

Equivalence between the ADM-Hamiltonian and the harmonic-coordinates approaches to the third post-Newtonian dynamics of compact binaries

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The third post-Newtonian approximation to the general relativistic dynamics of two point-mass systems has been recently derived by two independent groups, using different approaches and different coordinate systems. By explicitly exhibiting the map between the variables used in the two approaches we prove their physical equivalence. Our map allows one to transfer all the known results of the Arnowitt-Deser-Misner approach to the harmonic-coordinates one: in particular, it gives the value of the harmonic-coordinates Lagrangian and the expression of the ten conserved quantities associated with global Poincaré invariance.

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I. MOTIVATION

Binary systems made of compact objects (neutron stars or black holes) are the most promising sources for the upcoming ground-based network of interferometric gravitational wave detectors such as the Laser Interferometric Gravitational Wave Observatory (LIGO), VIRGO, or GEO. Because of their higher signal-to-noise ratio, the first detections are likely to involve massive binary black-hole systems, with total mass $m_1 + m_2 \gtrsim 30M_\odot$. Such systems emit most of their useful signal at the end of their inspiral phase, near the last stable (circular) orbit. This makes it very important to have the best possible analytical control of the general relativistic dynamics of two-body systems.

For many years the equations of motion of binary systems have been known only up to the 5/2 post-Newtonian (2.5PN) approximation [1–7]. Recently, Jaranowski and Schäfer [8,9] and Damour, Jaranowski, and Schäfer [10,11] succeeded in deriving the third post-Newtonian (3PN) dynamics of binary point-mass systems within the canonical formalism of Arnowitt, Deser, and Misner (ADM). More recently, Blanchet and Faye [12,13] succeeded in deriving the 3PN equations of motion of binary point-mass systems in harmonic coordinates relying on an independent framework. The purpose of this paper is to compare and relate these two sets of results.

For the present investigation it is sufficient to consider the conservative part of the dynamics; i.e., we shall drop the dissipative 2.5PN part which is the leading order radiation-damping level. This has the advantage that for the remaining part there exists an autonomous Hamiltonian (depending only on particle variables) and a conserved energy. The non-autonomous parts of the ADM Hamiltonian, up to the 3.5PN level (which is the next order radiation-damping level after 2.5PN), are given in Ref. [14].

II. REGULARIZATION AMBIGUITIES

Before tackling the comparison between the two sets of 3PN results several remarks are in order. First, let us empha-

size that both approaches to the 3PN dynamics have found that the use of Dirac-delta-function sources to model the two-body system causes the appearance of both badly divergent integrals and badly defined “contact terms,” which (contrary to what happened at the 2.5PN [2,4] and 3.5PN [14] levels) cannot be unambiguously regularized. More precisely, when Refs. [8,9] derived the relative-motion 3PN ADM Hamiltonian $H(\mathbf{x}, \mathbf{p})$, in the center-of-mass frame of the binary, they introduced two arbitrary *dimensionless* parameters, $\omega_k (\equiv \omega_{\text{kinetic}})$ and $\omega_s (\equiv \omega_{\text{static}})$, to formalize the presence of irreducible ambiguities in the regularization of the Hamiltonian. The regularization ambiguity parameter ω_k concerned a momentum-dependent contribution $\propto G^3 c^{-6} (\mathbf{p}^2 - 3(\mathbf{n} \cdot \mathbf{p})^2) r^{-3}$, while ω_s concerned a momentum-independent contribution $\propto G^4 c^{-6} r^{-4}$. Reference [11], on the one hand, generalized the work of [8] by deriving the 3PN ADM Hamiltonian $H(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2)$ in an arbitrary reference frame and, on the other hand, proved that ω_k was uniquely determined to have the value $\omega_k = 41/24$ by requiring global Poincaré invariance of the 3PN dynamics (see Ref. [15] for details of why ω_k is not fixable in the center-of-mass frame). Therefore, finally, the 3PN ADM Hamiltonian¹ contains only *one* regularization ambiguity (parametrized by the *dimensionless* parameter ω_s):

$$H(\mathbf{x}_a, \mathbf{p}_a; \omega_s) = \sum_a m_a c^2 + H_N(\mathbf{x}_a, \mathbf{p}_a)$$

¹Note that we are considering here the ordinary 3PN Hamiltonian, obtained (following a result of [10]) by a well-defined shift of phase-space coordinates, designed to *reduce* the *higher-order* Hamiltonian $\tilde{H}_{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a, \dot{\mathbf{x}}_a, \dot{\mathbf{p}}_a)$ defined by eliminating the field variables $h_{ij}^{\text{TT}}, \dot{h}_{ij}^{\text{TT}}$ in the “Routh functional” $R_{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \dot{h}_{ij}^{\text{TT}})$ of [8].

$$\begin{aligned}
& + \frac{1}{c^2} H_{1\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) + \frac{1}{c^4} H_{2\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) \\
& + \frac{1}{c^6} \left[H_{3\text{PN}}^{\omega_s=0}(\mathbf{x}_a, \mathbf{p}_a) + \omega_s \frac{G^4 m_1^2 m_2^2 (m_1 + m_2)}{r_{12}^4} \right]. \quad (2.1)
\end{aligned}$$

See Ref. [11] for the explicit expressions of $H_{3\text{PN}}^{\omega_s=0}$ and of the well-known lower-order (Newtonian, 1PN, and 2PN) contributions.

On the other hand, Ref. [13], in deriving the 3PN equations of motion in harmonic coordinates, introduced four arbitrary *dimensionfull* parameters s_1 , s_2 , r'_1 , and r'_2 (with dimensions of length; s_1 , s_2 correspond to the intermediate ADM regularization length scales l_1 , l_2 discussed below, which disappear in the final $H_{3\text{PN}}$). In addition, though these authors developed some formal generalization of the theory of distributions to deal with the badly divergent integrals appearing at 3PN [16], they could not prove the uniqueness of their prescriptions, and, in fact, they used two different prescriptions, the most recent of them [13] introducing a new *dimensionless* arbitrary parameter K . (However, they did prove that both versions of their regularization prescriptions finally lead to gauge-equivalent equations of motion.) In [17] these authors also introduced a modification of their regularization procedures, aimed at yielding (“in principle”) Lorentz-invariant equations of motion. The net result of using the set of regularization recipes developed in [16,17,13] is the derivation of 3PN two-body equations of motion, in harmonic coordinates, which depend on the *five* parameters s_1 , s_2 , r'_1 , r'_2 , and K and which are, generically, neither Lorentz-invariant² nor deducible from an action (because they do not lead to a conserved energy as any autonomous action-based equations of motion would). Then, the authors of [13] impose the triple requirement of (i) Lorentz invariance, (ii) the existence of a conserved energy, and (iii) polynomiality in m_1 and m_2 . They show that (i) uniquely determines K to have the value $K=41/160$ and (ii) imposes one constraint relating the four length scales s_1 , s_2 , r'_1 , and r'_2 , namely

$$m_2 \left[\ln \left(\frac{r'_2}{s_2} \right) + \frac{783}{3080} \right] = m_1 \left[\ln \left(\frac{r'_1}{s_1} \right) + \frac{783}{3080} \right]. \quad (2.2)$$

Note that, when they use their older version of their regularization prescriptions, the rational number appearing in Eq. (2.2) becomes $-159/308$. By further imposing the requirement (iii), they conclude that the two length scales s_1, s_2 can be expressed in terms of the two other scales r'_1 , r'_2 , and of a new *dimensionless* parameter λ , through

$$\ln \left(\frac{r'_1}{s_1} \right) = -\frac{783}{3080} + \lambda \frac{m_1 + m_2}{m_1}, \quad (2.3a)$$

$$\ln \left(\frac{r'_2}{s_2} \right) = -\frac{783}{3080} + \lambda \frac{m_1 + m_2}{m_2}. \quad (2.3b)$$

Finally, the 3PN equations of motion for the harmonic coordinates $\mathbf{y}_a(t)$ of the two point masses contain *three* regularization ambiguities (parametrized by the two scales r'_1 , r'_2 and the dimensionless parameter λ) and have the form $\ddot{\mathbf{y}}_a = \mathbf{A}_a(\mathbf{y}_b, \mathbf{v}_b)$, where $\mathbf{v}_a \equiv \dot{\mathbf{y}}_a$, with

$$\begin{aligned}
\mathbf{A}_a(\mathbf{y}_b, \mathbf{v}_b) &= \mathbf{A}_{aN}(\mathbf{y}_b, \mathbf{v}_b) + \frac{1}{c^2} \mathbf{A}_{a1\text{PN}}(\mathbf{y}_b, \mathbf{v}_b) \\
&+ \frac{1}{c^4} \mathbf{A}_{a2\text{PN}}(\mathbf{y}_b, \mathbf{v}_b) + \frac{1}{c^5} \mathbf{A}_{a2.5\text{PN}}(\mathbf{y}_b, \mathbf{v}_b) \\
&+ \frac{1}{c^6} \left[\mathbf{A}_{a3\text{PN}}^{(0)}(\mathbf{y}_b, \mathbf{v}_b) + \ln \left(\frac{r'_{12}}{r'_1} \right) \mathbf{A}_{a3\text{PN}}^{(1)}(\mathbf{r}_{12}^h, \mathbf{v}_{12}) \right. \\
&\left. + \ln \left(\frac{r'_{12}}{r'_2} \right) \mathbf{A}_{a3\text{PN}}^{(2)}(\mathbf{r}_{12}^h, \mathbf{v}_{12}) + \lambda \mathbf{A}_{a3\text{PN}}^{(3)}(\mathbf{r}_{12}^h) \right], \quad (2.4)
\end{aligned}$$

where $\mathbf{r}_{12}^h \equiv \mathbf{y}_1 - \mathbf{y}_2$ and $\mathbf{v}_{12}^h \equiv \mathbf{v}_1 - \mathbf{v}_2$ denote the (harmonic) relative position and velocity, respectively. It was, however, shown in [13] that the ambiguities linked to r'_1 , r'_2 can be gauged away, so that the physical ambiguity of the harmonic equations of motions is described by the *sole* (dimensionless) parameter λ . For simplicity, we shall work here with the equations of motion explicitly displayed in [13] which, in fact, corresponds to their older regularization prescription [with $783/3080$ being replaced by $-159/308$ in Eq. (2.2)]. See Eq. (7.16) of [13] for the explicit expression of the 3PN contributions to the harmonic equations of motion (as well as of the well-known lower-order contributions [1,2]). We shall only note here the fact that $\mathbf{A}_{a3\text{PN}}^{(1)}$ and $\mathbf{A}_{a3\text{PN}}^{(2)}$ depend only on the relative positions and velocities, and that the λ term reads (for $a=1$; $\mathbf{n}_{12}^h \equiv \mathbf{r}_{12}^h/r_{12}^h$)

$$\lambda \mathbf{A}_{13\text{PN}}^{(3)}(\mathbf{r}_{12}^h) = -\frac{44}{3} \lambda \frac{G^4 m_1 m_2^2 (m_1 + m_2)}{(r_{12}^h)^5} \mathbf{n}_{12}^h. \quad (2.5)$$

Even before any detailed calculation, it is clear that this λ contribution derives from a potential energy

$$\lambda V^{(3)} \equiv -\frac{11}{3} \lambda \frac{G^4 m_1^2 m_2^2 (m_1 + m_2)}{c^6 (r_{12}^h)^4},$$

so that, if the two different 3PN dynamics can be shown to be somehow equivalent, the “harmonic” regularization ambiguity λ must be related to the “ADM” one ω_s by $-\frac{11}{3} \lambda = \omega_s + \text{const.}$

²At least if one follows [13] in using the new “correct” derivative involving the parameter K .

III. ORIGIN OF THE REGULARIZATION AMBIGUITIES

As the presence of the regularization ambiguities at the 3PN level is very striking³ and physically momentous, let us discuss in more detail the origin of the ambiguities present in the two approaches, and the differences between them.

In the ADM approach, one computes a (spatially) *global scalar* quantity, the Hamiltonian $H(\mathbf{x}_a, \mathbf{p}_a)$ of the system. Essentially⁴ the global scalar $H(\mathbf{x}_a, \mathbf{p}_a)$ can be explicitly expressed as an integral over space of an integrand of the generic form

$$\mathcal{H}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) = \mathcal{H}_c^{(D)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + \mathcal{H}_f^{(D)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + \partial_i D^i(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a). \quad (3.1)$$

Here, $\mathcal{H}_c^{(D)}$ is made only of ‘‘contact terms,’’ i.e. of terms proportional to the delta functions modeling the sources, say $\mathcal{H}_c^{(D)} = \sum_a S_a(\mathbf{x}, \mathbf{x}_b) \delta(\mathbf{x} - \mathbf{x}_a)$, where S_a is constructed from field quantities, $\mathcal{H}_f^{(D)}$ is a ‘‘field-like’’ term, i.e. an ‘‘energy density’’ constructed from field quantities and distributed all over space, and the last term is a pure divergence, which formally gives a vanishing contribution⁵ to the integrated Hamiltonian. As indicated by the superscript notation, the explicit values of the ‘‘contact’’ and ‘‘field’’ terms depend on the choice of the divergence term $\partial_i D^i$. In other words, we can, by ‘‘operating by parts,’’ shuffle terms between $\mathcal{H}_c^{(D)}$ and $\mathcal{H}_f^{(D)}$, at the price of changing D^i . Note that, when so shuffling terms, one freely uses Einstein’s field equations (with delta-function sources) and one assumes the validity of the usual rules of functional calculus,⁶ such as Leibniz’ rule [$\partial_i(AB) = (\partial_i A)B + A(\partial_i B)$], and the commutativity of repeated derivatives ($\partial_i \partial_j A = \partial_j \partial_i A$).

The ambiguities in the determination of the value of $H \equiv \int d^3x \mathcal{H}$ come from two separate (but related) facts. First, the ‘‘contact’’ contribution

$$\begin{aligned} H_c^{(D)} &\equiv \int d^3x \mathcal{H}_c^{(D)} \\ &= \int d^3\mathbf{x} \sum_a S_a(\mathbf{x}, \mathbf{x}_b) \delta(\mathbf{x} - \mathbf{x}_a) \end{aligned}$$

is formally infinite because the (field-constructed) quantity $S_a(\mathbf{x}, \mathbf{x}_b)$ is generically singular as $\mathbf{x} \rightarrow \mathbf{x}_a$. To give a meaning to $H_c^{(D)}$ one must choose a specific regularization prescription to define the limit $\lim_{\mathbf{x} \rightarrow \mathbf{x}_a} S_a(\mathbf{x}, \mathbf{x}_b)$. Second, the ‘‘field’’ contribution $H_f^{(D)} \equiv \int d^3x \mathcal{H}_f^{(D)}$ is also formally infinite because the integrand $\mathcal{H}_f^{(D)}$ is generically too singular as $\mathbf{x} \rightarrow \mathbf{x}_a$ to be locally integrable. To give a meaning to $H_f^{(D)}$ one must choose a specific regularization prescription for such singular integrals. Finally for each choice of D^i , one defines the regularized value of the Hamiltonian as $H^{(D)\text{reg}} \equiv H_c^{(D)\text{reg}} + H_f^{(D)\text{reg}}$.

In Refs. [8,9] the following specific regularization prescriptions were adopted: (i) for contact terms $[\lim_{\mathbf{x} \rightarrow \mathbf{x}_a} S_a(\mathbf{x}, \mathbf{x}_b)]^{\text{reg}}$ is defined as Hadamard’s ‘‘partie finie’’ of $S_a(\mathbf{x}, \mathbf{x}_b)$, $\text{Pf}_a S_a(\mathbf{x}, \mathbf{x}_b)$, defined in Appendix B of [8] as the angle-averaged finite term in the Laurent expansion of $S_a(\mathbf{x}_a + \mathbf{r}_a, \mathbf{x}_b)$ in powers of $r_a \equiv |\mathbf{r}_a| \equiv |\mathbf{x} - \mathbf{x}_a|$ (as $\mathbf{r}_a \rightarrow 0$), and (ii) for field terms the regularized value I^{reg} of a singular integral $I = \int d^3x F(\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2)$ is defined as follows. First, one regularizes separately the divergences near each particle, i.e. the integrals $I_a \equiv \int_{V_a} d^3x F(\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2)$ where V_a is a volume which contains \mathbf{x}_a but not \mathbf{x}_b , with $b \neq a$. (Evidently, one can always decompose $I = I_1 + I_2 + I_{\text{compl}}$ with two local volumes V_1, V_2 and a regular complement.) Second, each local integral, say I_1 near particle 1, is regularized in the manner of Riesz, i.e. by analytic continuation (AC) in ϵ_1 of

$$I_1(\epsilon_1) \equiv \int_{V_1} d^3x \left(\frac{r_1}{l_1} \right)^{\epsilon_1} F(\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2),$$

where $r_1 = |\mathbf{x} - \mathbf{x}_1|$ and where l_1 is a certain length scale.⁷ Most integrands F lead to functions of $\epsilon_1, I_1(\epsilon_1)$, which are analytically continuable into the complex ϵ_1 plane down to $\epsilon_1 = 0$. In such a case this continuation $\text{AC}_{\epsilon_1 \rightarrow 0} I_1(\epsilon_1)$ uniquely defines the regularized value of I_1 . However, a limited subclass of ‘‘dangerous’’ integrals gives rise to a (simple) pole as $\epsilon_1 \rightarrow 0$: $I_1(\epsilon_1) = Z_1(\epsilon_1^{-1} + \ln(R_1/l_1)) + A_1$, where R_1 is an ‘‘infrared’’ length scale associated with the choice of local volume V_1 . For such integrals, one is naturally led (following the usual ‘‘minimal subtraction’’ prescription of quantum field theory) to defining the regularized value of $I_1(\epsilon_1)$ as the limit of $I_1(\epsilon_1) - Z_1/\epsilon_1$ as $\epsilon_1 \rightarrow 0$, i.e. as $I_1^{\text{reg}} \equiv Z_1 \ln(R_1/l_1) + A_1$. Note that this regularization pre-

³Though it was anticipated in [2]; see pp. 107 and 116 there.

⁴After applying the double ‘‘reduction’’ process of eliminating the field variables and reducing the order of the Hamiltonian [8,10].

⁵It has been checked in the ADM approach that the ‘‘surface term at infinity’’ associated with $\partial_i D^i$ is not causing any ambiguity. Indeed, most pieces in $\oint dS_i D^i$ decay like some inverse power of r at infinity, while the ones which might be problematic [like the one associated with the $\mathcal{O}(r)$ part of $h_{(6)ij}^{\text{TT}}$] have been explicitly shown to give a vanishing contribution to $\oint dS_i D^i$. The ambiguities come only from the singular behavior of the integrand near each particle, i.e. as the field point \mathbf{x} tends to either \mathbf{x}_1 or \mathbf{x}_2 .

⁶In the explicit computations of the Hamiltonian done in Refs. [8], [10], and [11] one has always chosen D^i ’s such that all the terms in $\mathcal{H}_f^{(D)}$ contain only one derivative (or its equivalent) acting on the elementary fields ($\phi_{(2)}, \pi_{(3)}^i, \phi_{(4)}, \dots$), so that there is no need to worry about using an improved distributional derivative. The distributional rule of differentiation of homogeneous functions described in Appendix B of [8] is, in fact, used only when gauging the ambiguities by computing the regularized value of $\int d^3x (\partial_i D^i)$, as explained in the Appendix A of [10].

⁷The ‘‘Riesz’’ prescription explained in the Appendix of B of [8] looks different from what we explain here (because it does not separate the integration volume in V_1, V_2 and the rest), but, as emphasized in [10], it is equivalent to the logically simpler prescription that we summarize here.

scription has introduced one arbitrary length scale: the regularization length l_1 . (The V_1 -related infrared length R_1 is easily seen to cancel out in $I=I_1+I_2+I_{\text{compl}}$.) However, as emphasized in Sec. IV of [8], a remarkable thing occurs in explicit calculations of the 3PN ADM Hamiltonian: the combination of dangerous integrals appearing in $H_{3\text{PN}}$ is such that all pole terms *exactly cancel*: $\Sigma Z_1=0$. In fact, one of the characteristics of the calculation of $H_{3\text{PN}}$ in the ADM formalism is that one finds it much safer (and simpler) to regularize, at once, the full integral, rather than to try (as in the harmonic-coordinate calculation [13]) to give a separate regularized value for each individual contribution to the equations of motion. Global cancellation of the poles shows that the combination of dangerous integrals appearing in $H_{3\text{PN}}$ is of a less dangerous type. A nice aspect of this cancellation is that, in the ADM approach, the two regularizing length scales l_1, l_2 completely cancel and do not appear in the regularized final $H_{3\text{PN}}$. This does not mean, however, that the final result is unambiguous. Indeed, it was emphasized in [8,10] that the regularized value of $H_{3\text{PN}}$ depends on the reshuffling of terms used to separate \mathcal{H} in $\mathcal{H}_c^{(D)}+\mathcal{H}_f^{(D)}+\partial_i D^i$. In other words, when operating by parts (which changes D^i , and $\mathcal{H}_c^{(D)}$ and $\mathcal{H}_f^{(D)}$) the regularized value of $H^{(D)}\equiv H_c^{(D)}+H_f^{(D)}$ is found to change.⁸ In addition to this D^i -dependent ambiguity, there is also the problem of the sensitivity of the contact contribution $H_c^{(D)}$ to the choice of prescription for defining the ‘‘partie finie’’ of $S_a(\mathbf{x},\mathbf{x}_b)$. It was emphasized in [9,10] that the definition of the ‘‘Hadamard partie finie’’ Pf_a becomes ambiguous at 3PN because it cannot be ‘‘threaded’’ through a product of field functions, i.e. that, in general, $\text{Pf}_a(f_1 f_2 \cdots) \neq (\text{Pf}_a f_1)(\text{Pf}_a f_2) \cdots$. (The prime, irreducible example of this ambiguity at 3PN comes from the fact that $\text{Pf}_a(\phi_{(2)}^4) \neq [\text{Pf}_a(\phi_{(2)}^2)]^2 = [\text{Pf}_a(\phi_{(2)})]^4$, where $\phi_{(2)}$ is the Newtonian potential.)

The attitude of Refs. [8–10] in relation to these regularization ambiguities has been the following: (i) one must acknowledge their existence, because there exists, as yet, no convincingly unique extension of distribution theory allowing one to select a preferred regularized value, and (ii) however, one can analyze in detail the structure of these ambiguities and show that they can be parametrized by only two (dimensionless) parameters: ω_k and ω_s . Indeed, after the pioneering work [8,9] which introduced these regularization ambiguity parameters, a systematic study of the ambiguities has been conducted in the Appendix A of [10] [by exploring all the possible operations by parts, as well as the effect of having $\text{Pf}(f_1 f_2) \neq \text{Pf}(f_1)\text{Pf}(f_2)$]. This study confirmed the existence of only two regularization ambiguities.⁹ As the

⁸In actual calculations (see especially Appendix A of [10]) one monitors the changes in $H^{(D)}\equiv H_c^{(D)}+H_f^{(D)}$ by computing the term-by-term *regularized* value of the full algebraic expansion of the divergence term $\int d^3x(\partial_i D^i)$.

⁹Reference [10] made an attempt at lessening the sources of ambiguity by choosing a D^i such that the contact terms $\mathcal{H}_c^{(D)}$ are absent. However, even in this ‘‘preferred’’ presentation, $\mathcal{H}_f^{(D)}$ gave rise to the two usual ADM ambiguities.

most recent work in the ADM formalism [11] has shown that the ‘‘kinetic’’ ambiguity ω_k was uniquely fixed by imposing global Poincaré invariance, the final conclusion is, as indicated in Eq. (2.1) above, that the ADM formalism introduces only *one* regularization ambiguity parameter: the ‘‘static’’ ambiguity ω_s .

It would take us too long to explain in detail why the harmonic-coordinates approach introduces more ambiguity parameters [four (s_1, s_2, r'_1, r'_2) or five $(s_1, s_2, r'_1, r'_2, K)$, depending on the regularization prescription, instead of two (ω_k, ω_s)] than the ADM one (see [16,17,13]). Let us only make a short list of the most significant differences between the two approaches: (i) Blanchet and Faye regularize separately many independent singular contributions to the spatial derivative of the gravitational field instead of working with the full scalar Hamiltonian as a block, (ii) when computing their ‘‘elementary integrals’’ by analytic continuation they can (after contracting free indices) use the *ordinary* Riesz formula [instead of the *generalized* Riesz formula of [8], necessary to deal with the denominators $\propto(r_1+r_2+r_{12})^\gamma$ that appear in the ADM Hamiltonian], (iii) they directly work with the full hierarchy of PN fields up to $g_{00}=-1+\cdots+2U_8/c^8$, while the ADM approach needs to work only with the contribution $\phi_{(6)}/c^6$ to the ‘‘scalar’’ potential, (iv) they get two (gauge-related) ambiguities of ‘‘logarithmic’’ type (involving two arbitrary length scales), and (v) they use a different coordinate system. It would be interesting to study whether a reworking of the harmonic-coordinates work along the more ‘‘global,’’ and more ‘‘PN order reduced’’ lines of the ADM approach would not simplify their results and get rid of several of their ambiguities.

IV. MATCHING THE TWO 3PN DYNAMICS

We shall now show in detail that the two 3PN dynamics are equivalent modulo a suitable shift of particle variables. Some time ago, Damour and Schäfer [3] studied the link, at the 2PN level, between the ADM dynamics and the harmonic-coordinates (or DeDonder-coordinates) one. They explicitly constructed the map between these two descriptions of the dynamics. Let us emphasize that this map acts on the ‘‘motions,’’ i.e. on the particle positions (and momenta or velocities) as functions of time. In other words, it gives either the transformation (with $\mathbf{v}_a \equiv \dot{\mathbf{y}}_a$)

$$\mathbf{y}_a(t) = \mathbf{Y}_a(\mathbf{x}_b(t), \mathbf{p}_a(t)), \quad (4.1a)$$

$$\mathbf{v}_a(t) = \mathbf{V}_a(\mathbf{x}_b(t), \mathbf{p}_b(t)), \quad (4.1b)$$

from the ADM variables $(\mathbf{x}_b, \mathbf{p}_b)$ to the harmonic ones $(\mathbf{y}_a, \mathbf{v}_a)$, or the inverse transformation

$$\mathbf{x}_a(t) = \mathbf{X}_a(\mathbf{y}_b(t), \mathbf{v}_b(t)), \quad (4.2a)$$

$$\mathbf{p}_a(t) = \mathbf{P}_a(\mathbf{y}_b(t), \mathbf{v}_b(t)). \quad (4.2b)$$

All variables in Eqs. (4.1), (4.2) are taken at the same value for their (respective) time argument. As explained in [3] it is always possible to express the looked for map in this form. One has to beware that the transformations (4.1) or (4.2) are

not the direct restriction of a coordinate transformation, $x'^{\mu} = x^{\mu} + \xi^{\mu}(x^{\lambda}, [\mathbf{x}_1], [\mathbf{x}_2])$ (where the brackets indicate *functional* dependence), to a field point x^{μ} on a particle world line, but that one must take into account the time shift ξ^0 to transform the coordinate shift ξ^i into the ‘‘motion’’ shift $\mathbf{x}'(t) - \mathbf{x}(t)$ [see Eqs. (3) of [3]].

Among the two forms (4.1) or (4.2) we found that it is simplest to work with Eqs. (4.1). Indeed, the necessary and sufficient conditions for Eqs. (4.1) to map the ADM dynamics onto the harmonic one is easily seen to be, simply,

$$\{\mathbf{Y}_a, H\} = \mathbf{V}_a, \quad (4.3a)$$

$$\begin{aligned} \{\mathbf{V}_a, H\} &= \{\{\mathbf{Y}_a, H\}, H\} \\ &= \mathbf{A}_a(\mathbf{Y}_b, \mathbf{V}_b). \end{aligned} \quad (4.3b)$$

All functions entering Eqs. (4.3a) and (4.3b) ($\mathbf{Y}_a, H, \mathbf{V}_a$) are functions of the ADM phase space coordinates $(\mathbf{x}_b, \mathbf{p}_b)$. The notation $\{\cdot, \cdot\}$ denotes the usual Poisson brackets:

$$\{A(\mathbf{x}_a, \mathbf{p}_a), B(\mathbf{x}_a, \mathbf{p}_a)\} \equiv \sum_a \sum_i \left(\frac{\partial A}{\partial x_a^i} \frac{\partial B}{\partial p_{ai}} - \frac{\partial A}{\partial p_{ai}} \frac{\partial B}{\partial x_a^i} \right).$$

Finally H denotes the full 3PN Hamiltonian (2.1) while $\mathbf{A}_a(\mathbf{y}_b, \mathbf{v}_b)$ denotes the harmonic equations of motion, Eq. (2.4). Note that Eq. (4.3a) explicitly determines $\mathbf{V}_a(\mathbf{x}_b, \mathbf{p}_b)$ in terms of $\mathbf{Y}_a(\mathbf{x}_b, \mathbf{p}_b)$. Therefore, the problem of the mapping between H and \mathbf{A}_a is reduced to solving Eq. (4.3b) as an equation for the two unknown phase-space vectorial functions $\mathbf{Y}_1(\mathbf{x}_b, \mathbf{p}_b)$ and $\mathbf{Y}_2(\mathbf{x}_b, \mathbf{p}_b)$. We tackled this problem by the method of undetermined coefficients, i.e. by writing the most general expression for the PN expansion of $\mathbf{Y}_a(\mathbf{x}_b, \mathbf{p}_b)$. We know that \mathbf{Y}_a differs from \mathbf{x}_a only at 2PN order, i.e.

$$\mathbf{Y}_a(\mathbf{x}_b, \mathbf{p}_b) = \mathbf{x}_a + \frac{1}{c^4} \mathbf{Y}_a^{2\text{PN}}(\mathbf{x}_b, \mathbf{p}_b) + \frac{1}{c^6} \mathbf{Y}_a^{3\text{PN}}(\mathbf{x}_b, \mathbf{p}_b). \quad (4.4)$$

The explicit expression of $\mathbf{Y}_a^{2\text{PN}}$ was given in Ref. [3] (we write it here for $a=1$, the expression for $a=2$ being obtained by a simple relabeling $1 \leftrightarrow 2$):

$$\begin{aligned} \mathbf{Y}_1^{2\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) &= Gm_2 \left\{ \left[\frac{5}{8} \frac{\mathbf{p}_2^2}{m_2^2} - \frac{1}{8} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_2^2} \right. \right. \\ &\quad \left. \left. + \frac{Gm_1}{r_{12}} \left(\frac{7}{4} + \frac{1}{4} \frac{m_2}{m_1} \right) \right] \mathbf{n}_{12} \right. \\ &\quad \left. + \frac{1}{2} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_2} \frac{\mathbf{p}_1}{m_1} - \frac{7}{4} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_2} \frac{\mathbf{p}_2}{m_2} \right\}. \end{aligned} \quad (4.5)$$

Actually, as a check on the algebraic manipulation programs (done with MATHEMATICA) that we wrote to solve Eq. (4.3b) we have explicitly checked that Eq. (4.5) is the unique (translation-and-rotation-invariant) solution of the 2PN matching.

At 3PN, we write (by using translation and rotation invariance) $\mathbf{Y}_a^{3\text{PN}}$ in terms of some scalar functions (here $\mathbf{n}_{ab} \equiv (\mathbf{x}_a - \mathbf{x}_b)/r_{ab}$; $r_{ab} \equiv |\mathbf{x}_a - \mathbf{x}_b|$)

$$\mathbf{Y}_a^{3\text{PN}}(\mathbf{x}_b, \mathbf{p}_b) = M_a \mathbf{n}_{ab} + \sum_b N_{ab} \mathbf{p}_b. \quad (4.6)$$

By imposing that the map reduces to the identity in the free-motion limit ($G \rightarrow 0$), it is enough to look for M_a and N_{ab} of the symbolic form:

$$M \propto p^4 + \frac{p^2}{r_{12}} + \frac{1}{r_{12}^2} + \frac{\ln r_{12}}{r_{12}^2},$$

$$N \propto p^3 + \frac{p}{r_{12}},$$

where ‘‘ p^n ’’ denotes all the scalars made with \mathbf{p}_1 , \mathbf{p}_2 and \mathbf{n}_{12} with homogeneity p^n , i.e.

$$\begin{aligned} p^n \propto \sum c_{n_1 n_2 n_3 n_4 n_5} (\mathbf{p}_1^2)^{n_1} (\mathbf{p}_2^2)^{n_2} \\ \times (\mathbf{p}_1 \cdot \mathbf{p}_2)^{n_3} (\mathbf{n}_{12} \cdot \mathbf{p}_1)^{n_4} (\mathbf{n}_{12} \cdot \mathbf{p}_2)^{n_5} \end{aligned}$$

with $2n_1 + 2n_2 + 2n_3 + n_4 + n_5 = n$. We find that $\mathbf{Y}_1^{3\text{PN}}$ *a priori* contains 52 unknown coefficients c_n (28 in M_1 , 12 in N_{11} , and 12 in N_{12}). We did not impose any *a priori* constraints on the mass dependence of the coefficients $c_n(m_1, m_2)$ entering $\mathbf{Y}_a^{3\text{PN}}$. (As a consequence we cannot make use of the $1 \leftrightarrow 2$ relabeling symmetry.) Writing in full Eq. (4.3b) gives a *linear* system of 512 equations for the $2 \times 52 = 104$ unknown coefficients c_n . In spite of this very high redundancy, we found that this system is compatible if and only if the arbitrary parameters ω_s and λ are related by

$$\lambda = -\frac{3}{11} \omega_s + \frac{1987}{3080}. \quad (4.7)$$

Then the solution is *unique* and reads (for $a=1$, the solution for $a=2$ being obtained by relabeling $1 \leftrightarrow 2$)

$$\begin{aligned} \mathbf{Y}_1^{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) &= Gm_2 \left\{ \left[Y_1^0 + \frac{Gm_1}{r_{12}} Y_1^1 + \left(\frac{Gm_1}{r_{12}} \right)^2 Y_1^2 \right] \mathbf{n}_{12} \right. \\ &\quad \left. + \left(Y_{11}^0 + \frac{Gm_1}{r_{12}} Y_{11}^1 \right) \frac{\mathbf{p}_1}{m_1} \right. \\ &\quad \left. + \left(Y_{12}^0 + \frac{Gm_1}{r_{12}} Y_{12}^1 \right) \frac{\mathbf{p}_2}{m_2} \right\}, \end{aligned} \quad (4.8)$$

where

$$\begin{aligned}
Y_1^0 = & -\frac{1}{8} \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2) \mathbf{p}_2^2}{m_1 m_2^3} - \frac{1}{8} \frac{(\mathbf{p}_2^2)^2}{m_2^4} - \frac{3}{8} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2) \mathbf{p}_2^2}{m_1 m_2^3} \\
& + \frac{3}{8} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2^3} - \frac{3}{16} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 \mathbf{p}_2^2}{m_2^4} \\
& + \frac{1}{8} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_1 m_2^3} + \frac{1}{16} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4}{m_2^4}, \quad (4.9a)
\end{aligned}$$

$$\begin{aligned}
Y_1^1 = & \frac{167}{48} \frac{\mathbf{p}_1^2}{m_1^2} - \frac{105}{16} \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + \left(\frac{13}{6} - \frac{65}{48} \frac{m_2}{m_1} \right) \frac{\mathbf{p}_2^2}{m_2^2} \\
& - \frac{25}{48} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^2} + \frac{9}{8} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \\
& - \left(\frac{25}{12} - \frac{25}{48} \frac{m_2}{m_1} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_2^2}, \quad (4.9b)
\end{aligned}$$

$$Y_1^2 = -\frac{28387}{2520} + \left(\frac{49}{36} - \frac{21}{32} \pi^2 \right) \frac{m_2}{m_1} + \frac{22}{3} \ln \frac{r_{12}}{r_1}, \quad (4.9c)$$

$$Y_{11}^0 = -\frac{1}{4} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2) \mathbf{p}_1^2}{m_1^2 m_2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2) \mathbf{p}_2^2}{m_2^3} - \frac{5}{12} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_2^3}, \quad (4.9d)$$

$$Y_{11}^1 = -\frac{73}{24} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)}{m_1} + \left(\frac{9}{8} - \frac{3}{2} \frac{m_2}{m_1} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_2}, \quad (4.9e)$$

$$\begin{aligned}
Y_{12}^0 = & -\frac{1}{8} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1) \mathbf{p}_2^2}{m_1 m_2^2} + \frac{1}{4} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2^2} - \frac{1}{8} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2) \mathbf{p}_2^2}{m_2^3} \\
& + \frac{3}{8} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1 m_2^2} + \frac{5}{12} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_2^3}, \quad (4.9f)
\end{aligned}$$

$$Y_{12}^1 = \frac{55}{16} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)}{m_1} + \left(\frac{17}{24} + \frac{221}{48} \frac{m_2}{m_1} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_2}. \quad (4.9g)$$

The results (4.8)–(4.9) give the explicit expression of the transformation $(\mathbf{x}_b, \mathbf{p}_b) \rightarrow \mathbf{y}_a = \mathbf{Y}_a(\mathbf{x}_b, \mathbf{p}_b)$. To complete the knowledge of the transformation between the phase-space variables of the two descriptions one also needs the explicit expression of the transformation $(\mathbf{x}_b, \mathbf{p}_b) \rightarrow \mathbf{v}_a = \mathbf{V}_a(\mathbf{x}_b, \mathbf{p}_b)$. This is straightforwardly obtained by inserting in Eq. (4.3a) the Hamiltonian of [11] and the results (4.4)–(4.9) for $\mathbf{Y}_a(\mathbf{x}_b, \mathbf{p}_b)$. As the explicit result is very lengthy we do not display it here. (Because of the availability of algebraic manipulation programs, it is safer for the interested reader to rederive it directly.)

Let us mention that, as a further check, we have also tried to map the Hamiltonian $H_{3\text{PN}}(\omega_s, \omega_k)$ containing *both* ω_s and ω_k to $\mathbf{A}_a^{\text{harmonic}}$ and that we found again that the mapping is possible only if $\omega_k = 41/24$, in complete agreement with [11].

It should be noted that the further ambiguity parameters r_1' and r_2' present in the harmonic equations of motion enter our result only through some logarithmic terms in M_1 [for $\ln(r_{12}/r_1')$] and M_2 [for $\ln(r_{12}/r_2')$]. This decoupling between

the two particle labels (r_1' entering only $\mathbf{Y}_1^{3\text{PN}}$, and r_2' only $\mathbf{Y}_2^{3\text{PN}}$) suggests (in confirmation of the discussion we gave above) that it is not a necessity, in the harmonic approach, to introduce the ambiguities r_1' and r_2' . Indeed, we see that they are locally (i.e. separately for each particle) introduced by the transformation of variables between our (more ambiguity-free ADM result) and the variables defined by the set of prescriptions of Refs. [16,17,13]. We note also that our result confirms the finding of [13] that r_1' and r_2' can be gauged away (by a harmonicity-preserving coordinate transformation).

Anyway, the most important result is that we have shown the physical equivalence (for invariant consequences of the dynamics) between the 3PN results of [8–11] of those of [12,16,17,13]. The invariants of the 3PN dynamics depend only on *one* ambiguity parameter, denoted ω_s in the ADM work, and λ in the harmonic-coordinates one. The change of notation between ω_s and λ is given in Eq. (4.7) which agrees with the conclusion of [12] which was restricted to the circular motion case.

Could have it been different? In view of the high redundancy of the linear system we had to solve, it may seem quasi-miraculous that the two independent results can be made to match. The compatibility we found is clearly a very useful check on the algebraic computations done by both groups. However, we want to point out that, as both groups had already checked the global Poincaré invariance of their results (see [11] for the ADM case and [17,13] for the harmonic one), the possible remaining discrepancies between the two dynamics were not very numerous. In fact we can count precisely the number of *irreducible* new coefficients entering all 3PN invariants of a Poincaré invariant dynamics. The simplest way to do that is to use the results of [18] on the 3PN “effective one body” dynamics [19]. Using Poincaré invariance we can reduce the dynamics to that of the *relative motion*. Following the results of [18] the number of irreducible new coefficients entering the relative dynamics at the n PN level is obtained by quotienting the arbitrariness in the (relative) Hamiltonian by that in a generic (relative) canonical transformation. This leaves only [computing the difference between Eq. (3.6) and Eq. (3.8) of [18]]

$$\left[\frac{(n+1)(n+2)}{2} + 1 \right] - \left[\frac{n(n+1)}{2} + 1 \right] = n+1 \quad (4.10)$$

irreducible coefficients at n PN. Moreover, one of the coefficients is trivial as it is given by the n PN-level expansion of the free-motion Hamiltonian $H_0 = \sqrt{m_1^2 + \mathbf{p}_1^2} + \sqrt{m_2^2 + \mathbf{p}_2^2}$. Finally, this leaves only n non-trivial irreducible coefficients at the n PN level, i.e. in particular only three coefficients at 3PN.

For instance, in the effective one body approach, these three coefficients are a_4 [the coefficient of $(GM/R)^4$ in $-g_{00}^{\text{eff}}$], d_3 [the coefficient of $(GM/R)^3$ in $-g_{00}^{\text{eff}} g_{RR}^{\text{eff}}$], and z_3 [the coefficient of $P_R^4 (GM/R)^2$ in the squared effective Hamiltonian]. These coefficients have been determined in [18], from $H_{3\text{PN}}^{\text{ADM}}$, and it was found that d_3 and z_3 are unambiguously determined (independently of the ω_s ambiguity) to be

$$d_3 = 2(3\nu - 26)\nu, \quad z_3 = 2(4 - 3\nu)\nu, \quad (4.11)$$

where $\nu \equiv m_1 m_2 / (m_1 + m_2)^2$ is the symmetric mass ratio, while a_4 turns out to depend on ω_s :

$$a_4 = \left(\frac{94}{3} - \frac{41}{32} \pi^2 + 2\omega_s \right) \nu. \quad (4.12)$$

In view of the sensitivity of a_4 to ω_s , a real difference between the ADM and the harmonic dynamics could then only have arisen as possible discrepancies in the values of the only two unambiguous 3PN irreducible coefficients d_3 and z_3 , Eq. (4.11). As a further check, we have in fact allowed for such differences in d_3 and z_3 by looking for the matching of the harmonic equations of motion to a modified $H_{3\text{PN}}^{\text{ADM}}$, containing two extra terms corresponding to variations in both d_3 and z_3 . The result of this generalized matching was that the variations in d_3 and z_3 had both to vanish for the matching to be possible.

V. CONSERVED QUANTITIES AND GENERALIZED LAGRANGIAN IN HARMONIC COORDINATES

Having explicitly obtained the transformation from ADM variables to harmonic ones which maps the two dynamics, we can use this map to transfer all the useful known results of the ADM approach to the harmonic one. For instance, Ref. [11] has explicitly computed the ten conserved quantities of the binary system associated to global Poincaré invariance: total energy $H(\mathbf{x}_a, \mathbf{p}_a)$, total momentum $P_i(\mathbf{x}_a, \mathbf{p}_a)$, total angular momentum $J_i(\mathbf{x}_a, \mathbf{p}_a)$, and the center-of-mass constant (boost vector) $J^{i0} \equiv K^i(\mathbf{x}_a, \mathbf{p}_a) \equiv G^i(\mathbf{x}_a, \mathbf{p}_a) - tP^i(\mathbf{x}_a, \mathbf{p}_a)$. Actually, to be able to express these conserved quantities within the harmonic framework one needs the *inverse* of the transformation $(x, p) \rightarrow (y, v)$; i.e., we need to know explicitly the functions

$$\mathbf{x}_a = \mathbf{X}_a(\mathbf{y}_b, \mathbf{v}_b), \quad (5.1a)$$

$$\mathbf{p}_a = \mathbf{P}_a(\mathbf{y}_b, \mathbf{v}_b). \quad (5.1b)$$

It is just a matter of (somewhat involved) algebraic manipulations to invert the PN-expanded map $(x, p) \rightarrow (y, v)$ to get Eqs. (5.1). By straightforward insertion of the formulas (5.1), we can then (if they are needed) explicitly compute the following quantities in harmonic coordinates:

$$E(\mathbf{y}_b, \mathbf{v}_b) = H^{\text{ADM}}(\mathbf{X}_a(\mathbf{y}_b, \mathbf{v}_b), \mathbf{P}_a(\mathbf{y}_b, \mathbf{v}_b)), \quad (5.2a)$$

$$P_i(\mathbf{y}_b, \mathbf{v}_b) = \sum_a P_{ai}(\mathbf{y}_b, \mathbf{v}_b), \quad (5.2b)$$

$$J_i(\mathbf{y}_b, \mathbf{v}_b) = \sum_a \varepsilon_{ikl} X_a^k(\mathbf{y}_b, \mathbf{v}_b) P_{al}(\mathbf{y}_b, \mathbf{v}_b), \quad (5.2c)$$

$$G_i(\mathbf{y}_b, \mathbf{v}_b) = G_i^{\text{ADM}}(\mathbf{X}_a(\mathbf{y}_b, \mathbf{v}_b), \mathbf{P}_a(\mathbf{y}_b, \mathbf{v}_b)), \quad (5.2d)$$

where the explicit expression of $G_i^{\text{ADM}}(\mathbf{x}_a, \mathbf{p}_a)$ is given in Ref. [11]. (Note that G_i is not conserved but satisfies $dG_i/dt = P_i$. It is a useful, and conserved, quantity in the center-of-mass frame where $P_i = 0$.) We shall not give here the explicit expressions of Eqs. (5.2) as modern computers make it safer for interested readers to do the manipulations

themselves starting from the formulas we give. We have checked that Eq. (5.2a) agrees with the 3PN conserved energy obtained in [13].

Our results allow us also to *prove* (without guesswork) that the harmonic equations of motion derive from a *generalized* Lagrangian $L^{\text{harmonic}}(\mathbf{y}_a, \dot{\mathbf{y}}_a, \ddot{\mathbf{y}}_a)$, depending on positions, velocities, and accelerations. Moreover, we can rather simply compute the Lagrangian L^{harmonic} by starting from the *phase-space* Lagrangian of our ADM Hamiltonian framework, namely

$$L^{\text{ADM}}(\mathbf{x}_a, \dot{\mathbf{x}}_a, \mathbf{p}_a) \equiv \sum_a \mathbf{p}_a \cdot \dot{\mathbf{x}}_a - H^{\text{ADM}}(\mathbf{x}_a, \mathbf{p}_a).$$

Indeed, it suffices (as one easily checks) to insert the transformation $(\mathbf{x}_a, \mathbf{p}_a) \rightarrow (\mathbf{y}_a, \dot{\mathbf{y}}_a)$ in $L^{\text{ADM}}(\mathbf{x}_a, \dot{\mathbf{x}}_a, \mathbf{p}_a)$. This gives

$$L^{\text{harmonic}}(\mathbf{y}_a, \dot{\mathbf{y}}_a, \ddot{\mathbf{y}}_a) = \sum_a \mathbf{P}_a(\mathbf{y}_b, \dot{\mathbf{y}}_b) \cdot \dot{\mathbf{X}}_a(\mathbf{y}_b, \dot{\mathbf{y}}_b, \ddot{\mathbf{y}}_b) - H^{\text{ADM}}(\mathbf{X}_a(\mathbf{y}_b, \dot{\mathbf{y}}_b), \mathbf{P}_a(\mathbf{y}_b, \dot{\mathbf{y}}_b)). \quad (5.3)$$

Note that the meaning of the time derivative \dot{X}_a^i in Eq. (5.3) is

$$\dot{X}_a^i \equiv \frac{dX_a^i(\mathbf{y}_b, \dot{\mathbf{y}}_b)}{dt} \equiv \sum_b \sum_j \left(\frac{\partial X_a^i}{\partial y_b^j} \dot{y}_b^j + \frac{\partial X_a^i}{\partial \dot{y}_b^j} \ddot{y}_b^j \right).$$

Therefore our *constructive* procedure for computing the harmonic Lagrangian automatically yields a Lagrangian which is *linear* in the accelerations $\ddot{\mathbf{y}}_a$. (It was shown in [3] that it was always possible, for perturbatively expanded generalized Lagrangians, to reduce their acceleration dependence to be linear.)

It should also be noted that such a linear-in-acceleration generalized Lagrangian is not unique, but is defined only modulo the addition of $dF(\mathbf{y}_a, \dot{\mathbf{y}}_a)/dt$, where $F(\mathbf{y}_a, \dot{\mathbf{y}}_a)$ is an arbitrary¹⁰ scalar function of positions and velocities (only).

When the 2PN-level generalized harmonic Lagrangian was first computed [20,2], use was made of the addition of some $F_{2\text{PN}}(\mathbf{y}_a, \dot{\mathbf{y}}_a)$ to simplify (in a somewhat arbitrary way) the expression of $L_{2\text{PN}}(\mathbf{y}_a, \dot{\mathbf{y}}_a, \ddot{\mathbf{y}}_a)$. As we are not here playing with the addition of \dot{F} we should not expect our constructive procedure (5.3) to yield a result which coincides with that of [20,2]. We have, however, checked that our 2PN-level result is indeed equivalent to the old one, modulo some $dF(\mathbf{y}_a, \dot{\mathbf{y}}_a)/dt$. Our explicit result for the 3PN-accurate generalized harmonic Lagrangian reads (here $\mathbf{v}_a \equiv \dot{\mathbf{y}}_a$ and $\mathbf{a}_a \equiv \ddot{\mathbf{y}}_a$)

¹⁰It is, however, convenient to restrict the arbitrariness in F so as to respect the symmetries of the problem: translations, rotations, space parity and time reversal.

$$\begin{aligned}
L^{\text{harmonic}}(\mathbf{y}_a, \mathbf{v}_a, \mathbf{a}_a) &= L_N(\mathbf{y}_a, \mathbf{v}_a) + \frac{1}{c^2} L_{1\text{PN}}(\mathbf{y}_a, \mathbf{v}_a) \\
&+ \frac{1}{c^4} L_{2\text{PN}}(\mathbf{y}_a, \mathbf{v}_a, \mathbf{a}_a) \\
&+ \frac{1}{c^6} L_{3\text{PN}}(\mathbf{y}_a, \mathbf{v}_a, \mathbf{a}_a). \tag{5.4}
\end{aligned}$$

The Newtonian and 1PN contributions to the Lagrangian (5.4) do not depend on accelerations. They equal

$$L_N(\mathbf{y}_a, \mathbf{v}_a) = \sum_a \frac{1}{2} m_a \mathbf{v}_a^2 + \frac{Gm_1 m_2}{r_{12}^h}, \tag{5.5}$$

$$\begin{aligned}
L_{1\text{PN}}(\mathbf{y}_a, \mathbf{v}_a) &= \frac{1}{8} m_1 (\mathbf{v}_1^2)^2 + \frac{Gm_1 m_2}{r_{12}^h} \left[\frac{3}{2} \mathbf{v}_1^2 - \frac{7}{4} (\mathbf{v}_1 \cdot \mathbf{v}_2) \right. \\
&\left. - \frac{1}{4} (\mathbf{n}_{12}^h \cdot \mathbf{v}_1) (\mathbf{n}_{12}^h \cdot \mathbf{v}_2) - \frac{1}{2} \frac{Gm_1}{r_{12}^h} \right] + (1 \leftrightarrow 2). \tag{5.6}
\end{aligned}$$

The 2PN acceleration-dependent Lagrangian $L_{2\text{PN}}$ reads

$$L_{2\text{PN}}(\mathbf{y}_a, \mathbf{v}_a, \mathbf{a}_a) = L_{2\text{PN}}^0(\mathbf{y}_a, \mathbf{v}_a) + L_{2\text{PN}}^1(\mathbf{y}_a, \mathbf{v}_a, \mathbf{a}_a) + (1 \leftrightarrow 2), \tag{5.7}$$

where

$$L_{2\text{PN}}^0 = \frac{1}{16} m_1 (\mathbf{v}_1^2)^3 + \frac{Gm_1 m_2}{r_{12}^h} \left[L_{2\text{PN}}^{01} + \frac{Gm_1}{r_{12}^h} L_{2\text{PN}}^{02} + \left(\frac{Gm_1}{r_{12}^h} \right)^2 L_{2\text{PN}}^{03} \right], \tag{5.8a}$$

$$\begin{aligned}
L_{2\text{PN}}^{01} &= \frac{7}{8} (\mathbf{v}_1^2)^2 - \frac{27}{8} \mathbf{v}_1^2 (\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{9}{16} \mathbf{v}_1^2 \mathbf{v}_2^2 + \frac{15}{8} (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 + \frac{13}{8} (\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2 (\mathbf{v}_1 \cdot \mathbf{v}_2) - \frac{5}{8} (\mathbf{n}_{12}^h \cdot \mathbf{v}_1) (\mathbf{n}_{12}^h \cdot \mathbf{v}_2) \mathbf{v}_1^2 \\
&- \frac{3}{4} (\mathbf{n}_{12}^h \cdot \mathbf{v}_1) (\mathbf{n}_{12}^h \cdot \mathbf{v}_2) (\mathbf{v}_1 \cdot \mathbf{v}_2) - \frac{3}{8} (\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2 \mathbf{v}_1^2 + \frac{3}{8} (\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^3 (\mathbf{n}_{12}^h \cdot \mathbf{v}_2) - \frac{3}{16} (\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2 (\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2, \tag{5.8b}
\end{aligned}$$

$$L_{2\text{PN}}^{02} = \frac{1}{4} \mathbf{v}_1^2 - \frac{7}{4} (\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{7}{4} \mathbf{v}_2^2 + \frac{7}{2} (\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2 - \frac{7}{2} (\mathbf{n}_{12}^h \cdot \mathbf{v}_1) (\mathbf{n}_{12}^h \cdot \mathbf{v}_2) + \frac{1}{2} (\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2, \tag{5.8c}$$

$$L_{2\text{PN}}^{03} = \frac{1}{2} + \frac{19}{8} \frac{m_2}{m_1}, \tag{5.8d}$$

$$L_{2\text{PN}}^1 = Gm_1 m_2 \left[\frac{3}{4} (\mathbf{n}_{12}^h \cdot \mathbf{v}_2) (\mathbf{v}_1 \cdot \mathbf{a}_1) - \frac{7}{4} (\mathbf{n}_{12}^h \cdot \mathbf{v}_1) (\mathbf{v}_2 \cdot \mathbf{a}_1) - \frac{7}{4} (\mathbf{v}_1 \cdot \mathbf{v}_2) (\mathbf{n}_{12}^h \cdot \mathbf{a}_1) + \frac{1}{2} \mathbf{v}_2^2 (\mathbf{n}_{12}^h \cdot \mathbf{a}_1) - \frac{1}{4} (\mathbf{n}_{12}^h \cdot \mathbf{v}_1) (\mathbf{n}_{12}^h \cdot \mathbf{v}_2) (\mathbf{n}_{12}^h \cdot \mathbf{a}_1) \right]. \tag{5.8e}$$

Finally, the 3PN contribution to the Lagrangian (5.4) reads

$$L_{3\text{PN}}(\mathbf{y}_a, \mathbf{v}_a, \mathbf{a}_a) = L_{3\text{PN}}^0(\mathbf{y}_a, \mathbf{v}_a) + L_{3\text{PN}}^1(\mathbf{y}_a, \mathbf{v}_a, \mathbf{a}_a) + (1 \leftrightarrow 2), \tag{5.9}$$

where

$$L_{3\text{PN}}^0 = \frac{5}{128} m_1 (\mathbf{v}_1^2)^4 + \frac{Gm_1 m_2}{r_{12}^h} \left[L_{3\text{PN}}^{01} + \frac{Gm_1}{r_{12}^h} L_{3\text{PN}}^{02} + \left(\frac{Gm_1}{r_{12}^h} \right)^2 L_{3\text{PN}}^{03} + \left(\frac{Gm_1}{r_{12}^h} \right)^3 L_{3\text{PN}}^{04} \right], \tag{5.10a}$$

$$L_{3\text{PN}}^1 = Gm_1 m_2 \left(L_{3\text{PN}}^{11} + \frac{Gm_1}{r_{12}^h} L_{3\text{PN}}^{12} \right), \tag{5.10b}$$

$$\begin{aligned}
L_{3\text{PN}}^{01} = & \frac{11}{16}(\mathbf{v}_1^2)^3 - \frac{47}{16}(\mathbf{v}_1^2)^2(\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{25}{16}(\mathbf{v}_1^2)^2\mathbf{v}_2^2 + \frac{23}{8}\mathbf{v}_1^2(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - \frac{65}{32}\mathbf{v}_1^2(\mathbf{v}_1 \cdot \mathbf{v}_2)\mathbf{v}_2^2 - \frac{3}{16}(\mathbf{v}_1 \cdot \mathbf{v}_2)^3 - \frac{17}{16}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)(\mathbf{v}_1^2)^2 \\
& - \frac{3}{8}(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2(\mathbf{v}_1^2)^2 + \frac{45}{16}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2\mathbf{v}_1^2(\mathbf{v}_1 \cdot \mathbf{v}_2) - \frac{1}{4}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)\mathbf{v}_1^2(\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{11}{8}(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2\mathbf{v}_1^2(\mathbf{v}_1 \cdot \mathbf{v}_2) \\
& - \frac{35}{16}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2\mathbf{v}_1^2\mathbf{v}_2^2 + \frac{41}{32}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)\mathbf{v}_1^2\mathbf{v}_2^2 - \frac{3}{4}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - \frac{15}{16}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 \\
& + \frac{19}{16}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^3(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)\mathbf{v}_1^2 + \frac{9}{16}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2\mathbf{v}_1^2 - \frac{15}{8}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^3\mathbf{v}_1^2 + \frac{19}{16}(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^4\mathbf{v}_1^2 \\
& - \frac{19}{16}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^4(\mathbf{v}_1 \cdot \mathbf{v}_2) - (\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^3(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)(\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{45}{32}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2(\mathbf{v}_1 \cdot \mathbf{v}_2) - \frac{5}{16}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^5(\mathbf{n}_{12}^h \cdot \mathbf{v}_2) \\
& - \frac{5}{16}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^4(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2 + \frac{15}{32}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^3(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^3, \tag{5.10c}
\end{aligned}$$

$$\begin{aligned}
L_{3\text{PN}}^{02} = & \frac{59}{8}(\mathbf{v}_1^2)^2 - \frac{199}{8}\mathbf{v}_1^2(\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{25}{3}\mathbf{v}_1^2\mathbf{v}_2^2 + \frac{493}{24}(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - \frac{113}{8}(\mathbf{v}_1 \cdot \mathbf{v}_2)\mathbf{v}_2^2 + \frac{45}{16}(\mathbf{v}_2^2)^2 - \frac{45}{8}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2\mathbf{v}_1^2 \\
& + 19(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)\mathbf{v}_1^2 - \frac{44}{3}(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2\mathbf{v}_1^2 - \frac{1}{4}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2(\mathbf{v}_1 \cdot \mathbf{v}_2) - \frac{61}{6}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)(\mathbf{v}_1 \cdot \mathbf{v}_2) \\
& + \frac{67}{4}(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2(\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{275}{24}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2\mathbf{v}_2^2 - \frac{33}{2}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)\mathbf{v}_2^2 + \frac{1}{4}(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2\mathbf{v}_2^2 - \frac{13}{3}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^4 \\
& + \frac{46}{3}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^3(\mathbf{n}_{12}^h \cdot \mathbf{v}_2) - \frac{137}{6}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2 + \frac{34}{3}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^3, \tag{5.10d}
\end{aligned}$$

$$\begin{aligned}
L_{3\text{PN}}^{03} = & \left[\frac{15611}{1260} + \left(\frac{41}{128}\pi^2 - \frac{305}{144} \right) \frac{m_2}{m_1} \right] \mathbf{v}_1^2 - \left[\frac{17501}{1260} + \left(\frac{41}{64}\pi^2 - \frac{439}{144} \right) \frac{m_2}{m_1} \right] (\mathbf{v}_1 \cdot \mathbf{v}_2) + \left[\frac{5}{4} + \left(\frac{41}{128}\pi^2 - \frac{305}{144} \right) \frac{m_2}{m_1} \right] \mathbf{v}_2^2 \\
& - \left[\frac{8243}{210} + \left(\frac{123}{128}\pi^2 - \frac{383}{48} \right) \frac{m_2}{m_1} \right] (\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2 + \left[\frac{15541}{420} + \left(\frac{123}{64}\pi^2 - \frac{889}{48} \right) \frac{m_2}{m_1} \right] (\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2) \\
& + \left[\frac{3}{2} + \left(\frac{383}{48} - \frac{123}{128}\pi^2 \right) \frac{m_2}{m_1} \right] (\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2 + \left[-\frac{22}{3}\mathbf{v}_1^2 + \frac{22}{3}(\mathbf{v}_1 \cdot \mathbf{v}_2) + 22(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2 - 22(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2) \right] \ln \frac{r_{12}^h}{r_1'}, \tag{5.10e}
\end{aligned}$$

$$L_{3\text{PN}}^{04} = -\frac{3}{8} - \left(\frac{9707}{420} + \omega_s \right) \frac{m_2}{m_1} + \frac{22}{3} \frac{m_2}{m_1} \ln \frac{r_{12}^h}{r_1'}, \tag{5.10f}$$

$$\begin{aligned}
L_{3\text{PN}}^{11} = & -\frac{15}{4}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{v}_1 \cdot \mathbf{v}_2)(\mathbf{v}_1 \cdot \mathbf{a}_1) + \frac{5}{2}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)\mathbf{v}_2^2(\mathbf{v}_1 \cdot \mathbf{a}_1) + \frac{7}{4}(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)\mathbf{v}_1^2(\mathbf{v}_1 \cdot \mathbf{a}_1) - \frac{1}{2}(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)(\mathbf{v}_1 \cdot \mathbf{v}_2)(\mathbf{v}_1 \cdot \mathbf{a}_1) \\
& - \frac{5}{8}(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)\mathbf{v}_2^2(\mathbf{v}_1 \cdot \mathbf{a}_1) - \frac{5}{8}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)(\mathbf{v}_1 \cdot \mathbf{a}_1) - \frac{3}{4}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2(\mathbf{v}_1 \cdot \mathbf{a}_1) + \frac{5}{12}(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^3(\mathbf{v}_1 \cdot \mathbf{a}_1) \\
& - \frac{15}{8}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)\mathbf{v}_1^2(\mathbf{v}_2 \cdot \mathbf{a}_1) + \frac{1}{2}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{v}_1 \cdot \mathbf{v}_2)(\mathbf{v}_2 \cdot \mathbf{a}_1) - \frac{1}{4}(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)\mathbf{v}_1^2(\mathbf{v}_2 \cdot \mathbf{a}_1) - \frac{1}{4}(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)(\mathbf{v}_1 \cdot \mathbf{v}_2)(\mathbf{v}_2 \cdot \mathbf{a}_1) \\
& - \frac{3}{4}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)\mathbf{v}_2^2(\mathbf{v}_2 \cdot \mathbf{a}_1) + \frac{5}{12}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^3(\mathbf{v}_2 \cdot \mathbf{a}_1) + \frac{3}{4}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)(\mathbf{v}_2 \cdot \mathbf{a}_1) - \frac{3}{8}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2(\mathbf{v}_2 \cdot \mathbf{a}_1) \\
& - \frac{15}{8}\mathbf{v}_1^2(\mathbf{v}_1 \cdot \mathbf{v}_2)(\mathbf{n}_{12}^h \cdot \mathbf{a}_1) + \frac{5}{4}\mathbf{v}_1^2\mathbf{v}_2^2(\mathbf{n}_{12}^h \cdot \mathbf{a}_1) + \frac{1}{4}(\mathbf{v}_1 \cdot \mathbf{v}_2)^2(\mathbf{n}_{12}^h \cdot \mathbf{a}_1) - \frac{3}{4}(\mathbf{v}_1 \cdot \mathbf{v}_2)\mathbf{v}_2^2(\mathbf{n}_{12}^h \cdot \mathbf{a}_1) + \frac{1}{4}(\mathbf{v}_2^2)^2(\mathbf{n}_{12}^h \cdot \mathbf{a}_1)
\end{aligned}$$

$$\begin{aligned}
& -\frac{5}{8}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)\mathbf{v}_1^2(\mathbf{n}_{12}^h \cdot \mathbf{a}_1) - \frac{3}{8}(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2\mathbf{v}_1^2(\mathbf{n}_{12}^h \cdot \mathbf{a}_1) + \frac{5}{4}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2(\mathbf{v}_1 \cdot \mathbf{v}_2)(\mathbf{n}_{12}^h \cdot \mathbf{a}_1) + \frac{3}{2}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)(\mathbf{v}_1 \cdot \mathbf{v}_2) \\
& \times (\mathbf{n}_{12}^h \cdot \mathbf{a}_1) - \frac{3}{8}(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2(\mathbf{v}_1 \cdot \mathbf{v}_2)(\mathbf{n}_{12}^h \cdot \mathbf{a}_1) - \frac{5}{4}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2\mathbf{v}_2^2(\mathbf{n}_{12}^h \cdot \mathbf{a}_1) + \frac{1}{4}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)\mathbf{v}_2^2(\mathbf{n}_{12}^h \cdot \mathbf{a}_1) \\
& + \frac{1}{4}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^3(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)(\mathbf{n}_{12}^h \cdot \mathbf{a}_1) + \frac{3}{8}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^2(\mathbf{n}_{12}^h \cdot \mathbf{a}_1) - \frac{1}{8}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)^3(\mathbf{n}_{12}^h \cdot \mathbf{a}_1), \tag{5.10g}
\end{aligned}$$

$$\begin{aligned}
L_{3\text{PN}}^{12} = & -\frac{19}{24}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{v}_1 \cdot \mathbf{a}_1) - \frac{185}{24}(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)(\mathbf{v}_1 \cdot \mathbf{a}_1) + \frac{205}{24}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{a}_1) + \frac{67}{6}\mathbf{v}_1^2(\mathbf{n}_{12}^h \cdot \mathbf{a}_1) - \frac{175}{12}(\mathbf{v}_1 \cdot \mathbf{v}_2)(\mathbf{n}_{12}^h \cdot \mathbf{a}_1) \\
& + \frac{3}{2}\mathbf{v}_2^2(\mathbf{n}_{12}^h \cdot \mathbf{a}_1) + \frac{91}{24}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2(\mathbf{n}_{12}^h \cdot \mathbf{a}_1) - \frac{17}{6}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)(\mathbf{n}_{12}^h \cdot \mathbf{a}_1) - \frac{5}{4}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{v}_1 \cdot \mathbf{a}_2) + \frac{235}{24}(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)(\mathbf{v}_1 \cdot \mathbf{a}_2) \\
& - \frac{31}{3}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{a}_2) - \frac{35}{4}\mathbf{v}_1^2(\mathbf{n}_{12}^h \cdot \mathbf{a}_2) + \frac{235}{24}(\mathbf{v}_1 \cdot \mathbf{v}_2)(\mathbf{n}_{12}^h \cdot \mathbf{a}_2) - \frac{21}{8}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)^2(\mathbf{n}_{12}^h \cdot \mathbf{a}_2) + \frac{17}{3}(\mathbf{n}_{12}^h \cdot \mathbf{v}_1)(\mathbf{n}_{12}^h \cdot \mathbf{v}_2)(\mathbf{n}_{12}^h \cdot \mathbf{a}_2). \tag{5.10h}
\end{aligned}$$

VI. CONCLUSIONS

To conclude, let us emphasize that the equivalence, established in this paper, between the existing independent approaches to 3PN dynamics is important because it confirms the basic soundness of *both* approaches. It shows that the quite different regularization procedures devised by the two groups are physically equivalent. None can claim to be mathematically “better” or “more correct” than the other one. We have, however, pointed out that the ADM regularization (i) is significantly simpler to define and apply in practice and (ii) leads to the introduction of a minimal set of regularization ambiguities (without extra gauge-related ambiguities).

So much for the good news. The bad news is that having proved the physical equivalence between the two approaches sheds no light on the problem of the “static ambiguity” ω_s . In fact, it is sobering to note that the enormous work which went into both 3PN investigations [8–10,15,11,18,12,16,17,13] and which led to the explicit evaluation of $\mathcal{O}(100)$ 3PN coefficients [not to mention the $\mathcal{O}(10^5)$ intermediate expressions which had to be computed and manipulated] succeeded in getting only two out of the three *irreducible* 3PN coefficients mentioned above (d_3 and z_3 , with a_4 staying ambiguous). This is all the more a pity that it was shown in [18] that the 3PN-level predictions for the physically most important quantities (dynamical behavior [18] and gravitational wave emission [21] near the transition between inspiral and plunge of a binary black hole) vary quite significantly when ω_s is allowed to vary within the

plausible range of $-10 \lesssim \omega_s \lesssim 10$. This makes it urgent to further clarify the origin of the “static ambiguity.”

Roughly speaking, it seems that this ambiguity is due to the breakdown, at 3PN, of the possibility of modeling extended (compact) objects (neutron stars or black holes) by delta-function sources. This is somewhat surprising because it was shown long ago that the extended nature of the objects (violation of the “effacing principle”) should show up only at 5PN (see Ref. [2], p. 86). One must probably use new techniques (or extend to 3PN-level old techniques, such as the matching technique used in [2]) to solve this problem.¹¹ We note that it would be nice if the “effective one body” approach [19], which is so efficient in condensing the invariant content of the dynamics to a few coefficients, could be developed into a calculational technique, giving explicit algorithmic recipes for directly computing the three 3PN irreducible coefficients.

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¹¹Let us recall that Ref. [18] found several independent arguments suggesting that $\omega_s \approx -9$ and maybe $\omega_s = -\frac{47}{3} + \frac{41}{64}\pi^2$.

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