# Back reaction problem in the inflationary universe

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We investigate the back reaction of cosmological perturbations on an inflationary universe using the renormalization-group method. The second-order zero mode solution which appears by the nonlinearity of the Einstein equation is regarded as a secular term of a perturbative expansion; we redefine the constants of integration contained in the background solution, and the secular terms are absorbed in these constants in a gauge-invariant manner. The resulting renormalization-group equation describes the back reaction effect of inhomogeneity on the background universe. For a scalar-type classical perturbation, by solving the renormalization-group equation, we find that the back reaction of the long wavelength fluctuation works as a positive spatial curvature, and the short wavelength fluctuation works as a radiation fluid. For the long wavelength quantum fluctuation, the effect of back reaction is equivalent to a negative spatial curvature.

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## I. INTRODUCTION

In the context of the standard cosmological perturbation approach [1,2], our Universe is treated as a homogeneous isotropic Friedmann-Robertson-Walker (FRW) model with small fluctuations in it. But due to the nonlinearity of Einstein's equation, the fluctuation has an effect on the evolution of the background spacetime, so we must solve the evolution of the fluctuation and the background in a self-consistent manner. This is the cosmological back reaction problem which has been studied by several authors [3–11]. The conventional approach to the problem is to construct an effective energy momentum tensor of the fluctuation. By adding this tensor to the right-hand side of the background Einstein equation, we can evaluate the effect of inhomogeneity on the evolution of the background FRW universe. For a large scale fluctuation in which the wavelength is larger than the Hubble horizon, the gauge dependence of the perturbation becomes conspicuous, and it is necessary to construct a gauge-invariant formalism of the back reaction problem. Towards this direction, Abramo and co-workers [8–11] derived the gauge-invariant effective energy momentum tensor of cosmological perturbations, which is invariant under the first-order gauge transformation. They applied their formalism to an inflationary universe and obtained the result that the back reaction effect of the long wavelength scalar type fluctuation is equivalent to a negative cosmological constant, and greatly reduces the inflationary expansion of the universe. But they did not derive solutions of an effective scale factor for the FRW universe with the back reaction.

Recently, the renormalization-group method [12–15] was applied to the cosmological back reaction problem [16]. Starting from a naive perturbative expansion of the solution of the original differential equation, this method gives an improved solution by renormalizing a secular term which appears by the nonlinearity of the equation. For a dust dominated universe, the second-order gauge-invariant zero mode metric is constructed by using the method of Abramo and co-workers. Then, by assuming that the second-order metric is secular, it is absorbed in a constant of integration contained in the background scale factor by using the renormalization-group method. The renormalized scale factor represents the effective dynamics of the FRW universe with the back reaction due to inhomogeneity. By solving the renormalization-group equation, it was found that perturbations of the scalar mode and the long wavelength tensor mode work as a positive spatial curvature, while the short wavelength tensor mode works as a radiation fluid.

In this paper, we aim to investigate the back reaction problem in the inflationary universe using the renormalization-group method. The advantage of this method is that we can obtain a solution of the back reaction equation directly by solving the renormalization-group equation. We do not need to solve the FRW equation with the effective energy momentum tensor, which is done in a conventional approach to the back reaction problem.

The plan of this paper is as follows. In Sec. II, we introduce the renormalization-group method by using a FRW model with a cosmological constant and perfect fluid. In Sec. III, the formulation of the back reaction problem based on the renormalization group-method is presented. In Sec. IV, we apply our formalism to the inflationary universe. Section V is devoted to a summary and discussion. We use the units in which $c=8\pi G=1$ throughout the paper.

## II. RENORMALIZATION-GROUP METHOD

To introduce the renormalization-group method, we consider a spatially flat FRW universe with a cosmological constant $\Lambda$ and perfect fluid. The Einstein equations are

$$\dot{\alpha}^2=\frac{\Lambda}{3}+\frac{\rho_1}{3}, \tag{1a}$$

$$\ddot{\alpha}+\frac{3}{2}\dot{\alpha}^2=-\frac{p_1}{2}, \tag{1b}$$

where $\alpha$ is a logarithm of a scale factor of the universe, and $\rho_1$ and $p_1$ are the energy density and the pressure of perfect

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fluid, respectively. The equation of state is assumed to be $p_1=(\Gamma-1)\rho_1$ where $\Gamma$ is a constant. The conservation equation for fluid is

$$\dot{\rho}_1+3\dot{\alpha}(\rho_1+p_1)=0, \tag{2}$$

and the solution is

$$\rho_1=\frac{c_1}{e^{3\Gamma\alpha}}, \tag{3}$$

where $c_1$ is a constant of integration. By substituting this solution in Eq. (1a), we obtain

$$\dot{\alpha}^2=\frac{\Lambda}{3}+\frac{c_1}{e^{3\Gamma\alpha}}. \tag{4}$$

We solve this equation perturbatively by assuming the second term of the right-hand side is small:

$$\alpha=\alpha_0+\alpha_1+\cdots. \tag{5}$$

The solution of the scale factor up to the first order of perturbation is given by

$$a(t)=a_0\,e^{H_0\,t}\left[1-\frac{c_1\,a_0^{-3\Gamma}}{2\Gamma\Lambda}(e^{-3\Gamma H_0 t}-e^{-3\Gamma H_0 t_0})\right], \tag{6}$$

where $H_0=\sqrt{\Lambda/3}$, $a_0$ and $t_0$ are constants of integration of the zeroth order and the first order, respectively. We regard the first-order solution as secular and apply the renormalization-group method [12–15].

We redefine the zeroth-order integration constant $a_0$ as

$$a_0=a_0^R(\mu)+\delta a_0(t_0;\mu), \tag{7}$$

where $\mu$ is a renormalization point and $\delta a_0$ is a counterterm which absorbs the secular term $\propto e^{-3\Gamma H_0 t}$ that diverges as $t\to 0$. The naive solution, Eq. (6), can be written as

$$\begin{aligned}a(t)&=e^{H_0\,t}\Bigg[a_0^R(\mu)+\delta a_0(t_0;\mu)-\frac{c_1\,(a_0^R)^{1-3\Gamma}}{2\Gamma\Lambda}\\&\quad\times(e^{-3\Gamma H_0 t}-e^{-3\Gamma H_0\mu}+e^{-3\Gamma H_0\mu}-e^{-3\Gamma H_0 t_0})\Bigg]\\&=e^{H_0\,t}\left[a_0^R(\mu)-\frac{c_1\,(a_0^R)^{1-3\Gamma}}{2\Gamma\Lambda}(e^{-3\Gamma H_0 t}-e^{-3\Gamma H_0\mu})\right],\end{aligned} \tag{8}$$

where we have chosen the counterterm $\delta a_0$ so as to absorb the $(e^{-3\Gamma H_0\mu}-e^{-3\Gamma H_0 t_0})$-dependent term:

$$\begin{aligned}\delta a_0(t_0;\mu)&=a_0^R(t_0)-a_0^R(\mu)\\&=\frac{c_1\,(a_0^R)^{1-3\Gamma}}{2\Gamma\Lambda}(e^{-3\Gamma H_0\mu}-e^{-3\Gamma H_0 t_0}).\end{aligned} \tag{9}$$

This defines the renormalization transformation

$$\begin{aligned}\mathcal{R}_{\mu,t_0}\;:\;& a_0^R(t_0)\mapsto a_0^R(\mu)\\&=a_0^R(t_0)-\frac{c_1\,(a_0^R)^{1-3\Gamma}}{2\Gamma\Lambda}(e^{-3\Gamma H_0\mu}-e^{-3\Gamma H_0 t_0}),\end{aligned} \tag{10}$$

and this transformation forms the Lie group up to the first order of the perturbation. We can obtain $a_0(\mu)$ for any arbitrary large value of $(e^{-3\Gamma H_0\mu}-e^{-3\Gamma H_0 t_0})$ by assuming the property of the Lie group, and this makes it possible to produce a globally uniform approximated solution of the original differential equation (4). The renormalization group equation is obtained by differentiating Eq. (9) with respect to $\mu$, and setting $t_0=\mu$:

$$\frac{\partial}{\partial\mu}a_0^R(\mu)=-\frac{c_1}{2\Gamma\Lambda}(a_0^R)^{1-3\Gamma}\frac{\partial}{\partial\mu}e^{-3\Gamma H_0\mu}. \tag{11}$$

The renormalized solution is obtained by equating $\mu=t$ in Eq. (8):

$$a^R(t)=e^{H_0 t}\left(\text{const}-\frac{3c_1}{2\Lambda}e^{-3\Gamma H_0 t}\right)^{1/(3\Gamma)}. \tag{12}$$

### III. RENORMALIZATION-GROUP APPROACH TO THE COSMOLOGICAL BACK REACTION PROBLEM

We treat the cosmological back reaction problem using a perturbation approach. Let us assume that the metric is expanded as follows:

$$g_{ab}=\overset{(0)}{g}_{ab}+\overset{(1)}{g}_{ab}+\overset{(2)}{g}_{ab}+\cdots. \tag{13}$$

$\overset{(0)}{g}_{ab}$ is the background FRW metric and represents a homogeneous and isotropic space. $g_{(1)ab}$ is the metric of the first-order linear perturbation. We assume that the spatial average of the first-order perturbation vanishes:

$$\langle\overset{(1)}{g}_{ab}\rangle=0, \tag{14}$$

where $\langle\cdots\rangle$ means the spatial average with respect to the background FRW metric. $\overset{(2)}{g}_{ab}$ is the second-order metric and contains nonlinear effects caused by the first-order linear perturbation. This nonlinearity produces homogeneous and isotropic zero modes as part of the second-order metric. That is,

$$\langle\overset{(2)}{g}_{ab}\rangle\neq 0. \tag{15}$$

Since we want to interpret the zero mode part of the metric as the background FRW metric, we must redefine the background metric as follows:

$$\overset{(0)}{g}_{ab}\to\overset{(0)}{g}_{ab}+\langle\overset{(2)}{g}_{ab}\rangle. \tag{16}$$

This is the back reaction caused by the nonlinearities of the fluctuation, and it changes the background metric. But in general, the second-order perturbation term will dominate the background metric over the course of time, and this simple prescription does not work to observe the long time behavior of the system. Furthermore, the meaning of the gauge invariance is not obvious in the second-order quantity, and we cannot adopt Eq. (16) as the definition of the background metric because the gauge transformation changes the definition of the background metric. By using the renormalization group method with the second-order gauge invariant quantities, we resolve these problems and can obtain the effective scale factor of the FRW universe with the back reaction in a gauge-invariant manner [16].

We consider a spatially flat FRW universe with scalar type first-order perturbation. As a matter field, we consider a minimally coupled scalar field $\chi$. The metric and scalar fields are

$$ds^2=-(1+2\phi+2\phi_2)dt^2+a^2(t)\times[(1-2\psi-2\psi_2)\delta_{ij}+2E_{,ij}]dx^idx^j,$$

$$\chi=\chi_0(t)+\chi_1+\chi_2, \tag{17}$$

where $a(t)$ is a scale factor of the background FRW universe, $\chi_0(t)$ is the background scalar field, $\phi$, $\psi$, $E$, and $\chi_1$ are the first-order variables, and $\phi_2$, $\psi_2$, and $\chi_2$ are the second-order zero mode variables. To obtain the back reaction on the FRW universe, it is sufficient to consider only the zero mode part of the second-order perturbation. We use a co-moving gauge in which the fluctuation of the scalar field vanishes: $\chi_1=\chi_2=0$. The second-order gauge invariant quantity which is invariant under the first-order gauge transformation is given by [16]

$$\langle \overset{(2)}{Q}_{ab}\rangle=\langle \overset{(2)}{g}_{ab}\rangle+\langle\mathcal{L}_X \overset{(1)}{g}_{ab}\rangle+\frac{1}{2}\langle\mathcal{L}_X^2 \overset{(0)}{g}_{ab}\rangle,$$

$$X^\mu=(0,-\delta^{ij}E_{,j}). \tag{18}$$

Its components are

$$\langle \overset{(2)}{Q}_{00}\rangle=-2\phi_2+a^2\sum_{\boldsymbol{k}} k^2\dot{E}_{\boldsymbol{k}}\dot{E}_{\boldsymbol{k}}^*, \tag{19a}$$

$$\langle \overset{(2)}{Q}_{ij}\rangle=a^2\left(-2\psi_2+\sum_{\boldsymbol{k}}\left(\frac{2k^2}{3}\psi_{\boldsymbol{k}}E_{\boldsymbol{k}}^*-\frac{k^4}{3}E_{\boldsymbol{k}}E_{\boldsymbol{k}}^*\right)\right)\delta_{ij}, \tag{19b}$$

where $\psi_{\boldsymbol{k}}$ and $E_{\boldsymbol{k}}$ are Fourier components of $\psi(t,\boldsymbol{x}),E(t,\boldsymbol{x})$:

$$\psi(t,\boldsymbol{x})=\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\psi_{\boldsymbol{k}}(t),\quad \psi_{\boldsymbol{k}}^*=\psi_{-\boldsymbol{k}},$$

$$E(t,\boldsymbol{x})=\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}E_{\boldsymbol{k}}(t),\quad E_{\boldsymbol{k}}^*=E_{-\boldsymbol{k}}. \tag{20}$$

The line element of the FRW universe obtained from the second-order gauge invariant variables is

$$\langle ds^2\rangle=-\left(1+2\phi_2-a^2\sum_{\boldsymbol{k}} k^2\dot{E}_{\boldsymbol{k}}\dot{E}_{\boldsymbol{k}}^*\right)dt^2+a^2(t)\left(1-2\psi_2+\sum_{\boldsymbol{k}}\left(\frac{2k^2}{3}\psi_{\boldsymbol{k}}E_{\boldsymbol{k}}^*-\frac{k^4}{3}E_{\boldsymbol{k}}E_{\boldsymbol{k}}^*\right)\right)d\boldsymbol{x}^2. \tag{21}$$

By the second-order coordinate transformation of time

$$t=T-\int dT\left(\phi_2-\frac{a^2}{2}\sum_{\boldsymbol{k}} k^2\dot{E}_{\boldsymbol{k}}\dot{E}_{\boldsymbol{k}}^*\right), \tag{22}$$

we obtain the metric of the FRW universe in a synchronous form as

$$\langle ds^2\rangle=-dT^2+(a^2+\delta a^2)d\boldsymbol{x}^2, \tag{23}$$

$$\delta a^2=a^2(T)\left(\sum_{\boldsymbol{k}}\left(\frac{2k^2}{3}\psi_{\boldsymbol{k}}E_{\boldsymbol{k}}^*-\frac{k^4}{3}E_{\boldsymbol{k}}E_{\boldsymbol{k}}^*\right)-2\psi_2-2H\int dT\left(\phi_2-\frac{a^2}{2}\sum_{\boldsymbol{k}} k^2\dot{E}_{\boldsymbol{k}}\dot{E}_{\boldsymbol{k}}^*\right)\right).$$

The second-order term $\delta a^2$ represents the back reaction of inhomogeneity on the FRW universe. In the context of perturbative expansion, the effective scale factor of the FRW universe with the back reaction can be written as

$$a_{\text{eff}}=a\left(1+\frac{\delta a^2}{2a^2}\right). \tag{24}$$

However, in general, the second-order correction $\delta a^2$ may dominate the background scale factor $a^2$ as time goes on, and the prescription of perturbation will break down. We can go beyond the perturbation by using the renormalization-group method. We absorb $\delta a^2$ to a constant of integration of the background scale factor, and this procedure yields the renormalization-group equation. By solving the renormalization group equation, we obtain a renormalized scale factor which gives more accurate late time behavior of the system than the naive perturbative expression, Eq. (24).

## IV. THE BACK REACTION IN THE INFLATIONARY UNIVERSE

In this section, we investigate the back reaction problem in the inflationary universe using the renormalization-group method. We work in the comoving gauge $\chi_1=\chi_2=0$. The background equations are

$$3H^2=\frac{1}{2}\dot{\chi}_0^2+V(\chi_0), \tag{25a}$$

$$2\dot{H}+3H^2=-\frac{1}{2}\dot{\chi}_0^2+V(\chi_0), \tag{25b}$$

$$\ddot{\chi}_0+3H\dot{\chi}_0+V'(\chi_0)=0. \tag{25c}$$

The first-order equations are

$$3H(\dot{\psi}_{\boldsymbol{k}}+H\phi_{\boldsymbol{k}})+\frac{k^2}{a^2}\psi_{\boldsymbol{k}}+k^2H\dot{E}_{\boldsymbol{k}}=\frac{\dot{\chi}_0^2}{2}\phi_{\boldsymbol{k}}, \tag{26a}$$

$$\dot{\psi}_{\boldsymbol{k}}+H\phi_{\boldsymbol{k}}=0, \tag{26b}$$

$$\ddot{\psi}_{\boldsymbol{k}}+H(\dot{\phi}_{\boldsymbol{k}}+3\dot{\psi}_{\boldsymbol{k}})+(3H^2+2\dot{H})\phi_{\boldsymbol{k}}=-\frac{\dot{\chi}_0^2}{2}\phi_{\boldsymbol{k}}, \tag{26c}$$

$$\ddot{E}_{\boldsymbol{k}}+3H\dot{E}_{\boldsymbol{k}}+\frac{1}{a^2}(\psi_{\boldsymbol{k}}-\phi_{\boldsymbol{k}})=0, \tag{26d}$$

$$2V'(\chi_0)\phi_{\boldsymbol{k}}-\dot{\chi}_0(\dot{\phi}_{\boldsymbol{k}}+3\dot{\psi}_{\boldsymbol{k}}+k^2\dot{E}_{\boldsymbol{k}})=0. \tag{26e}$$

Equations (26a), (26b), (26c), and (26d) are the first-order Einstein equations, and Eq. (26e) is the first-order equation of motion of the scalar field. The spatial curvature perturbation $\psi$ obeys the following single equation:

$$\ddot{\psi}_{\boldsymbol{k}}+\left(3H-\frac{2\dot{H}}{H}+\frac{2\ddot{\chi}_0}{\dot{\chi}_0}\right)\dot{\psi}_{\boldsymbol{k}}+\frac{k^2}{a^2}\psi_{\boldsymbol{k}}=0. \tag{27}$$

The second order equations for the zero mode variables $\phi_2,\psi_2$ are

$$6H\dot{\psi}_2+2(3H^2+\dot{H})\phi_2=\sum_{\boldsymbol{k}}\Bigg[-\frac{2k^4}{a^2}E_{\boldsymbol{k}}\psi_{\boldsymbol{k}}^*-\frac{5k^2}{a^2}\psi_{\boldsymbol{k}}\psi_{\boldsymbol{k}}^*+2k^2\dot{E}_{\boldsymbol{k}}\dot{\psi}_{\boldsymbol{k}}^*+3\dot{\psi}_{\boldsymbol{k}}\dot{\psi}_{\boldsymbol{k}}^*-12H(\psi_{\boldsymbol{k}}\dot{\psi}_{\boldsymbol{k}}^*-\phi_{\boldsymbol{k}}\dot{\psi}_{\boldsymbol{k}}^*)$$
$$+4(3H^2+\dot{H})\phi_{\boldsymbol{k}}\phi_{\boldsymbol{k}}^*-4H(k^4E_{\boldsymbol{k}}\dot{E}_{\boldsymbol{k}}^*-k^2\phi_{\boldsymbol{k}}\dot{E}_{\boldsymbol{k}}^*+k^2\psi_{\boldsymbol{k}}\dot{E}_{\boldsymbol{k}}^*+k^2E_{\boldsymbol{k}}\dot{\psi}_{\boldsymbol{k}}^*)\Bigg], \tag{28a}$$

$$2(\dot{H}+3H^2)\phi_2+2H(\dot{\phi}_2+3\dot{\psi}_2)+2\ddot{\psi}_2=\sum_{\boldsymbol{k}}\Bigg[-\frac{2k^2}{3a^2}\left(k^2E_{\boldsymbol{k}}(\psi_{\boldsymbol{k}}^*-\phi_{\boldsymbol{k}}^*)+\phi_{\boldsymbol{k}}\phi_{\boldsymbol{k}}^*-3\phi_{\boldsymbol{k}}\psi_{\boldsymbol{k}}^*+\frac{5}{2}\psi_{\boldsymbol{k}}\psi_{\boldsymbol{k}}^*\right)$$
$$-\frac{2k^2}{3}\dot{E}_{\boldsymbol{k}}(k^2\dot{E}_{\boldsymbol{k}}^*+\dot{\psi}_{\boldsymbol{k}}^*-\dot{\phi}_{\boldsymbol{k}}^*)+2\dot{\psi}_{\boldsymbol{k}}\dot{\phi}_{\boldsymbol{k}}^*-\dot{\psi}_{\boldsymbol{k}}\dot{\psi}_{\boldsymbol{k}}^*+4(2H^2+\dot{H})\phi_{\boldsymbol{k}}\phi_{\boldsymbol{k}}^*$$
$$-\frac{4k^2}{3}\ddot{E}_{\boldsymbol{k}}(k^2E_{\boldsymbol{k}}^*+\psi_{\boldsymbol{k}}^*-\phi_{\boldsymbol{k}}^*)-4\ddot{\psi}_{\boldsymbol{k}}\left(\frac{k^2}{3}E_{\boldsymbol{k}}^*+\psi_{\boldsymbol{k}}^*-\phi_{\boldsymbol{k}}^*\right)-4H(-H\phi_{\boldsymbol{k}}\phi_{\boldsymbol{k}}^*-2\phi_{\boldsymbol{k}}\dot{\phi}_{\boldsymbol{k}}^*$$
$$-3\phi_{\boldsymbol{k}}\dot{\psi}_{\boldsymbol{k}}^*+3\psi_{\boldsymbol{k}}\dot{\psi}_{\boldsymbol{k}}^*)-4k^2H(k^2E_{\boldsymbol{k}}\dot{E}_{\boldsymbol{k}}^*-\phi_{\boldsymbol{k}}\dot{E}_{\boldsymbol{k}}^*+\psi_{\boldsymbol{k}}\dot{E}_{\boldsymbol{k}}^*+E_{\boldsymbol{k}}\dot{\psi}_{\boldsymbol{k}}^*)\Bigg], \tag{28b}$$

$$-\dot{\chi}_0(\dot{\phi}_2+3\dot{\psi}_2)-2(\ddot{\chi}_0+3H\dot{\chi}_0)\phi_2=-\dot{\chi}_0\sum_{\boldsymbol{k}}\Bigg[-\frac{8k^4}{3}E_{\boldsymbol{k}}\dot{E}_{\boldsymbol{k}}^*+k^2(3\dot{E}_{\boldsymbol{k}}\phi_{\boldsymbol{k}}^*+E_{\boldsymbol{k}}\dot{\phi}_{\boldsymbol{k}}^*)-4k^2(\dot{E}_{\boldsymbol{k}}\psi_{\boldsymbol{k}}^*+E_{\boldsymbol{k}}\dot{\psi}_{\boldsymbol{k}}^*)$$
$$+4\phi_{\boldsymbol{k}}\dot{\phi}_{\boldsymbol{k}}^*+3\dot{\phi}_{\boldsymbol{k}}\psi_{\boldsymbol{k}}^*+9\phi_{\boldsymbol{k}}\dot{\psi}_{\boldsymbol{k}}^*-12\psi_{\boldsymbol{k}}\dot{\psi}_{\boldsymbol{k}}^*+4\left(3H+\frac{\ddot{\chi}_0}{\dot{\chi}_0}\right)\phi_{\boldsymbol{k}}\phi_{\boldsymbol{k}}^*\Bigg]. \tag{28c}$$

Equations (28a) and (28b) are the second-order Einstein equations, and Eq. (28c) is the second-order equation of motion of the scalar field.

We solve these equations under the condition of slow-rolling inflation,

$$\left|\frac{\dot{H}}{H^2}\right|\ll 1,\quad \left|\frac{\ddot{\chi}_0}{H\dot{\chi}_0}\right|\ll 1. \tag{29}$$

Under this condition,

$$H^2\approx\frac{1}{3}V(\chi_0),\quad \dot{\chi}_0\approx-\frac{V'(\chi_0)}{3H}. \tag{30}$$

### A. Long wavelength mode

For the long wavelength mode, of which wavelength is larger than the horizon scale $1/H$, the growing mode solution of Eq. (27) is given by

$$\psi_{\boldsymbol{k}}=\psi_0(k)\left[1+\frac{k^2}{2a^2H^2}\right]+O(k^4), \tag{31}$$

where $\psi_0(k)$ is an arbitrary function of $k$ and satisfies $\psi_0(k)=0$ as $k\to 0$ to ensure $\langle\psi\rangle=0$. If we take only the $O(k^0)$ term $\psi_0(k)$, the right-hand sides of the second-order equations become zero. We must account for the $O(k^2)$ term to get the back reaction effect. By using this as the first order solution, other first-order quantities are given by

$$\phi \approx \psi_0 \frac{k^2}{a^2H^2}, \quad E \approx \frac{\psi_0}{2}\frac{1}{a^2H^2},$$

$$\dot{\psi} \approx -\psi_0 \frac{k^2}{a^2H}, \quad \dot{\phi} \approx -2\psi_0 \frac{k^2}{a^2H}, \quad \dot{E} \approx -\psi_0 \frac{1}{a^2H}. \tag{32}$$

Then, the second-order equations become

$$\dot{\psi}_2 + H\phi_2 \approx \frac{11}{6a^2H}\sum_{\boldsymbol{k}} k^2|\psi_0|^2, \tag{33a}$$

$$\ddot{\psi}_2 + H\dot{\phi}_2 + 3H(\dot{\psi}_2 + H\phi_2) \approx \frac{11}{6a^2}\sum_{\boldsymbol{k}} k^2|\psi_0|^2, \tag{33b}$$

$$6H\phi_2 + \dot{\phi}_2 + 3\dot{\psi}_2 \approx \frac{10}{a^2H}\sum_{\boldsymbol{k}} k^2|\psi_0|^2, \tag{33c}$$

and the second-order zero mode solution is

$$\psi_2 \approx \frac{4}{3a^2H^2}\sum_{\boldsymbol{k}} k^2|\psi_0|^2, \quad \phi_2 \approx \frac{9}{2a^2H^2}\sum_{\boldsymbol{k}} k^2|\psi_0|^2. \tag{34}$$

The metric of the FRW universe, Eq. (23), is given by

$$\langle ds^2 \rangle = -dT^2 + a^2(T)\left(1 + \frac{7}{3a^2H^2}\sum_{\boldsymbol{k}} k^2|\psi_0|^2\right)d\boldsymbol{x}^2. \tag{35}$$

At this stage, we apply the renormalization-group method to obtain the effective scale factor which includes the back reaction effect. The background scale factor can be written as

$$a(T) = a_0\,\tilde{a}(T), \tag{36}$$

where $a_0$ is a constant of integration which reflects the freedom of rescaling of the scale factor. We redefine the constant $a_0$ so as to absorb the second-order correction of the scale factor. The renormalization-group equation is given by

$$\frac{\partial a_0^2}{\partial(1/\tilde{a}^2)} = \frac{7}{3H^2}\sum_{\boldsymbol{k}} k^2|\psi_0|^2, \tag{37}$$

and the solution is

$$a_0(T) = \left(\text{const} + \frac{7}{3\tilde{a}^2H^2}\sum_{\boldsymbol{k}} k^2|\psi_0|^2\right)^{1/2}. \tag{38}$$

The renormalized metric is

$$\langle ds^2 \rangle = -dT^2 + (a^R(T))^2 d\boldsymbol{x}^2,$$

$$a^R(T) = \tilde{a}(T)\left(\text{const} + \frac{7}{3\tilde{a}^2H^2}\sum_{\boldsymbol{k}} k^2|\psi_0|^2\right)^{1/2}. \tag{39}$$

Comparing this with the analysis of Sec. II, the renormalized scale factor is the same as that of the FRW universe with $\Gamma = \frac{2}{3}, c_1 < 0$. We conclude that the back reaction of the long wavelength scalar perturbation on the FRW universe is equivalent to a positive spatial curvature. But because of the inflationary expansion of the universe, the back reaction effect decays as $e^{-2Ht}$ and becomes negligible.

### B. Short wavelength mode

For the short wavelength mode, of which the wavelength is smaller than the horizon scale $1/H$, the first-order solution of $\psi$ is given by the WKB form

$$\psi \approx \frac{\psi_0}{a}\cos(k\eta), \tag{40}$$

where $\psi_0 \equiv H/\dot{\chi}_0 C(k)$, $C(k)$ is an arbitrary function of $k$, and $\eta = \int(dt/a)$. In the inflationary universe, $\psi_0$ becomes approximately constant in time. Other first-order quantities are

$$\dot{E} \approx -\frac{1}{Ha^2}\psi, \quad k^2E \approx 3\psi + \frac{\dot{\psi}}{H}. \tag{41}$$

Using these solutions, the second-order equation becomes

$$\dot{\psi}_2 + H\phi_2 \approx 0, \tag{42a}$$

$$\ddot{\psi}_2 + H\dot{\phi}_2 + 3H(\dot{\psi}_2 + H\phi_2) \approx 0, \tag{42b}$$

$$\dot{\phi}_2 + 3\dot{\psi}_2 + 6H\phi_2 \approx \frac{8}{3a^4H}\sum_{\boldsymbol{k}} k^2|\psi_0|^2, \tag{42c}$$

where we have omitted oscillatory terms in the right-hand sides because they do not contribute to the secular behavior of the zero mode solution. The second-order zero mode metric is given by

$$\phi_2 \approx -\frac{8}{3H^2a^4}\sum_{\boldsymbol{k}} k^2|\psi_0|^2, \quad \psi_2 \approx -\frac{2}{3H^2a^4}\sum_{\boldsymbol{k}} k^2|\psi_0|^2. \tag{43}$$

Therefore, the metric of the FRW universe, Eq. (23), becomes

$$\langle ds^2 \rangle = -dT^2 + a^2(T)\left(1 - \frac{7}{24a^4H^2}\sum_{\boldsymbol{k}} k^2|\psi_0|^2\right)d\boldsymbol{x}^2. \tag{44}$$

The renormalization group equation becomes

$$\frac{\partial a_0^2}{\partial(1/\tilde{a}^4)} = -\frac{7}{24 a_0^2 H^2}\sum_{\boldsymbol{k}} k^2|\psi_0|^2, \tag{45}$$

and the solution is

$$a_0(T) = \left(\text{const} - \frac{7}{24\tilde{a}^4 H^2}\sum_{\boldsymbol{k}} k^2|\psi_0|^2\right)^{1/4}. \tag{46}$$

The renormalized metric is

$$\langle ds^2\rangle = -dT^2 + (a^R(T))^2 d\boldsymbol{x}^2,$$

$$a^R(T) = \tilde{a}(T)\left(\text{const} - \frac{7}{24\tilde{a}^4 H^2}\sum_{\boldsymbol{k}} k^2|\psi_0|^2\right)^{1/4}. \tag{47}$$

Comparing this with the results of Sec. II, the renormalized scale factor is the same as that of the FRW universe with $\Gamma = \frac{4}{3}, c_1 > 0$. The effect of the back reaction of the short wavelength scalar perturbation is the same as radiation fluid.

### C. Long wavelength quantum fluctuation

We consider the back reaction due to the quantum fluctuation in the inflationary universe. By quantizing the first order perturbation, the normalization of the mode function is determined. For the quantum fluctuation, the operation of the spatial average $\langle\cdots\rangle$ must be replaced with an expectation value of a suitable vacuum state. Introducing a variable $\delta\chi \equiv \dot{\chi}_0/H\psi$, Eq. (27) becomes [17]

$$\delta\ddot{\chi}_{\boldsymbol{k}} + 3H\delta\dot{\chi}_{\boldsymbol{k}} + \left(\frac{k^2}{a^2} + V'' + \frac{2\dot{H}}{H}\left(3H - \frac{\dot{H}}{H} + \frac{2\ddot{\chi}_0}{\dot{\chi}_0}\right)\right)\delta\chi_{\boldsymbol{k}} = 0. \tag{48}$$

We quantize the variable $\delta\chi$ as

$$\delta\hat{\chi}(\boldsymbol{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}}[\hat{a}_{\boldsymbol{k}}\delta\chi_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{a}_{\boldsymbol{k}}^\dagger\delta\chi_{\boldsymbol{k}}^*(t)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}], \tag{49a}$$

$$[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}] = [\hat{a}_{\boldsymbol{k}}^\dagger, \hat{a}_{\boldsymbol{k}'}^\dagger] = 0, \quad [\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}^\dagger] = \hbar\,\delta^3(\boldsymbol{k}-\boldsymbol{k}'), \tag{49b}$$

$$\delta\chi_{\boldsymbol{k}}\,\delta\dot{\chi}_{\boldsymbol{k}}^* - \delta\chi_{\boldsymbol{k}}^*\,\delta\dot{\chi}_{\boldsymbol{k}} = \frac{i}{a^3}. \tag{49c}$$

Under the slow-rolling condition, Eq. (48) is approximately the same as the mode equation of the scalar field on a fixed de Sitter background, and the solution of the mode function is given by

$$\delta\chi_{\boldsymbol{k}}(t) \approx \frac{\sqrt{\pi}}{2} H\eta^{3/2}[c_1(k)H_\nu^{(1)}(k\eta) + c_2(k)H_\nu^{(2)}(k\eta)], \tag{50}$$

$$\eta \approx -\frac{1}{aH}, \quad \nu \approx \frac{3}{2} - \frac{V''(\chi_0)}{3H^2}, \quad |c_2(k)|^2 - |c_1(k)|^2 = 1,$$

where $H_\nu^{(1,2)}$ is a Hankel function. We choose a Bunch-Davies vacuum with $c_2 = 1, c_1 = 0$. The power spectrum for $\delta\chi$ in the long wavelength is given by

$$P_{\delta\chi}(k,t) \equiv \hbar\frac{k^3}{2\pi^2}|\delta\chi_{\boldsymbol{k}}(t)|^2 \approx \hbar\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu}, \tag{51}$$

and

$$\langle\delta\chi^2\rangle = \hbar\sum_{\boldsymbol{k}}|\delta\chi_{\boldsymbol{k}}|^2 = \int_H^{aH} d\ln k\, P_{\delta\chi}(k,t) \approx \frac{3\hbar H^4}{8\pi^2 V''(\chi_0)^2}\left[1 - \exp\left(-\frac{2V''(\chi_0)^2}{3H}t\right)\right],$$

$$\langle(\nabla\delta\chi)^2\rangle = \hbar\sum_{\boldsymbol{k}} k^2|\delta\chi_{\boldsymbol{k}}(t)|^2 = \int_H^{aH} d\ln k\, k^2 P_{\delta\chi}(k,t) \approx \frac{\hbar}{2}\left(\frac{H}{2\pi}\right)^2 H^2(a^2-1), \tag{52}$$

where we cut off the infrared and ultraviolet contributions, which is a conventionally used regularization. Therefore,

$$\langle(\nabla\psi)^2\rangle = \hbar\left(\frac{H}{\dot{\chi}_0}\right)^2\sum_{\boldsymbol{k}} k^2|\delta\chi_{\boldsymbol{k}}|^2 \approx \frac{\hbar}{2}\left(\frac{H}{\dot{\chi}_0}\right)^2\left(\frac{H}{2\pi}\right)^2 H^2(a^2-1), \tag{53}$$

and the metric of the FRW universe, Eq. (35), becomes

$$\langle ds^2\rangle = -dT^2 + a^2(T)\left(1 + \frac{7}{3a^2H^2}\langle(\nabla\psi)^2\rangle\right)d\boldsymbol{x}^2 = -dT^2 + a^2(T)\left[1 + \frac{7\hbar}{6}\left(\frac{H}{\dot{\chi}_0}\right)^2\left(\frac{H}{2\pi}\right)^2\left(1 - \frac{1}{a^2}\right)\right]d\boldsymbol{x}^2. \tag{54}$$

The renormalization-group equation is

$$\frac{\partial a_0^2}{\partial(1/\tilde{a}^2)} = -\frac{7\hbar}{6}\left(\frac{H}{\dot{\chi}_0}\right)^2\left(\frac{H}{2\pi}\right)^2, \tag{55}$$

and the renormalized metric is

$$\langle ds^2\rangle = -dT^2 + (a^R(T))^2 d\mathbf{x}^2,$$
$$a^R(T) = \tilde{a}(T)\left(\text{const} - \frac{7\hbar}{6\tilde{a}^2}\left(\frac{H}{\dot{\chi}_0}\right)^2\left(\frac{H}{2\pi}\right)^2\right)^{1/2}. \tag{56}$$

Comparing this with the results of Sec. II, the renormalized scale factor is the same as that of the FRW universe with $\Gamma=\frac{2}{3}, c_1<0$. The effect of the back reaction of the quantum fluctuation is equivalent to a negative spatial curvature.

## V. SUMMARY AND DISCUSSION

We have investigated the back reaction problem in the inflationary universe using the renormalization-group method. By renormalizing the second-order gauge-invariant zero mode metric, which appears by the nonlinear effect of Einstein's equation, we obtained the effective scale factor which includes the back reaction of the scalar-type fluctuation. For the long wavelength classical perturbation, the effect of the back reaction is equivalent to a positive spatial curvature. For the short wavelength perturbation, the back reaction is equivalent to radiation fluid. For the long wavelength quantum fluctuation, the effect of the back reaction is same as a negative spatial curvature. In any case, the effect of the back reaction quickly decays and becomes negligible, and does not alter the expansion of the inflationary universe.

Our result on the back reaction of the long wavelength perturbation is different from the analysis of Abramo and co-workers [8–11]. They used the longitudinal gauge and obtained the result which stated that the back reaction due to the long wavelength perturbation works as a negative cosmological constant. To understand why this discrepancy occurs, we have performed the calculation of the back reaction using the longitudinal gauge (see Appendix). The crucial point is the form of the first-order solution, which is used to evaluate the back reaction effect. In the long wavelength limit, the growing mode solution of $\psi$ in the longitudinal gauge is given by

$$\psi = \frac{H}{a}\int_{t_0}^{t} dt\, a\frac{\dot{H}}{H^2} \approx -\left(\frac{a}{H}\right)_{t=t_0}\frac{H}{a} + \frac{\dot{H}}{H^2} + \cdots. \tag{57}$$

They used $\dot{H}/H^2$ as the first order solution. But the obtained second order solution is canceled to zero by choosing an appropriate homogeneous solution of the second-order equation. Hence, we should take $H/a$ as the first order solution to observe the back reaction effect. Our calculation in the longitudinal gauge shows that we have the same back reaction effect as the comoving gauge case.

The back reaction becomes important in the preheating stage of the universe. For a massive scalar field, the scalar field oscillates around the minimum of the potential and the scale factor grows as $a\sim t^{2/3}$. We calculated the back reaction of the long wavelength perturbation at this stage, and the renormalized scale factor is given by

$$a^R(t) = T^{2/3}\left(\text{const} - c\, T^{2/3}\sum_{\mathbf{k}} k^2|\psi_0(k)|^2\right)^{1/2}, \tag{58}$$

where $c$ is a numerical factor. The back reaction of the long wavelength scalar perturbation is the same as the effect of a positive spatial curvature, and this is the same as in the case of a dust dominated universe [16]. In the preheating stage of the universe, the back reaction slows down the expansion of the universe, so its effect cannot be negligible.

The renormalization-group method can be viewed as a tool of system reduction. The renormalzation-group equation corresponds to the amplitude equation, which describes slow motion dynamics in the original system. We can describe complicated dynamics contained in the original equation by extracting a simpler representation using the renormalization-group equation. For the cosmological back reaction problem, we can reduce the Einstein equation to the FRW equation, and this is nothing but a back reaction equation. In this paper, we perturbatively solved the Einstein equation and obtained the solution for the FRW equation with the back reaction effect, though we do not derive the back reaction equation. It is possible to derive the back reaction equation by applying the renormalization-group method to the equation of motion directly. We will report this subject in a separate publication.

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## APPENDIX: BACK REACTION IN LONGITUDINAL GAUGE

In this appendix, we present the calculation of the back reaction using the longitudinal gauge $\phi=\psi, E=0$ to check the gauge independence of the back reaction effect. The metric is

$$ds^2 = -(1+2\psi+2\phi_2)dt^2 + a^2(1-2\psi-2\psi_2)d\mathbf{x}^2. \tag{A1}$$

The first-order equations are

$$3H(\dot{\psi}_{\mathbf{k}} + H\psi_{\mathbf{k}}) + \frac{k^2}{a^2}\psi_{\mathbf{k}} = \frac{1}{2}(-\dot{\chi}_0\dot{\chi}_{1\,\mathbf{k}} - V'(\chi_0)\chi_{1\,\mathbf{k}} + \dot{\chi}_0^2\psi_{\mathbf{k}}), \tag{A2a}$$

$$\dot{\psi}_{\mathbf{k}} + H\psi_{\mathbf{k}} = \frac{1}{2}\dot{\chi}_0\chi_{1\,\mathbf{k}}, \tag{A2b}$$

$$\ddot{\psi}_{\mathbf{k}} + 4H\dot{\psi}_{\mathbf{k}} + (2\dot{H}+3H^2)\psi_{\mathbf{k}} = -\frac{1}{2}(\dot{\chi}_0^2\psi_{\mathbf{k}} - \dot{\chi}_0\dot{\chi}_{1\,\mathbf{k}} + V'(\chi_0)\chi_{1\,\mathbf{k}}), \tag{A2c}$$

$$\ddot{\chi}_{1\,\mathbf{k}}+3H\dot{\chi}_{1\,\mathbf{k}}+V''(\chi_0)\chi_{1\,\mathbf{k}}+\frac{k^2}{a^2}\chi_{1\,\mathbf{k}}+2V'(\chi_0)\psi_{\mathbf{k}}-4\dot{\chi}_0\dot{\psi}_{\mathbf{k}}=0. \tag{A2d}$$

The evolution equation of $\psi$ becomes

$$\ddot{\psi}_{\mathbf{k}}+\left(H-\frac{2\ddot{\chi}_0}{\dot{\chi}_0}\right)\dot{\psi}_{\mathbf{k}}+\left(2\dot{H}-2H\frac{\ddot{\chi}_0}{\dot{\chi}_0}+\frac{k^2}{a^2}\right)\psi_{\mathbf{k}}=0. \tag{A3}$$

The second-order equations are

$$\begin{aligned}6H\dot{\psi}_2+2(3H^2+\dot{H})\phi_2+\dot{\chi}_0\dot{\chi}_2+V'(\chi_0)\chi_2 &=\sum_{\mathbf{k}}\left[-\frac{5k^2}{a^2}\psi_{\mathbf{k}}\psi^*_{\mathbf{k}}+4(3H^2+\dot{H})\psi_{\mathbf{k}}\psi^*_{\mathbf{k}}+3\dot{\psi}_{\mathbf{k}}\dot{\psi}^*_{\mathbf{k}}\right.\\ &\quad-\frac{1}{2}\dot{\chi}_{1\,\mathbf{k}}\dot{\chi}^*_{1\,\mathbf{k}}-\frac{k^2}{2a^2}\chi_{1\,\mathbf{k}}\chi^*_{1\,\mathbf{k}}+2\dot{\chi}_0\psi_{\mathbf{k}}\dot{\chi}^*_{1\,\mathbf{k}}\\ &\quad\left.-\frac{V''}{2}\chi_{1\,\mathbf{k}}\chi^*_{1\,\mathbf{k}}\right],\end{aligned} \tag{A4a}$$

$$\begin{aligned}2\ddot{\psi}_2+6H\dot{\psi}_2+2H\dot{\phi}_2+2(\dot{H}+3H^2)\phi_2+V'(\chi_0)\chi_2-\dot{\chi}_0\dot{\chi}_2 &=\sum_{\mathbf{k}}\left[-\frac{k^2}{3a^2}\psi_{\mathbf{k}}\psi^*_{\mathbf{k}}+4(3H^2+\dot{H})\psi_{\mathbf{k}}\psi^*_{\mathbf{k}}+8H\psi_{\mathbf{k}}\dot{\psi}^*_{\mathbf{k}}\right.\\ &\quad+\dot{\psi}_{\mathbf{k}}\dot{\psi}^*_{\mathbf{k}}+\frac{1}{2}\dot{\chi}_{1\,\mathbf{k}}\dot{\chi}^*_{1\,\mathbf{k}}-\frac{k^2}{6a^2}\chi_{1\,\mathbf{k}}\chi^*_{1\,\mathbf{k}}-2\dot{\chi}_0\psi_{\mathbf{k}}\dot{\chi}^*_{1\,\mathbf{k}}\\ &\quad\left.-\frac{V''}{2}\chi_{1\,\mathbf{k}}\chi^*_{1\,\mathbf{k}}\right],\end{aligned} \tag{A4b}$$

$$\begin{aligned}\ddot{\chi}_2+3H\dot{\chi}_2+V''(\chi_0)\chi_2-\dot{\chi}_0\left(6H\phi_2+\dot{\phi}_2+\frac{2\ddot{\chi}_0}{\dot{\chi}_0}\phi_2+3\dot{\psi}_2\right) &=\sum_{\mathbf{k}}\left[-\frac{V'''(\chi_0)}{2}\chi_{1\,\mathbf{k}}\chi^*_{1\,\mathbf{k}}-\frac{2k^2}{a^2}\psi_{\mathbf{k}}\chi^*_{1\,\mathbf{k}}+6H\psi_{\mathbf{k}}\dot{\chi}^*_{1\,\mathbf{k}}\right.\\ &\quad+4\dot{\chi}_{1\,\mathbf{k}}\dot{\psi}^*_{\mathbf{k}}+2\psi_{\mathbf{k}}\ddot{\chi}^*_{1\,\mathbf{k}}\\ &\quad\left.-4\dot{\chi}_0\left[\psi_{\mathbf{k}}\dot{\psi}^*_{\mathbf{k}}+\left(\frac{\ddot{\chi}_0}{\dot{\chi}_0}+3H\right)\psi_{\mathbf{k}}\psi^*_{\mathbf{k}}\right]\right].\end{aligned} \tag{A4c}$$

In this gauge, $\phi_2$ and $\psi_2$ are invariant under the first-order gauge transformation. The metric of the FRW universe is given by

$$\begin{aligned}\langle ds^2\rangle &=-(1+2\phi_2)dt^2+a^2(t)(1-2\psi_2)d\mathbf{x}^2\\ &=-dT^2+a^2(T)\left(1-2\psi_2-2H\int dt\,\phi_2\right)d\mathbf{x}^2,\end{aligned} \tag{A5}$$

where

$$t=T-\int dT\,\phi_2. \tag{A6}$$

### 1. Long wavelength case

The growing mode solution of Eq. (A3) in the long wavelength limit is given by

$$\psi_{\mathbf{k}}=D(k)\left[-1+\frac{H}{a}\int_{t_0}^{t}dt\,a\right]\approx D(k)\frac{\dot{H}}{H^2}+C(k)\frac{H}{a}, \tag{A7}$$

where $C(k)$ and $D(k)$ are arbitrary functions of $k$ which satisfy $C(k)=0$ and $D(k)=0$ as $k\to 0$ to ensure $\langle\psi\rangle=0$. Using this solution, the second-order solution is given by

$$\phi_2\approx\frac{1}{2}(\epsilon-3\delta)\epsilon\sum_{\mathbf{k}}|D|^2+\frac{H}{2a}(\epsilon+7\delta)\sum_{\mathbf{k}}(CD^*+C^*D)-\frac{5H^2}{2a^2}\sum_{\mathbf{k}}|C|^2, \tag{A8a}$$

$$\psi_2\approx\frac{1}{2}(\epsilon-3\delta)\epsilon\sum_{\mathbf{k}}|D|^2+\frac{H}{2a}(-3\epsilon+7\delta)\times\sum_{\mathbf{k}}(CD^*+C^*D)-\frac{5H^2}{2a^2}\sum_{\mathbf{k}}|C|^2, \tag{A8b}$$

$$\chi_2\approx\frac{1}{2}(\epsilon+9\delta)\frac{\dot{\chi}_0}{H}\sum_{\mathbf{k}}|D|^2+\frac{\dot{\chi}_0 H}{2a^2}\sum_{\mathbf{k}}|C|^2, \tag{A8c}$$

where $\epsilon$ and $\delta$ are slow roll parameters

$$\epsilon=\frac{\dot{H}}{H^2},\qquad \delta=\frac{\ddot{\chi}_0}{H\dot{\chi}_0}, \tag{A9}$$

and they can be treated as small constants under the considering orders of the approximation.

It is important to notice that the second-order equation has the following homogeneous solution:

$$\psi_2^{(\mathrm{homo})}=\phi_2^{(\mathrm{homo})}\approx D_2\frac{\dot{H}}{H^2}+C_2\frac{H}{a}, \tag{A10}$$

where $C_2$, and $D_2$ are arbitrary constants. By using the freedom of the homogeneous solution and the second-order gauge transformation $t\to t+B_2(H/a)$ where $B_2$ is a constant, it is always possible to reduce the second-order solution to the following form:

$$\psi_2=\phi_2\approx-\frac{5H^2}{2a^2}\sum_{\mathbf{k}}|C|^2,\qquad \chi_2\approx\frac{\dot{\chi}_0}{2}\frac{H}{a^2}\sum_{\mathbf{k}}|C|^2. \tag{A11}$$

The metric of the FRW universe is given by

$$\langle ds^2\rangle = -dT^2 + a^2(T)\left(1+\frac{5H^2}{2a^2}\sum_{\boldsymbol{k}}|C|^2\right)d\boldsymbol{x}^2. \tag{A12}$$

This expression is the same as in Eq. (35), and the effect of the back reaction is equivalent to a positive spatial curvature. This result is consistent with the calculation using the comoving gauge.

We can read the component of the gauge invariant effective energy momentum tensor from (A4a) and (A4b). (A4a) is the time-time component and (A4b) is the space-space component of the second-order Einstein equation. Hence, for the first-order solution $\phi_1 = \psi_1 \approx C(H/a)$,

$$\overset{(2)}{\rho} = -\frac{15H^4}{a^2}\sum_{\boldsymbol{k}}|C|^2, \quad \overset{(2)}{p} = \frac{5H^4}{a^2}\sum_{\boldsymbol{k}}|C|^2, \tag{A13}$$

and this gives the equation of state $\overset{(2)}{p} = -\frac{1}{3}\overset{(2)}{\rho}, \overset{(2)}{\rho}.<0$. This corresponds to a positive spatial curvature, and is consistent with the analysis using the renormalization-group method.

#### 2. Short wavelength case

The first-order solutions are

$$\psi \approx -2C\dot{\chi}_0 \sin(k\eta), \quad \chi_1 \approx -C\frac{4k}{a}\cos(k\eta),$$

$$\left(\eta \equiv \int \frac{dt}{a}\right). \tag{A14}$$

The second-order equations are

$$6H\dot{\psi}_2 + 6H^2\phi_2 + \dot{\chi}_0\dot{\chi}_2 + V'\chi_2 \approx -8\frac{k^4}{a^4}\sum_{\boldsymbol{k}}|C|^2, \tag{A15a}$$

$$2\ddot{\psi}_2 + 6H\dot{\psi}_2 + 2H\dot{\phi}_2 - \dot{\chi}_0\dot{\chi}_2 + V'\chi_2 \approx \frac{8}{3}\frac{k^4}{a^4}\sum_{\boldsymbol{k}}|C|^2, \tag{A15b}$$

$$\ddot{\chi}_2 + 3H\dot{\chi}_2 + V''\chi_2 - \dot{\chi}_0(6H\phi_2 + \dot{\phi}_2 + 3\dot{\psi}_2) \approx 0. \tag{A15c}$$

The second-order solutions are

$$\psi_2 \approx -\frac{2}{3H^2a^4}\sum_{\boldsymbol{k}}k^4|C|^2, \quad \phi_2 \approx -\frac{4}{H^2a^4}\sum_{\boldsymbol{k}}k^4|C|^2,$$

$$\chi_2 \approx 0. \tag{A16}$$

The metric of the FRW universe is

$$\langle ds^2\rangle = -dT^2 + a^2(T)\left(1-\frac{4}{15H^2a^4}\sum_{\boldsymbol{k}}k^4|C|^2\right)d\boldsymbol{x}^2. \tag{A17}$$

This result is the same as in Eq. (44), and the effect of the back reaction is equivalent to radiation. The component of the effective energy momentum tensor is

$$\overset{(2)}{\rho} = \frac{8}{a^4}\sum_{\boldsymbol{k}}k^4|C|^2, \quad \overset{(2)}{p} = \frac{8}{3a^4}\sum_{\boldsymbol{k}}k^4|C|^2, \tag{A18}$$

and the equation of state is that of radiation $\overset{(2)}{p} = \frac{1}{3}\overset{(2)}{\rho}, \overset{(2)}{\rho} > 0$.


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