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Qualitative dynamical properties of a spatially closed FRW universe conformally coupled to a scalar field

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In this work the dynamics of a spatially closed Friedmann-Robertson-Walker model for the universe conformally coupled to a scalar field is studied. It is proven that this dynamics, formulated in terms of the proper time t (or cosmic time) is very simple. For arbitrary initial conditions, we prove that the universe will ultimately collapse in a finite time. We also show that there is no inflation at any stage of the evolution. When represented in phase space all trajectories of the system are unbounded, owing to a divergence that appears at the singularity, and showing that there is no chaos.

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I. INTRODUCTION

The dynamics of a spatially closed[1] Friedmann-Robertson-Walker (FRW) universe conformally coupled to a scalar field has been studied in many papers. Several authors have reached the conclusion that this dynamics is chaotic ([2–4]). On the other hand, it is well known that time is not well defined in general relativity, in the sense that time can be considered as a coordinate. So we have many possible times and it is also known that chaotic behavior depends on time [5]. This fact reflects a great difference between the usual study of chaos in classical mechanical systems and the one in general relativity. In the former case we have a unique classical time and no reason, whatsoever, to change it. In the latter case our conclusions depend on the time we choose and therefore, if we make a bad choice, the results may have doubtful physical relevance.

Essentially, we have two candidate times: conformal time η and proper time t, related by the equation $d\eta = dt/a$, where a is the radius (or scale) of the universe. As the equations with conformal coupling are simpler using η, most of the study has been made with this time. This is the case of the quoted papers [2–4]. But real physical time is proper time t, since this is the one measured by atomic and geodesic clocks[2] [6]. Moreover, in the classical limit, proper time is the one that becomes the classical time of classical dynamical systems.[3] Thus, if we must choose a physical time to study dynamical evolution of cosmological models, it must be proper time t.

So it is quite necessary to restudy the conclusions of the papers [2–4] where chaos was found.[4] It turns out that in using t, chaos disappears and the dynamics is very simple. This will be the main conclusion of the paper. The idea is the following: Suppose that we want to study the asymptotic behavior of a dynamical system, defined in terms of the natural proper time t, with a direct physical meaning. It is possible in this case to introduce a new time t' defined by the relation $dt' = f(t)dt$, where $f(t) \epsilon C^1$. If $f(t)$ is a nonvanishing function of t, then the conclusions about the behavior of the system would be the same if we use t or t'. But if this condition is not satisfied, the new dynamical system obtained after the change of time is not topologically equivalent to the original one. If the scale factors change sign or vanish, as in the case of closed cosmological systems, we cannot use conformal time η to study the asymptotic behavior of the model because the results will be different. Therefore, in this work, we only use proper time t to analyze the dynamics of the system.

As in paper [1], we do not perform the numerical integration of the equations of motion, the perturbative calculations, or the search of exact closed form solutions. We only study the qualitative dynamical behavior in a rigorous way.

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[1]For the case $k=0$, see [1].

[2]Geodesic time is the time measured by the oscillation of a light pulse bouncing between two parallel curves. Therefore, it is the simplest and more natural time that can be defined in general relativity.

[3]I.e., when velocity is much smaller than light velocity. In the case of conformal time we would need this condition plus $\dot{a} \simeq 0$ to obtain the same limit, and the second condition is not fulfilled in many cases of interest.

[4]Some criticism is already anticipated in [7].

II. THE MODEL

In this work we study a simple spatially closed FRW model conformally coupled to a scalar field. The metric equation is given by

$$ds^2=dt^2-a^2(t)\left(\frac{dr^2}{1-kr^2}+r^2d\Omega^2\right), \tag{1}$$

where t is the proper time (or cosmic time) and $k=1$. The Lagrangian density of the system is

$$\mathcal{L}=L_G+\mathcal{L}_M, \tag{2}$$

where $\mathcal{L}_G=-\frac{1}{12}R$ is the gravitational Lagrangian density, and

$$\mathcal{L}_M=-\frac{1}{2}\partial_\mu\psi\partial^\mu\psi+\frac{1}{12}R\psi^2+\frac{m^2}{2}\psi^2 \tag{3}$$

is the matter Lagrangian density. Here R is the Ricci scalar, related to the scale factor through

$$\frac{R}{6}=\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2+\frac{1}{a^2}. \tag{4}$$

We have chosen units where $4\pi G/3=c=1$ and the overdot symbolize the t derivative. In terms of cosmic time t the field equations read

$$\ddot{\psi}+3\frac{\dot{a}}{a}\dot{\psi}+\frac{R}{6}\psi+m^2\psi=0,$$

$$\ddot{a}+\frac{\dot{a}^2}{a}+\frac{1}{a}-m^2a\psi^2=0. \tag{5}$$

The total energy of the system is

$$\mathcal{H}=\left(\frac{\dot{a}}{a}\right)^2+\frac{1}{a^2}-\left(\dot{\psi}+\frac{\dot{a}}{a}\psi\right)^2-\frac{\psi^2}{a^2}-m^2\psi^2. \tag{6}$$

This expression is the first integral of Eq. (5). At this point we must recall that, because we have relinquished the gauge freedom in writing the metric equation (1) (one of Einstein's equations), the Hamiltonian constraint $\mathcal{H}=0$ is missing. Anyhow, we can reintroduce this constraint as a restriction on the set of allowable initial conditions. Namely, we will only consider initial conditions with zero total energy, i.e., $\mathcal{H}=0$ [8].

It is convenient to recast the field equations in terms of the rescaled variable $\phi=a\psi$. In this way, Eqs. (5) and (6) become

$$a^2\ddot{\phi}+a\dot{a}\dot{\phi}+(1+m^2a^2)\phi=0, \tag{7}$$

$$a\ddot{a}+\dot{a}^2-m^2\phi^2+1=0, \tag{8}$$

$$a^2\dot{\phi}^2+\phi^2+m^2a^2\phi^2-(1+\dot{a}^2)a^2=0. \tag{9}$$

This set of equations is invariant under the transformation

$$m\to\alpha m,\quad \phi\to\frac{1}{\alpha}\phi,\quad a\to\frac{1}{\alpha}a,\quad t\to\frac{1}{\alpha}t, \tag{10}$$

where α is an arbitrary parameter. Therefore, without loss of generality, we can take $m=1$. The only excluded case is $m=0$, but it is a trivial case where the coupling between the geometry and the rescaled field disappears.

This model has been analyzed by several authors [2–4], but always employing conformal time η. These authors have found a chaotic behavior of the dynamics of the model.

III. THE IMPOSSIBILITY OF INFLATION

Equations (7) and (8) can be written as a first order system in terms of phase space variables a, $\dot{a}$, ϕ, and $\dot{\phi}$. It is easy to verify that the resulting first order system has no equilibrium points.

Let us now begin our qualitative analysis of the dynamics of the system. First, we will prove that inflationary behavior is not possible at any stage of the evolution. From Eqs. (8) and (9) we obtain

$$a\ddot{a}=-\frac{1}{a^2}(a^2\dot{\phi}^2+\phi^2)<0. \tag{11}$$

We will consider only the physical region where $a>0$. Then, from the last equation we obtain

$$\ddot{a}<0. \tag{12}$$

Therefore, an inflationary behavior, characterized by condition $\ddot{a}>0$, is not possible at any stage of the evolution.

IV. EVOLUTION OF THE SCALE FACTOR

From conditions $a\geqslant 0$ and $\ddot{a}<0$, we have only three possible behaviors for the scale factor a:

(i) $a>0$, $\ddot{a}<0$, $\dot{a}>0$, $\lim_{t\to+\infty}a(t)=+\infty$,

(ii) $a>0$, $\ddot{a}<0$, $\dot{a}>0$, $\lim_{t\to+\infty}a(t)=a_0$, where a_0 is a finite constant, and

(iii) $\lim_{t\to+t_1}a(t)=0$, where t_1 is a certain finite time.

A. Case (i)

We will show that case (i), which corresponds to a monotonous expansion, is impossible. Let us define, for case (i) the function $U=\frac{1}{2}(\dot{\phi}^2+(1+1/a^2)\phi^2)$. From Eq. (7) we obtain $\dot{U}=-(\dot{a}/a)(\dot{\phi}^2+\phi^2/a^2)$. Since we consider that $\dot{a}>0$, we have $\dot{U}\leqslant 0$. Therefore, U is a non-negative decreasing function of t. Then we have $\lim_{t\to+\infty}U=\alpha$, where α is a non-negative constant. First suppose that α is positive. Then, from the expression of U we deduce that ϕ and $\dot{\phi}$ are finite when $t\to+\infty$. Dividing Eq. (9) by a^2 we deduce that $\dot{a}$ is also finite when $t\to+\infty$. More precisely, from the discussion above we obtain $\dot{a}^2(t)\simeq 2\alpha-1$ for $t\to+\infty$, and then $\ddot{a}(t)\simeq 0$ in this limit. From these results, and dividing Eq. (7) by a^2, we obtain $\ddot{\phi}+\phi\simeq 0$ for $t\simeq+\infty$.

Then we have

$$\phi(t) \simeq A \cos t + B \sin t,$$

$$\dot{\phi}(t) \simeq -A \sin t + B \cos t$$

for $t \simeq +\infty$, where A, B are constant parameters with $A^2 + B^2 = 2\alpha$.

Replacing the above results in Eq. (8), we see that this equation cannot be satisfied for $t \simeq +\infty$. Then, the case $\alpha > 0$ is not possible. Let us now consider the case $\alpha = 0$.

From the definition of U, we deduce that $\lim_{t \to +\infty} \phi(t) = \lim_{t \to +\infty} \dot{\phi}(t) = 0$. If we introduce the positive variable $u = a^2$, from Eq. (8), we obtain $\ddot{u} = -2 + 2\dot{\phi}^2$ (recall that we have taken $m = 1$). For $t \simeq \infty$, we have $\ddot{u} \simeq -2$ because $\dot{\phi}(t) \simeq 0$. Integrating this equality in the asymptotic region and conserving only the dominant behavior we obtain $u(t) \simeq -t^2$, which is in contradiction with the positive character of $u(t)$. Therefore, we have proved that case (i) is impossible.

B. Case (ii)

It is easy to show that case (ii) is also not possible. As $\lim_{t \to +\infty} a(t) = a_0$, we show that $\lim_{t \to +\infty} \dot{a}(t) = 0$, $\lim_{t \to +\infty} \ddot{a}(t) = 0$. From Eq. (8) we obtain $\lim_{t \to +\infty} \dot{\phi}^2 = +1$, and then $\lim_{t \to +\infty} \dot{\phi}(t) = 0, \lim_{t \to +\infty} \ddot{\phi}(t) = 0$. If we replace these results in Eq. (7), we see that this equation is not satisfied.

C. Case (iii)

Let us consider now case (iii). Since we have put aside cases (i) and (ii), the only possibility that is compatible with conditions $a \geqslant 0$ and $\ddot{a} < 0$ is case (iii), i.e., a collapse in a certain finite time t_1. We will now show that there is a singularity at t_1, with $\lim_{t \to t_1} \dot{a}(t) = -\infty$. Suppose that $\lim_{t \to t_1} \dot{a}(t) = \kappa$, where κ is a finite constant. From Eq. (9) we obtain

$$\lim_{t \to t_1} \phi^2(t) = \lim_{t \to t_1} a^2(t) \dot{\phi}^2 = 0. \tag{13}$$

Then, for $t \simeq t_1$, $t < t_1$we have

$$\ddot{u} \simeq -2, \quad \dot{u} \simeq 0, \quad u \simeq 0, \tag{14}$$

where $u = a^2$, as defined above.

From these conditions we obtain $u(t) \simeq -(t_1 - t)^2$ for $t < t_1$and $t \simeq t_1$. This is in contradiction with the positive character of u. Therefore, we have proved that $\lim_{t \to t_1} \dot{a}(t) = -\infty$. If we represent the trajectories of the system in phase space a, $\dot{a}$, ϕ, and $\dot{\phi}$, we conclude that all trajectories are unbounded so there is no chaos. We say that a dynamical system has a chaotic behavior if it has a positive Kolmogorov-Sinai entropy, i.e., exponential separation of trajectories, and a bounded phase-space so that folding takes place. Chaos is related to the asymptotic complex behavior of solutions that remain in a bounded subset of phase space. Obviously, we consider only trajectories that originated from physical initial conditions with $a > 0$.

V. BEHAVIOR OF THE HUBBLE FUNCTION IN THE NEIGHBORHOOD OF THE SINGULARITY

We will now analyze the limit value of $a\dot{a}$ when $t \to t_1$. We will show that this limit is given by a finite, nonzero constant value. Let us consider the function $F = \frac{1}{2}(a^2 + a^2 \dot{a}^2)$. From Eq. (8) we obtain $\dot{F} = a\dot{a}\dot{\phi}^2$. For $t \simeq t_1$, we have $\dot{a} < 0$. Hence, F is a positive, always decreasing function for $t \simeq t_1$. Therefore, $\lim_{t \to t_1} F < \infty$, and then $\lim_{t \to t_1} a\dot{a} < \infty$.

We will now that this limit is nonzero. In fact, if this limit were be zero, we would deduce from Eq. (9) that $\lim_{t \to t_1} \phi = 0$. Then, as before, we have

$$\ddot{u} \simeq -2, \quad \dot{u} \simeq 0, \quad u \simeq 0, \tag{15}$$

for $t \simeq t_1$, $t < t_1$.

We have seen above that these conditions are incompatible with $u > 0$. Therefore, we have proved that

$$\lim_{t \to t_1} a\dot{a} = C, \tag{16}$$

where C is a nonzero, negative finite constant.

Since the Hubble function is $H = \dot{a}/a$, using Eq. (16) we obtain

$$\lim_{t \to t_1} H = -\infty.$$

VI. CONCLUSIONS

Summing up, we have obtained the following results: (a) for arbitrary initial conditions that satisfy Eq. (9) and the condition $a > 0$, the universe will ultimately collapse in a finite time; (b) all trajectories in phase space a, $\dot{a}$, ϕ, and $\dot{\phi}$ are unbounded, owing to the divergence of $\dot{a}$ at the singularity; (c) $\lim_{t \to t_1} H = -\infty$; (d) $\ddot{a} < 0$, i.e., inflationary behavior is not possible at any stage of the evolution of the model; and (e) from (a) and (b), it follows that there is no chaos in the model.

Even if the dynamical evolution of the model is controlled by highly nonlinear equations, we have been able to obtain, in a rigorous way, the most relevant dynamical properties of the model by using only qualitative arguments employed in the theory of dynamical systems.

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