New holographic entropy bound from quantum geometry

Saurya Das*
Center for Gravitational Physics and Geometry, Penn State University, University Park, Pennsylvania 16802-6300

Romesh K. Kaul† and Parthasarathi Majumdar‡
The Institute of Mathematical Sciences, Chennai 600 113, India
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A new entropy bound, tighter than the standard holographic bound due to Bekenstein, is derived for space-times with nonrotating isolated horizons from the quantum geometry approach, in which the horizon is described by the boundary degrees of freedom of a three dimensional Chern-Simons theory.

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The holographic principle (HP) [1–8] and the holographic entropy bound (EB) have been the subject of a good deal of attention lately. In its original form [1,2], the HP asserts that the maximum possible number of degrees of freedom within a macroscopic bounded region of space is given by a quarter of the area (in units of Planck area) of the boundary. This takes into account that a black hole for which this boundary is (a spatial slice of) its horizon has an entropy which obeys the Bekenstein-Hawking area law and also the generalized second law of black hole thermodynamics [4]. Given the relation between the number of degrees of freedom and entropy, this translates into a holographic EB valid generally for space-times with boundaries.

The basic idea underlying both these concepts is a network at whose vertices are variables that take only two values (''binary,'' ''Boolean'' or ''pixel''), much like a lattice with spin one-half variables at its sites. Assuming that the spin value at each site is *independent* of that at any other site (i.e., the spins are *randomly* distributed on the sites), the dimensionality of the space of states of such a network is simply 2^p for a network with p vertices. In the limit of arbitrarily large p, such a network can be taken to approximate the macroscopic surface alluded to above, a quarter of whose area bounds the entropy contained in it. Thus any theory of quantum gravity in which space-time might acquire a discrete character at length scales of the order of Planck scale is expected to conform to this counting and hence to the HP.

Let us consider now a slightly altered situation: one in which the binary variables at the vertices of the network considered are no longer distributed randomly but according to some other distribution. Typically, for example, one could distribute them *binomially*, assuming, without loss of generality, a large lattice with an even number of vertices. Consider now the number of cases for which the binary variable acquires one of its two values, at exactly $p/2$ of the p vertices. In case of a lattice of spin 1/2 variables which can either point ''up'' or ''down,'' this corresponds to a situation of net spin zero, i.e., an equal number of spin-ups and spin-downs. Using standard formulas of binomial distributions, this number is

$$N\left(\frac{p}{2}|a\right)=2^p\left(\begin{array}{c}p\\ p/2\end{array}\right)[a(1-a)]^{p/2}, \tag{1}$$

where a is the probability of an occurrence of a spin-up at any given vertex. Clearly, this number is maximum when the probability of occurrence $a=1/2$; it is given by $p!/((p/2)!)^2$. Thus the number of degrees of freedom is now no longer 2^p but a smaller number. This obviously leads to a lowering of the entropy. For very large p corresponding to a macroscopic boundary surface, this number is proportional to $2^p/p^{1/2}$. The new EB can therefore be expressed as

$$S_{max}=\ln\left(\frac{\exp S_{BH}}{S_{BH}^{1/2}}\right), \tag{2}$$

where $S_{BH}=A_H/4l_P^2$ is the Bekenstein-Hawking entropy. This is a tighter bound than that of Ref. [4] mentioned above. The ''tightening'' of holographic EB is the subject of this paper. We shall show below that, in the quantum geometry framework, it is possible to have an even tighter bound than that depicted in Eq. (2).

There are, of course, examples of situations where the EB is violated [5,6] and must be generalized. However, generalizations proposed so far [6] appear to be tied to fixed classical background space-times, and may not hold when gravitational fluctuations are taken into account [7]. In this note, we restrict ourselves to the older version of the EB appropriate to stationary space-times, but with allowance for the existence of radiation in the vicinity of the boundary. In this sense, the appropriate conceptual framework is that of the Isolated Horizon [9]. We consider generic $3+1$ dimensional isolated horizons without rotation, on which one assumes an appropriate class of boundary conditions. These boundary conditions require that the gravitational action be augmented by the action of an $SU(2)$ Chern-Simons theory living on the isolated horizon [9]. Boundary states of the Chern-Simons theory contribute to the entropy. These states correspond to conformal blocks of the two-dimensional Wess-Zumino model that lives on the spatial slice of the horizon, which is a 2-sphere of area A_H. The dimensionality of the boundary Hilbert space has been calculated thus [10–12] by counting the number of conformal blocks of two-dimensional $SU(2)_k$ Wess-Zumino model for arbitrary level

*Email address: das@gravity.phys.psu.edu
†Email address: kaul@imsc.ernet.in
‡Email address: partha@imsc.ernet.in

k and number of punctures p on the 2-sphere. We shall show, from the formula for the number of conformal blocks specialized to macroscopic black holes characterized by large k and p [12], that the restricted situation described above ensues, thus realizing a more stringent EB. We may mention that similar ideas relating the quantum geometry approach to the HP and EB have been pursued by Smolin [7], although, as far as we understand, the issue of tightening the Bekenstein bound has not been addressed.

We start with the formula for the number of conformal blocks of two-dimensional $SU(2)_k$ Wess-Zumino model that lives on the punctured 2-sphere. For a set of punctures $\mathcal{P}$ with spins $\{j_1, j_2, \ldots, j_p\}$ at punctures $\{1,2,\ldots,p\}$, this number is given by [10]

$$N^{\mathcal{P}}=\frac{2}{k+2}\sum_{r=0}^{k/2}\frac{\prod_{l=1}^{p}\sin\left(\frac{(2j_l+1)(2r+1)\pi}{k+2}\right)}{\left[\sin\left(\frac{(2r+1)\pi}{k+2}\right)\right]^{p-2}}. \tag{3}$$

Observe now that Eq. (3) can be rewritten as a multiple sum,

$$N^{\mathcal{P}}=\left(\frac{2}{k+2}\right)\sum_{l=1}^{k+1}\sin^2\theta_l \times\sum_{m_1=-j_1}^{j_1}\cdots\sum_{m_p=-j_p}^{j_p}\exp\left\{2i\left(\sum_{n=1}^{p}m_n\right)\theta_l\right\}, \tag{4}$$

where $\theta_l\equiv\pi l/(k+2)$. Expanding the $\sin^2\theta_l$ and interchanging the order of the summations, this becomes

$$N^{\mathcal{P}}=\sum_{m_1=-j_1}^{j_1}\cdots\sum_{m_p=-j_p}^{j_p}\left[\overline{\delta}_{(\Sigma_{n=1}^{p}m_n),0}-\frac{1}{2}\overline{\delta}_{(\Sigma_{n=1}^{p}m_n),1}-\frac{1}{2}\overline{\delta}_{(\Sigma_{n=1}^{p}m_n),-1}\right], \tag{5}$$

where we have used the standard resolution of the periodic Kronecker deltas in terms of exponentials with period $k+2$,

$$\overline{\delta}_{(\Sigma_{n=1}^{p}m_n),m}=\left(\frac{1}{k+2}\right)\sum_{l=0}^{k+1}\exp\left\{2i\left[\left(\sum_{n=1}^{p}m_n\right)-m\right]\theta_l\right\}. \tag{6}$$

Our interest focuses on the limit of large k and p, appropriate to macroscopic black holes of large area. Observe, first of all, that as $k\to\infty$, the periodic Kronecker delta's in Eq. (6) reduce to ordinary Kronecker deltas,

$$\lim_{k\to\infty}\overline{\delta}_{m_1+m_2+\cdots+m_p,m}=\delta_{m_1+m_2+\cdots+m_p,m}. \tag{7}$$

In this limit, the quantity $N^{\mathcal{P}}$ counts the number of $SU(2)$ singlet states, rather than $SU(2)_k$ singlets states. For a given set of punctures with $SU(2)$ representations on them, this number is larger than the corresponding number for the affine extension. This is desirable for the purpose of deducing an upper bound on the number of degrees of freedom in any space-time.

Next, recall that the eigenvalues of the area operator for the horizon, lying within one Planck area of the classical horizon area A_H, are given by

$$\hat{A}_H\Psi_S=8\pi\beta\, l_P^2\sum_{l=1}^{p}[j_l(j_l+1)]^{1/2}\Psi_S, \tag{8}$$

where l_P is the Planck length, j_l is the spin on the lth puncture on the 2-sphere, and β is the Barbero-Immirzi parameter [13]. We consider a large fixed classical area of the horizon, and ask what the largest value of number of punctures p should be, so as to be consistent with Eq. (8); this is clearly obtained when the spin at *each* puncture assumes its lowest nontrivial value of 1/2, so that the relevant number of punctures p_0 is given by

$$p_0=\frac{A_H}{4l_P^2}\frac{\beta_0}{\beta}, \tag{9}$$

where $\beta_0=1/\pi\sqrt{3}$. We are, of course, interested in the case of very large p_0.

Now, with the spins at all punctures set to 1/2, the number of states for this set of punctures $\mathcal{P}_0$ is given by

$$N^{\mathcal{P}_0}=\sum_{m_1=-1/2}^{1/2}\cdots\sum_{m_{p_0}=-1/2}^{1/2}\left[\delta_{(\Sigma_{n=1}^{p_0}m_n),0}-\frac{1}{2}\delta_{(\Sigma_{n=1}^{p_0}m_n),1}-\frac{1}{2}\delta_{(\Sigma_{n=1}^{p_0}m_n),-1}\right] \tag{10}$$

The summations can now be easily performed, with the result

$$N^{\mathcal{P}_0}=\binom{p_0}{p_0/2}-\binom{p_0}{(p_0/2-1)}. \tag{11}$$

There is a simple intuitive way to understand the result embodied in Eq. (11). This formula simply counts the number of ways of making $SU(2)$ singlets from p_0 spin 1/2 representations. The first term corresponds to the number of states with net J_3 quantum number $m=0$ constructed by placing $m=\pm 1/2$ on the punctures. However, this term by itself *overcounts* the number of $SU(2)$ singlet states, because even nonsinglet states (with net integral spin, for p is an even integer) have a net $m=0$ sector. Besides having a sector with total $m=0$, states with net integer spin have, of course, a sector with overall $m=\pm 1$ as well. The second term basically eliminates these nonsinglet states with $m=0$ by counting the number of states with net $m=\pm 1$ constructed from $m=\pm 1/2$ on the p_0 punctures. The difference then is the net number of $SU(2)$ singlet states that one is interested in for that particular set of punctures.

To get to the entropy from the counting of the number of conformal blocks, we need to calculate $N_{bh}=\Sigma_{\mathcal{P}}N^{\mathcal{P}}$, where, the sum is over all sets of punctures. Then, $S_{bh}=\ln N_{bh}$.

It may be pointed out that the first term in Eq. (11) also has another interpretation. It represents the counting of boundary states for an effective $U(1)$ Chern-Simons theory. It counts the number of ways unit positive and negative $U(1)$ charges can be placed on the punctures to yield a vanishing total charge. This would then correspond to an entropy bound given by the same formula (2) above for binomial distribution of charges.

On the other hand, the combination of both terms in Eq. (11), which corresponds to counting of states in the $SU(2)$ Chern-Simons theory, yields an even tighter bound for entropy than that in Eq. (2). One can show that [14,15], the contribution to N_{bh} for this set of punctures $\mathcal{P}_0$ with all spins set to 1/2, is by far the dominant contribution; contributions from other sets of punctures are far smaller in comparison. Thus the entropy of an isolated horizon is given by the formula derived in Ref. [12]. We may mention that very recently Carlip [16,17] has presented compelling arguments that this formula may possibly be of a universal character. Here, the formula follows readily from Eq. (11) and Stirling approximations for factorials of large integers. The number of punctures p_0 is rewritten in terms of area A_H through Eq. (9) with the identification $\beta=\beta_0 \ln 2$. This allows us to write the entropy of an isolated horizon in terms of a power series in horizon area A_H:

$$S_{bh}=\ln N^{\mathcal{P}_0}=\frac{A_H}{4l_p^2}-\frac{3}{2}\ln\left(\frac{A_H}{4l_p^2}\right)-\frac{1}{2}\ln\left(\frac{\pi}{8(\ln 2)^3}\right)-O(A_H^{-1}). \tag{12}$$

Notice that the constant term here is negative and so is the order A_H^{-1} term. This then implies that the entropy is bound from above by a tighter bound which can be written in terms of Bekenstein-Hawking entropy ($S_{BH}=A_H/4l_p^2$) as

$$S_{max}=\ln\left(\frac{\exp S_{BH}}{S_{BH}^{3/2}}\right). \tag{13}$$

Inclusion of other than spin 1/2 representations on the punctures does not affect this bound. For example, we may place spin 1 on one or more punctures and spin 1/2 on the rest. The number of ways singlets can be made from this set of representations can be computed in a straightforward way. Adding these new states to the already counted ones above just changes the constant and order A_H^{-1} terms in formula (12). However, these additional terms continue to be negative, and hence the entropy bound (13) still holds.[1]

The steps leading to the EB now follows the standard route of deriving the Bekenstein bound (see, e.g., Ref. [7]): We assume, for simplicity, that the spatial slice of the boundary of an asymptotically flat space-time has the topology of a 2-sphere on which is induced a spherically symmetric 2-metric. Let this space-time contain an object whose entropy exceeds the bound. Certainly, such a space-time cannot have an isolated horizon as a boundary, since then, its entropy would have been subject to the bound. But, in that case, its energy should be less than that of a black hole which has the 2-sphere as its (isolated) horizon. Let us now add energy to the system, so that it does transform adiabatically into a black hole with the said horizon, but without affecting the entropy of the exterior. But we have already seen above that a black hole with such a horizon must respect the bound; it follows that the starting assumption that the object, to begin with, had an entropy violating the bound is not tenable.

There is, however, an important caveat in the foregoing argument. Strictly speaking, there is as yet no derivation of the second law of black hole mechanics within the framework of the isolated horizon. However, this is perhaps not a conceptual roadblock as far as deriving the EB is concerned. One has to assume that if matter or radiation crosses the isolated horizon adiabatically in small enough amounts, the *isolated* character of the horizon will not be seriously affected. This is perhaps not too drastic an assumption. Thus, for a large class of space-times, one may propose Eq. (13) as the new holographic entropy bound.

Finally, we should mention that we prefer to think of the above holographic principle and the consequent entropy bound as ''weak'' rather than ''strong'' in the sense of Smolin [7].

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[1] Using the Cardy formula with the prefactor (á la Carlip [17]) appears [18] to lead to entropy corrections for certain black holes not in accord with Eq. (13) [although the bound (2) is indeed respected]. This could be an artifact of the application of the Cardy formula. We refrain from further comment on these works, since the precise relation of the Cardy formula approach to the present framework is not clear.

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