0}$, thus the product of the two factors transforms covariantly at $(x_0,\mathbf{x})$, as e^{-tD_0}.

In particular, by taking $t=\beta$ in the previous formula, one obtains the identity

$$e^{-\beta D_0}=e^{-\beta\partial_0}\Omega, \tag{27}$$

where

$$\Omega(x)=T\exp\left(-\int_{x_0}^{x_0+\beta}A_0(x_0',\mathbf{x})dx_0'\right). \tag{28}$$

(Again $e^{-\beta\partial_0}\Omega$ is to be understood as the product of two operators.) Beyond the interval $[0,\beta]$ $A_0(x)$ is defined as a periodic function of the time, so $\Omega(x)$ is also periodic. Although $\Omega(x)$ is nonlocal in terms of A_0, it behaves as a local field which takes values on the gauge group. In particular it transforms covariantly at x:

$$\Omega^U(x)=U^{-1}(x)\Omega(x)U(x). \tag{29}$$

The matrices $\Omega(x)$ at different values of x_0, but equal $\mathbf{x}$, are related by similarity transformations and their trace, the Polyakov loop, is independent of x_0. Another important property is

$$\mathcal{D}_0\Omega=[D_0,\Omega]=0. \tag{30}$$

On the other hand, the effect of the operator $\exp(-\beta\partial_0)$ is to produce the shift $x_0\to x_0-\beta$, therefore it is equivalent to the identity operator on the space of periodic functions in which we are working (as noted the periodic wave function $|0\rangle$ appears regardless of whether we are considering bosons of fermions). So in this space

$$e^{-\beta D_0}=\Omega. \tag{31}$$

This produces the manifestly gauge invariant expression

$$\mathrm{TT}=\mathrm{tr}\langle 0|\varphi(\Omega_1,\mathcal{D}_{01},\mathcal{D}_{02})XY|0\rangle. \tag{32}$$

[The label 1 in Ω indicates to put this operator in position 1. The relative order between $\mathcal{D}_{01}$ and Ω_1 is immaterial due to Eq. (30).]

It can be noted that all previous manipulations, starting from Eq. (16), hold also without taking $\mathrm{tr}\langle 0|\ |0\rangle$. Inside $\mathrm{tr}\langle 0|\ |0\rangle$ integration by parts implies that $\mathcal{D}_{01}$ is equivalent to $-\mathcal{D}_{02}$ (or equivalently, that $D_{03}=D_{01}$), and so we have the final formula

$$\mathrm{TT}=\mathrm{tr}\langle 0|\varphi(\Omega_1,\mathcal{D}_{02})XY|0\rangle, \tag{33}$$

where

$$\varphi(\omega,y)=\varphi(\omega,-y,y). \tag{34}$$

The whole point of these manipulations was to end up with a manifestly gauge covariant and multiplicative operator so

that $\mathrm{tr}[\varphi(\Omega_1,\mathcal{D}_{02})XY]$ is just a gauge invariant function of x, constructed with the fields $A_\mu(x)$ and $M(x)$:

$$\mathrm{TT}=\int d^{d+1}x\ \mathrm{tr}[\varphi(\Omega_1,\mathcal{D}_{02})XY]. \tag{35}$$

The fact that $g(x,y,z)$ is periodic is essential to produce the gauge covariant operator $\Omega(x)$. This periodicity property would be lost in an expansion in powers of D_0. We should remark that no restriction has been put on the field configuration, which is completely general (may be non-Abelian and nonstationary) and also no choice of gauge has been needed.

Perhaps it should be emphasized how exactly the lack of periodicity would break gauge invariance. To see this it is sufficient to consider $0+1$-dimensional fermions in presence of an Abelian configuration [10]. The corresponding effective action is of the form $\beta g(a)$ with $a=\int_0^\beta dx_0 A_0$. The quantity a is invariant under topologically small gauge transformations, $A_0^U=A_0+\partial_0\Lambda$ [$\Lambda(x_0)$ being a periodic function] but under a large gauge transformation, e.g., $A_0^U=A_0+2\pi i n/\beta$ [which corresponds to $U(x_0)=\exp(2\pi i n x_0/\beta)$], a changes by an integer multiple of $2\pi i$, so $g(a)$ will not be invariant in general. When g is periodic the effective action becomes $\beta\varphi(\Omega)$ with $\Omega=\exp(-a)$ and it is invariant under all gauge transformations. See Sec. IV A for further remarks.

D. Relation with the calculation fixing the gauge

In Refs. [23] and [17] the kind of calculation just described was carried out for fermions but fixing the gauge through the gauge condition $\partial_0 A_0=0$. (The idea was that the two operators treated not perturbatively, ∂_0 and A_0, are then commuting.) No loss of generality is actually implied by this approach since such a gauge always exists [23]. However, because it is not unique, it is necessary to find all remaining gauge transformations allowed within the A_0-stationary gauge, and then check that all of them produce the same result. This was shown to be equivalent to the periodicity condition that follows from summing over Matsubara frequencies, Eq. (20). All this is unnecessary in the present approach since the gauge has not been fixed. Using gauge invariance, the results obtained within the A_0-stationary gauge can directly be taken over as follows. When $\partial_0 A_0=0$, the field Ω becomes $e^{-\beta A_0}$, so it is only necessary to replace $e^{-\beta A_0}$ of the calculation in A_0-stationary gauge by $\Omega(x)$ to obtain the result expressed in an arbitrary gauge. Further comments are made in Sec. IV A.

E. Expansion in space-time derivatives at finite temperature

As noted, expanding in powers of D_0 breaks periodicity and hence gauge invariance, however, in principle nothing prevents from expanding in powers of $\mathcal{D}_0$ in Eq. (35), namely,

$$\mathrm{TT}=\sum_{n=0}^{\infty}\int d^{d+1}x\ \mathrm{tr}[\varphi_n(\Omega)X\mathcal{D}_0^n Y], \tag{36}$$

where

$$\varphi(\omega,y)=\sum_{n=0}^{\infty}\varphi_n(\omega)y^n. \tag{37}$$

The interest of doing that is, of course, that, at least at lower orders, the result is simpler than the full result. This can be regarded as the finite temperature generalization of the usual derivative expansion at zero temperature. Recall that $\boldsymbol{D}$ already was restricted to appear in commutators only, so this is really an expansion in powers of $\mathcal{D}_\mu$. As usual, higher orders are increasingly ultraviolet convergent. It can be noted that in the Abelian and stationary case $\mathcal{D}_0$ vanish identically [on multiplicative operators, such as X and Y in Eq. (36)] therefore the zeroth order in the above expansion becomes exact.

Nevertheless, it should be noted that such an expansion is not as well defined as for instance the expansion in powers of $\boldsymbol{D}$. The latter is defined from the functional itself, since it corresponds to spatial dilatations of the fields. No such transformation is known for the expansion in powers of $\mathcal{D}_0$. So in principle it should be expected that different ways of expressing the functional in terms of Ω and $\mathcal{D}_0$ would yield different expansions, only the sum of all orders being unambiguously defined. This can be seen more clearly as follows. Recalling that inside $\langle 0||0\rangle$ the operators D_{03} and D_{01} are equivalent, the typical term considered above, Eq. (18), takes the form

$$\mathrm{TT}=\mathrm{tr}\langle 0|g(D_{01},D_{02})XY|0\rangle \tag{38}$$

[with $g(x,y)=g(x,y,x)$]. Using the final form Eq. (33), it is easily established that it can also be written as

$$\mathrm{TT}=\mathrm{tr}\langle 0|g(D_{02},D_{01})YX|0\rangle, \tag{39}$$

because in Eq. (33) all operators are multiplicative and therefore integration by parts and the cyclic symmetry can be used. Now, in the frequent case of contributions where $X=Y$, this implies that only the symmetric part of the function $g(x,y)$ is actually contributing. However, it is easy to write purely antisymmetric and periodic functions $g(x,y)$ such that when used in Eq. (36), each order is nonvanishing, although of course their full contribution vanish when summed to all orders. This particular kind of ambiguity can be fixed by imposing a symmetry restriction on $g(x,y)$ before carrying out the expansion in $\mathcal{D}_0$. This ambiguity is further discussed in Sec. IV B.

F. Illustration of the method

To illustrate the previous manipulations in a practical case, we will consider the quantity

$$C[m,A]=-\frac{1}{4}\mathrm{Tr}_b\left[\left(\frac{1}{-D_\mu^2+m^2}\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}\right)^2\right]+O(F_{\mu\nu}^3), \tag{40}$$

which will appear later in the study of the scalar field in $2+1$ dimensions. In this expression $F_{\mu\nu}=[D_\mu,D_\nu]$ and $\sigma_{\mu\nu}=\frac{1}{2}[\gamma_\mu,\gamma_\nu]$ (where γ_μ are Hermitian Dirac matrices in $2+1$ dimensions). m is a c number.

$C[m,A]$ is ultraviolet finite and so it is a well-defined and unambiguous quantity. We will compute it through second order in an expansion in the number of spatial covariant derivatives. Clearly the expansion starts at second order and it is sufficient to retain the explicit term in Eq. (40) since terms of $O(F_{\mu\nu}^3)$ must contain four or more spatial indices.

First, the symbols method, Eq. (11), is applied. Afterwards, taking the trace in Dirac space and keeping just terms with two spatial covariant derivatives, produces

$$C_2[m,A]=\frac{1}{2}\int\frac{d^2\boldsymbol{p}}{(2\pi)^2}\frac{1}{\beta}\times\sum_{p_0}\langle 0|\mathrm{tr}\left[\left(\frac{1}{-(D_0+p_0)^2+\boldsymbol{p}^2+m^2}\boldsymbol{E}\right)^2\right]|0\rangle, \tag{41}$$

where $\boldsymbol{p}^2=-p_i^2$, $p_0=2\pi i n/\beta$, and $\boldsymbol{E}=[D_0,\boldsymbol{D}]$.

To proceed to the integration over momenta and sum over frequencies we use the trick of adding a label 1 or 2 to the operators D_0 to indicate their actual position in the expression, namely

$$\begin{aligned}C_2[m,A]&=\frac{1}{2}\int\frac{d^2\boldsymbol{p}}{(2\pi)^2}\frac{1}{\beta}\times\sum_{p_0}\langle 0|\mathrm{tr}\left[\frac{1}{-(D_{01}+p_0)^2+\boldsymbol{p}^2+m^2}\times\frac{1}{-(D_{02}+p_0)^2+\boldsymbol{p}^2+m^2}\boldsymbol{E}^2\right]|0\rangle\\&:=\langle 0|\mathrm{tr}[g(D_{01},D_{02})\boldsymbol{E}^2]|0\rangle.\end{aligned} \tag{42}$$

The momentum integration yields

$$g(x_1,x_2)=-\frac{1}{8\pi}\frac{1}{\beta}\sum_{p_0}\frac{1}{(x_1+p_0)^2-(x_2+p_0)^2}\times\log\left(\frac{m^2-(x_1+p_0)^2}{m^2-(x_2+p_0)^2}\right). \tag{43}$$

In order to sum over frequencies, it is convenient to reduce the expression to a rational form. This is achieved by derivating with respect m and then integrating back [using that $g(x_1,x_2)$ vanishes as $m\to\infty$]. Then the identity

$$\sum_n\left(\frac{1}{x_1+i\pi n}-\frac{1}{x_2+i\pi n}\right)=\coth(x_1)-\coth(x_2), \tag{44}$$

can be applied. This produces

$$g(x_1,x_2)=-\frac{1}{16\pi}\frac{1}{x_1-x_2}\int_m^\infty dt\left(\frac{\coth\left[\frac{\beta}{2}(t+x_1)\right]}{2t+x_1-x_2}-\frac{\coth\left[\frac{\beta}{2}(t+x_2)\right]}{2t-x_1+x_2}\right)+\text{p.p.c.}, \tag{45}$$

where p.p.c. (which stands for pseudoparity conjugate) refers to the same expression with the replacements $x_1\to -x_1$ and $x_2\to -x_2$.

Correspondingly,

$$C_2[m,A]=\int d^3x\ \mathrm{tr}[\varphi(\Omega_1,\mathcal{D}_{02})\boldsymbol{E}^2] \tag{46}$$

with

$$\varphi(\omega,y)=\frac{1}{16\pi}\frac{1}{y}\int_m^\infty dt\left(\frac{1}{2t-y}\frac{e^{\beta t}+\omega}{e^{\beta t}-\omega}-\frac{1}{2t+y}\frac{e^{\beta(t+y)}+\omega}{e^{\beta(t+y)}-\omega}\right)+\text{p.p.c.}; \tag{47}$$

p.p.c. corresponds to $y\to -y$ and $\omega\to\omega^{-1}$. This is the final expression which contains all contributions to $C[m,A]$ with two spatial Lorentz indices and any number of zeroth indices. As expected at finite temperature, it is nonlocal in time but local in $\boldsymbol{x}$. Note that $C_2[m,A]$ is an even function of m.[4]

We can now consider a further expansion in powers of $\mathcal{D}_0$ since it respects gauge invariance. An explicit computation shows that $\varphi(\omega,y)$ is an analytic function of y. At leading (zeroth) order in $\mathcal{D}_0$ the result from Eq. (47) is

$$C_2[m,A]=\frac{1}{16\pi}\frac{1}{2m}\int d^3x\ \mathrm{tr}\left[\left(\frac{e^{\beta m}+\Omega}{e^{\beta m}-\Omega}+\frac{e^{\beta m}+\Omega^{-1}}{e^{\beta m}-\Omega^{-1}}\right)\boldsymbol{E}^2\right]+O(\mathcal{D}_0). \tag{48}$$

Note that this result, unlike the full result in Eqs. (46) and (47), does not contain an integral over the mass ($\int_m^\infty dt$). This property holds to all orders in $\mathcal{D}_0$.

G. The effective action

In this subsection we summarize some properties of the effective action which will be needed later.

The effective action. The Euclidean effective action is defined as minus the logarithm of the partition functional. For noninteracting fields it takes the form

[4]This can be shown by noting that $\varphi(\omega,y;m)-\varphi(\omega,y;-m)$ is given by the same formula (47) with replacement $m\to-\infty$, and then showing that the integrand is convergent and odd as a function of t.

$$W[M,A]=c\ \mathrm{Tr}\log[K(M,D)]=c\ \log\mathrm{Det}[K(M,D)], \tag{49}$$

where $K(M,D)$ is a differential operator [e.g., the Klein-Gordon operator as in Eq. (2) or the Dirac operator] and c some constant.

Ultraviolet ambiguities. The previous expression needs to be regularized, and a number of methods can be used to obtain a renormalized version of it. The key observation is that all renormalized versions of the effective action must yield the same ultraviolet finite contributions and so two such versions can differ at most by a term which is a local polynomial in the external fields and their derivatives, with canonical dimension no more than $d+1$ (in $d+1$ dimensions). Note that for this discussion a mass m plays the role of an external scalar field which happens to take a constant c-number configuration and so, in particular, the ambiguity in the renormalized action will depend polynomically on m. This implies that any sensible (that is, correctly describing the ultraviolet finite contributions) regularization plus renormalization prescription can be used to make the effective action finite; the actual effective action describing the physical system at hand will correspond to adding the appropriate local polynomial action to the previous result. Another consequence is that formal identities can be applied so long as a violation of them is allowed in the form of a local polynomial of degree $d+1$ or less.

The ζ-function method. The effective action can be defined through the ζ-function prescription, namely

$$\mathrm{Tr}\log(K)=\left.\frac{d}{ds}\mathrm{Tr}(K^s)\right|_{s=0}. \tag{50}$$

(An analytical extension in s is understood from sufficiently negative values of s.) When K admits a complete set of eigenvectors, $\mathrm{Tr}(K^s)=\Sigma_n\lambda_n^s$, λ_n being the eigenvalues of K. If the calculation has to be made using some expansion it is convenient to use the following formula [25]:

$$\mathrm{Tr}(K^s)=\mathrm{Tr}\int_\Gamma\frac{dz}{2\pi i}z^s\frac{1}{z-K}, \tag{51}$$

where the path Γ encloses counterclockwise the eigenvalues of K but not $z=0$. This method is practical in actual calculations combined with the symbols method: after applying Eq. (11) to expand the functional trace, it is straightforward to make an explicit expansion in M and $\boldsymbol{D}$, for instance. This method has been used for fermions in Ref. [21] and for a nonlocal Dirac operator in Ref. [22] at zero temperature, and at finite temperature for odd dimensional fermions in Ref. [23] and even dimensional fermions in Ref. [17].

Anomalies. If the effective action breaks a symmetry of the action there is an anomaly. In general the anomaly can be defined as the difference between the effective action of the original and the transformed configurations (of the external fields) and by construction is a local polynomial. It may happen that the symmetry can be restored by adding an appropriate local polynomial to the effective action. In this case the breaking is an unessential anomaly. When not all symmetries can be restored simultaneously the theory presents an essential anomaly. All symmetries which are implemented through a similarity transformation of $K(M,D)$ leave the spectrum invariant thus because the ζ-function prescription defines the determinant of a differential operator by regularizing the product of its eigenvalues, it follows that for these symmetries the ζ-function version of $\mathrm{Det}(K)$ is always free of anomalies. This applies in particular to vector gauge invariance. (Axial gauge transformations and scale transformations, for instance, are not implemented by similarity transformations and they can be anomalous.) Therefore the partition functional is always invariant under gauge transformations. On the other hand the effective action can change by integer multiples of $2\pi i$, since the logarithm is a many-valued function. (We are assuming that no zero modes are involved. They would induce changes multiples of $i\pi$ in the effective action.) By continuity, this can only happen for topologically large gauge transformations. For a scalar field the multivaluation cannot occur since the corresponding ζ-function renormalized effective action is purely real. For fermions the multivaluation may take place depending on the topological numbers of the gauge transformation and the gauge field configuration and this indicates the presence of topological pieces in the effective action [1,4,5]. Such multivaluation is indeed found in the explicit calculation for $2+1$-dimensional fermions of Ref. [23].

Locality and finite temperature. In previous subsections we have considered operators of the form $f(M,D)$ with f one-valued and ultraviolet convergent. Actually, one wants to compute the effective action which contains multivaluation and ultraviolet divergences, Eq. (49). In many expansions (perturbative, derivative, $1/m$, etc.) higher orders are ultraviolet finite and thus they are also free of multivaluation. For those terms all our previous considerations hold directly. In particular, we find a remarkable result, namely, that the effective action at finite temperature can be written as [cf. Eq. (36)]

$$W[M,A]=\sum_n\int d^{d+1}x\ \mathrm{tr}[\varphi_n(\Omega)\mathcal{O}_n], \tag{52}$$

where φ_n are some functions and $\mathcal{O}_n$ are gauge covariant and local operators constructed out of $\mathcal{D}_\mu$ and M. In this sense the theory at finite temperature is local in the usual sense (i.e., the effective action admits an expansion in $\mathcal{D}_\mu$) provided that the field $\Omega(x)$ is regarded as local.

For the ultraviolet divergent terms, some oddities appear which are necessary in order to accommodate the existence of anomalies, topological terms and multivaluation, all these issues being related. For instance, when the expansion in spatial covariant derivatives is computed for fermions in $2+1$ dimensions [23] the functions φ_n of lower orders are many-valued (a property belonging to the exact result in $0+1$ dimensions [10] and in $2+1$ dimension for Abelian and stationary configurations [12]). In addition, negative powers of $\mathcal{D}_0$ may appear when one goes beyond the Abelian and stationary case. This was handled in Ref. [23] by introducing the fields $\boldsymbol{\mathcal{A}}(x)$, defined as any solution of the equation $\mathcal{D}_0\boldsymbol{E}=\mathcal{D}_0^2\boldsymbol{\mathcal{A}}$. There it was shown that the ambiguity in the

definition of $\mathcal{A}$ cancels and all solutions yield the same effective action. In terms of the $\mathcal{A}(x)$ and $\Omega(x)$ the finite temperature effective action remains local.

III. THE SCALAR FIELD IN 2+1 DIMENSIONS

In what follows we will apply the previous ideas to the computation of the effective action of $2+1$-dimensional scalar particles at finite temperature and density. The Euclidean action of the system is that of Eq. (1). There, the field $\phi(x)$ is a Lorentz scalar and a vector in the internal symmetry space which will collectively be referred to as flavor space, with dimension N_f. The covariant derivative is defined as $D_\mu=\partial_\mu+A_\mu$, where the gauge field $A_\mu(x)$ is an anti-Hermitian matrix in flavor space. Correspondingly, the gauge transformations $U(x)$ are unitary matrices. The mass m is a space-time constant and a real c number in flavor space. The more general case of an arbitrary scalar field $M(x)$ replacing m will not be considered here. The effective action is given by Eq. (2).

Relevant symmetries of the problem are pseudoparity and gauge transformations. Pseudoparity corresponds to changing the sign of every Lorentz zeroth index, i.e., $(x_0,\boldsymbol{x})\rightarrow(-x_0,\boldsymbol{x})$ and $A_0\rightarrow -A_0$. Since the spectrum of the Klein-Gordon operator $-D_\mu^2+m^2$ is unchanged under this transformation the ζ-function regularization prescription provides a pseudoparity preserving effective action. Such Euclidean effective action contains only contributions with an even number of Lorentz zeroth indices, it does not contain the Levi-Civita pseudotensor, and thus it is purely real. Any other renormalization prescription can only produce imaginary contributions which are local polynomials. As noted, the ζ-function regularized effective action will be strictly gauge invariant since it is real. In fact no essential anomalies are present in the case of scalar fields in $2+1$ dimensions (scale anomalies are absent in odd dimensions [26]).

There is a third symmetry, namely, the transformation $m\rightarrow -m$ which is trivial for scalar particles and again free from anomalies using the ζ-function regularization. Within other renormalization schemes there can appear terms breaking this symmetry but they will be removable by adding a local polynomial. In the case of odd-dimensional fermions neither pseudoparity nor the transformation $m\rightarrow -m$ are symmetries, however, their product gives the parity transformation. Parity is a symmetry of the fermionic action but is not a similarity transformation of the Dirac operator, so it is not guaranteed to be preserved by the ζ-function renormalization prescription. As is well known, parity for odd-dimensional fermions is in general in conflict with invariance under large gauge transformations and if the latter invariance is enforced, parity may present an anomaly, depending on the number of flavors [4].

A. The 0+1-dimensional model

The above-mentioned remarks can be illustrated with the $0+1$-dimensional version of the system. The corresponding effective action has been computed in Ref. [18]. Perhaps the simplest way to derive this effective action is by computing the partition function of the associated 0-dimensional Hamiltonian system in a gauge where A_0 is time independent. The energy spectrum is then obtained directly from the Klein-Gordon equation written as

$$-\partial_0\phi=(\pm m+A_0)\phi. \tag{53}$$

Thus for each flavor there are two single particle levels $\epsilon_{\pm,a}=\pm m+A_{0,a}$ ($a=1,\ldots,N_f$ labelling the eigenvalues of the matrix A_0 in flavor space). The standard textbook result for the partition function of a system of noninteracting bosons then applies:

$$Z_s[m,A]=\prod_{\sigma=\pm}\prod_{a=1}^{N_f}\sum_{n=0}^{\infty}e^{-\beta(\sigma m+A_{0,a})n}, \tag{54}$$

or equivalently

$$W_s[m,A]=\mathrm{tr}\log[(1-e^{-\beta(m+A_0)})(1-e^{-\beta(-m+A_0)})]. \tag{55}$$

The trace refers to flavor space. This result can be rewritten as

$$W_s[m,A]=-\beta\,\mathrm{Tr}(A_0)+\Gamma_s[m,A]. \tag{56}$$

The first term is the $0+1$-dimensional Chern-Simons action which breaks pseudoparity and can be removed by a local polynomial counterterm. The second term is (up to a constant)

$$\Gamma_s[m,A]=\mathrm{tr}\log\left[4\sinh\left(\frac{\beta}{2}(m+A_0)\right)\sinh\left(\frac{\beta}{2}(m-A_0)\right)\right]. \tag{57}$$

This effective action is an even function of m and A_0, so it preserves parity and pseudoparity. It can be written in a manifestly gauge invariant form as

$$\Gamma_s[m,A]=\mathrm{tr}\log[e^{\beta m}+e^{-\beta m}-\Omega-\Omega^{-1}]. \tag{58}$$

As in the case of fermions [10], periodicity of the effective action as a function of βA_0 would be lost within a perturbative expansion, i.e., an expansion in powers of A_0.

Since $\Gamma_s[m,A]$ enjoys all symmetries of the action it coincides with the ζ-function regularized effective action, up to a constant (since any other local polynomial must be of degree one in m or A_0 and would break parity). The result in Ref. [18] corresponds to $\Gamma_s[m,A]-\Gamma_s[m,0]$ in Minkowski space. It is noteworthy that the partition function defined directly from the Hamiltonian breaks pseudoparity [due to the Chern-Simons term in Eq. (56)] even if no ultraviolet divergences are introduced in $0+1$ dimensions in the canonical formalism.

B. Computation of the effective action in 2+1 dimensions: Relation to the fermionic case

The effective action of the $2+1$-dimensional model cannot be computed in closed form for arbitrary space-time and internal symmetry configurations. Our approach will be to

expand the effective action in the number of spatial covariant derivatives, or equivalently, in the number of spatial Lorentz indices. The computation will be carried out through second order, that is, we will keep terms with zero or two Lorentz indices. (There are no odd order terms in the expansion.) The zeroth Lorentz index dependency is treated exactly since this guarantees the preservation of the periodicity condition which is essential for gauge invariance.

The calculation can be done by applying the ζ-function regularization prescription with the help of the symbols method, as described in Ref. [23], however, it is more economical to use the results already established for fermions in that reference. This can be done as follows. The effective action for fermions is

$$W_f[m,A] = -\mathrm{Tr}_f \log(\gamma_\mu D_\mu + m). \tag{59}$$

The functional trace is taken in the space of antiperiodic wave functions and includes Dirac degrees of freedom, in addition to space time and flavor degrees of freedom. The gamma matrices are Hermitian and satisfy $\gamma_\mu\gamma_\nu = \delta_{\mu\nu} + \sigma_{\mu\nu}$. Actually, there are two inequivalent irreducible representations of the Dirac algebra which are distinguished by the label $\eta = \pm 1$ in the relation $\gamma_\mu\gamma_\nu\gamma_\rho = i\eta\epsilon_{\mu\nu\rho}$. So if γ_μ is one of the representations, $-\gamma_\mu$ provides another inequivalent representation of the Dirac algebra. The label η is attached to the Levi-Civita pseudotensor and thus a change in η is equivalent to a pseudoparity transformation. Therefore the fermionic effective action can be split into two components:

$$W_f[m,A] = W_f^+[m,A] + \eta W_f^-[m,A], \tag{60}$$

where W_f^+ is real and even under pseudoparity and W_f^- is imaginary and pseudoparity odd. (Of course, this relation can be violated by adding a local polynomial.) Next, note that the formal identity $\mathrm{Tr}\log(AB) = \mathrm{Tr}\log(A) + \mathrm{Tr}\log(B)$ holds for the functional trace up to ultraviolet divergent contributions and so it holds modulo local polynomial terms. This implies that

$$W_f^+[m,A] = -\frac{1}{2}\mathrm{Tr}_f \log[(\gamma_\mu D_\mu + m)(-\gamma_\mu D_\mu + m)]$$

$$= -\frac{1}{2}\mathrm{Tr}_f \log\left[-D_\mu^2 + m^2 - \frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}\right],$$

$$(F_{\mu\nu} = [D_\mu, D_\nu]). \tag{61}$$

So $W_s[m,A]$ [cf. Eq. (2)] is closely related to $W_f^+[m,A]$. The differences between both expressions are (i) the Dirac degree of freedom which is absent in the scalar case, (ii) the different (periodic versus antiperiodic) boundary conditions, and (iii) the extra term $-\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}$ which is not present in the Klein-Gordon operator. In addition, a local polynomial action can be further added at the end.[5]

The first correction amounts to dividing the fermionic result by the trace of unity in Dirac space which is two in the $2+1$-dimensional calculation in Ref. [23]. The second correction can also be tackled straightforwardly. The Matsubara frequency (related to ∂_0 in D_0) takes discrete values $2\pi i(n + \frac{1}{2})/\beta$ for fermions and $2\pi i n/\beta$ for bosons, thus the functional trace computed in the fermionic Hilbert space is related to the bosonic one after the replacement $A_0 \to A_0 - i\pi/\beta$. This is equivalent to $\Omega(x) \to -\Omega(x)$. [$\Omega(x)$ has been defined in Eq. (28).]

$$W_s[m,A] = -W_f^+\left[m, A_0 - \frac{i\pi}{\beta}, \boldsymbol{A}\right] - C[m,A]. \tag{62}$$

The term $C[m,A]$ takes into account the spurious contributions coming from $-\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}$, which have to be removed from the Fermionic result.

This formula can be illustrated in the $0+1$-dimensional model, where it reads $W_s[m,A] = -2W_f^+[m,A_0 - i\pi/\beta]$ (note that Dirac space is one-dimensional in $0+1$ dimensions so the factor 2 is not canceled in this case, and also $C[m,A]=0$). The simplest way to obtain the $0+1$-dimensional fermionic effective action is again using the Hamiltonian formalism (fixing $\partial_0 A_0 = 0$). Since there is a single-particle level with energy $\epsilon_a = \eta m + A_{0,a}$ for each flavor (where $\eta = \pm 1$ is the Dirac matrix γ_0 in $0+1$ dimensions) it follows that

$$W_f[m,A] = -\log \prod_{a=1}^{N_f} \sum_{n=0,1} e^{-\beta(\eta m + A_{0,a})n}$$

$$= -\mathrm{tr}\log[1 + e^{-\beta(\eta m + A_0)}]. \tag{63}$$

The result in Ref. [10] corresponds (up to a local polynomial) to $W_f[m,A] - W_f[m,0]$ with $\eta = 1$. This version of the effective action does not directly satisfy Eq. (60), i.e., the pseudoparity transformation $A_0 \to -A_0$ is not equivalent to the transformation $\eta \to -\eta$ in the previous formula. However, subtracting an appropriate η-dependent polynomial[6] yields

$$W_f'[m,A] = -\mathrm{tr}\log[1 + e^{-\beta(m + \eta A_0)}]$$

$$= -\mathrm{tr}\log[1 + e^{-\beta m}\Omega^\eta], \tag{64}$$

which does satisfy Eq. (60). This is the ζ-function result [23]. It is readily verified that $\Gamma_s[m,A]$ in Eq. (58) coincides with $-2W_f'^+[m,A_0 - i\pi/\beta]$ plus a polynomial, $\int dx_0\, \mathrm{tr}(m)$.

[5]Alternatively, one can choose to change $m \to -m$ instead of $\gamma_\mu \to -\gamma_\mu$ in the second factor in the logarithm in Eq. (61), and then relate $W_s[m,A]$ to $W_f[m,A] + W_f[-m,A]$. Up to a local polynomial, this procedure is equivalent to the one used in the text.

[6]To wit, $\theta(-\eta)\int dx_0\, \mathrm{tr}(\eta m + A_0)$, which is temperature independent. This is consistent with the fact that the finite temperature does not introduce new ultraviolet divergences.

In $2+1$ dimensions the subtraction $C[m,A]$ can be computed in an expansion in powers of $\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}$. The term of first order does not contribute (since the trace of $\sigma_{\mu\nu}$ in Dirac space vanishes) thus the leading term is that with two powers of $F_{\mu\nu}$, namely, the expression in Eq. (40). Because $C[m,A]$ is ultraviolet finite it is free from anomalies, i.e., it enjoys all symmetries of the bosonic action. This term has been computed, up to two spatial derivatives, in Sec. II F.

Let us quote the results for the pseudoparity even component of the effective action of fermions in $2+1$ dimensions [23]. At zeroth order in the number of spatial covariant derivatives the result is

$$W_{f,0}[m,A]=\frac{1}{4\pi}\mathrm{tr}\langle 0|\left[\left(\frac{2}{\beta}\right)^2 m\phi_1\left(\frac{\beta}{2}(m-D_0)\right)-\left(\frac{2}{\beta}\right)^3\phi_2\left(\frac{\beta}{2}(m-D_0)\right)\right]|0\rangle+\text{p.p.c.} \tag{65}$$

At second order

$$\begin{aligned}W^+_{f,2}[m,A]=-\frac{1}{8\pi}\mathrm{tr}\langle 0|\Bigg\{&\left[\frac{1}{2}\left(\frac{2}{\beta}\right)^2\phi_1\left(\frac{\beta}{2}(m-D_{01})\right)-\frac{2}{\beta}\phi_0\left(\frac{\beta}{2}(m-D_{01})\right)\left(\frac{1}{4}(D_{02}-D_{01})+\frac{m}{2}\right)\right.\\&\left.-\int_m^\infty dt\,\tanh\left(\frac{\beta}{2}(t-D_{01})\right)\times\frac{\left[\frac{1}{4}(D_{02}-D_{01})^2+m^2\right]}{2t+D_{02}-D_{01}}\right]\frac{1}{(D_{02}-D_{01})^3}\boldsymbol{E}^2\Bigg\}|0\rangle\\&+X_{12}+\text{p.p.c.}\end{aligned} \tag{66}$$

In these formulas, p.p.c. means pseudoparity conjugate, $D_0\to -D_0$, and X_{12} means the same expression exchanging the labels 1 and 2.

The functions $\phi_n(z)$ are given by

$$\phi_n(\omega)=P_{n+1}(\omega)-\int_\omega^{+\infty}dz\,\frac{(\omega-z)^n}{n!}[\tanh(z)-1],\qquad n=0,1,2,\ldots,\quad \mathrm{Re}(\omega)>0, \tag{67}$$

where the integration path runs parallel to the real positive axis towards $+\infty$. The $P_n(\omega)$ are polynomials of degree n,

$$P_1(\omega)=\omega,\qquad P_2(\omega)=\frac{1}{2}\omega^2-\frac{1}{6}\left(\frac{i\pi}{2}\right)^2,\qquad P_3(\omega)=\frac{1}{6}\omega^3-\frac{1}{6}\left(\frac{i\pi}{2}\right)^2\omega. \tag{68}$$

These formulas for $\phi_n(\omega)$ refer to $\mathrm{Re}(\omega)>0$. When $\mathrm{Re}(\omega)<0$ the property $\phi_n(-\omega)=(-1)^n\phi_n(\omega)$ can be used.

C. The effective action

We can now use Eq. (62) to obtain the effective action of the $2+1$-dimensional scalar field. Up to two Lorentz indices, $C[m,A]$ equals $C_2[m,A]$ which is given in Eq. (46). $W_f^+[m,A]$ is given in Eqs. (65) and (66). In these latter expressions the functions $\phi_n(z)$ are made explicit using Eq. (67), and the variables Ω and $\mathcal{D}_0$ are used instead of D_{01} and D_{02} (cf. Sec. II C). In addition, we introduce a chemical potential by means of the replacement $A_0\to A_0-\mu$ [8] where μ is a real constant c number (recall that A_0 is anti-Hermitian). This shift is gauge invariant and it is equivalent to $\Omega\to e^{\beta\mu}\Omega$.

The effective action up to two spatial covariant derivatives at finite temperature and density is thus

$$W_{s,0}[m,A]=\int d^3x\,\mathrm{tr}[\,\varphi_0(e^{\beta\mu}\Omega)], \tag{69}$$

$$W_{s,2}[m,A]=\int d^3x\,\mathrm{tr}[\,\varphi_2(e^{\beta\mu}\Omega_1,\mathcal{D}_{02})\boldsymbol{E}^2]. \tag{70}$$

The functions φ_0 and φ_2 are given by

$$\varphi_0(\omega)=-\frac{1}{4\pi}\left(\frac{1}{3}|m|^3+\int_{|m|}^{+\infty}dt(t^2-m^2)\frac{\omega}{e^{\beta t}-\omega}\right)+\text{p.p.c.}, \tag{71}$$

$$\begin{aligned}\varphi_2(\omega,y)=-\frac{1}{8\pi}\frac{1}{y^3}\Bigg\{&\frac{1}{2}|m|y+\frac{1}{2}\left(\frac{1}{4}y^2-m^2\right)\times\log\left(\frac{2|m|+y}{2|m|-y}\right)+\int_{|m|}^\infty dt\Bigg[\frac{2\omega}{e^{\beta t}-\omega}\\&\times\left(\frac{t^2-m^2}{2t-y}-\frac{1}{2}y\right)-\frac{2\omega}{e^{\beta(t+y)}-\omega}\times\left(\frac{t^2-m^2}{2t+y}+\frac{1}{2}y\right)\Bigg]\Bigg\}+\text{p.p.c.}\end{aligned} \tag{72}$$

In these formulas p.p.c. corresponds to $y\to -y$ and $\omega\to\omega^{-1}$.

The formulas (69) and (70), expanded in Eqs. (71) and (72), constitute the main result of this section. They are necessarily complicated looking since an infinite number of Feynman graphs (with any number of temporal gauge fields, cf. Sec. IV B) are being added, and the effective action is nonlocal in time at finite temperature. A much simpler formula is presented below if only the lowest order is retained in a further expansion in the number of temporal covariant derivatives.

Some remarks are in order. In writing the formula we have already used the fact that the effective action is an even function of m (since this is already true for $W_f^+[m,A]$). The

condition $\mathrm{Re}(\omega)>0$ in Eq. (67), implies that the previous formulas refer to the physically relevant case $|\mu|<|m|$ only. All polynomial contributions introduced by $P_n(\omega)$ in Eq. (67) combine in such a way that all the polynomial dependence on D_{01} cancels and the result is a periodic function of D_{01}. This cancellation requires the explicit as well as the p.p.c. terms to hold and it is a nontrivial check of the formulas. (Of course, the periodicity holds also for the fermionic effective actions in the pseudoparity even sector, $W_f^+[m,A]$.) The argument of the logarithm in $\varphi_2(\omega,y)$ is to be taken in the interval $(-\pi,\pi)$ and t runs on the real positive axis. (Note that y represents $\mathcal{D}_0$ and thus it is purely imaginary.)

D. Expansion in space-time derivatives at finite temperature

A simpler expression for $W_{s,2}[m,A]$ is obtained retaining only the leading order in an expansion in $\mathcal{D}_0$. This corresponds to expand in powers of y in the function $\varphi_2(\omega,y)$. As noted this expansion does not break any symmetry and is a natural one in the present context. This produces

$$W_{s,2}[m,A]=\int d^3x\,\mathrm{tr}[\,\varphi_{2,0}(e^{\beta\mu}\Omega)\boldsymbol{E}^2]+O(\mathcal{D}_0), \quad (73)$$

with

$$\varphi_{2,0}(\omega)=-\frac{1}{96\pi}\frac{1}{m}\frac{e^{2\beta m}-\omega^2-2\beta m e^{\beta m}\omega}{(e^{\beta m}-\omega)^2}+\text{p.p.c.}$$

$$=-\frac{1}{96\pi}\frac{1}{m}\frac{d}{dm}\left[m\frac{e^{\beta m}+\omega}{e^{\beta m}-\omega}\right]+\text{p.p.c.} \quad (74)$$

Note that $\varphi_{2,0}(\omega)$ is an even function of m. This formula is manifestly invariant under all symmetries of the action. Two nontrivial checks of the calculation are (i) that negative powers of y have canceled and (ii) that $\varphi_{2,0}(\omega)$ no longer contains parametric integrals (i.e., $\int_{|m|}^{\infty}dt$). As noted, this expression is exact for $W_{s,2}[m,A]$ in the Abelian and stationary case.

E. Zero temperature limit

The zero temperature limit of Eqs. (71) and (72) is straightforward: because $|\mu|<|m|$ and Ω is unitary, the term $e^{\beta t}$ always dominates, thus the limit $\beta\to\infty$ just removes the terms with $\int_{|m|}^{\infty}dt$ and the dependence in μ is also canceled (we are assuming μ fixed as $T\to 0$). That is,

$$W_{s,0}[m,A]=-\frac{1}{6\pi}\int d^3x\,\mathrm{tr}[|m|^3], \quad (T=0), \quad (75)$$

$$W_{s,2}[m,A]=-\frac{1}{8\pi}\int d^3x\,\mathrm{tr}\left\{\boldsymbol{E}\left[|m|\mathcal{D}_0+\left(\frac{1}{4}\mathcal{D}_0^2-m^2\right)\times\log\left(\frac{2|m|+\mathcal{D}_0}{2|m|-\mathcal{D}_0}\right)\right]\frac{1}{\mathcal{D}_0^3}\boldsymbol{E}\right\}. \quad (76)$$

At zero temperature Lorentz invariance is a symmetry of the action. Such symmetry is not obvious in $W_{s,2}[m,A]$ since it sums all orders in $\mathcal{D}_0$ but not in $\mathcal{D}_i$, $i=1,2$. However, considering an expansion in inverse powers of m (a Lorentz invariant expansion) gives

$$W_{s,2}[m,A]=-\frac{1}{48\pi}\frac{1}{|m|}\int d^3x\,\mathrm{tr}[\boldsymbol{E}^2]+O(m^{-3}). \quad (77)$$

The same result follows from Eq. (73). This expression admits a unique Lorentz invariant completion, namely

$$W_s[m,A]=-\int d^3x\,\mathrm{tr}\left[\frac{1}{6\pi}|m|^3+\frac{1}{96\pi}\frac{1}{|m|}F_{\mu\nu}^2\right]+O(m^{-3}). \quad (78)$$

This formula comes also from a direct application of the results in Ref. [27] thereby being a check for our formulas. At next order in $1/m$ several Lorentz invariant operators of dimension 6 can appear and Eq. (76) puts a constraint on their coefficients. Note that the last two formulas hold also at finite temperature since the temperature-dependent corrections are $O(e^{-\beta|m|})$ (see also Sec. IV C).

F. The partition function

The effective action at finite temperature and density is directly related to the grand-canonical potential, namely $W_s[m,A;\beta,\mu]=\beta\Omega(\beta,\mu)$. (Note that μ introduced by the replacement $\partial_0\to\partial_0-\mu$ couples to the charge and not to the number of particles which is not conserved in the relativistic case [28]). However, strictly speaking, a system at equilibrium with temperature T and chemical potential μ can only be stationary. In addition, the physical effect of a (negative) constant external scalar potential is indistinguishable from that of a (positive) chemical potential, since both add to the energy for positive charges and subtract for negative ones, thus for the partition function A_0 should not be Wick rotated, and A_0 is real instead of imaginary. All expressions depend only on the combination $A_0-\mu$. (It would be algebraically inconsistent not to rotate A_0 to its Euclidean version in the general case, but not within the subset of stationary configurations.) Note that the fact that A_0 is real or imaginary does not affect the functional form; in any case the functional derivative of the effective action with respect to A_0 yields the charge density and the derivative with respect to μ yields the total charge:

$$\rho(x)=\frac{\delta W_s}{\delta A_0(x)}, \quad Q=-\frac{1}{\beta}\frac{\partial W_s}{\partial\mu}. \quad (79)$$

In view of the relation between partition function and effective action, it follows that $W_{s,0}$ describes a two-dimensional relativistic ideal gas in presence of an almost space-time constant scalar potential A_0. An explicit calculation of the charge density using $W_{s,0}$ in Eq. (71) and integrating by parts yields

$$\rho_0=\frac{N_f}{4\pi}\int_{|m|}^{\infty}dt\,t\left[\coth\left(\frac{\beta}{2}(t-\mu)\right)-\coth\left(\frac{\beta}{2}(t+\mu)\right)\right], \tag{80}$$

$N_f=\mathrm{tr}(1)$ is the number of flavors. We are assuming that μ couples equally to all flavors and we have dropped A_0 since it can be recovered from μ. This formula can rewritten in the standard form [28]

$$\rho_0=\int\frac{d^2k}{(2\pi)^2}\left(\frac{N_f}{e^{\beta[\omega(k)-\mu]}-1}-\frac{N_f}{e^{\beta[\omega(k)+\mu]}-1}\right), \tag{81}$$

where $\omega(k)=\sqrt{k^2+m^2}$.

Likewise, the term $W_{s,2}$ adds a contribution to the density coupled to A_0. Let us consider the Abelian case (in addition to stationary), then $\mathcal{D}_0$ vanishes and Eq. (73) becomes exact. There will be two contributions to the charge density, one coming from the explicit dependence on A_0 and another through the dependence in $\boldsymbol{E}$. The latter is a total derivative and does not contribute to the total charge. Both contribution can be combined to give

$$\rho_2(x)=-\frac{N_f}{96\pi}\frac{1}{m}\frac{d}{dm}\left\{m\left[-\frac{\beta}{2}\mathrm{cosech}^2\left(\frac{\beta}{2}(m-\mu+A_0)\right)\boldsymbol{E}^2+2\coth\left(\frac{\beta}{2}(m-\mu+A_0)\right)\boldsymbol{\nabla E}\right]\right\}-\text{p.p.c.} \tag{82}$$

(Note that A_0, μ, and $\boldsymbol{E}$ are pseudoparity odd.) At zero temperature (assuming $|m|>|\mu-A_0|$, or else at finite temperature but large mass) this simplifies to

$$\rho_2(x)=-\frac{N_f}{24\pi}\frac{1}{|m|}\boldsymbol{\nabla E}, \tag{83}$$

which also follows from Eq. (77).

It is also interesting to note the relation of our results with other formulas using polylogarithms.[7] The functions $\phi_n(\omega)$ introduced in Ref. [23] and Sec. III B are directly related to polylogarithms, namely

$$\phi_n(\omega)=P_{n+1}(\omega)-(-2)^{-n}\mathrm{Li}_{n+1}(-e^{-2\omega}). \tag{84}$$

This relation is easily established from Eq. (67) by noting that it verifies the following defining properties of $\mathrm{Li}_n(z)$:

$$\mathrm{Li}_0(z)=\frac{z}{1-z},\quad z\frac{d}{dz}\mathrm{Li}_n(z)=\mathrm{Li}_{n-1}(z),\quad \mathrm{Li}_n(0)=0. \tag{85}$$

In this notation, $W_{s,0}$ in Eqs. (69) and (71) becomes

$$W_{s,0}[m,A]=-\frac{1}{4\pi}\int d^3x\,\mathrm{tr}\left[\frac{|m|^3}{3}+2T^3[\beta|m|\mathrm{Li}_2(z)+\mathrm{Li}_3(z)]\right]+\text{p.p.c.}, \tag{86}$$

where $z=e^{-\beta(|m|-\mu+A_0)}$ is the fugacity, and the corresponding density becomes

$$\rho_0=\frac{N_fT^2}{2\pi}[\beta|m|\mathrm{Li}_1(z)+\mathrm{Li}_2(z)]-\text{p.p.c.} \tag{87}$$

in agreement with Ref. [30].

As a final comment related to the partition function, we note that the relation Eq. (62) can also be written using the chemical potential instead of the scalar potential A_0. That is, if $\Omega_b(\beta,\mu)$ and $\Omega_f(\beta,\mu)$ represent the grand-canonical potentials of a system of noninteracting particles in presence of external fields, treated as bosons or fermions, respectively, then

$$\Omega_f(\beta,\mu)=-\Omega_b\left(\beta,\mu+\frac{i\pi}{\beta}\right). \tag{88}$$

The minus sign comes because the functional integral with Grassman variables gives the determinant of the quadratic form instead of the inverse determinant. The shift $\mu\to\mu+i\pi/\beta$ accounts for the different boundary conditions. Of course a shift $\mu\to\mu+2\pi i/\beta$ must leave the partition function invariant, since μ is coupled to an integer-quantized charge. This is another manifestation of gauge invariance. For interacting particles Eq. (88) can be extended using the well-known prescription of adding a minus sign for each particle loop [Eq. (88) corresponds to the particular case of one-loop].[8]

IV. FURTHER GENERAL REMARKS ON THE METHOD

A. Remarks on gauge invariance

As noted in the Introduction, topologically large gauge transformations have played a prominent role in the development of this field by putting severe constraints on the allowable forms of the effective action functional. On the other hand it seems that in our present approach such a role is played instead by the periodicity constraint which prompts the appearance of the Polyakov loop $\Omega(x)$. In this subsection we will make some remarks to try to clarify the relation between both concepts.

[7]The polylogarithms are defined as [29]

$$\mathrm{Li}_n(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^n}.$$

[8]An equivalent procedure would be to compute the grand-canonical potential for n replicas of the particles and then set n to -1 at the end. That is, in the notation of Ref. [31] the Hamiltonian becomes

$$H=\sum_{\alpha,\beta}h_{\alpha\beta}\sum_{\sigma=1}^{n}a^{\dagger}_{\alpha\sigma}a_{\beta\sigma}+\sum_{\alpha,\beta,\gamma,\delta}v_{\alpha\beta\gamma\delta}\sum_{\sigma_1,\sigma_2=1}^{n}a^{\dagger}_{\alpha\sigma_1}a^{\dagger}_{\beta\sigma_2}a_{\delta\sigma_2}a_{\gamma\sigma_1}. \tag{89}$$

Let G be the gauge group, and let Γ be the relevant homotopy group controlling the existence of topologically large gauge transformations at finite temperature. In $0+1$ dimensions $\Gamma=\pi_1(G)$, therefore there are large gauge transformations for the Abelian group U(1) but not for simply connected groups such as SU(n). In $d+1$ dimensions (with d different from zero), let us assume for this discussion that the spatial boundary conditions are such that the space becomes effectively compactified to a sphere. Hence Γ contains classes of mappings from the space-time manifold $S^1\times S^d$ into the gauge group G, where the factor S^1 corresponds to the compactified Euclidean time and S^d to the d-dimensional space. (Note that, unless otherwise stated, we will require all functions to be continuous on the space-time manifold, and in particular, periodic as a function of time.) Theorem 4.4 of Ref. [32] then implies that[9]

$$\Gamma/\pi_{d+1}(G)=\pi_1(G)\times\pi_d(G). \tag{90}$$

Let us consider the case $d=2$. Because the group $\pi_2(G)$ is always trivial for any (compact) Lie group G, this simplifies to $\Gamma/\pi_3(G)=\pi_1(G)$. Two typical cases are as follows.

(i) $G=\mathrm{U}(1)$. In this case π_3 is trivial and π_1 is $\mathbb{Z}$, so $\Gamma=\mathbb{Z}$. There are nontrivial gauge transformations which wind n times around the S^1 factor of the space-time. They can be realized by space-independent, but time-dependent, gauge transformations.

(ii) $G=\mathrm{SU}(n)$ $(n\geqslant 2)$. In this case π_1 is trivial but $\pi_3=\mathbb{Z}$ and so once again $\Gamma=\mathbb{Z}$. In this case the corresponding large gauge transformations must be space-time dependent.

Note that time-independent gauge transformations are controlled by the homotopy group $\pi_2(G)$ which is trivial, and so they are always topologically small in two spatial dimensions.

Let us now turn to the point of view used in this work. The periodicity constraint refers to the fact that a gauge invariant functional must depend on $\Omega(x)$ and not just on $\log[\Omega(x)]$. The cleanest way to formalize this is by working on the gauge $\partial_0 A_0=0$, in which $\Omega=\exp[-\beta A_0(\boldsymbol{x})]$. Thus both Ω and A_0 are time independent in this gauge. Let us remark that taking A_0 to be time independent is not merely a restriction on the set of possible gauge field configurations, it is a choice of gauge in the sense that every gauge field configuration admits a gauge transformed configuration which is A_0 stationary [23]. In this gauge the periodicity constraint expresses that a gauge invariant functional must be a periodic functional of A_0. To be concrete consider a generic expression of the form [cf. Eq. (52)]

$$\Gamma[M,A]=\sum_n \int d^{d+1}x\ \mathrm{tr}[g_n(A_0)\mathcal{O}_n], \tag{91}$$

where $\mathcal{O}_n$ are gauge covariant local operators, then gauge invariance requires the functions $g_n(z)$ to be periodic with period $2\pi i/\beta$. The necessity of this requirement follows immediately from considering the following class of gauge transformations

$$U(x)=\exp[x_0\Lambda(\boldsymbol{x})], \tag{92}$$

where the time-independent function $\Lambda(\boldsymbol{x})$ takes values on the Lie algebra of G and is restricted by the following conditions

$$[A_0(\boldsymbol{x}),\Lambda(\boldsymbol{x})]=0,\quad \exp[\beta\Lambda(\boldsymbol{x})]=1. \tag{93}$$

The second condition means that the eigenvalues of $\Lambda(\boldsymbol{x})$ are of the form $\lambda_j=2\pi i n_j/\beta$, for integer n_j (such integers are $\boldsymbol{x}$ independent by continuity) and it ensures that the corresponding $U(x)$ is periodic in the temporal direction. Under such a gauge transformation

$$A_0^U(\boldsymbol{x})=A_0(\boldsymbol{x})+\Lambda(\boldsymbol{x}), \tag{94}$$

i.e., the spectrum of A_0 is shifted by multiples of $2\pi i/\beta$ and the functions $g_n(z)$ must be periodic.

Before proceeding, an important point should be noted regarding the approach used in this work. Namely, at the price of working with asymptotic expansions, we can afford to derive formulas which are ''universal'' in the sense that no restriction is put on the algebraic properties of the internal space, in particular, the formulas must hold for any gauge group. This means for instance that the functions $g_n(z)$ above are the same for all theories and configurations. The requirement of universality puts stronger constraints on the functionals that cannot be appreciated when working with concrete theories only [for instance, in particular theories some of the operators $\mathcal{O}_n$ can vanish identically and so the corresponding function $g_n(z)$ does not play a role].

The transformations introduced in Eq. (92), subjected to the conditions Eq. (93), have been named discrete transformations associated to $A_0(\boldsymbol{x})$ in Ref. [23] since in general they form a discrete set due to the condition $\lambda_j=2\pi i n_j/\beta$. Note that this set depends on the particular time-independent field A_0, through the condition $[A_0,\Lambda]=0$. Clearly, these transformations leave invariant the gauge condition $\partial_0 A_0=0$. Likewise, the gauge condition is also preserved by time-independent gauge transformations. In Ref. [23] it is proven that these two kinds of transformations are the most general ones which preserve the A_0-stationary condition.[10] This means that within this gauge a functional such as $\Gamma[M,A]$ above is gauge invariant if and only if the functions $g_n(z)$ are periodic. Then it can be written in a manifestly gauge invariant form (without gauge fixing) as in the right-hand side of Eq. (52) with $\varphi_n(e^{-\beta z})=g_n(z)$. Note that invariance

[9]When $d=0$ the theorem is consistent with $\Gamma=\pi_1(G)\times\pi_1(G)$ which follows from $S^0=\{1,-1\}$. Physically, we want the spatial manifold at $d=0$ to be just $\{1\}$ and so $\Gamma=\pi_1(G)$.

[10]This is the generic case which we will assume. It holds whenever $\exp[\beta A_0(\boldsymbol{x})]$ is either nowhere degenerated or at least the regions of degeneracy are sufficiently small that a unique eigenbasis can be selected (up to normalization) by continuity [23]. In this case $A_0(\boldsymbol{x})$ is also nowhere degenerated and thus $\Lambda(\boldsymbol{x})$ is completely determined by its eigenvalues. So generically the discrete transformations form a discrete set.

under time-independent gauge transformations does not impose further constraints on the $g_n(z)$.

The previous discussion suggests a comparison between large gauge transformations, on the one hand, and discrete transformations, on the other. Two questions pose themselves at this point. Are the discrete transformations large in the topological sense? Is it necessary to rely on large gauge invariance in order to arrive to an A_0-stationary gauge?

The first question can be answered as follows [23]. For a multiply connected group such as U(1), the nontrivial discrete transformations are topologically large since they loop once or more on the temporal circle S^1. On the other hand, for a simply connected group such as SU(n) (in more than one space-time dimension) the discrete transformations associated to some gauge configuration may be large or small, depending on $A_0(\boldsymbol{x})$. For instance if $A_0(\boldsymbol{x})$ is everywhere diagonal, $\Lambda(\boldsymbol{x})$ is also diagonal and in fact a constant. In this case the discrete transformation describes a single loop on the gauge group for all x and it is homotopically trivial. In general, however, discrete transformations can be topologically large. It might seem that the form of the discrete gauge transformations in Eq. (92) factorizes time and space and so it is always classified by the homotopy group $\pi_1(G)$ being always small for a simply connected group. This is true when $\Lambda(\boldsymbol{x})$ is constant or homotopic to a constant, but in general this is not the case. The reason is that although $\Lambda(\boldsymbol{x})$ is a map from S^2 into the Lie algebra of G (a vector space, and thus contractile), it cannot be contracted to a point since the spectrum of $\Lambda(\boldsymbol{x})$ is constrained to be in $(2\pi i/\beta)\mathbb{Z}$. An explicit SU(2) example in $2+1$ dimensions is provided in Ref. [23], namely,

$$U(x)=\exp\left(\frac{2\pi n x_0}{\beta} i\boldsymbol{\tau}\boldsymbol{x}\right), \quad n\in\mathbb{Z}, \tag{95}$$

where $\boldsymbol{\tau}$ are the Pauli matrices and $\boldsymbol{x}$ lies on the unit sphere S^2 in $\mathbb{R}^3$. $U(x)$ covers SU(2) $2n$ times and thus it is homotopically large for nonvanishing n.

Regarding the second question, whether a given gauge configuration can be brought to a A_0-stationary gauge using only small transformations, it also depends on the group and the configuration [23]. For the group U(1), any gauge configuration is in the same homotopy class as one which is A_0 stationary. For a simply connected group such as SU(n), it depends on the initial $A_0(x)$.[11]

An apparent paradox arises here. As emphasized in Refs. [11] and [13], although perturbation theory breaks large gauge invariance, it respects invariance under small transformations, or equivalent, under infinitesimal ones. (Actually, this is strictly correct for Abelian theories only. In non-Abelian theories infinitesimal gauge transformations mix different orders, however, the mixing is mild since it only involves finite sets of Feynman graphs.) On the other hand, we have just seen that expanding a functional such as $\Gamma[M,A]$ in powers of A_0 (which is a perturbative expansion) destroys periodicity and thus gauge invariance under discrete transformations. This looks paradoxical since the breaking occurs even if the discrete transformations are topologically small, and no breaking was expected in this case.

The resolution comes from the observation that the right-hand side of Eq. (91) refers solely to configurations in the gauge $\partial_0 A_0=0$. Within this gauge the only allowed infinitesimal transformations are the time-independent ones, for which no breaking occurs. Nontrivial discrete gauge transformations, even small ones, cannot be reached continuously within the A_0-stationary gauge. In this sense they are always topologically large. It should be realized that the concept of homotopically trivial is a relative one. A transformation which is topologically large within the gauge group G can become small if G is regarded as a subgroup of a larger group G' and deformations within G' are allowed. Conversely, a small discrete gauge transformation becomes homotopically nontrivial if one insists on preserving the gauge condition $\partial_0 A_0=0$. The usefulness of the concept of small and large transformations remains: perturbation theory provides a functional valid in the region of small fields, this region is preserved by infinitesimal transformations and thus the perturbative functional must be small gauge invariant. Large transformations, on the other hand, necessarily move the gauge configuration away from the perturbative region and thus the response of the perturbative functional under large transformations is not trustworthy. As a consequence invariance under large transformations provides useful nonperturbative information and puts nontrivial constraints on the functional. This holds whether the transformations are large from the point of view of G or from the point of view of the submanifold of A_0-stationary configurations.

For another argument, we can recall our previous remark that a functional such as $\Gamma[M,A]$ in Eq. (91) must hold for all theories at the same time. It is not surprising to find a breaking of gauge invariance under discrete transformations in a perturbative expansion, when such transformations happen to be large, and conclude that nontrivial nonperturbative conditions, namely, periodicity, are required to avoid the breaking. However, as we have emphasized, the property of being topologically large or small depends on the group, whereas the formula must hold in all cases, and so periodicity must follow in all cases too.

In this subsection we have considered a gauge fixing condition in order to deal with the quantity $\log\Omega$ in a simple way. We must recall, however, that the expressions derived with the method studied in this work are all fully gauge invariant, provided some regularity conditions are met, since they depend on Ω. This does not mean that large or discrete gauge transformations play no role whatsoever. This is because the regularity conditions fail for the ultraviolet divergent pieces of the effective action. This translates into the fact that the functions $\varphi_n(\Omega)$ [cf. Eq. (52)], can be many

[11]This can be seen as follows. Let $A_0(\boldsymbol{x})$ be some A_0-stationary configuration for which all its discrete transformations are small, and let us further assume that time-independent transformations are also small (for instance $d=2$). It follows that all A_0-stationary configurations related to the previous one by a gauge transformation are in the same homotopy class. Thus, if $A_0^U(x)$ is a gauge transformed configuration with $U(x)$ large, no small transformation will bring it to the A_0-stationary gauge.

valued (although $\Omega=0$ needs not be one of the branching points). A typical example is the effective action of $0+1$-dimensional fermions, Eq. (64). Further let us consider the Abelian U(1) case, which admits large transformations and they coincide with the nontrivial discrete transformations in the A_0-stationary gauge. Choosing $\eta=+1$ the branching point is at $\Omega=-\exp(\beta m)$. Under a large transformation, the argument of the logarithm may go into a different Riemann sheet (depending on the sign of m), yet the functional is such that it changes, at most, by an integer multiple of $2\pi i$, reflecting the fact that the partition functional is one-valued. This is a general property which puts constraints on the functions $\varphi_n(\Omega)$.

Perhaps this is a good place to remark that the gauge invariance implied by the use of the ζ-function prescription, Eq. (50), has two levels [23]. One the one hand, there is the somewhat trivial gauge invariance implied by the fact that the regularization depends solely on the spectrum. Since the spectrum of the operator K is left unchanged by gauge (or more generally similarity) transformations, gauge invariance follows. However, the definition of the ζ-function introduces a branch cut in the manifold of operators K, each singular operators being a branching point on such a manifold. On a given Riemann sheet, the effective action functional may display a jump discontinuity along the branch cut. Because the determinant of K, being the regularized product of eigenvalues, is a smooth functional, it follows that the jump must be an integer multiple of $2\pi i$. This is a tighter constraint on top of the trivial gauge invariance noted above. For instance, we have noted in the Introduction that perturbation theory for fermions at finite temperature yields a Chern-Simons term which is renormalized by a temperature-dependent coefficient. Under large gauge transformations this would introduce an unacceptable change in the effective action by a quantity which is not a multiple of $2\pi i$. It would be tempting to ''solve'' the problem by simply replacing the Chern-Simons term by a suitable gauge invariant version of it, namely, the η invariant. This prescription restores gauge invariance but introduces jumps which again are proportional to the temperature-dependent coefficient and thus it can be ruled out. The exact result known in particular but nontrivial cases [10–12] shows that this is not the correct mechanism and that the determinant is a continuous functional, with no jumps.

B. Feynman graphs and large gauge invariance

We have already noted in Sec. III C that expressions such as those in Eqs. (69) and (70) involve an infinite number of Feynman graphs. It seems interesting to understand which Feynman graphs are being added and gain some insight on how preservation of full gauge invariance is related to this.

To this end, we will first consider the simpler case of $0+1$-dimensional fermions [10]. The corresponding exact effective action is given in Eq. (64) and that formula holds for arbitrary gauge fields which need not be Abelian nor stationary. As noted, when the gauge group is not simply connected it supports large gauge transformations, which augment the value of the effective action by $2\pi ikn$, $n,k\in\mathbb{Z}$, where n is the winding number of the gauge transformation and k depends on the gauge group and the sign of the mass (in higher spatial dimensions it may also depend on the homotopic class of the gauge field configuration).

Because the $0+1$-dimensional formula is exact, all Feynman graphs are included in this case. This suggests that all graphs are required in order to reconstruct the Polyakov loop Ω appearing in the formula. We conclude that large gauge invariance does not act selecting a certain subset of graphs. The same conclusion is expected to hold in higher dimensional formulas since the dependence on Ω there is qualitatively similar to that of the one-dimensional case.

To further discuss this point let us restrict ourselves to the case of an Abelian gauge group. In this case large gauge transformations correspond to discrete shifts of $a=-\log\Omega=\int dx_0 A_0$ and

$$W_f(a+2\pi in)=W_f(a)+2\pi ikn, \tag{96}$$

where the integer constant k is known. This equation contains all the information on large gauge transformations, and it is completely equivalent to the statement

$$W_f(a)=P(a)+ka, \quad P(a+2\pi in)=P(a). \tag{97}$$

Therefore large gauge invariance is equivalent to the strict periodicity of the function $P(a)=W_f(a)-ka$. Feynman graphs correspond to expand in powers of a, and looking for large gauge invariance in terms of Feynman graphs corresponds to detect periodicity of a function from its Taylor expansion. This seems to be a difficult task.

A related issue is studying to what extent large gauge invariance determines the effective action. In the previous $0+1$-dimensional Abelian case we have seen that to comply with gauge invariance, $P(a)$ must be periodic, i.e.,

$$P(a)=\sum_{n\in\mathbb{Z}} c_n e^{na}, \tag{98}$$

for some (infinite number of) coefficients c_n. No further information can be extracted from gauge invariance, and the coefficients c_n are not determined. In order to achieve further restrictions on the function $P(a)$, more information has to be provided. If, for instance, one knows that the partition function $Z(a)=\exp[-W_f(a)]$ (a periodic function) contains only a finite number of periodic modes, the corresponding Fourier coefficients can then be determined from a few perturbative terms. This is actually the case in $0+1$ dimensions [19], and can be traced back to the fact that the corresponding Hamiltonian contains a finite number of states, namely, the vacuum or the one-fermion state (cf. Sec. III B). In higher dimensions, besides the number of fermions, there is a momentum quantum number and a corresponding kinetic energy contributing to the eigenvalues of the Hamiltonian, thus in general the partition function will contain all kinds of Fourier modes. To see how this works, it is sufficient to consider fermions in the Abelian case with $A_i=0$ and A_0 a space-time constant (such A_0 cannot be gauged away at finite temperature). Let ϵ_k^0 denote the single-particle levels of the Hamiltonian when A_0 is set to zero (that is, the kinetic energy only) and let ϵ_k

be the levels when A_0 is switched on. Clearly, $\epsilon_k = \epsilon_k^0 + A_0$, and the label k is related to the momentum of the fermion. The partition function is

$$Z[m,a] = \prod_k \, (1 + e^{-\beta\epsilon_k}) = \prod_k \, (1 + \Omega e^{-\beta\epsilon_k^0}), \quad \Omega = e^{-a}. \tag{99}$$

$Z[m,a]$ will be periodic in a but with many periodic modes, unless $d=0$. When $d=2$, integration over the label k yields a particular case of the formula for $W_{f,0}[m,A]$ in Eq. (65).

In more than one dimension we have proposed an expansion in the number of spatial covariant derivatives. The zeroth order [e.g., Eq. (69)], which contains no spatial indices, corresponds to the sum of all Feynman graphs with the spatial momenta p_i (of the external gauge fields) set to zero and no legs with spatial A_i, but any number of A_0 external legs and the full dependence in the frequency p_0. There are no odd-order terms in the expansion. The second order [e.g., Eq. (70)] corresponds to all graphs with two A_i or p_i, that is, (i) graphs with $p_i=0$ and two spatial gauge fields, plus (ii) graphs with one A_i and p_i kept up to first order in a Taylor expansion of the Green functions, plus (iii) graphs with no legs A_i and p_i kept up to second order in a Taylor expansion around zero spatial momentum. Note that the method assumes that the space has topology $\mathbb{R}^d$. The contribution of higher orders follows a similar pattern. Eventually, all contributions, all Feynman graphs, are added up.

The previous conclusions follow from inspection of the formulas or else from making the expansion by introducing a bookkeeping parameter in order to count spatial indices, as explained in Sec. II C. For the simpler formulas obtained by further expanding in powers of $\mathcal{D}_0$, [e.g., Eq. (73)], we have noted that this expansion does not seem to follow from insertion of a bookkeeping parameter and so different expansions can be obtained, all of them being equivalent when added to all orders. The problem and its interpretation in terms of Feynman graphs can be seen in a particularly simple case: let us assume that the gauge group is Abelian, that A_0 is a space-time constant and A_i are space independent although time dependent. In this case the Green functions depend only on the frequencies $\kappa_n = 2\pi i n/\beta$ of the fields A_i:

$$W = f_0(\Omega) + \sum_n \, f_2(\Omega,\kappa_n) A_{i,n} A_{i,-n} + \cdots,$$
$$A_i(x_0) = \frac{1}{\beta} \sum_n \, e^{\kappa_n x_0} A_{i,n}\,. \tag{100}$$

In addition, $\mathcal{D}_0$ is equivalent to ∂_0 and so an expansion in $\mathcal{D}_0$ is just a Taylor expansion in powers of κ_n, to be made on top of the expansion in powers of A_i. This gives the interpretation in terms of Feynman graphs. Because κ_n is a discrete variable, the function f_2 (and similarly for higher orders) is only well-defined at those discrete values. The ambiguity comes when it is smoothly extended to continuous values of κ_n in order to carry out the Taylor expansion. Presumably this ambiguity can be removed by using the choice suggested by Carlson's theorem.

C. Large mass expansions

Large mass expansions of the effective action can be also considered, as done in Ref. [13] for fermions using the heat-kernel technique. Inspection of our formulas in Secs. III C and III E, show that as the mass becomes large the temperature dependence is exponentially suppressed, being $O(e^{-\beta|m|})$, thus the large mass expansion is an asymptotic expansion with temperature-independent coefficients. This is consistent with the fact that the coefficients are local operators independent of the global topology of the space-time manifold [13]. Therefore in order to carry out a large mass expansion one can simplify and start with a zero temperature theory. The simplification of working at zero temperature is enormous to the point that this problem can be considered a solved one. There is a very large body of work on this subject, both for bosons and for fermions, largely summarized in Ref. [33] and references therein. Large mass expansions for fermions with arbitrary Dirac operators and arbitrary dimension, computed along the lines of the method discussed in this paper for finite temperature, can be found in Ref. [21]. The large mass expansion can also be used as a check of the finite temperature formulas. In Sec. III E we have already noted that Eq. (78) is consistent with the result more straightforwardly obtained from the zero temperature method presented in Ref. [27]. Likewise, for $2+1$-dimensional fermions, and starting from the full finite temperature calculation, one finds [23]

$$W^+ = -\frac{1}{48\pi}\frac{1}{|m|}\int d^3x \, \mathrm{tr}(F_{\mu\nu}^2) + O\left(\frac{1}{m^3}\right),$$
$$W^- = \eta\sigma\Theta(-\sigma m) W_{\mathrm{CS}} - \frac{i\eta}{8\pi}\frac{\varepsilon(m)}{12m^2}\int d^3x \, \epsilon_{\mu\nu\alpha} \, \mathrm{tr}(F_{\beta\mu}\mathcal{D}_\alpha F_{\beta\nu}) + O\left(\frac{1}{m^3}\right) \tag{101}$$

for the pseudoparity even and odd components, respectively. (In these formulas W_{CS} is the Chern-Simons term, Θ and ε denote the step and sign functions, respectively, $\eta=\pm 1$ depends on the irreducible representation of the Dirac gamma matrices taken, and $\sigma=\pm 1$ distinguish the two possible ζ-function definitions of the effective action, depending on the branch cut in the function z^s.) These results also derive more directly from the zero temperature formulas in Ref. [21]. The term with $F_{\mu\nu}^2$ in W^+ is that with H_4 in Eq. (4.11) of Ref. [13], already noted there, whereas the term with $F_{\beta\mu}\mathcal{D}_\alpha F_{\beta\nu}$ is that with P_5 in Eq. (4.12) of the same reference.

V. SUMMARY AND CONCLUSIONS

Our findings can be summarized as follows.

(1) The one-loop effective action at finite temperature and density, for bosonic or fermionic particles in presence of background fields (both gauge and nongauge) with arbitrary

internal symmetry group and arbitrary space-time dependence, can be written as a sum (an asymptotic series in general) of terms ordered by the number of spatial Lorentz indices. Each term is well defined from the effective action functional itself and is separately gauge invariant under all gauge transformations.

(2) These terms are amenable to explicit computation using a combination of ζ-function and symbols method. We have shown that this kind of calculation can be carried out preserving full gauge invariance throughout, without assuming particular internal symmetry groups or special space-time configurations for the background fields. The same arguments show that previous calculations done fixing the gauge through the condition $\partial_0 A_0 = 0$ can be repeated lifting this condition, and this is equivalent to rewrite the final original result in a manifestly gauge invariant way.

(3) A further gauge invariant expansion can be taken in the number of temporal covariant derivatives in the adjoint representation. Within this expansion, all ultraviolet finite terms and more generally, all terms not related to essential anomalies, can be written as a sum of gauge invariant local operators (i.e., constructed with $\mathcal{D}_\mu = [D_\mu,]$ and M) times a function of the field $\Omega(x)$, which also transforms locally under gauge transformations. For those terms containing anomalies, topological pieces and multivaluation, the effective action still looks local in terms of a suitable gauge covariant version of $\boldsymbol{A}(x)$, in addition to M, $\mathcal{D}_\mu$, and Ω.

(4) The method is explicitly applied to the problem of relativistic scalar particles in $2+1$ dimensions. The corresponding effective action is computed up to terms with two spatial Lorentz indices. The result is checked against the known result at zero temperature and also the known partition function of a relativistic Bose gas. The corrections to the density are also computed. Finally, a simple rule is noted relating the bosonic and fermionic versions of the grand-canonical potentials of ideal or interacting systems.

ACKNOWLEDGMENTS

This work was supported in part by funds provided by the Spanish DGICYT grant no. PB98-1367 and Junta de Andalucía grant No. FQM-225.

[1] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. (N.Y.) **140**, 372 (1982).
[2] S.S. Chern and J. Simons, Ann. Math. **99**, 48 (1974).
[3] G.V. Dunne, presented at Les Houches Summer School on Theoretical Physics: Session 69: Topological Aspects of Low-dimensional Systems, Les Houches, France, 1998, hep-th/9902115.
[4] A.N. Redlich, Phys. Rev. Lett. **52**, 18 (1984); Phys. Rev. D **29**, 2366 (1984).
[5] A.J. Niemi and G.W. Semenoff, Phys. Rev. Lett. **51**, 2077 (1983).
[6] L. Alvarez-Gaumé and E. Witten, Nucl. Phys. **B234**, 269 (1984); R. Jackiw, presented at Les Houches Summer School on Theoretical Physics: Relativity Groups and Topology, Les Houches, France, 1983.
[7] T. Matsubara, Prog. Theor. Phys. **14**, 351 (1955).
[8] N.P. Landsman and C.G. van Weert, Phys. Rep. **145**, 141 (1987).
[9] A.J. Niemi, Nucl. Phys. **B251**, 155 (1985); K. Ishikawa and T. Matsuyama, *ibid.* **B280**, 523 (1987); K.S. Babu, A. Das, and P. Panigrahi, Phys. Rev. D **36**, 3725 (1987); E.R. Poppitz, Phys. Lett. B **252**, 417 (1990); L. Moriconi, Phys. Rev. D **44**, 2950 (1991); M. Burgess, *ibid.* **44**, 2552 (1991); I.J. Aitchison, C.D. Fosco, and J.A. Zuk, *ibid.* **48**, 5895 (1993); I.J. Aitchison and J.A. Zuk, Ann. Phys. (N.Y.) **242**, 77 (1995); R.D. Pisarski, Phys. Rev. D **35**, 664 (1987); N. Bralic, C.D. Fosco, and F.A. Schaposnik, Phys. Lett. B **383**, 199 (1996); D. Cabra, E. Fradkin, G.L. Rossini, and F.A. Schaposnik, *ibid.* **383**, 434 (1996); A.N. Sissakian, O.Y. Shevchenko, and S.B. Solganik, Nucl. Phys. **B518**, 455 (1998).
[10] G. Dunne, K. Lee, and C. Lu, Phys. Rev. Lett. **78**, 3434 (1997).
[11] S. Deser, L. Griguolo, and D. Seminara, Phys. Rev. Lett. **79**, 1976 (1997).
[12] C. Fosco, G.L. Rossini, and F.A. Schaposnik, Phys. Rev. Lett. **79**, 1980 (1997); **79**, 4296(E) (1997).
[13] S. Deser, L. Griguolo, and D. Seminara, Phys. Rev. D **57**, 7444 (1998).
[14] A. Das and A.J. da Silva, Phys. Rev. D **59**, 105011 (1999).
[15] C.D. Fosco, R.E. Gamboa Saravi, and F.A. Schaposnik, hep-th/9902002.
[16] G. Dunne and J. Feinberg, Phys. Lett. B **477**, 474 (2000).
[17] L.L. Salcedo, Phys. Rev. D **58**, 125007 (1998).
[18] J. Barcelos-Neto and A. Das, Phys. Rev. D **59**, 087701 (1999).
[19] A. Das, G. Dunne, and J. Frenkel, Phys. Lett. B **472**, 332 (2000).
[20] J. Barcelos-Neto and A. Das, Phys. Rev. D **58**, 085022 (1998).
[21] L.L. Salcedo and E. Ruiz Arriola, Ann. Phys. (N.Y.) **250**, 1 (1996).
[22] E. Ruiz Arriola and L.L. Salcedo, Phys. Lett. B **450**, 225 (1999).
[23] L.L. Salcedo, Nucl. Phys. **B549**, 98 (1999).
[24] N.G. Pletnev and A.T. Banin, Phys. Rev. D **60**, 105017 (1999).
[25] R.T. Seeley, *Collection: Singular Integrals*, Proceedings of the Symposia on Pure Mathematics, Chicago, IL, 1966 (American Mathematical Society, Providence, RI, 1967), pp. 288–307; J.S. Dowker and R. Critchley, Phys. Rev. D **13**, 3224 (1976); S.W. Hawking, Commun. Math. Phys. **55**, 133 (1977); E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko, and S. Zerbini, *Zeta Regularization Techniques with Applications* (World Scientific, Singapore, 1994).
[26] S. Blau, M. Visser, and A. Wipf, Nucl. Phys. **B310**, 163 (1988); N.D. Birrel and P.C.W. Davis, *Quantum Fields in a Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[27] L. Chan, Phys. Rev. Lett. **57**, 1199 (1986).
[28] H.E. Haber and H.A. Weldon, Phys. Rev. Lett. **46**, 1497 (1981); Phys. Rev. D **25**, 502 (1982).

[29] L. Lewin, *Polylogarithms and Associated Functions* (North-Holland, New York, 1981).
[30] H. Blas, B.M. Pimentel, and J.L. Tomazelli, Phys. Rev. E **60**, 6164 (1999).
[31] J.W. Negele and H. Orland, *Quantum Many-Particle Systems* (Addison-Wesley, New York, 1988).
[32] G.H. Whitehead, *Elements of Homotopy Theory* (Springer-Verlag, New York, 1995).
[33] R.D. Ball, Phys. Rep. **182**, 1 (1989).