

Gauge dual and noncommutative extension of an $\mathcal{N}=2$ supergravity solution

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We investigate some properties of a recent supergravity solution of Pilch and Warner, which is dual to the $\mathcal{N}=4$ gauge theory softly broken to $\mathcal{N}=2$. We verify that a D3-brane probe has the expected moduli space and its effective action can be brought to $\mathcal{N}=2$ form. The kinetic term for the probe vanishes on an enhançon locus, as in earlier work on large- N $\mathcal{N}=2$ theories, though for the Pilch-Warner solution this locus is a line rather than a ring. On the gauge theory side we find that the probe metric can be obtained from a perturbative one-loop calculation; this principle may be useful in obtaining the supergravity dual at more general points in the $\mathcal{N}=2$ gauge theory moduli space. We then turn on a B -field, following earlier work on the $\mathcal{N}=4$ theory, to obtain the supergravity dual to the noncommutative $\mathcal{N}=2$ theory.

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I. INTRODUCTION

It is an important direction to extend the AdS conformal field theory (CFT) duality of Maldacena [1] to nonconformal systems with less supersymmetry. One way to do this is by perturbing the Hamiltonian, which is equivalent to perturbing the boundary conditions on the AdS space [2,3].

The understanding of the resulting solutions is still limited. One approach, beginning with Refs. [4,5], is to reduce to five-dimensional gauged supergravity. This has been very useful, but it has limitations. For one, only rather special states can be obtained in this way. The solutions have only a finite number of integration constants, whereas a gauge theory moduli space has of order N parameters. For a second, the full ten-dimensional geometry is in general quite complicated in the reduced directions. This is encoded in the five-dimensional geometry through the algebraic magic of consistent truncation, but it is necessary to lift the solution to ten dimensions to see its full structure. A related issue is that most solutions are singular. While there have been attempts to identify allowed singularities in a purely five-dimensional picture [6], the ten-dimensional structure is crucial for a full understanding.

For $\mathcal{N}=4$ broken to $\mathcal{N}=1$ or $\mathcal{N}=0$ by mass terms the full ten-dimensional geometries have recently been found [7]. Here there is the simplifying feature that the 3-brane charge dominates the dynamics, so the solution can be treated as a perturbation of the Coulomb branch (black 3-brane [8]) solution. However, this approximation was found to break down in some interesting regimes. In particular, it becomes

less useful for phases with many 5-branes.

For $\mathcal{N}=4$ broken to $\mathcal{N}=2$ by mass terms (the $\mathcal{N}=2^*$ theory) there is a moduli space. Pilch and Warner (PW) [9] have recently found the ten-dimensional supergravity solution on a one-parameter subspace of the moduli space. It is the purpose of this paper to analyze some of the physics of the PW solution. The PW theory has the same *massless* content, pure $\mathcal{N}=2$ gauge theory, as for D7-branes wrapped on K3; the latter was studied in Ref. [10]. In that case the naive supergravity solution had a naked singularity, which was resolved by an interesting stringy phenomenon. The constituent D7 branes were forced to lie on a ring of finite radius, the *enhançon*. This mechanism involved states becoming massless when the K3 on which the branes were wrapped became small, and as such it may be a much more general phenomenon. The PW solution has a feature resembling the *enhançon*, and we would like to make the connection more precise.

In Sec. II we study the PW supergravity background. We first discuss its symmetries, and also remark on a recent $\mathcal{N}=1$ ten-dimensional solution [11]. We then study a probe in the PW geometry. The basic constituents of the PW solution are the D3-branes, and so these are the natural probes to consider. We find that the probe potential vanishes on a two-dimensional plane in the transverse space, which is the correct moduli space, and that the low energy action for the probe can be put in the expected $\mathcal{N}=2$ form by an appropriate choice of coordinates; these are checks on the PW solution. In addition, we will determine the precise configuration of branes that the solution of PW represents. We find that it is a different part of moduli space than that studied in Ref. [10]—the branes lie on a line segment rather than in a ring.

In Sec. III we discuss the gauge theory side of the correspondence. We identify the $\mathcal{N}=2^*$ gauge theory vacuum corresponding to linear *enhançon* of the PW geometry and compute from the field theory perspective the moduli space

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metric of a D3 probe. The $\mathcal{N}=2^*$ supersymmetric gauge theory was solved by Donagi and Witten [12]. As we argue below, matching the supergravity probe computation is essentially perturbative in the gauge theory, so we will not really use the nonperturbative tools of Seiberg-Witten theory. The gravity and the gauge theory computations of the moduli space metric agree up to $1/N$ corrections. This provides another check on the proposed correspondence.

Because the gauge theory calculation is perturbative, it can be extended to any point on the moduli space. Thus the gauge side gives some information about the general supergravity solution. There is a further simplifying feature that the gauge theory is close to the $\mathcal{N}=4$ theory, in the sense that the masses from gauge symmetry breaking are large compared to the masses from explicit $\mathcal{N}=4$ breaking; this may allow more of the supergravity solution to be extracted. By using information from the gauge theory side it may be possible to find the supergravity solution at all points on moduli space.

In Sec. IV we find the noncommutative generalization of the Pilch-Warner solution, extending to PW solution the construction of Refs. [13,14]. Although the PW solution is much more complicated than $\text{AdS}_5 \times S^5$, the same strategy can be used to generate the solution. That is, take the T -dual on a T^2 , turn on a constant B -field on the T^2 , and T -dualize back. The resulting solution should be dual to the noncommutative $\mathcal{N}=2^*$ gauge theory. As a check, we find that the D3-probe moduli space is unaffected by the noncommutativity, a result which is expected from the gauge theory side.

II. THE SUPERGRAVITY SIDE

In this section will examine the physics of the PW background by using a D3-brane probe in order to elucidate its properties. The PW background is complicated: all the type IIB supergravity fields are nontrivial. An essentially new feature of their solution is that the ten-dimensional dilaton-axion field depends on the radial coordinate, and it also depends (perhaps surprisingly) on two angular transverse coordinates as well.

A. The PW solution and its symmetries

We begin by recalling the necessary pieces of the PW solution [9]. The ten-dimensional Einstein frame metric is

$$ds_E^2 = \frac{(cX_1X_2)^{1/4}}{\rho^3} \left\{ \frac{k^2\rho^6}{c^2-1} dx_{\parallel}^2 - \frac{L^2}{\rho^6(c^2-1)^2} dc^2 - L^2 \left[\frac{1}{c} d\theta^2 + \frac{\sin^2\theta}{X_2} d\phi^2 + \rho^6 \cos^2\theta \left(\frac{1}{cX_2} \sigma_3^2 + \frac{1}{X_1} (\sigma_1^2 + \sigma_2^2) \right) \right] \right\}; \quad (2.1)$$

the five-form field strength is

$$\tilde{F}_{(5)} = \mathcal{F} + \star \mathcal{F},$$

$$\mathcal{F} = 4 dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dw(r, \theta); \quad (2.2)$$

(the factor of 4 results from conventions, to be explained in Sec. IV A) and the dilaton-axion is

$$\tau = \frac{\tau_0 - \bar{\tau}_0 B}{1 - B}, B = e^{2i\phi} \frac{\sqrt{cX_1} - \sqrt{X_2}}{\sqrt{cX_1} + \sqrt{X_2}} \quad (2.3)$$

where $\tau_0 = \theta_s/2\pi + i/g_s$ is the asymptotic value. PW's abbreviations are¹

$$X_1 = \cos^2\theta + c\rho^6 \sin^2\theta, \quad X_2 = c \cos^2\theta + \rho^6 \sin^2\theta,$$

$$w(r, \theta) = \frac{k^4 \rho^6 X_1}{4g_s(c^2 - 1)^2}, \quad (2.4)$$

and

$$\rho^6 = c + (c^2 - 1) \left[\gamma + \frac{1}{2} \ln \left(\frac{c-1}{c+1} \right) \right]. \quad (2.5)$$

The σ_i are the differentials $\sigma_1 = \frac{1}{2}(\cos\alpha d\psi + \sin\alpha \sin\psi d\beta)$, $\sigma_2 = \frac{1}{2}(-\sin\alpha d\psi + \cos\alpha \sin\psi d\beta)$, $\sigma_3 = \frac{1}{2}(d\alpha + \cos\psi d\beta)$. That is, the angles α, β, ψ parametrize a 3-sphere, which we can also describe by an $SU(2)$ matrix g where

$$\sigma_i = \text{tr}(g^{-1} \tau_i dg). \quad (2.6)$$

The above set of coordinates may be unfamiliar, so for orientation purposes we note the behavior of various coordinates and functions of interest in the gauge theory UV where we get back the $\mathcal{N}=4$ symmetry. The AdS_5 metric goes as $-dr^2 + e^{2r/L} dx_{\parallel}^2 = -(L/\tilde{r})^2 d\tilde{r}^2 + (\tilde{r}/L)^2 dx_{\parallel}^2$, where $L = (4\pi g_s N)^{1/4} \alpha'^{1/2}$ is the radius of curvature of AdS_5 and $\tilde{r} = Le^{r/L}$ is the usual isotropic radial coordinate appearing in the 3-brane harmonic function. The asymptotic region $r \rightarrow \infty$, $\tilde{r} \rightarrow \infty$ matches the metric (2.1) as $c \rightarrow 1^+$ (so $\rho^6, X_1, X_2 \rightarrow 1$), with $\tilde{r} = Le^{r/L} = kL/\text{arccosh}(c)$.

The solution contains two parameters. The parameter k is proportional to the symmetry-breaking mass perturbation m . One way to see this is to note that k and x_{\parallel} appear in the background only in the combination kx_{\parallel} , while the gauge theory physics depends only on the combination mx_{\parallel} . More precisely, Eq. (63) of Ref. [7] shows that deviations from AdS become large at $\tilde{r} \sim mL^2$ (where r in that reference is \tilde{r} here), while the deviation here becomes large when $c-1 = O(1)$ or $\tilde{r} \sim kL$. Thus $k = mL$ times a constant of order 1.

¹Note that this equation for w corrects a typo in PW equation (4.9), and also extends it to general g_s .

The parameter γ defines a family of distinct solutions. In PW, the interpretation is given that $\gamma \ll 0$ corresponds to being on the $\mathcal{N}=4$ Coulomb branch while $\gamma > 0$ is unphysical. The solution $\gamma = 0$ appears to have an enhancement, and so we are most interested in this value but we keep γ general for now.

The $\mathcal{N}=4 \rightarrow \mathcal{N}=2$ gauge theory has an $SU(2) \times U(1)$ R-symmetry. The symmetry breaking gives equal masses to two of the four Weyl fermions λ_i . The $SU(2)$ acts on the two massless fermions, and the $U(1) = SO(2)$ mixes the two massive fermions. This symmetry is evident in the metric (2.1) as $g \rightarrow e^{i\zeta\tau_3}gh$ with $h \in SU(2)$. The six scalars transform as the combinations $\lambda_{[i}\lambda_{j]}$ [$\mathbf{6} = (\mathbf{4} \times \mathbf{4})_{\text{antisym}}$], and two of these are invariant under $SU(2) \times U(1)$. Thus the supergravity solution has a fixed plane where the radius of the transverse (squashed) two-sphere goes to zero. At long distance this is the equator $\theta = \pi/2$, but there is a second coordinate patch where $\rho = 0$. This fixed plane will play an important role.

Note that the $SU(2) \times U(1)$ does not act on the coordinate ϕ . Thus it is not surprising that the dilaton (2.3) has a complicated ϕ -dependence. Rather, what is surprising is that the ϕ -dependence of B is so simple, and even more surprising is that ϕ -translation is a Killing vector of the metric (2.1). This can be understood as follows. The $SL(2, \mathbf{Z})$ multiplies the fermion bilinears by a complex number; if we extend this to $SL(2, \mathbf{R})$, then there is a $U(1)$ subgroup which multiplies the bilinear by a phase. A combination of this $U(1)$ and a $U(1) \subset SO(6)$ leaves the mass perturbation invariant; call this combination $U(1)'$. Supergravity without branes is invariant under $SL(2, \mathbf{R})$. The boundary conditions are invariant under $U(1)'$, and so in fact is the full PW solution. Since the metric does not transform under $SL(2, \mathbf{R})$ it is invariant under ϕ -translation; the field B transforms by a phase under $U(1) \subset SL(2, \mathbf{R})$ and so by a phase under ϕ -translation.

The $\mathcal{N}=2$ PW solution implicitly contains D3-branes, as we will see, but these are invariant under $SL(2, \mathbf{R})$ and so do not affect the forgoing argument. Thus, the $U(1)'$ is an accidental symmetry of the gauge theory as long as we restrict attention to states and observables that only involve the supergravity fields and D3-branes on the supergravity side.

It is interesting to compare the more recent $\mathcal{N}=1$ solution of Pilch and Warner (PW2) [11]. Here one does expect 5-branes on the supergravity side [7]. Again there is a $U(1)'$ symmetry in the supergravity and in the boundary conditions, and some puzzling features of the $\mathcal{N}=1$ solution can be understood if this is a symmetry of the full solution. Namely, if there is a D5-brane as in Ref. [7], then the $U(1)'$ will carry this into a $(\cos \phi, \sin \phi)$ -brane at angle ϕ : the solution appears to contain a continuous distribution of such branes on an $S^2 \times S^1$. Now, $(\cos \phi, \sin \phi)$ -branes may sound unfamiliar, but supergravity does not know that (p, q) 5-brane charges are quantized, and so it admits such sources. Thus it might seem that the PW2 solution is illegitimate in string theory. However, it can be obtained as a limit of the multi-5-brane phases described in Ref. [7]. Namely,

for large enough n , $n(\cos \phi, \sin \phi)$ can be approximated by integers, and such 5-branes can be distributed around the ϕ direction.

B. Probing the solution

We have in general for a Dp -brane probe

$$S_{\text{probe}} = S_{\text{DBI}} + S_{\text{WZ}} \\ = -\mu_p \int d^{p+1}y e^{-\Phi} \sqrt{-\det(\mathbf{P}[G+B]_{ab} + 2\pi\alpha' F_{ab})} \\ + \mu_p \int \mathbf{P}[\exp(2\pi\alpha' F_{(2)} + B_{(2)}) \wedge \oplus_n C_{(n)}]. \quad (2.7)$$

where $\mu_p^{-1} = (2\pi)^p \alpha'^{(p+1)/2}$ and \mathbf{P} denotes pullback to the world volume of the bulk fields.

We take the directions parallel to the probe to be x_{\parallel} ; the transverse coordinates x^i in the conventions of PW are (r, θ, ϕ) plus the three coordinates on the (squashed) sphere generated by the $SU(2)$ R-symmetry of the $\mathcal{N}=2$ gauge theory. In the PW background, the components of the Neveu-Schwarz–Neveu-Schwarz (NS-NS) or Ramond-Ramond (R-R) two-form potentials parallel to the D3-brane probe and the world-volume field strength all vanish. It follows that the only remaining terms in the probe action come from the R-R four-form potential and from a combination of the metric and dilaton. The supergravity metric in [9] is given in Einstein frame. Conveniently, for the case of the D3-brane this is precisely the combination² $g_{\mu\nu} = e^{-\Phi/2} G_{\mu\nu}$ that appears in the probe action. The Chern-Simons terms in $F_{(5)}$ do not contribute to the longitudinal components, and so we can read the longitudinal components of $C_{(4)}$ directly from Eq. (2.2) with $\mathcal{F} = dC_{(4)\parallel}$.

The probe action in static gauge becomes

$$S_{\text{probe}} = \mu_3 \int d^4y [-g_s^{-1} \sqrt{-\det \mathbf{P}[g_{ab}] + \mathbf{P}[C_{(4)}]}] \\ = \frac{\mu_3}{g_s} \int d^4y [-\sqrt{-\det g_{ab}(1 - v^i v^j |g_{ij}/g_{00})}]^{1/2} \\ + w(r, \theta)]. \quad (2.8)$$

Inserting the PW solution, we find the potential energy density to be

$$V = \frac{\tau_3 k^4 \rho^6}{(c^2 - 1)^2} (\sqrt{cX_1 X_2 - X_1}), \quad (2.9)$$

with $\tau_3 = \mu_3/g_s$. Similarly, the kinetic energy density is

² G is the string metric.

$$\begin{aligned}
T = & \tau_3 \frac{k^2 L^2}{2} \sqrt{\frac{c X_1 X_2}{(c^2 - 1)}} \left(\frac{1}{\rho^6 (c^2 - 1)^2} (v^c)^2 \right. \\
& + \frac{1}{c} (v^\theta)^2 + \frac{\sin^2 \theta}{X_2} (v^\phi)^2 \\
& \left. + \rho^6 \cos^2 \theta \left[\frac{1}{c X_2} (v^3)^2 + \frac{1}{X_1} [(v^1)^2 + (v^2)^2] \right] \right), \quad (2.10)
\end{aligned}$$

where $v^{c,\theta,\phi,1,2,3}$ are the velocities of the probe in each of the six transverse directions.

For comparison to the enhancement physics of [10], we are interested in the moduli space, where the potential vanishes. There are two solutions to the condition $V=0$,

$$(I) \quad c X_2 = X_1 \Rightarrow \cos \theta = 0, \quad (II) \quad \rho^6 = 0. \quad (2.11)$$

Let us determine the dimensionality of these pieces of moduli space by inspecting the kinetic terms on loci I and II.

On locus I, the kinetic term is independent of γ ,

$$T_I = \tau_3 \frac{k^2 L^2}{2} \frac{c}{(c^2 - 1)} \left[\frac{1}{(c^2 - 1)^2} (v^c)^2 + (v^\phi)^2 \right] \quad (2.12)$$

and we see that the locus I is the (c, ϕ) plane. Note that the potential term (2.9) in the D3 probe Lagrangian has a particularly simple expansion about locus I,

$$V_I = 0 + \tau_3 \frac{k^4 \rho^6}{2(c^2 - 1)} \left(\theta - \frac{\pi}{2} \right)^2 + \dots \quad (2.13)$$

Locus II does not exist for $\gamma > 0$: the function ρ is positive on the entire range $1 < c < \infty$. For $\gamma < 0$ there is a unique value $c_0(\gamma)$ such that $\rho(c_0) = 0$ in Eq. (2.5), and this defines locus II. Locus II is then parameterized by (θ, ϕ) , and the moduli space metric there is

$$T_{II}(\gamma < 0) = \tau_3 \frac{k^2 L^2}{2} \frac{1}{(c_0^2 - 1)} [\cos^2 \theta (v^\theta)^2 + \sin^2 \theta (v^\phi)^2]. \quad (2.14)$$

As noted in PW, the dilaton-axion bulk field is trivial on locus II.

For $\gamma > 0$, there is only locus I, where $c \rightarrow 1^+$ is the AdS boundary and $c \rightarrow \infty$ is a singularity. For $\gamma < 0$, locus I is defined by $1 < c \leq c_0$, $\theta = \pi/2$; locus II is defined by $c = c_0$, $0 \leq \theta \leq \pi$. These fit together to form a plane if we identify $\theta \equiv \pi - \theta$ in locus II. In the limit $\gamma = 0$, $c_0 \rightarrow \infty$ and locus II becomes singular: the moduli space metric vanishes while the dilaton field blows up.

The moduli space is two-dimensional in accordance with expectation from $\mathcal{N}=2$ gauge theory. This is the same as the fixed plane of the $SU(2) \times U(1)$ R -symmetry, consistent with the fact that this symmetry is unbroken on the moduli space.

To identify the $\mathcal{N}=2$ structure in the moduli space metric we need also the gauge field action. Expanding the probe action in powers of the field strength, the coefficient of the kinetic term is $e^{-\Phi}$ and that of $F \wedge F$ is $C_{(0)}$, so that

$$\tau_{YM} = \tau_{\text{sugra}}. \quad (2.15)$$

In the natural coordinates on the $\mathcal{N}=2$ moduli space, the kinetic term for the transverse scalars is the imaginary part of τ_{YM} :

$$T(Y) = \frac{1}{2} \mu_3 e^{-\Phi} v^Y v^{\bar{Y}} \quad (2.16)$$

where Y is a complex coordinate encoding the two-dimensional moduli space.

We focus now on locus I. From Eq. (2.3), the dilaton is

$$e^{-\Phi} = \frac{c}{g_s |\cos \phi + i c \sin \phi|^2}. \quad (2.17)$$

In order to find the coordinate Y we first identify the obvious isotropic coordinate r' in the metric (2.12) via $dc/(c^2 - 1) = -dr'/r'$. Then in terms of

$$z = r' e^{-i\phi} = e^{-i\phi} \sqrt{(c+1)/(c-1)} \quad (2.18)$$

the locus I metric becomes

$$\tau_3 \frac{k^2 L^2}{2} \frac{c}{(c+1)^2} v^z v^{\bar{z}}. \quad (2.19)$$

Equating the metrics (2.16) and (2.19), Y is analytic in z with

$$\left| \frac{\partial Y}{\partial z} \right|^2 = k^2 L^2 \left| \frac{\cos \phi + i c \sin \phi}{c+1} \right|^2 = \frac{k^2 L^2}{4} \left| 1 - \frac{1}{z^2} \right|^2. \quad (2.20)$$

Thus

$$Y = \frac{kL}{2} (z + z^{-1}) \quad (2.21)$$

and

$$\tau = \frac{\tau_0 z^2 - \bar{\tau}_0}{z^2 - 1} = \frac{i}{g_s} \left(\frac{Y^2}{Y^2 - k^2 L^2} \right)^{1/2} + \frac{\theta_s}{2\pi}. \quad (2.22)$$

This is holomorphic, as expected from supersymmetry; note that B is simply z^{-2} .

Notice that this function has a branch cut emanating from $Y = \pm kL$. The real line segment $-kL \leq Y \leq kL$ maps to the circle $z=1$ and thence to $c=\infty$. Thus the branch cut is present only for $\gamma \geq 0$, and runs along the real axis where $c = \infty$. This form for τ is the main result that we will need in the next section. Note that $kL = \zeta m L^2$ for some constant ζ which we determine explicitly in the next section.

We would now like to know what kind of brane distribution would give rise to the function τ which we found above. We expect that the source D3-branes are distributed on the Coulomb branch, and we can infer their distribution in a number of ways. For example, the metric components g_{\parallel} vanish as one approaches a D3-brane distribution, provided that the branes are not spread in too many dimensions. In terms of an appropriate radial coordinate y , the metric behaves as $y^{(4-k)/2} dx_{\parallel}^2 + y^{-(4-k)/2} dy^2$. In the metric (2.1), g_{\parallel}

vanishes only near locus II, $\rho=0$. Since ρ vanishes linearly at c_0 , the metric behaves as $(c_0-c)^{1/2}dx_{\parallel}^2+(c_0-c)^{-3/2}dc^2$. For $y=(c_0-c)^{1/2}$ this is of the expected form with $k=2$, consistent with the two-dimensionality of locus II. Thus, when $\gamma<0$ the branes are spread over locus II. PW make the identification that $\gamma<0$ corresponds to the Coulomb branch of the $\mathcal{N}=4$ theory. This will be true for very negative γ ; for smaller values the effect of the soft breaking parametrized by m will be less negligible.

In the limit $\gamma=0$ this locus collapses to the line segment found above. Curiously, the metric $c^{-2}(dx_{\parallel}^2+dc^2)$ is of $k=0$ form with $y=1/c$, which is more singular than expected for a one-dimensional distribution. Evidently the effect of the perturbation on the D3-brane metric cannot be ignored in this case. Notice that we are at a different point in moduli space than the setup of [10], where the branes of the enhançon lay on a circle in the ‘‘natural’’ coordinates. If we start from $\gamma=0$ and turn on a slightly negative γ , the source brane distribution will turn from a line segment into a very squashed disk.

Finally, for $\gamma>0$ both g_{\parallel}^2 and the string metric G_{\parallel}^2 diverge at $c=\infty$, which appears to be unphysical [6,9].

The linear D3-brane distribution at $\gamma=0$ is reminiscent of the $\mathcal{N}=2^*$ limit of the $\mathcal{N}=1^*$ theory, where the 5-brane collapses into a line as one mass is taken to zero [7]. However, the length of the distribution here is $O(kL)=O(m\sqrt{g_s N\alpha'})$ in the isotropic coordinate, where the limit of a D5-brane has length $O(mN\alpha')$ and that of an NS5-brane has length $O(mg_s N\alpha')$. The latter are both larger, reflecting the fact that the 5-branes in the $\mathcal{N}=1^*$ theory are large compared with the radius at which the perturbation becomes large. Possibly the PW solution could be obtained as a limit of a configuration with a large number of 5-branes.

Let us make more precise the relation of the branch cut on the real axis to the enhançon of Ref. [10]. The enhançon is a distribution of D-branes on a curve where the gauge kinetic term $e^{-\Phi}$ vanishes.³ Any further contraction of the distribution would lead to a negative kinetic term. From Eq. (2.17), the limit $c\rightarrow\infty$, which again is a line segment in the natural coordinates, has vanishing kinetic term, and this is where the D3-branes are located when $\gamma=0$. The shape of the distribution is dependent on where one is on moduli space.

Let us also remark on magnetic Wilson lines. This is relevant to the enhançon physics in the following way. In the $d=2+1$ case of [10], the $\mathcal{N}=2$ gauge theory setup comes from D6-branes wrapped on a K3. A D0-brane probe of this system feels a force, but the more interesting aspect of its behavior is that the coefficient of its kinetic term goes to zero at the enhançon, and by duality one can see that it becomes the gauge boson of the enhanced $SU(2)$ symmetry. We would like to investigate the analog of this for the PW system. The most direct analogy is to consider a D-string par-

allel to the D3-branes; for a static configuration we need to hang a D-string in from infinity, and so the gauge theory dual is the magnetic Wilson line.

We wish to concentrate on that part of the D-string worldsheet parallel to the D3-brane. Starting with the action (2.2) for a general Dp -brane probe, we need to extract the terms which are turned on in the PW background. Since we are interested only in the kinetic piece of the action we can ignore all the Chern-Simons-type terms in S_{WZ} . Therefore, let us consider the DBI piece

$$S_{D1}=\tau_1\int d^2ye^{-\Phi}\sqrt{-\det\mathbf{P}(G_{ab}+B_{ab})}. \quad (2.23)$$

In static gauge where we allow only time dependence of the transverse coordinates, the pullback of the NS-NS B -field is zero, and so we need only the metric. In the Einstein frame, expanding to lowest order in velocities gives

$$T_{D1}=e^{-\Phi/2}v^iv^j|g_{ij}/g_{00}|\sqrt{-\det(g_{ab})}=e^{-\Phi/2}v^iv^jg_{ij}, \quad (2.24)$$

where in the last equality we used the fact that $g_{11}=-g_{00}$ in the PW coordinates. The effective mass, the coefficient of the invariant velocity, goes to zero when the dilaton blows up. This is the fact we were after to make the connection to enhançon physics.

Let us make a few remarks about the resolution of singularities by brane expansion. When all D3-branes are at the origin it appears that the supergravity solutions are singular there. When they are sufficiently spread out then the origin is like an ordinary point and it is possible to connect the non-normalizable perturbation from infinity with the normalizable solution at the origin. An enhançon distribution is one that is as compact as possible. For the $\mathcal{N}=1^*$ theory the same principle holds but there is no moduli space; the branes are expanded by the dielectric mechanism [15].

III. GAUGE THEORY

Via the AdS-CFT correspondence, the supergravity solution of [9] corresponds to softly broken $\mathcal{N}=4$, large N $SU(N)$ Yang-Mills theory at a specific point on the Coulomb branch of the $\mathcal{N}=2$ supersymmetric Yang-Mills theory with a massive adjoint hypermultiplet. In this section we discuss the gauge theory part of the correspondence. We identify the $\mathcal{N}=2$ gauge theory vacuum corresponding to linear enhançon of the PW geometry at $\gamma=0$ and compute from the field theory perspective the moduli space metric of a D3 probe. The supergravity calculation matches to a one loop gauge theory calculation, a result that is likely to be useful in understanding the supergravity solution at more general points in moduli space.

In the language of four-dimensional $\mathcal{N}=1$ supersymmetry, the mass deformed $\mathcal{N}=4$ $SU(N)$ Yang-Mills theory consists of a vector multiplet V , an adjoint chiral superfield Φ related by $\mathcal{N}=2$ supersymmetry to the gauge field, and two additional adjoint chiral multiplets Q and \bar{Q} which form the $\mathcal{N}=2$ hypermultiplet. In addition to the usual gauge-invariant kinetic terms for these fields, the theory has addi-

³In Ref. [10], the gauge potential term was proportional to $V-V_*$, where V was the volume of K3 and V_* the self-dual volume, so that $V=V_*$ defined the enhançon.

tional interactions and hypermultiplet mass term summarized in the superpotential⁴

$$W = 2 \sqrt{\frac{2}{g_{YM}^2}} \text{tr}([Q, \bar{Q}]\Phi) + \frac{m}{g_{YM}^2} (\text{tr} Q^2 + \text{tr} \bar{Q}^2). \quad (3.1)$$

The theory has a moduli space of Coulomb vacua parametrized by expectation values of the adjoint scalar

$$\Phi = \text{diag}(a_1, a_2, \dots, a_N), \quad \sum_i a_i = 0, \quad (3.2)$$

in the Cartan subalgebra of the gauge group. For generic values of the moduli a_i , the gauge symmetry is broken to that of the Cartan subalgebra $U(1)^{N-1}$, up to the permutation of individual $U(1)$ factors. At the semi-classical level, non-generic values of the moduli may yield a larger symmetry group. One of the fundamental results of the Seiberg-Witten theory [17]⁵ is that in the full quantum theory, such larger residual gauge symmetry groups do not survive quantization, so that the theory is always in the Coulomb phase. The entire low energy effective action \mathcal{L} of the $N-1$ Abelian $U(1)$ $\mathcal{N}=2$ vector multiplets is completely determined in terms of the single prepotential $\mathcal{F} \equiv \mathcal{F}(\tau, m; \{a_i\})$ which depends holomorphically on the microscopic parameters [the gauge coupling $\tau = \theta/2\pi + i(4\pi/g_{YM}^2)$ and the hypermultiplet mass m] and the Coulomb branch moduli $\{a_i\}$

$$8\pi\mathcal{L} = -g_{i\bar{j}}(D_\mu a^i D^\mu \bar{a}^{\bar{j}} + i\bar{\psi}\bar{\sigma}^\mu D_\mu \psi^j) + \text{Re} \left\{ \tau_{ij} \left(\frac{i}{2} F_{\mu\nu}^i F^{j\mu\nu} + \frac{1}{2} F_{\mu\nu}^i \tilde{F}^{j\mu\nu} - 2\bar{\lambda}^i \bar{\sigma}^\mu D_\mu \lambda^j \right) \right\} \quad (3.3)$$

with

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j}, \quad g_{i\bar{j}} = \text{Im}[\tau_{ij}]. \quad (3.4)$$

In Eq. (3.3) ψ 's and λ 's are fermionic superpartners of the scalars and gauge bosons respectively. The covariant derivative D_μ is taken with respect to the Levi-Civita connection Γ_{jk}^i of the scalar metric $g_{i\bar{j}}$

$$D_\mu a^i = \partial_\mu a^i + \Gamma_{jk}^i a^j \partial_\mu a^k. \quad (3.5)$$

Classically, the prepotential is given by

$$\mathcal{F}_{\text{class}} = \frac{1}{2} \tau \sum_i a_i^2. \quad (3.6)$$

The full quantum prepotential receives both perturbative and nonperturbative corrections

$$\mathcal{F} = \mathcal{F}_{\text{class}} + \mathcal{F}_{\text{pert}} + \mathcal{F}_{\text{non-pert}}. \quad (3.7)$$

⁴The classical Kähler potential is normalized $(2/g_{YM}^2) \text{tr}[\bar{\Phi}\Phi + \bar{Q}Q + \bar{\tilde{Q}}\tilde{Q}]$.

⁵See [18] for an introduction and extensive list of references.

The perturbative contribution is one-loop exact [16] and is determined by the standard quantum field theory computation

$$\mathcal{F}_{\text{pert}} = \frac{i}{8\pi} \left[\sum_{i \neq j} (a_i - a_j)^2 \ln \frac{(a_i - a_j)^2}{\mu^2} - \sum_{i \neq j} (a_i - a_j + m)^2 \ln \frac{(a_i - a_j + m)^2}{\mu^2} \right]. \quad (3.8)$$

From the Wilsonian effective action viewpoint it is generated by integrating out electrically charged gauge bosons and the charged components of the adjoint hypermultiplet. Finally, the nonperturbative prepotential is generated by instantons. The nonperturbative part of the prepotential can, in principle, be extracted from the exact solution of the theory [12]. In practice the computation is very difficult to carry out explicitly other than for gauge groups of small rank. Nonperturbative corrections become important in regions of moduli space with light BPS states. Semi-classically, monopoles are expected to have masses $4\pi v/g_{YM}^2$, where v is a characteristic scale of the Higgs field $v \sim |a_i - a_j|$. In the $N \rightarrow \infty$ limit we scale the gauge coupling $g_{YM}^2 \rightarrow 0$ while keeping the 't Hooft coupling fixed $N g_{YM}^2 \rightarrow O(1)$. So unless the spacing between eigenvalues of Φ is $O(1/N)$ or smaller, instanton corrections do not survive in this limit.

The moduli space of a D3-brane probing the PW supergravity background is dual to the projection of the Coulomb branch vacua of $SU(N+1) \rightarrow U(1) \times U(1)^{N-1}$ to that of the probe $U(1)$. If u is the modulus of the $U(1)$ representing the probe, the perturbative parametrization of the full moduli space (3.2) is given by

$$\Phi = \text{diag}(u, a_1 - u/N, a_2 - u/N, \dots, a_N - u/N),$$

$$\sum_i a_i = 0. \quad (3.9)$$

If $|u - a_i| \gg 1/N$, the instanton corrections to the metric on the probe moduli space are exponentially suppressed and the complete answer (in the large N limit) is determined by the perturbative prepotential. From Eqs. (3.6) and (3.8) we find

$$\tau(u) = \frac{i}{g_s} + \frac{\theta_s}{2\pi} + \frac{i}{2\pi} \sum_i \ln \frac{(u - a_i - u/N)^2}{(u - a_i - u/N)^2 - m^2}. \quad (3.10)$$

We would like to match Eq. (3.10) and the metric on the moduli space of the D3 probe (2.22) in the large N limit for a specific Coulomb vacuum $\{a_1, a_2, \dots, a_n\}$ of the “ $U(1)^{N-1}$ background.”

Recall that the D3 probe computation in the previous section suggests that $\mathcal{N}=2$ supergravity flow with $\gamma=0$ corresponds to the Coulomb branch vacuum in which the background branes form a \mathbf{Z}_2 symmetric linear enhançon singularity around the origin of the probe moduli space. The size of the enhançon in the variable $a = Y/2\pi\alpha'$ is

$$\frac{kL}{2\pi\alpha'} = \zeta \frac{mL^2}{2\pi\alpha'} = \zeta \frac{m\sqrt{g_s N}}{\sqrt{\pi}} \equiv a_0. \quad (3.11)$$

In particular, the characteristic scale of moduli space is large compared to m and so we can approximate

$$\tau(u) = \frac{i}{g_s} + \frac{\theta_s}{2\pi} + \frac{i}{2\pi} \sum_i \frac{m^2}{(u-a_i)^2}. \quad (3.12)$$

Away from the enhançon singularity nonperturbative corrections are suppressed and the probe metric is given by the continuous limit of this

$$\tau(u) = \frac{i}{g_s} + \frac{\theta_s}{2\pi} + \frac{i}{2\pi} \int_{-a_0}^{a_0} da \rho(a) \frac{m^2}{(u-a)^2} \quad (3.13)$$

where $\rho(a)$ is a linear density of the background branes eigenvalues normalized as

$$\int_{-a_0}^{a_0} da \rho(a) = N. \quad (3.14)$$

This is to be equal to the supergravity result (2.22),

$$\tau(u) = \frac{i}{g_s} \left(\frac{u^2}{u^2 - a_0^2} \right)^{1/2} + \frac{\theta_s}{2\pi}. \quad (3.15)$$

Equating the discontinuities across the enhançon branch cut gives

$$m^2 \rho'(u) = -\frac{2}{g_s} \frac{u}{\sqrt{a_0^2 - u^2}}, \quad \rho(\pm a_0) = 0. \quad (3.16)$$

This integrates to

$$\rho(u) = \frac{2}{m^2 g_s} \sqrt{a_0^2 - u^2}, \quad (3.17)$$

and the normalization condition fixes $a_0^2 = m^2 g_s N / \pi$ or $\zeta = 1$.

The one loop metric (3.12) should apply everywhere on moduli space—except of course when it goes negative in some region. Seiberg and Witten [17] showed that instanton corrections make the metric positive everywhere. The lesson of Ref. [10] is that at large N these corrections turn on sharply on the boundary where the metric changes sign, that is, the enhançon. Outside the enhançon the metric is perturbative. The effect of nonperturbative corrections is that the constituent D3-branes are expanded from their perturbative positions and dissolved in the enhançon.

Thus the gauge theory one loop calculation gives information about the supergravity solution anywhere on the $SU(N)$ moduli space. Essentially, it determines the dilaton and metric on a two-dimensional plane. Finding the solution on the full six-dimensional transverse space may then be possible, with some ingenuity. It may also be possible to extract information about the full solution from the gauge theory. Off the moduli space plane supersymmetry is broken and so the gauge theory less constrained. However, there is a simplification in the problem, which we have used in deriving Eq. (3.12). That is, the splitting of $\mathcal{N}=4$ multiplets by the mass term is small compared to the masses from gauge

symmetry breaking. This should restrict the renormalization of the perturbative effective action even off the plane where supersymmetry is unbroken.

One physical interest in studying exactly solvable $\mathcal{N}=2$ gauge theories is that upon deformation to $\mathcal{N}=1$ one hopes to get a new handle in the mystery of confinement. We have noted in Sec. II that the PW solution is not the $\mathcal{N}=2$ limit of the confining vacuum of Ref. [7]. It would be very interesting to find the exact supergravity flows corresponding to the linearized solutions of that paper. Constructing first the relevant $\mathcal{N}=2$ solution of the mass deformed $\mathcal{N}=4$ Yang-Mills theory [12] might be a way of approaching this problem.

IV. TURNING ON A CONSTANT B-FIELD

Recently there has been a revival of interest in quantum field theories formulated on noncommutative spaces, in particular those that emerge as various limits of M-theory compactifications. Gauge theories are especially interesting: the limit of large noncommutativity is similar to the large N limit of ordinary gauge theories. In the previous section we reconstructed the low-energy effective action on the one complex dimensional submanifold of the moduli space of the mass deformed $\mathcal{N}=4$ gauge theory from its supergravity dual. In this section we construct the deformation of the PW flow by turning on a B -field on the world-volume of the D3 branes. We propose that this deformation is the dual gravity description of the noncommutative $\mathcal{N}=2$ gauge theory with massive adjoint hypermultiplet.

After constructing the solution we consider the same observable as in the commutative case, namely the moduli space metric for a probe D3-brane. In fact this metric should be the same as in the commutative case. The classical supergravity description is dual to the large- N limit of the gauge theory, and planar graphs in the noncommutative theory differ from those of the commutative theory only by a phase factor, which is trivial for the two-derivative terms which define the moduli space metric.

The section is organized as follows. After fixing conventions, we review the gravity flow dual to the noncommutative $\mathcal{N}=4$ gauge theory constructed in [13,14]. In the third part we present the deformation of the PW flow and study the dynamics of a D3 probe in the deformed PW geometry.

A. Type IIB equations and conventions

We use mostly negative conventions for the signature $(+ - \dots -)$ and $\epsilon^{1 \dots 10} = +1$. The type IIB equations consist of the following [19].

The Einstein equations

$$R_{MN} = T_{MN}^{(1)} + T_{MN}^{(3)} + T_{MN}^{(5)}, \quad (4.1)$$

where the energy momentum tensors of the dilaton or axion field, B , the three index antisymmetric tensor field, $F_{(3)}$, and the self-dual five-index tensor field, $F_{(5)}$, are given by

$$T_{MN}^{(1)} = P_M P_N^* + P_N P_M^*, \quad (4.2)$$

$$T_{MN}^{(3)} = \frac{1}{8} \left(G^{PQ} G_M^* G_{PQN}^* + G^{*PQ} G_M G_{PQN} - \frac{1}{6} g_{MN} G^{PQR} G_{PQR}^* \right) \quad (4.3)$$

$$T_{MN}^{(5)} = \frac{1}{6} F^{PQRS} F_{PQRS}. \quad (4.4)$$

In the unitary gauge B is a complex scalar field and

$$P_M = f^2 \partial_M B, \quad Q_M = f^2 \text{Im}(B \partial_M B^*) \quad (4.5)$$

with

$$f = \frac{1}{(1 - BB^*)^{1/2}} \quad (4.6)$$

while the antisymmetric tensor field $G_{(3)}$ is given by

$$G_{(3)} = f(F_{(3)} - BF_{(3)}^*). \quad (4.7)$$

The Maxwell equations:

$$(\nabla^P - iQ^P)G_{MNP} = P^P G_{MNP}^* - \frac{2}{3} i F_{MNPQR} G^{PQR}. \quad (4.8)$$

The dilaton equation:

$$(\nabla^M - 2iQ^M)P_M = -\frac{1}{24} G^{PQR} G_{PQR}. \quad (4.9)$$

The self-dual equation:

$$F_{(5)} = \star F_{(5)}. \quad (4.10)$$

In addition, $F_{(3)}$ and $F_{(5)}$ satisfy Bianchi identities which follow from the definition of those field strengths in terms of their potentials:

$$\begin{aligned} F_{(3)} &= dA_{(2)} \\ F_{(5)} &= dA_{(4)} - \frac{1}{8} \text{Im}(A_{(2)} \wedge F_{(3)}^*). \end{aligned} \quad (4.11)$$

The above supergravity potentials do not transform simply under T -dualities. Let Φ be a dilaton, $B_{(2)}$ NSNS two-form and $C_{(n)}$ RR forms, as conventionally defined in D-brane physics. For the type IIB theory one would have $C_{(n)}$ with $n=0,2,4,6,8$. This range of n is consistent with the implicit summation over RR potentials in the D-brane world-volume action. Recall however, that these are not all independent, but rather appear in dual pairs. Following [7], we define the ‘‘modified’’ field strengths

$$\begin{aligned} \tilde{F}_{(1)} &= dC_{(0)} \\ \tilde{F}_{(3)} &= dC_{(2)} + C_{(0)} dB_{(2)} \\ \tilde{F}_{(5)} &= dC_{(4)} + C_{(2)} \wedge dB_{(2)} \\ \tilde{F}_{(7)} &= dC_{(6)} + C_{(4)} \wedge dB_{(2)} \\ \tilde{F}_{(9)} &= dC_{(8)} + C_{(6)} \wedge dB_{(2)}. \end{aligned} \quad (4.12)$$

The duality constraint is then implemented as

$$\star \tilde{F}_{(n+1)} = (-)^{n(n-1)/2} \tilde{F}_{(9-n)}. \quad (4.13)$$

Comparing the Einstein equations (4.1) with those of [7] we identify

$$\begin{aligned} C_{(0)} + i e^{-\Phi} &= i \frac{1+B}{1-B} \\ A_{(2)} &= C_{(2)} + i B_{(2)} \\ A_{(4)} &= \frac{1}{4} \left(C_{(4)} + \frac{1}{2} B_{(2)} \wedge C_{(2)} \right). \end{aligned} \quad (4.14)$$

Now we wish to recall the T -duality transformations of these supergravity fields. T -duality acts on the Neveu-Schwarz fields as [20]

$$\begin{aligned} \tilde{G}_{yy} &= \frac{1}{G_{yy}} & e^{2\tilde{\phi}} &= \frac{e^{2\phi}}{G_{yy}} \\ \tilde{G}_{\mu\nu} &= G_{\mu\nu} - \frac{G_{\mu y} G_{\nu y} - B_{\mu y} B_{\nu y}}{G_{yy}} & \tilde{G}_{\mu y} &= \frac{B_{\mu y}}{G_{yy}} \end{aligned} \quad (4.15)$$

$$\tilde{B}_{\mu\nu} = B_{\mu\nu} - \frac{B_{\mu y} G_{\nu y} - G_{\mu y} B_{\nu y}}{G_{yy}} \quad \tilde{B}_{\mu y} = \frac{G_{\mu y}}{G_{yy}}$$

where we defined the string metric by $(G_{\alpha\beta})_{\text{string}} = e^{\Phi/2} (g_{\alpha\beta})_{\text{Einstein}}$. In Eq. (4.15), y denotes the Killing coordinate with respect to which the T -dualization is applied, while μ, ν denote any coordinate directions other than y . If y is identified on a circle of radius R , i.e., $y \sim y + 2\pi R$, then after T -duality the radius becomes $\tilde{R} = \alpha'/R = l^2/R$. The string coupling is also shifted as $\tilde{g} = gl/R$.

T -duality transforms the type IIB theory into the type IIA theory and vice versa, through its action on the world-sheet spinors [21,22]. This aspect of T -duality is then apparent in the transformations of the RR fields. The odd-form potentials of the type IIA theory are traded for even-form potentials in the type IIB theory and vice versa. Using the conventions adopted above, the transformation rules for the RR potentials are [23,15]

$$\begin{aligned} \tilde{C}_{(n)\mu\cdots\nu\alpha y} &= C_{(n-1)\mu\cdots\nu\alpha} - (n-1) \frac{C_{(n-1)\mu\cdots\nu|y} G_{|\alpha|y}}{G_{yy}} \\ \tilde{C}_{(n)\mu\cdots\nu\alpha\beta} &= C_{(n+1)\mu\cdots\nu\alpha\beta y} + n C_{(n-1)[\mu\cdots\nu\alpha} B_{\beta]y} \\ &\quad + n(n-1) \frac{C_{(n-1)[\mu\cdots\nu|y} B_{|\alpha|y} G_{|\beta|y}}{G_{yy}}. \end{aligned} \quad (4.16)$$

B. Pure $\mathcal{N}=4$ flow

The type IIB supergravity background dual to noncommutative $\mathcal{N}=4$ supersymmetric Yang-Mills theory was constructed in [13,14]. We briefly review this analysis here.

Generalization of quantum commutative field theories to theories on noncommutative spaces involves adding infinitely many higher derivative terms, which renders the

theory nonlocal. It is thus very hard to provide a proof of the quantum consistency of such theory using the familiar renormalization tools of local field theories [24]. String theory provides a way to obtain noncommutative gauge theories by considering the decoupling limit of $D(p-2)$ -branes in type II string theories on T^2 with a background NSNS 2-form field $B_{\mu\nu}$ polarized along the T^2 [25,26]. The fact that noncommutative supersymmetric gauge theory is obtained in the decoupling limit of string theory suggests that it should be consistent at the quantum level. In the specific example of [25,26] the noncommutative gauge theory has 16 supercharges. In general, we would expect the quantum consistency of any gauge theory (even with less supersymmetry as in the example below), provided it can be realized in the limit of string theory where one decouples gravity.

Following [13], consider a large number of D3-branes in weakly coupled type IIB theory, oriented along the x^0, x^1, x^2, x^3 directions. Decoupling the stringy excitations by sending $\alpha' \rightarrow 0$ results in $\mathcal{N}=4$ supersymmetric Yang-Mills theory on the world-volume of the D3 branes. This theory has an $\text{AdS}_5 \times S^5$ supergravity dual, describing the near horizon geometry of the D3-branes [1]. In the string frame the metric and the dilaton is given by⁶

$$ds_s^2 = \frac{r^2}{L^2} \eta_{\mu\nu} dx^\mu dx^\nu - \frac{L^2}{r^2} dr^2 - ds_5^2$$

$$e^\Phi = g_s \quad (4.17)$$

where $L^4 = 4\pi g_s N \alpha'^2$. Consider compactifying the x^2, x^3 directions on the square torus T^2 . The system of D3-branes extending along x^0, x^1 and wrapping the T^2 is T -dual to D1-branes oriented along x^0, x^1 directions. The near horizon geometry of the string frame solution in the presence of D1-branes and their images coming from the T^2 compactification is given by the T -dual of Eq. (4.17) [8]

$$ds_s^2 = \frac{r^2}{L^2} [d(x^0)^2 - d(x^1)^2] - \frac{L^2}{r^2} [d(x^2)^2 + d(x^3)^2]$$

$$- \frac{L^2}{r^2} dr^2 - ds_5^2$$

$$e^\Phi = g_s \frac{L^2}{r^2}. \quad (4.18)$$

Let us turn on a constant $B_{(2)}$ -field polarized along T^2 . According to [25,26] we end up in the decoupling limit with the noncommutative $\mathcal{N}=4$ Yang-Mills theory. More specifically, it was argued in [26,27] that in order to get a finite noncommutative scale one should take

$$B_{(2)} \rightarrow \infty, \quad \alpha' \rightarrow 0 \quad (4.19)$$

while keeping $B_{(2)}\alpha'$ fixed. The constant $B_{(2)}$ -field does not act as a source for other supergravity fields, as $dB_{(2)}=0$, so Eq. (4.18) with the background NSNS two form

$$\delta B_{(2)} = -\frac{\Delta^2}{\alpha'} dx^2 \wedge dx^3 \quad (4.20)$$

is still a solution. T -duality on the T^2 produces finally the supergravity background dual to the noncommutative $\mathcal{N}=4$ Yang-Mills theory [13,14]:

$$ds_s^2 = \frac{r^2}{L^2} [d(x^0)^2 - d(x^1)^2] - \frac{r^2}{L^2 h} [d(x^2)^2 + d(x^3)^2]$$

$$- \frac{L^2}{r^2} dr^2 - ds_5^2$$

$$e^\Phi = g_s / h^{1/2}$$

$$\delta B_{(2)} = \frac{\Delta^2 r^4}{\alpha' L^4 h} dx^2 \wedge dx^3 \quad (4.21)$$

where

$$h = 1 + \frac{\Delta^4 r^4}{\alpha'^2 L^4}. \quad (4.22)$$

The solution (4.22) reduces to the $\text{AdS}_5 \times S^5$ solution for small r , which corresponds to the IR regime of the gauge theory. This is consistent with the field-theoretical expectations [28]: the commutative $\mathcal{N}=4$ gauge theory does not have UV divergences, so its noncommutative deformation does not change the IR physics (in any case UV-IR mixing would show up only in nonplanar effects).

C. Deformed PW flow

In constructing the gravity dual of the noncommutative $\mathcal{N}=2$ gauge theory we follow the strategy of [13], reviewed above. The starting point is the PW supergravity solution compactified on a square torus T^2 along x^2, x^3 directions. The bosonic background can be written schematically as

$$ds_E^2 = g_1^2 \eta_{\mu\nu} dx^\mu dx^\nu - g_2^2 [d(x^2)^2 + d(x^3)^2] - ds_6^2$$

$$A_{(2)} = c_2 + i b_2$$

$$F_{(5)} = d\chi_4 + \star d\chi_4,$$

$$\chi_4 = w dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \quad (4.23)$$

where ds_6^2 is the transverse metric and $A_{(2)}$ has nonvanishing components only along S^5 . The various functions and forms depend on the six transverse coordinates and are independent of x^0, \dots, x^3 . In this initial solution the functions g_1 and g_2 are equal. Note that the metric is given in Einstein frame.

T -duality transformations are most conveniently expressed in fields conventional in D-brane physics. Using Eq. (4.14) we find

⁶We discuss RR potentials in detail in a more general setting in the next section.

$$e^\Phi = \frac{(1-B)(1-\bar{B})}{1-B\bar{B}}, \quad C_{(0)} \equiv c = i \frac{B-\bar{B}}{(1-B)(1-\bar{B})}$$

$$B_{(2)} = b_2, \quad C_{(2)} = c_2$$

$$\tilde{F}_{(5)} = 4F_{(5)}. \quad (4.24)$$

We also define the string frame metric according to

$$ds_s^2 = G_1^2 \eta_{\mu\nu} dx^\mu dx^\nu - G_2^2 [d(x^2)^2 + d(x^3)^2] - dS_6^2 \quad (4.25)$$

$$G_1 = e^{\Phi/4} g_1, \quad G_2 = e^{\Phi/4} g_2, \quad dS_6^2 = e^{\Phi/2} dS_6^2. \quad (4.26)$$

Since 1-, 3-, and 5-form field strengths are nonzero, there will be nonvanishing 8-, 6-, and 4-form potentials as well

$$C_{(4)} = 4w dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 + \alpha_4$$

$$C_{(6)} = f_2 \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

$$C_{(8)} = p_4 \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \quad (4.27)$$

where the 2-form f_2 and 4-forms α_4 and p_4 have only transverse components, are independent of x^0, \dots, x^3 , and satisfy

$$\star dc = (dp_4 + f_2 \wedge db_2) \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

$$\star (dc_2 + cdb_2) = -(df_2 + 4wdb_2) \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

$$\star (dw \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3) = \frac{1}{4} (d\alpha_4 + c_2 \wedge db_2). \quad (4.28)$$

Equations (4.28) reflect the duality constraints (4.13).

Using the transformations rules (4.15) and (4.16), T -duality first along x^3 and then along x^2 produces the following configuration, denoted by tildes:

$$e^{2\tilde{\Phi}} = e^{2\Phi} / G_2^4$$

$$\tilde{G}_1 = G_1, \tilde{G}_2 = 1/G_2, d\tilde{S}_6^2 = dS_6^2$$

$$\tilde{B}_{(2)} = B_{(2)}$$

$$\tilde{C}_{(0)} = 0$$

$$\tilde{C}_{(2)} = c dx^3 \wedge dx^2 + 4w dx^0 \wedge dx^1$$

$$\tilde{C}_{(4)} = c_2 \wedge dx^3 \wedge dx^2 + f_2 \wedge dx^0 \wedge dx^1$$

$$\tilde{C}_{(6)} = \alpha_4 \wedge dx^3 \wedge dx^2 + p_4 \wedge dx^0 \wedge dx^1$$

$$\tilde{C}_{(8)} = 0. \quad (4.29)$$

It is straightforward to verify that given Eq. (4.28), the field strengths constructed from the R-R potentials of Eq. (4.29) satisfy the duality constraints (4.13).

As in the case of the supergravity flow corresponding to the $\mathcal{N}=4$ Yang-Mills theory, to generate a background dual to the noncommutative $\mathcal{N}=2$ gauge theory we now turn on a constant NS-NS 2-form potential on the T^2 :

$$\delta \tilde{B}_{(2)} = -\frac{\Delta^2}{\alpha'} dx^2 \wedge dx^3. \quad (4.30)$$

Again, since the corresponding field strength vanishes, $\delta \tilde{B}_{(2)}$ is a modulus.

After turning on $\tilde{B}_{(2)}$, T -duality along x^2 and then along x^3 directions, followed by the decompactification of T^2 produces the gravitational dual on the noncommutative $\mathcal{N}=2$ gauge theory with massive adjoint hypermultiplet. We denote this final configuration with primes:

$$e^{2\Phi'} = e^{2\Phi} / h, \quad h = 1 + \frac{\Delta^4 G_1^4}{\alpha'^2} \quad (4.31)$$

$$G_1' = G_1, \quad G_2' = G_1 / h^{1/2}, \quad dS_6'^2 = dS_6^2 \quad (4.32)$$

$$B_{(2)'} = b_2 + \frac{\Delta^2 G_1^4}{\alpha' h} dx^2 \wedge dx^3$$

$$C_{(0)'} = c$$

$$C_{(2)'} = c_2 + 4 \frac{\Delta^2 w}{\alpha'} dx^0 \wedge dx^1 - \frac{\Delta^2 G_1^4 c}{\alpha' h} dx^2 \wedge dx^3$$

$$C_{(4)'} = \frac{4w}{h} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 + \alpha_4 + \frac{\Delta^2}{\alpha'} f_2$$

$$\wedge dx^0 \wedge dx^1 - \frac{\Delta^2 G_1^4}{\alpha' h} c_2 \wedge dx^2 \wedge dx^3$$

$$C_{(6)'} = \frac{1}{h} f_2 \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 + \frac{\Delta^2}{\alpha'} p_4 \wedge dx^0$$

$$\wedge dx^1 - \frac{\Delta^2 G_1^4}{\alpha' h} \alpha_4 \wedge dx^2 \wedge dx^3$$

$$C_{(8)'} = \frac{1}{h} p_4 \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \quad (4.33)$$

where we used the fact that in the original metric $G_1 = G_2$. We have checked that the field strengths produced by the RR potentials of Eq. (4.33) satisfy the duality constraints (4.13).

In the remainder of this section we show that a D3 probe in the background (4.33) has the same moduli space as that in the PW geometry. Furthermore, the metric on this moduli space is the same, as expected.

For convenience, we reproduce the action of a D3 probe

$$S = -\mu_3 \int d^4 y e^{-\Phi} \sqrt{-\det(\mathbf{P}[G+B]_{ab} + 2\pi\alpha' F_{ab})} \\ + \mu_3 \int \mathbf{P}[\exp(2\pi\alpha' F_{(2)} + B_{(2)}) \wedge \oplus_n C_{(n)}], \quad (4.34)$$

where a, b denote directions parallel to the world volume of the probe. For a probe oriented along x^0, \dots, x^3 directions, the potential energy density in the background (4.33) is

$$V = \mu_3 e^{-\Phi'} \sqrt{-\det(\mathbf{P}[G' + B']_{ab})} \\ - \mu_3 \mathbf{P}[C_{(4)'} + C_{(2)'} \wedge B_{(2)'}]_{0123} \\ = \mu_3 (g_s^{-1} G_1^4 - 4w), \quad (4.35)$$

where to get the second line we used the transformation rules (4.33). This is identical to the potential for the original PW solution in Sec. II. Thus, the moduli space of a D3 probe in the gravity background dual to the noncommutative $\mathcal{N}=2$ gauge theory coincides with its commutative counterpart.

Further, the metric of this space is the same. Letting the probe coordinates x^i have a slow dependence on y^a , the relevant part of the probe action is

$$S = -\mu_3 \int d^4 y e^{-\Phi'} \sqrt{-\det(\mathcal{G}_{ab} + g'_{ij} \partial_a x^i \partial_b x^j)} \\ \xrightarrow{O(\partial^2)} -\frac{\mu_3}{2} \int d^4 y e^{-\Phi'} \sqrt{-\det(\mathcal{G}_{ab})} \mathcal{G}^{ab} g'_{ij} \partial_a x^i \partial_b x^j \quad (4.36)$$

where

$$\mathcal{G}_{ab} = \mathbf{P}[G' + B']_{ab}. \quad (4.37)$$

Using the properties

$$e^{-\Phi'} \sqrt{-\det(\mathcal{G}_{ab})} \mathcal{G}^{(ab)} = e^{-\Phi} \sqrt{-\det(G_{ab})} G^{ab}, \quad g'_{ij} = g_{ij} \quad (4.38)$$

of the solution (4.33), it follows that the metric on moduli space is the same as in the commutative case. We have explained earlier why this should be true from the gauge theory point of view. Note that in the supergravity description this result is obvious in the T -dual tilted picture, where a probe D1-brane does not couple to the transverse $\delta B_{(2)}$.

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