Quantum corrections for an (anti)evaporating black hole

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In this paper we analyze the quantum correction for a Schwarzschild black hole in the Unruh state in the framework of the spherically symmetric gravity (SSG) model. SSG is a two-dimensional dilaton model that is obtained by spherically symmetric reduction from four-dimensional theory. We find the one-loop geometry of the (anti)evaporating black hole and corrections for mass, entropy, and apparent horizon.

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I. INTRODUCTION

The two-dimensional spherically symmetric gravity (SSG) model is interesting for many reasons. This model is obtained from a four-dimensional (4D) Einstein-Hilbert action coupled minimally to scalar fields by spherically symmetric reduction of metric and scalar fields. The reduction is done in the spirit of string theory, via the introduction of a dilaton field Φ, assuming that the line element is of the form

$$ds^2_{(4)}=g_{\mu\nu}\,dx^\mu\,dx^\nu+e^{-2\Phi}(\sin^2\theta\,d\phi^2+d\theta^2), \qquad (1)$$

where $\mu,\nu=0,1$. The action for this model is given by Eq. (3) below, and it has a Schwarzschild black hole as a static vacuum solution.

One reason that makes this model interesting is that the quantum effective action for scalar fields can be calculated to one-loop order. This gives the possibility of obtaining the back reaction effects of quantized matter to gravity analytically (in the case of a black hole solution this is the back reaction of the Hawking radiation). These analytic 2D calculations can then be compared with the numerical 4D estimates, as the effective action cannot be obtained analytically in 4D. This analysis was done in great detail for the Hartle-Hawking vacuum state of matter [1–3]. Summarizing, one can say that the main drawback of the SSG model is that it gives negative luminosity of the black hole. It is argued in the literature [4] that this result is a consequence of the fact that only the radial modes of the scalar field are counted in the expectation value of the energy density while the angular modes are omitted. Formally, the negative luminosity is not a surprising result as the scalar field and the dilaton are strongly coupled at spatial infinity, as can be seen from the action (3). There are also some attempts to improve the Lagrangian of the model [5–8].

2D dilaton gravity is also interesting by itself from the heuristic point of view. Dilaton couplings are present in all theories that are obtained by dimensional reduction from string theories. Furthermore, the one-loop effective actions are nonlocal. One possibility to deal with such actions is their conversion to the local form by introduction of auxiliary fields. The local form of action is rather handy for calculations (e.g., for equations of motion or the energy-momentum tensor). On the other hand, the fact that auxiliary fields describe nonlocal effects implies that they are dynamical, and it is unclear *a priori* how to fix the arbitrary constants (or functions) in the solutions. It is also not known whether all solutions have physical meaning. In the case of the SSG model the properties of auxiliary fields are rather well established for the Hartle-Hawking vacuum state. In the present paper we extend the analysis to the Unruh vacuum. We think that it is of importance to understand the ways of describing nonlocal effects by auxiliary fields. SSG is important as it provides us with an example of effective action that is tractable, but, as we shall see, in some respects more complicated than the (usually discussed) Polyakov-Liouville action.

A complementary way of discussing different vacuum states was developed in the very instructive paper [2] by Balbinot and Fabbri. Their analysis is based on the conformal properties of fields under change of the conformal vacuum state. In this method, the initial step is to identify the energy-momentum tensor (EMT) of one vacuum state (e.g., Boulware). Then one can find the expectation values of the EMT in other states from the conformal transformation properties of fields.

The organization of the paper is as follows. In Sec. II we solve the equations of motion for the auxiliary fields in the Unruh vacuum and obtain the value of the energy-momentum tensor. In order to fix the arbitrary functions in the solution we use the conditions of regularity of the EMT on the future horizon. For comparison, the energy-momentum tensor is found by the Balbinot-Fabbri procedure. The differences between the Polyakov-Liouville action and the SSG action are also discussed. In Sec. III we find the influence of the Hawking radiation on the geometry in one-loop order. In order to fix the integration constants in the metric, we impose the condition that the emitted flux of radiation is constant. We calculate the Arnowitt-Deser-Misner (ADM) mass of the black hole. In Sec. IV we obtain the position of the apparent horizon and entropy. Furthermore, we analyze the behavior of the entropy along the line of the apparent horizon and find that the second law of thermodynamics is satisfied.

II. ENERGY-MOMENTUM TENSOR AND AUXILIARY FIELDS

The Einstein-Hilbert action with minimally coupled N scalar fields f_i $(i=1,\ldots,N)$ in 4D is given by

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$$\Gamma_0^{(4)}=\frac{1}{16\pi G}\int d^4x\sqrt{-g^{(4)}}R^{(4)} -\frac{1}{8\pi}\sum_i\int d^4x\sqrt{-g^{(4)}}(\nabla f_i)^2. \tag{2}$$

After spherically symmetric reduction (1), from the action (2) we get the two-dimensional classical action Γ_0:

$$\Gamma_0=\frac{1}{4G}\int d^2x\sqrt{-g}\left\{e^{-2\Phi}[R+2(\nabla\Phi)^2+2e^{2\Phi}] -2Ge^{-2\Phi}\sum_i(\nabla f_i)^2\right\}, \tag{3}$$

where g and R denote the two-dimensional metric and curvature. The Schwarzschild black hole is the classical vacuum solution of the equations of motion that follow from the action (3). This solution is given by

$$ds^2=-f(x^1)(dx^0)^2+\frac{1}{f(x^1)}(dx^1)^2,$$

$$\Phi=-\log x^1,$$

$$f_i=0\quad(\text{ except at the point } x^1=0), \tag{4}$$

where $f(x^1)=1-a/x^1$. The constant a is the radius of the event horizon, $a=2MG$, and M is the mass of the Schwarzschild black hole.

When we add the one-loop quantum correction for the matter fields f_i to the classical action (3), we get the nonlocal effective action. Its one-loop part is given by [9–15]

$$\overline{\Gamma_1}=-\frac{N}{96\pi}\int d^2x\sqrt{-g}\left(R\frac{1}{\Box}R-12R\frac{1}{\Box}(\nabla\Phi)^2+12R\Phi\right), \tag{5}$$

which describes the quantum effects of the scalar matter fields. Calculations can be simplified if the nonlocal correction part $\overline{\Gamma_1}$ is rewritten in the local form using two auxiliary fields ψ and χ [1]:

$$\Gamma_1=-\frac{N}{96\pi}\int d^2x\sqrt{-g}[2R(\psi-6\chi)+(\nabla\psi)^2 -12(\nabla\psi)(\nabla\chi)-12\psi(\nabla\Phi)^2+12R\Phi]. \tag{6}$$

The additional fields ψ and χ satisfy the equations of motion

$$\Box\psi=R, \tag{7}$$

$$\Box\chi=(\nabla\Phi)^2. \tag{8}$$

Γ_1 and $\overline{\Gamma_1}$ are equivalent in the following sense. If we introduce Eqs. (7),(8) into the local form of the action Γ_1, we will get the nonlocal action $\overline{\Gamma_1}$ up to boundary terms.[1] This difference does not influence the equations of motion. The analysis of the boundary terms can be postponed until the calculation of ADM mass and it was done carefully in [16].

The form of the action we will use is

$$\Gamma=\Gamma_0+\Gamma_1=\frac{1}{4G}\int d^2x\sqrt{-g}[r^2R+2(\nabla r)^2+2] -\frac{\kappa}{4G}\int d^2x\sqrt{-g}\left[(\nabla\psi)^2+2R\psi-12(\nabla\psi)(\nabla\chi) -12\psi\frac{(\nabla r)^2}{r^2}-12R\chi-12R\log r\right], \tag{9}$$

where $\kappa=NG\hbar/24\pi$. Instead of the dilaton Φ we introduced the new variable $r=e^{-\Phi}$. Varying the action (9) we obtain the equations of motion [1]:

$$\Box\psi=R, \tag{10}$$

$$\Box\chi=\frac{(\nabla r)^2}{r^2}, \tag{11}$$

$$2\Box r-rR=-6\kappa\left(2\psi\frac{\Box r}{r^2}+2\frac{(\nabla\psi)(\nabla r)}{r^2}-2\psi\frac{(\nabla r)^2}{r^3}+\frac{R}{r}\right), \tag{12}$$

$$g_{\mu\nu}[\Box r^2-(\nabla r)^2-1]-2r\nabla_\mu\nabla_\nu r =2GT_{\mu\nu}=\kappa\left[g_{\mu\nu}\left(2R+6\psi\frac{(\nabla r)^2}{r^2}-\frac{1}{2}(\nabla\psi)^2 +6(\nabla\psi)(\nabla\chi)-12\frac{\Box r}{r}\right)+\nabla_\mu\psi\nabla_\nu\psi -12\nabla_\mu\psi\nabla_\nu\chi-2\nabla_\mu\nabla_\nu\psi+12\nabla_\mu\nabla_\nu\chi +12\frac{\nabla_\mu\nabla_\nu r}{r}-12(1+\psi)\frac{\nabla_\mu r\nabla_\nu r}{r^2}\right]. \tag{13}$$

First, let us note that $r=x^1$ ($\Phi=-\log x^1$) remains the solution of the quantum-corrected equations of motion (10)–(13), so we see that the field r has the meaning of a radius. We will use the following notation for the coordinates: $x^1=r$, $x^0=t$.

We want to find the quantum correction of the geometry of a 2D black hole for the case when the black hole evaporates. This means that the black hole is in the Unruh state. Our calculation is perturbative in the orders of κ, which is a small parameter. All quantities will be calculated to the first order in κ, as the effective action is also calculated to this precision only. The ansatz for the one-loop metric is

$$ds^2=-F(r,\tilde v)e^{2\kappa\varphi}\,d\tilde v^2+2e^{\kappa\varphi}d\tilde v\,dr, \tag{14}$$

and we solve the equations in Eddington-Finkelstein $r,\tilde v$ coordinates:

[1]We would like to thank D. Vassilevich for discussion about this point.

$$\tilde{v}=t+r_*=t+r+a\log\left(\frac{r}{a}-1\right). \tag{15}$$

The function F is taken in the form

$$F(r,\tilde{v})=f(r)+\frac{\kappa m(r,\tilde{v})}{r}=1-\frac{a}{r}+\frac{\kappa m(r,\tilde{v})}{r}. \tag{16}$$

Introducing the ansatz (14) into Eqs. (12),(13), we get that the equations for unknown functions m and φ in the first order in κ take the simple forms

$$\kappa\partial_r\varphi=G\frac{T_{rr}}{r}, \tag{17}$$

$$\kappa\partial_r m=2Ge^{-\kappa\varphi}T_{r\tilde{v}}, \tag{18}$$

$$\kappa\partial_{\tilde{v}}m=-2G(FT_{r\tilde{v}}+e^{-\kappa\varphi}T_{\tilde{v}\tilde{v}}), \tag{19}$$

where T_{rr}, $T_{r\tilde{v}}$, and $T_{\tilde{v}\tilde{v}}$ are the corresponding components of the energy-momentum tensor defined by Eq. (13). The EMT is a quantity of the first order in κ, so in order to determine it with the necessary precision we need the zeroth order solution for metric and auxiliary fields.

Let us briefly review how the solutions were found previously, in [1]. In the Hartle-Hawking state ψ and χ are time independent, as they describe the black hole in thermal equilibrium with the Hawking radiation. Therefore, the solutions of Eqs. (10),(11) are

$$\psi=Cr+Ca\log\frac{r-a}{a}-\log\frac{r-a}{r}, \tag{20}$$

$$\chi'=\frac{2Dr^2-2r+a}{2r(r-a)}. \tag{21}$$

The assumption of regularity of EMT on the classical horizon $r=a$ in the free-falling frame gives the values of the integration constants: $C=1/a$, $D=1/2a$.

We will now solve Eqs. (10),(11) in the general case. As mentioned, we need the zeroth order metric:

$$ds^2=g_{\mu\nu}\,dx^\mu\,dx^\nu=-fd\tilde{v}^2+2d\tilde{v}\,dr. \tag{22}$$

The other quantities entering Eqs. (10),(11) are

$$R=-\frac{d^2f}{dr^2},\quad \frac{(\nabla r)^2}{r^2}=\frac{f}{r^2}. \tag{23}$$

Introducing these values, the equation for ψ becomes

$$\Box\psi=\partial_r(2\partial_{\tilde{v}}\psi+f\partial_r\psi)=-\frac{d^2f}{dr^2}, \tag{24}$$

and it reduces to the linear partial differential equation

$$2\partial_{\tilde{v}}\psi+f\partial_r\psi=-\frac{df}{dr}+\tilde{\mathcal{G}}(\tilde{v}). \tag{25}$$

In order to find the general solution of Eq. (25) one has to find two independent integrals $\alpha(\tilde{v},r,\psi)=\text{const}$ and $\beta(\tilde{v},r,\psi)=\text{const}$ of the system

$$\frac{d\tilde{v}}{2}=\frac{dr}{f}=\frac{d\psi}{\tilde{\mathcal{G}}(\tilde{v})-\partial_r f}; \tag{26}$$

the general solution of Eq. (25) is then an arbitrary function of α and β. In our case, the independent integrals are

$$\alpha=r_*-\frac{\tilde{v}}{2},\quad \beta=\psi+\log f-\frac{1}{2}\int\tilde{\mathcal{G}}(\tilde{v})d\tilde{v}. \tag{27}$$

Therefore, the general solution for ψ can be written in the form

$$\psi=-\log\left(1-\frac{a}{r}\right)+\mathcal{G}(\tilde{v})+\mathcal{C}\left(r_*-\frac{\tilde{v}}{2}\right), \tag{28}$$

where $r_*=r+a\log(r/a-1)$, while $\mathcal{G}(\tilde{v})=\frac{1}{2}\int\tilde{\mathcal{G}}(\tilde{v})d\tilde{v}$ and $\mathcal{C}(r_*-\tilde{v}/2)$ are arbitrary functions. Similarly, the equation for χ,

$$\Box\chi=\partial_r(2\partial_{\tilde{v}}\chi+f\partial_r\chi)=\frac{f}{r^2}, \tag{29}$$

reduces to the system

$$\frac{d\tilde{v}}{2}=\frac{dr}{f}=\frac{d\chi}{\tilde{\mathcal{H}}(\tilde{v})+(a-2r)/2r^2}. \tag{30}$$

The general solution for χ is

$$\chi=-\frac{1}{2}\log\frac{r(r-a)}{a^2}+\mathcal{H}(\tilde{v})+\mathcal{D}\left(r_*-\frac{\tilde{v}}{2}\right), \tag{31}$$

where $\mathcal{H}(\tilde{v})$ and $\mathcal{D}(r_*-\tilde{v}/2)$ are arbitrary functions. The functions $\mathcal{G}(\tilde{v})$, $\mathcal{C}(r_*-\tilde{v}/2)$, $\mathcal{H}(\tilde{v})$, and $\mathcal{D}(r_*-\tilde{v}/2)$ describe various quantum states of matter. To recover the static Hartle-Hawking vacuum solution we have to put all functions linear in their arguments in order to cancel t terms. This combined with the condition of regularity gives $\mathcal{C}(r_*-\tilde{v}/2)=(1/a)(r_*-\tilde{v}/2)$, $\mathcal{G}(\tilde{v})=(1/2a)\tilde{v}$, $\mathcal{H}(\tilde{v})=(1/4a)\tilde{v}$, and $\mathcal{D}(r_*-\tilde{v}/2)=(1/2a)(r_*-\tilde{v}/2)$.

We now pass to the case of the Unruh vacuum. It is most naturally discussed in the null coordinates u,v:

$$v=\tilde{v},u=\tilde{v}-2r_*=\tilde{v}-2\left[r+a\log\left(\frac{r}{a}-1\right)\right]. \tag{32}$$

The Unruh vacuum state is defined as the state that has the EMT regular on the future event horizon $u\to\infty$, $v=\text{const}$. The conditions of regularity in the free-falling frame read [17]

$$T_{vv}<\infty,\quad \frac{T_{uv}}{f}<\infty,\quad \frac{T_{uu}}{f^2}<\infty. \tag{33}$$

Components of the energy-momentum tensor in the u,v coordinates can be found from the relations

$$T_{rr}=4\left(\frac{r}{r-a}\right)^2 T_{uu}, \tag{34}$$

$$T_{r\tilde{v}}=-2\frac{r}{r-a}(T_{uu}+T_{uv}), \tag{35}$$

$$T_{\tilde{v}\tilde{v}}=T_{uu}+2T_{uv}+T_{vv}. \tag{36}$$

Along with the condition of regularity of the EMT, we will impose that at spatial infinity $r\to\infty$ the outgoing flux T_{uu} has a constant nonvanishing value, while the ingoing flux T_{vv} tends to 0. When we introduce the solutions (28),(31) for the components of the EMT we get

$$T_{uv}=\frac{a}{24\pi}\frac{r-a}{r^4}, \tag{37}$$

$$T_{vv}=\frac{(a-r)^2}{16\pi r^4}\log\frac{r-a}{r}+\frac{1}{48\pi}(\mathcal{G}'^2-12\mathcal{G}'\mathcal{H}'-2\mathcal{G}''+12\mathcal{H}'')$$
$$-\frac{1}{192\pi r^4}[-3a^2+4ar+12(a-r)^2\mathcal{C}+12(a-r)^2\mathcal{G}$$
$$+(12ar^2-24r^3)\mathcal{G}'], \tag{38}$$

$$T_{uu}=\frac{(a-r)^2}{16\pi r^4}\log\frac{r-a}{r}+\frac{1}{48\pi}(\mathcal{C}'^2-12\mathcal{C}'\mathcal{D}'-2\mathcal{C}''+12\mathcal{D}'')$$
$$-\frac{1}{192\pi r^4}[-3a^2+4ar+12(a-r)^2\mathcal{C}+12(a-r)^2\mathcal{G}$$
$$+(6ar^2-12r^3)\mathcal{C}'] \tag{39}$$

(primes denote derivatives of the functions with respect to their arguments).

There is no information about the unknown functions contained in T_{uv}. Further, it can be seen that T_{vv} is regular on the horizon. The condition that $T_{vv}\to 0$ as $r\to\infty$ means that, in this limit,

$$\mathcal{G}'^2-12\mathcal{G}'\mathcal{H}'-2\mathcal{G}''+12\mathcal{H}''=0. \tag{40}$$

The solution of the last equation, which is in accordance with the radiation law, is given by linear functions $\mathcal{G},\mathcal{H}$:

$$\mathcal{G}(\tilde{v})=g\,\tilde{v},\quad \mathcal{H}(\tilde{v})=h\tilde{v}, \tag{41}$$

with

$$g(g-12h)=0, \tag{42}$$

i.e., either $g=0$ or $g=12h$.

Similarly, the condition that $T_{uu}\to$const as $r\to\infty$ gives that the functions $\mathcal{C}$ and $\mathcal{D}$ are linear in their arguments,

$$\mathcal{C}(x)=cx,\quad \mathcal{D}(x)=dx. \tag{43}$$

Nonsingularity of T_{uu}/f^2 on the horizon gives us the values of the constants: $c=1/a$, $d=1/2a$. Introducing c and d in Eq. (39) we see that the luminosity has the Hartle-Hawking value $-5/192\pi a^2$. The 2D black hole antievaporates. This is because we took into account the contribution of the s modes of the radiation only.

To conclude our reasoning, let us observe that one arbitrariness remains, and that is the dependence of the EMT on the constant g. This arbitrariness can be naturally fixed by choosing the $g=0$ solution of the condition (42). Note also that the value of the constant h does not enter the EMT, and therefore we can fix it freely, e.g., $h=1/4$. Finally we have the solution for ψ,χ in the zeroth order,

$$\psi=\frac{r}{a}+\log\frac{r}{a}-\frac{v}{2a}, \tag{44}$$

$$\chi=\frac{r}{2a}-\frac{1}{2}\log\frac{r}{a}. \tag{45}$$

We just mention briefly that it can be shown that for $g=0$ the value of h does not influence the ADM mass.

We can now perform the Balbinot-Fabbri procedure [2] and compare the values of the EMT. If the vacuum state of matter is defined in such a way that the ingoing and outgoing modes have positive frequency with respect to the coordinates u,v, the EMT corresponds to the Boulware state:

$$\langle u,v|\hat{T}_{uv}|u,v\rangle=-\frac{1}{12\pi}(\partial_v\partial_u\rho+3\partial_v\Phi\;\partial_u\Phi-3\partial_v\;\partial_u\Phi), \tag{46}$$

$$\langle u,v|\hat{T}_{vv}|u,v\rangle=-\frac{1}{12\pi}(\partial_v\rho\partial_v\rho-\partial_v^2\rho)$$
$$+\frac{1}{2\pi}\left(\rho(\partial_v\Phi)^2+\frac{1}{2}\frac{\partial_v}{\partial_u}(\partial_v\Phi\partial_u\Phi)\right)$$
$$-\frac{1}{4\pi}[-2(\partial_v\rho)(\partial_v\Phi)+\partial_v^2\Phi], \tag{47}$$

$$\langle u,v|\hat{T}_{uu}|u,v\rangle=-\frac{1}{12\pi}(\partial_u\rho\partial_u\rho-\partial_u^2\rho)$$
$$+\frac{1}{2\pi}\left(\rho(\partial_u\Phi)^2+\frac{1}{2}\frac{\partial_u}{\partial_v}(\partial_+\Phi\partial_-\Phi)\right)$$
$$-\frac{1}{4\pi}[-2(\partial_u\rho)(\partial_u\Phi)+\partial_u^2\Phi], \tag{48}$$

where $\rho=\frac{1}{2}\log(1-a/r)$ is a conformal factor.

The conformal transformation to the other conformal state $|\tilde{u},\tilde{v}\rangle$ defined by the other set of null coordinates $\tilde{u}=\tilde{u}(u)$, $\tilde{v}=\tilde{v}(v)$, gives

$$\langle\tilde{u},\tilde{v}|\hat{T}_{uv}|\tilde{u},\tilde{v}\rangle=\langle u,v|\hat{T}_{uv}|u,v\rangle, \tag{49}$$

$$\langle\tilde{u},\tilde{v}|\hat{T}_{vv}|\tilde{u},\tilde{v}\rangle=\langle u,v|\hat{T}_{vv}|u,v\rangle+\frac{1}{24\pi}\left(\frac{G''}{G}-\frac{1}{2}\frac{G'^2}{G^2}\right)$$
$$+\frac{1}{4\pi}\left((\partial_v\Phi)^2\log(FG)\right.$$
$$\left.+\frac{G'}{G}\int du\;\partial_v\Phi\;\partial_u\Phi\right), \tag{50}$$

$$\langle\tilde{u},\tilde{v}|\hat{T}_{uu}|\tilde{u},\tilde{v}\rangle=\langle u,v|\hat{T}_{uu}|u,v\rangle+\frac{1}{24\pi}\left(\frac{F''}{F}-\frac{1}{2}\frac{F'^2}{F^2}\right)+\frac{1}{4\pi}\left((\partial_u\Phi)^2\log(FG)+\frac{F'}{F}\int dv(\partial_u\Phi)(\partial_u\Phi)\right), \tag{51}$$

where $F(u)=du/d\tilde{u},\ G(v)=dv/d\tilde{v}$.

The Unruh vacuum state is the state $|U,v\rangle$, U being the Kruskal coordinate $U=-2ae^{u/2a}$. Using Eqs. (50) and (51) after simple calculation, we get the value of the EMT in the Unruh state ($1/24\pi=\kappa/G$):

$$T_{uv}=\frac{\kappa}{G}\left(1-\frac{a}{r}\right)\frac{a}{r^3}, \tag{52}$$

$$T_{uu}=\frac{\kappa}{G}\left[\frac{3a^2-4ar}{8r^4}-\frac{5}{8a^2}-\frac{3}{2a}\left(\frac{a}{2r^2}-\frac{1}{r}\right)+\frac{3}{2r^2}\left(1-\frac{a}{r}\right)^2\left(\frac{v}{2a}-\frac{r}{a}-\log\frac{r}{a}\right)\right], \tag{53}$$

$$T_{vv}=\frac{\kappa}{G}\left[\frac{3a^2-4ar}{8r^4}+\frac{3}{2r^2}\left(1-\frac{a}{r}\right)^2\left(\frac{v}{2a}-\frac{r}{a}-\log\frac{r}{a}\right)\right]. \tag{54}$$

These expressions are the same as those previously given [Eqs. (38)–(39)] with fixed integration functions.

Let us give one final comment on the values of the EMT (52),(53). The values obtained have v dependence, i.e., t dependence. This dependence does not show up in the asymptotic behavior of the EMT and it was considered in [18] as an unwished property of the energy-momentum tensor. In fact, in [18] the auxiliary fields were constrained in such a way that the time dependence of ψ,χ would not produce any time dependence in the EMT. We think that a condition like this is too stringent and unnecessary. It holds, though, in the ''minimal coupling'' case, i.e., in the case when the effective action is given by the Polyakov-Liouville term only, as can easily be seen. That is, it is known [3] that in this case the change of the conformal frame produces in the EMT only the additional term proportional to the Schwarzian derivative of the transformation of coordinates:

$$\langle\tilde{u},\tilde{v}|\hat{T}_{vv}|\tilde{u},\tilde{v}\rangle=\langle u,v|\hat{T}_{vv}|u,v\rangle+\frac{1}{24\pi}\left(\frac{F''}{F}-\frac{1}{2}\frac{F'^2}{F^2}\right). \tag{55}$$

For exponential mappings, which are typical for the transformation to Kruskal coordinates, the Schwarzian derivative is constant. This means that, if we start with the time-independent EMT for, e.g., the Hartle-Hawking vacuum, we will get the time-independent EMT for all other conformal vacuums. But this is a special property of the Polyakov-Liouville effective action. In the SSG case the structure of the additional terms is more complicated and this causes the time dependence in the Unruh vacuum state. The fact that this dependence is linear is in accordance with the expected property that the black hole in the Unruh vacuum radiates at a constant rate, $dM(t)/dt=$const. The meaning of the mass $M(t)$ will be discussed in more detail after we solve the back reaction equations for the metric and identify the ADM mass of the solution.

III. BACK REACTION AND CORRECTED GEOMETRY

The equations that determine the one-loop correction of the metric can now easily be integrated. The solution is

$$\varphi=\frac{5}{ar}+3\frac{a-2v}{4ar^2}+\frac{3}{r^2}\log\frac{r}{a}-\frac{5}{2a^2}\log\frac{r}{l}+C_1, \tag{56}$$

$$m=\frac{5r}{2a^2}+\frac{a+6v}{2ar}+\frac{11a-6v}{4r^2}-\frac{5v}{4a^2}-3\frac{2r-a}{r^2}\log\frac{r}{a}+\frac{5}{2a}\log\frac{r}{l}+C_2. \tag{57}$$

We see that the functions $m(v,r)$ and $\varphi(v,r)$ depend linearly on v, i.e., on time. There are two independent integration constants C_1 and C_2. The expression for the ADM energy was found in [16]. We would like to mention the interesting paper [19] where the authors calculated the ADM mass for a static black hole. The value of the energy is given by the value of the boundary term that has to be added to the canonical Hamiltonian in order to have a well defined theory. It is given by

$$\Delta=-\delta H_b, \tag{58}$$

where

$$\begin{aligned}4G\Delta=&\sqrt{\frac{-g}{g_{11}}}(4Br'\,\delta r-2\kappa\psi'\,\delta\psi+12\kappa\psi'\,\delta\chi+12\kappa\chi'\,\delta\psi)\\&+\frac{2}{\sqrt{-g}}\delta\left(\frac{-g}{g_{11}}\right)(Arr'-\kappa\psi'+6\kappa\chi')\\&+\frac{2}{\sqrt{-g}}\left(\frac{-g}{g_{11}}\right)'(Ar\,\delta r-\kappa\,\delta\psi+6\kappa\,\delta\chi)\\&+4G\pi^{11}\left(2\,\delta g_{01}-\frac{g_{01}}{g_{11}}\delta g_{11}\right)\\&+4G\frac{g_{01}}{g_{11}}(\pi_r\delta r+\pi_\psi\delta\psi+\pi_\chi\delta\chi).\end{aligned} \tag{59}$$

δ denotes the variation in the chosen class of field configurations, described in more detail in [16]. A and B are $A=1+6\kappa/r^2$ and $B=1+6\kappa\psi/r^2$. Of course, in order to identify the real value of the energy, we have to find it in a coordinate system that is asymptotically Minkowskian. As we have solved the equations for m and φ, we can now write the corrected values of the components of the metric:

$$g_{00}=-\left[1-\frac{a}{r}+\frac{\kappa m}{r}+2\kappa\left(1-\frac{a}{r}\right)\varphi\right],$$

$$g_{01}=-\kappa\left(\frac{m}{r-a}+\varphi\right),$$

$$g_{11}=\frac{r}{r-a}-\kappa\frac{mr}{(r-a)^2}, \tag{60}$$

so we see that, unlike the static case, the metric is not diagonal in the first order in κ.

In order to find a coordinate system $\tilde{t},\tilde{r}$ in which the asymptotic values of the metric are

$$\tilde{g}_{00}\to-1+O\left(\frac{1}{L}\right),\quad \tilde{g}_{01}\to 0 \tag{61}$$

(it is not really necessary to assume $\tilde{g}_{11}\to 1$ also, as we are interested only in the value of the energy), we introduce the transformation of coordinates

$$\tilde{t}=t+\kappa\alpha(t,r),\quad \tilde{r}=r. \tag{62}$$

Under this transformation, the metric transforms as

$$\tilde{g}_{00}=g_{00}\left(1-2\kappa\frac{\partial\alpha}{\partial t}\right),$$

$$\tilde{g}_{01}=g_{01}-\kappa\frac{\partial\alpha}{\partial r}g_{00},$$

$$\tilde{g}_{11}=g_{11}. \tag{63}$$

In accordance with the asymptotic relations (61) the function α should be chosen in the form

$$\alpha(t,r)=F_1r+F_2t+F_3rt+F_4t^2, \tag{64}$$

where

$$F_1=\frac{5}{4a^2}+\frac{9}{aL}-\frac{5}{2a^2}\log\frac{L}{l}-\frac{5}{4aL}\log\left(\frac{L}{a}-1\right)+\frac{LC_1}{L-a}+\frac{LC_2}{(L-a)^2}, \tag{65}$$

$$F_2=\frac{15}{8a^2}+\frac{45}{8aL}-\frac{5}{2a^2}\log\frac{L}{l}-\frac{5}{8aL}\log\frac{L}{a}+C_1+\frac{C_2}{2(L-a)}, \tag{66}$$

$$F_3=-\frac{5}{4a^2L}, \tag{67}$$

$$F_4=\frac{5}{32a^2(L-a)}-\frac{5}{16a^2L}. \tag{68}$$

The coordinate transformation induces the following change in the boundary term:

$$4G\tilde{\Delta}=4G\Delta-2\kappa r\left[\frac{\partial\alpha}{\partial t}\delta\left(-\frac{g}{g_{11}}\right)+2\left(-\frac{g}{g_{11}}\right)\delta\left(\frac{\partial\alpha}{\partial t}\right)\right]. \tag{69}$$

Introducing the solutions obtained for ψ, χ, g we get for the value of $\tilde{\Delta}$

$$4G\tilde{\Delta}=-2\delta a+\kappa\left(\frac{21}{4a^2}-\frac{11L}{2a^3}-\frac{C_2(a)}{L-a}-\frac{5}{a^2}\log\frac{L}{l}+\frac{5}{a^3}t\right)\delta a. \tag{70}$$

The corresponding value of energy is

$$\tilde{H}_b=-\frac{1}{4G}\int 4G\tilde{\Delta}=M+\frac{\kappa}{4G}\left(\frac{21}{4a}-\frac{11L}{4a^2}-\frac{5}{a}\log\frac{L}{l}+\frac{5}{2a^2}t+\int\frac{C_2}{L-a}da\right). \tag{71}$$

The first term in Eq. (71) is the classical mass of the black hole, while the second one is the quantum correction of the mass. We can take that $C_2=0$. One immediately notes the time dependence of the ADM mass, which is in agreement with the radiation law of the black hole, namely,

$$\frac{d\tilde{H}_b}{dt}=T_{uu}|_{r\to L}=-\frac{5}{192\pi a^2}. \tag{72}$$

The increase of the mass corresponds to the fact that the outgoing flux is negative at large distances, i.e., that the black hole antievaporates. It is important to mention that the mass increases only if we consider large but finite volumes L. If we take the limit $L\to\infty$, the t term in the expression for energy (71) can be neglected in comparison with the larger terms proportional to $\log L$ and L, so we have the conservation of the energy of the whole system, $\dot{\tilde{H}}_b=0$. Notice that the ‘‘mass function’’ $M(r,v)=M-\kappa m(r,v)/2$ satisfies the condition $\dot{M}(r,v)=-5/192\pi a^2$.

IV. APPARENT HORIZON AND ENTROPY

The apparent horizon is the boundary of the trapped surfaces. In 2D dilaton gravity it is defined by [20]

$$g^{\mu\nu}\,\partial_\mu r\,\partial_\nu r=0. \tag{73}$$

If we define the one-loop corrected null coordinates by

$$ds^2=-e^{2\rho}\,d\bar{u}\,d\bar{v}, \tag{74}$$

the condition (73) is reduced to $\partial_{\bar{u}}r=0$ and $\partial_{\bar{v}}r=0$. We will take $\bar{v}=v=t+r_*$. The other null coordinate $\bar{u}$ can be found easily. The first step is to rewrite the metric (14) in the form

$$ds^2 = -Fe^{2\kappa\varphi}\left(d\bar{v} - \frac{2}{F}e^{-\kappa\varphi}\, dr\right)d\bar{v}$$
$$= -\frac{Fe^{2\kappa\varphi}}{\mu}\left(\mu\, d\bar{v} - \frac{2\mu}{F}e^{-\kappa\varphi}\, dr\right)d\bar{v}, \qquad (75)$$

where μ is the integration factor. Therefore, the conformal coordinate $\bar{u}$ satisfies

$$d\bar{u} = \mu\, d\bar{v} - \frac{2\mu}{F}e^{-\kappa\varphi}\, dr. \qquad (76)$$

We will not solve the previous equation for $\bar{u}$, but just use it to find the position of the apparent horizon. From Eq. (76) we get

$$dr = \frac{1}{2}e^{\kappa\varphi}F\left(d\bar{v} - \frac{1}{\mu}d\bar{u}\right). \qquad (77)$$

The last equation, if we use $\partial_{\bar{u}}r = 0$ and $\partial_{\bar{v}}r = 0$, implies $e^{\kappa\varphi}F = 0$ on the horizon. This means that the equation of the apparent horizon is

$$(1+\kappa\varphi)\left(1 - \frac{a}{r} + \frac{\kappa m}{r}\right) = 0. \qquad (78)$$

The position of the apparent horizon is found perturbatively by taking $r_{AH} = a + \kappa r_1$, where r_1 is the first-order correction. From Eq. (78) we get

$$r_{AH} = a - \kappa\left(\frac{23}{4a} + \frac{5}{2a}\log\frac{a}{l} + \frac{1}{4a^2}\bar{v}\right). \qquad (79)$$

The intersection point between the line of singularity and the apparent horizon is the end point of the Hawking radiation. It is given by

$$\bar{u}_{int} = \infty, \quad \bar{v}_{int} = 4a^2\left(\frac{a}{\kappa} - \frac{23}{4a} - \frac{5}{2a}\log\frac{a}{l}\right) \approx \frac{4a^3}{\kappa}. \qquad (80)$$

As we can take the $\bar{v}$ coordinate as the time, we see that the (anti)evaporation of the black hole is very long but finite.

In order to calculate the entropy of the quantum-corrected solution, we use the Wald technique [21]. Note that the conical singularity method is defined for static configurations only and therefore cannot be used here. In Refs. [22–24] it was shown that for Lagrangians of the form $L = L(f_m, \nabla f_m, g_{\mu\nu}, R_{\mu\nu\rho\sigma})$ (f_m are the matter fields) the entropy is given by

$$S = -2\pi\epsilon_{\alpha\beta}\epsilon_{\chi\delta}\frac{\partial L}{\partial R_{\alpha\beta\chi\delta}}\bigg|_H,$$

evaluated on the horizon. In our case we find

$$S = \frac{\pi}{G}[r^2 - \kappa(2\psi - 12\chi - 12\log r)]|_{AH}$$
$$= \frac{\pi}{G}\left[r^2 + \kappa\left(4\frac{r}{a} - 8\log\frac{r}{a} + 12\log\frac{r}{l} + \frac{1}{a}\bar{v}\right)\right]_{AH}$$
$$= \frac{\pi}{G}\left[a^2 - \kappa\left(\frac{15}{2} + 5\log\frac{a}{l} - \frac{\bar{v}}{2a}\right)\right]. \qquad (81)$$

Now we will show that the entropy increases along the line of apparent horizon. To this end we will find the equation for the $\bar{u}$ coordinate. The integration factor, which we introduced in Eq. (76), is of the form

$$\mu = 1 + \kappa R(r) + \kappa V(\bar{v}), \qquad (82)$$

where $R(r)$ and $V(\bar{v})$ are unknown functions. If we introduce the ansatz (82) in the condition of integrability of Eq. (76),

$$\frac{\partial\mu}{\partial r}\bigg|_{\bar{v}} = -2\frac{\partial}{\partial\bar{v}}\left(\frac{\mu}{F}e^{-\kappa\varphi}\right)\bigg|_r, \qquad (83)$$

we obtain the following expressions:

$$V(\bar{v}) = \alpha\bar{v}, \qquad (84)$$

$$R(r) = -2\alpha r - \frac{1}{2a(r-a)} - \frac{4a^3\alpha + 5}{2a^2}\log(r-a), \qquad (85)$$

where α is the integration constant. On the other hand, if we start from

$$\frac{\partial\bar{u}}{\partial\bar{v}}\bigg|_r = 1 + \kappa R(r) + \kappa V(\bar{v}), \qquad (86)$$

$$\frac{\partial\bar{u}}{\partial r}\bigg|_{\bar{v}} = -\frac{2\mu}{F}e^{-\kappa\varphi}, \qquad (87)$$

we get

$$\bar{u} = \bar{v} + \kappa\bar{v}R(r) + \frac{1}{2}\kappa\alpha\bar{v}^2 + G(r). \qquad (88)$$

Therefore the function $G(r)$ is determined by the equation

$$\frac{dG}{dr} = -\frac{2r}{r-a}\Bigg[1 + \kappa\left(-2\alpha r - \frac{1}{2a(r-a)} - \frac{5+4\alpha a^3}{2a^2}\log(r-a)\right) - \kappa\left(\frac{5}{ar} + \frac{3}{4r^2} + \frac{3}{r^2}\log\frac{r}{a} - \frac{5}{2a^2}\log\frac{r}{a} + C_1\right) - \frac{\kappa}{r-a}\left(\frac{5r}{2a^2} + \frac{1}{2r} + \frac{11a}{4r^2} - 3\frac{2r-a}{r^2}\log\frac{r}{a} + \frac{5}{2a}\log\frac{r}{l}\right)\Bigg], \qquad (89)$$

which can easily be integrated. The derivative of the entropy along the apparent horizon is determined by

$$t^a\ \partial_a S = \left(\frac{\partial}{\partial \overline{v}} + \frac{d\overline{u}_{AH}}{d\overline{v}}\frac{\partial}{\partial \overline{u}}\right) S, \tag{90}$$

where t^a is the tangent vector of the apparent horizon. The expression (79) for the apparent horizon and Eqs. (84)–(89) give

$$t^a\ \partial_a S = \frac{\kappa \pi}{2aG} > 0. \tag{91}$$

So the entropy increases along the line of the apparent horizon. This shows that the second law of thermodynamics is satisfied in the framework of the SSG model.

V. CONCLUSIONS

In this paper we calculated the back reaction effects of the Hawking radiation in the Unruh state of the Schwarzschild black hole. The effect was discussed in the framework of the SSG model. The calculation was simplified using the formalism of auxiliary fields. It was shown that the definition of the Unruh state fixes the integration functions and that the corresponding EMT coincides with the EMT calculated by other methods. The position of the apparent horizon was found and the evaporation of the black hole discussed. The obtained duration of the evaporation is large (proportional to $1/\kappa$). Unfortunately, at the intersecting point of the line of singularity and apparent horizon the singularity becomes naked, which prevents us from predicting the future evolution of the black hole. The discussion of the static remnant of the black hole is an interesting question and will be the subject of further investigation. The entropy of the black hole–radiation system was obtained and shown to increase during the evolution. The quantum corrections of the energy of the system were calculated using the ADM procedure. We found that the flux of radiation through the large spherical surface of radius L is in accordance with the radiation law. In the limit $L \to \infty$ though, the energy of the whole system is conserved, as one would expect.

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