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Giant gravitons, Bogomol'nyi-Prasad-Sommerfield bounds, and noncommutativity

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It has been recently suggested that gravitons moving in $AdS_m \times S^n$ spacetimes along the S^n blow up into spherical $(n-2)$-branes whose radius increases with increasing angular momentum. This leads to an upper bound on the angular momentum, thus ''explaining'' the stringy exclusion principle. We show that this bound is present only for states which saturate a BPS-like condition involving the energy E and angular momentum $J, E \geqslant J/R$, where R is the radius of S^n. Restriction of motion to such states lead to a noncommutativity of the coordinates on S^n. As an example of motions which do not obey the exclusion principle bound, we show that there are finite action instanton configurations interpolating between two possible BPS states. We suggest that this is consistent with the proposal that there is an effective description in terms of supergravity defined on noncommutative spaces and noncommutativity arises here because of imposing supersymmetry.

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I. INTRODUCTION AND SUMMARY

One of the striking consequences of the holographic correspondence in $AdS_m \times S^n$ spacetimes [1] is the stringy exclusion principle [2]. It has been also argued that the stringy exclusion principle means that dual supergravity should reside on a noncommutative spacetime, e.g., quantum deformations of AdS×S [3]. The principle states that the maximum angular momentum of single particle Bogomol'nyi-Prasad-Sommerfield (BPS) states (in spacetime) is bounded by N, where N is the flux of the n-form magnetic field strength on the sphere. Recently, McGreevy, Susskind, and Toumbas [4] have provided a novel explanation of this phenomenon. According to their proposal, a single trace operator of the holographic theory is well described as single-particle supergravity modes (we call generically call them gravitons) for low values of the angular momenta. However, for large angular momenta, these blow up into spherical $(n-2)$-branes moving on the S^n. In some cases this blowup follows qualitatively from the Myers effect [5].[1] It is then demonstrated that the radius of these spherical branes grows with angular momentum for a restricted class of motion where the radius of the brane does not change with time and in addition there are no waves along the brane. Since the radius of the brane cannot exceed the radius of the sphere on which it moves, there is a bound on the angular momentum.

As we will see, it is important to emphasize that for consistency there should be only one BPS state for a given angular momentum even though it might appear that there is both a graviton and a ''giant graviton'' or a wrapped brane. The point is that the former is a valid description for small angular momenta, while the latter is the correct description for large angular momenta.

A natural question arises immediately: What about non-BPS states? There are of course a large number of non-BPS states in the gauge theory which are still represented by single-trace operators and should therefore correspond to single-particle or single-brane states in the dual description. In this paper we show that the Hamiltonian for brane motions on spheres leads to a BPS-type bound; viz., there is a lower bound on the energy for a given angular momentum. The restricted type of motion considered in [4] which saturate this bound. If we are working in a supersymmetric theory, it is natural to expect that these are also configurations which preserve some of the supersymmetries. Thus non-BPS states correspond to other motions, e.g., oscillations and changes of the radius. From this bound it is straightforward to see that the radius of the brane increases with the angular momentum for $n>3$. For such motions the potential has two minima—one at zero radius and the other at a radius which scales as $J^{1/(n-3)}$. We show that there are finite action instanton configurations interpolating between the two minima. The implication of this is, however, less clear since the description of the system as a brane with some Dirac-Born-Infeld (DBI) action fails when the size of the brane is small.

Since the dynamics of D-branes is defined on a commutative configuration space, one might wonder whether noncommutativity is visible in the ''giant graviton'' scenario. We show that this is in fact true. More precisely, we show that the restriction of the motion of the branes to those which saturate the BPS bound and which can be implemented in terms of supersymmetry generators implies a set of second-class constraints in the phase space. This implies that the Dirac brackets of two coordinates on the S^n transverse to the $(n-2)$-brane are nonzero, which should imply that in the

[1]This connection has been explored in [6].

quantum problem these operators do not commute. We show that the Dirac brackets in fact become singular at $n=3$, which is consistent with the fact that in this case all such states have the same (maximal) angular momentum. One could arrive at the same conclusion by considering directly the quantum commutators of the coordinate operators projected on to the subspace of states implied by the restriction on brane motion. Thus in reduced phase space, it appears that two space directions do not commute. This is quite similar to the problem of a charge particle moving on a two-plane in the presence of a constant magnetic field. In that case, restriction to the lowest Landau level implies that the two spacelike coordinates do not commute. Connections of the bound on the angular momenta of giant gravitons with noncommutativity have been heuristically discussed earlier in [7]. However, the precise origin of noncommutativity is rather unclear in this treatment.

The results of this paper were reported in Ref. [8]. While this paper was under preparation, Refs. [9] and [10] appeared on the internet which have some overlap with Secs. II and III of our work. In Sec. IV we use some results of [9] and [10] to describe the origin of noncommutative space in the theory.

II. BRANE MOTION ON SPHERES

We consider spacetimes of the form of $\mathrm{AdS}_m \times \mathrm{S}^n$ where $m+n=10$ in string theory and $m+n=11$ in M theory. The radius of S^n is R, which is also the scale of the AdS spacetime and there is a constant n-form flux on S^n with N quanta of flux. Let us consider the sphere S^n embedded in R^{n+1} with coordinates $X^1 \cdots X^{n+1}$,

$$(X^1)^2 + \cdots + (X^{n+1})^2 = R^2, \tag{2.1}$$

and we choose coordinates on the sphere as follows:

$$X^1 = \sqrt{R^2 - r^2}\cos\phi,$$
$$X^2 = \sqrt{R^2 - r^2}\sin\phi, \tag{2.2}$$

where $0 \leq r \leq R$. The remaining $X^3 \cdots X^{n+1}$ are chosen to satisfy

$$(X^3)^2 + \cdots + (X^{n+1})^2 = r^2. \tag{2.3}$$

These may be written in terms of $p=(n-2)$ angles $\theta_1 \cdots \theta_p$ and r in the form of standard spherical polar coordinates in $p+1$ dimensions. Then the metric on S^n becomes

$$ds^2 = \frac{R^2}{R^2 - r^2} dr^2 + (R^2 - r^2) d\phi^2 + r^2 d\Omega_p^2, \tag{2.4}$$

where $d\Omega_p^2$ is the volume element on a unit p-sphere.

We consider p-branes, with $p=n-2$, wrapped on the S^p embedded in S^n, moving entirely in the S^n and sitting at the center of global coordinates in AdS_m spacetime. The time coordinate in AdS is denoted by t. In the $(p+1)$-dimensional world volume of the brane with coordinates $\tau, \sigma_1, \ldots, \sigma_p$, we choose a static gauge

$$\tau = t, \quad \sigma_i = \theta_i \quad (i = 1, \ldots, p). \tag{2.5}$$

The dynamical coordinates are now $r(t,\theta_i)$ and $\phi(t,\theta_i)$.

We will look at motions of the brane where there are no oscillations; i.e., r, ϕ are independent of the angles θ_i. Then the brane Lagrangian is given by

$$L = -\lambda[r^p(1 - g_{rr}(r)\dot{r}^2 - g_{\phi\phi}\dot{\phi}^2)^{1/2} - r^{p+1}\dot{\phi}^{1/2}], \tag{2.6}$$

where

$$\lambda = \frac{N}{R^{p+1}},$$
$$g_{rr}(r) = \frac{R^2}{R^2 - r^2},$$
$$g_{\phi\phi}(r) = R^2 - r^2. \tag{2.7}$$

The first term is the DBI term. The coefficient is a rewriting of the tension of the brane in terms of N and R. This follows from the corresponding classical supergravity solution. It is crucial in what follows that we have exactly the same coefficient in the second term—the Chern-Simons (CS) term. This is the coupling of the brane with the n-form field strength and the precise coefficient follows from standard flux quantization.

In the following we will set $R=1$ so that all dimensional quantities are in units of R. We will restore R at the very end.

The canonical momenta for r and ϕ are p_r and p_ϕ, respectively, and are given by

$$p_r \equiv \lambda P = \frac{\lambda r^p g_{rr}(r)\dot{r}}{(1 - g_{rr}(r)\dot{r}^2 - g_{\phi\phi}\dot{\phi}^2)^{1/2}},$$
$$p_\phi \equiv \lambda j = \frac{\lambda r^p g_{\phi\phi}(r)\dot{\phi}}{(1 - g_{rr}(r)\dot{r}^2 - g_{\phi\phi}\dot{\phi}^2)^{1/2}} + \lambda r^{p+1}. \tag{2.8}$$

The momentum p_ϕ is an angular momentum and is conserved. p_r is not conserved. From Eqs. (2.8) one gets

$$(1 - g_{rr}(r)\dot{r}^2 - g_{\phi\phi}\dot{\phi}^2)^{1/2} = r^p \left[r^{2p} + \frac{P^2}{g_{rr}(r)} + \frac{(j - r^{p+1})^2}{g_{\phi\phi}(r)} \right]^{-1/2}. \tag{2.9}$$

The canonical Hamiltonian can be now derived in a standard fashion and becomes

$$H = \lambda \left[r^{2p} + \frac{P^2}{g_{rr}(r)} + \frac{(j - r^{p+1})^2}{g_{\phi\phi}(r)} \right]^{1/2}. \tag{2.10}$$

A. BPS bounds

Motion can be labeled by the quantum number j. It is easy to show that for some given j there is a lower bound on the energy—a BPS bound. This is not immediately obvious from the form of the Hamiltonian (2.10). However, a straightforward algebra allows us to rewrite H in the following form:

$$H=\lambda\left[j^2+\frac{P^2}{g_{rr}(r)}+\frac{(jr-r^p)^2}{g_{\phi\phi}(r)}\right]^{1/2}. \tag{2.11}$$

Since $g_{rr}(r)=(1-r^2)^{-1}$ and $g_{\phi\phi}(r)=1-r^2$ (in $R=1$ units) are positive, it is clear that

$$H\geqslant\lambda j. \tag{2.12}$$

This is the BPS bound.

In deriving the form of the Hamiltonian given in Eq. (2.11), it is absolutely crucial that the relative coefficient between the DBI term and the Chern-Simons term is what it is. This happens because the *n*-form flux is quantized in the standard way. Furthermore, the exact form of the metric on the sphere is also crucial. All the details of working in a consistent supergravity background has entered in the calculation.

B. BPS saturated states and angular momentum bounds

The bound is saturated when

$$p_r=0 \tag{2.13}$$

and

$$jr=r^p. \tag{2.14}$$

The latter has two solutions for $p\neq 0$:

$$r=r_1=0,\quad r=r_2=j^{(1/p-1)}. \tag{2.15}$$

Thus BPS motions have constant *r*, which is the size of the brane.

The potential energy, for such motion,

$$V(r)=\frac{(jr-r^p)^2}{1-r^2}. \tag{2.16}$$

For $p=0$ the potential does not vanish at $r=0$. There is a minimum for $j<1$ in the physical range of *r*. However, the potential is nonvanishing at the minimum and this does not correspond to a BPS state. When $j>1$ there is one minimum where $V(r)=0$ and the location of the minimum moves to smaller values of *r* as *j* increases. Thus, for $p=0$, BPS states must have $j>1$.

For $p=1$ and $j\neq 1$, there is no nontrivial minimum for $r<1$. For $j=1$ the potential is zero everywhere. Thus all BPS states have $j=1$.

For $p\geqslant 2$ the potential has two minima with a maximum in between. These minima are precisely r_1 and r_2 given in Eqs. (2.15). Thus there are two kinds of BPS states: the one which corresponds to zero-size branes and the other with branes with sizes scaling as $j^{1/(p-1}$. Since the range of *r* is between 0 and 1, this immediately implies that there is an upper bound for *j*:

$$j\leqslant R^{p+1}, \tag{2.17}$$

where we have restored *R*. The physical angular momentum is

$$p_\phi=N. \tag{2.18}$$

For a BPS state, $H=\lambda j=p_\phi/R$. This is the *same* dispersion relation as that of a massless graviton which is moving purely on the sphere. What is surprising is that states of branes, which are by themselves heavy objects, can lead to a light state. The reason behind this is of course the coupling to the *n*-form field strength. The effect of this canceled the effect of brane tension.

In the above discussion, we have used the phrase ''BPS configuration'' in its original sense. In a supersymmetric theory one would expect that these configurations also preserve some of the supersymmetries.[2]

III. TUNNELING BETWEEN VACUA

We saw that for $p\geqslant 2$ there are two minima. Strictly speaking, the minimum at $r=0$ is in a regime where we cannot trust our picture. The description of brane physics in terms of a DBI-CS Lagrangian is valid when the size of the brane is much larger than the basic length scale of the theory and clearly a zero-size brane cannot be described in this fashion. On the other hand, for sufficiently large *j* the other minimum lies in the domain of validity of our calculation. This is consistent with the overall picture implied in [4]. For low values of the angular momentum, the perturbative graviton is a good description of the state. For large values of angular momentum, this description fails and one should consider the states as wrapped branes.

It is nevertheless of some interest to ask whether there are finite action tuneling configurations between the two vacua. We now want to consider motion in the *r* direction, for a given value of *j* in Euclidean signature. The Hamiltonian for such motion is given by

$$H=\lambda\left[U(r)+\frac{p_r^2}{\lambda^2 g_{rr}(r)}\right]^{1/2}, \tag{3.1}$$

where

$$U(r)=j^2+V(r). \tag{3.2}$$

The corresponding Lagrangian is then

$$L=-\lambda[U(r)]^{1/2}[1-g_{rr}(r)\dot{r}^2]^{1/2}, \tag{3.3}$$

so that the Euclidean action is

$$S_E=\int dt[U(r)]^{1/2}[1+g_{rr}(r)\dot{r}^2]^{1/2}, \tag{3.4}$$

while the Euclidean Hamiltonian is

$$H_E=-\lambda\left[U(r)-\frac{p_E^2}{\lambda^2 g_{rr}(r)}\right]^{1/2}, \tag{3.5}$$

where

[2] This has in fact been shown in [9,10].

$$p_E=\lambda\frac{g_{rr}(r)\dot{r}[U(r)]^{1/2}}{[1+g_{rr}(r)\dot{r}^2]^{1/2}}. \tag{3.6}$$

To construct instanton solutions interpolating between the two minima of $U(r)$, we need solutions of the Euclidean equations of motion with Euclidean energy $H_E^2=\lambda^2 j^2$. It is easily seen that such motion obeys

$$\dot{r}=\pm\frac{1}{j}r(j-r^{p-1}), \tag{3.7}$$

the two signs corresponding to instantons and anti-instantons. Using Eqs. (3.7) and (3.4), it is easy to check that the Euclidean action for this solution is

$$S'_{\rm ins}=\frac{\lambda}{j}\int dt\, U(r). \tag{3.8}$$

This action may be easily seen to be infinite. However, one must remember that the energy of the states between which tuneling is occurring is nonzero and equal to λj. This has itself an action

$$S_0=\lambda j\int dt. \tag{3.9}$$

Thus the true instanton action is

$$S_{\rm ins}=S'_{\rm ins}-S_0, \tag{3.10}$$

and this is in fact finite.

For example, for $p=2$ the solution to Eq. (3.7) is

$$\frac{r}{j-r}=e^t \tag{3.11}$$

and the action $S'_{\rm ins}$ is

$$S'_{\rm ins}=\lambda\int_0^j dr\left[\frac{j^2}{r(j-r)}+\frac{r(j-r)}{1-r^2}\right]. \tag{3.12}$$

The first term in the integral is clearly divergent, while the second term is finite. However, using Eq. (3.7), again one sees that this term is in fact

$$\lambda j\int dt, \tag{3.13}$$

which is exactly S_0. Thus the subtracted quantity $S_{\rm ins}$ is finite:

$$S_{\rm ins}=\lambda\left[j+\frac{1}{2}(1-j)\log(1-j)-\frac{1}{2}(1+j)\log(1+j)\right]. \tag{3.14}$$

We have shown that there are finite action instanton configurations which interpolate between the two minima of the potential. However, as emphasized above, the meaning of this is not very clear since the configuration with zero-sized branes clearly lies outside the validity of our description. In fact for a given angular momentum there is only one state: for low angular momentum this is a pointlike state represented by a graviton and for large angular momentum this is an extended brane. For intermediate angular momenta the description is probably complicated.

IV. MULTIPLE-BRANE STATES

The $N=4$ super Yang-Mills theory that arises from D-3-branes has chiral operators of the form tr$[\Phi^{i_1}\cdots\Phi^{i_n}]$, where we symmetrize in the indices i_k. But it has also been argued that there exist multitrace operators that are also chiral primaries [11]. These operators are of the form (for two traces)

$$\mathrm{tr}[\Phi^{i_1}\cdots\Phi^{i_m}]\mathrm{tr}[\Phi^{i_{m+1}}\cdots\Phi^{i_n}], \tag{4.1}$$

with the indices i_k again symmetrized. In a similar manner we can make operators with more traces. These operators are expected to be dual to multiparticle states in the dual string theory. In the case of $AdS_3\times S^3\times M^4$, it was found in [12] that multiparticle states in supergravity were needed to account for the elliptic genus computed from the dual conformal field theory (CFT).

The existence of multiparticle chiral primaries raises the following issue for the stringy exclusion principle. To be able to get these states, we must be able to construct multiparticle states where the interactions between the particles exactly ''cancel out,'' giving energy equal to the sum of R charges. At the same time we should not be able to increase the number of such quanta without bound, since when the total charge exceeds the limit set by the exclusion principle then the state should not be BPS.

Let us examine the consequence of this fact for the giant gravitons. The n-trace operators in the gauge theory would presumably be dual to n-branes placed in the dual spacetime. If this state is to be BPS, then the interactions between these branes must ''cancel out.'' This n-particle state is different from the single-particle state with the same charges, so we do not require that these configurations mix to produce one effective state. (Note, however, that operator refinitions mix single-particle and multiparticle states in some cases [13].)

On the other hand, when the total charge on the branes exceeds the limit set by the exclusion principle, we should no longer be able to make a BPS state. Thus consider the giant graviton that expands in the AdS direction rather than along the sphere, and let the angular momentum L exceed the limit set by the exclusion principle. Then there can be a tunneling from this state to one where there are say two giant gravitons, with angular momenta $L_1, L-L_1$. Pictorially, we imagine a large sphere pinching in the middle and separating into two spheres.

While we have not computed the action for such an instanton, there appears to be no reason why it should diverge. In the case of tunneling from a finite brane to a point, the latter configuration was singular and one could worry about corresponding divergences in the amplitude. But the tunneling on hand is between two regular configurations, and we can make interpolating configurations that have finite contribution from the Born-Infeld and Wess-Zumino terms.

Thus the picture of tunneling may be more complex than that noted in [9] and [10]. Apart from the pointlike graviton and the two giant gravitons, we have a host of multibrane states that have the same quantum numbers. The exclusion principle requires some of these to exist, while the others (with L larger than the exclusion bound) may disappear by tunnelings that involve all the configurations mentioned above.

V. SUPERSYMMETRY AND NONCOMMUTATIVITY

We have seen that for BPS motions wrapped branes have a bound on the angular momentum, thus providing a new perspective on the stringy exclusion principle.

Bounds on angular momentum appear naturally for particle motions on noncommutative spaces, e.g., fuzzy spheres or quantum spheres. One might wonder whether the dynamics discussed above, which is entirely based on a commutative space, implies an (effective) dynamics in a noncommutative space.

The important point here is that it is only for BPS states that there is a bound on the angular momentum, not for other motions such as changes of the size of the brane or oscillations of the brane. Likewise from the point of view of holography, there are non-BPS states which can have any value of the angular momentum. Consider, for example $AdS_5\times S^5$ spacetime where the holographic theory is $(3+1)$- dimensional $N=4$ Yang-Mills theory. In terms of the Higgs fields Φ^i, $i=1,...,6$ of this theory chiral primary operators are of the form

$$\mathrm{Tr}_S[\Phi^{i_1}\cdots\Phi^{i_n}], \tag{5.1}$$

where the subscript S means that we have to symmetrize with respect to the indices $i_1,\ldots,i_n$ and subtract the trace. Supergravity modes lie in the chiral primary multiplet obtained from Eq. (5.1) by acting with supersymmetry charges. Clearly Eq. (5.1) has a SO(6) angular momentum equal to n. The rank N of the gauge group SU(N) is in fact the quantized flux of the five-form field strength on S^5. Such operators and their supersymmetry partners can have a maximum angular momentum N. There are, however, operators which involve derivatives of Φ which do not create BPS states: e.g.,

$$\mathrm{Tr}_S[\Phi^{i_1}\partial\Phi^{i_2}\partial\Phi^{i_3}\cdots\Phi^{i_n}]. \tag{5.2}$$

Clearly, there is no bound for the angular momenta of these operators since we can have higher and higher derivatives and $\partial^n\Phi$ is a different matrix than Φ.

Knowing that BPS states respect half of the supersymmetries, we are now led impose the condition $Q=0$ on the phase space. This can be done since Q represents symmetry generators of the theory. The imposition of this symmetry strongly on the phase space will lead, as we will now argue to a noncommutative space.

We therefore need to know what kind of motion of the brane respects half the supersymmetries of the background. The question we ask is the converse of the question answered in [9] and [10], where it was shown that the giant graviton with no motion in the r direction respects half the supersymmetries. On the world volume action of the brane with fermionic coordinates Θ^α, we need to choose a κ-symmetry gauge condition $(1+\Gamma)\Theta=0$, where Γ is the pullback:

$$\Gamma=\frac{1}{(p+1)!}\epsilon^{i_1\cdots i_{p+1}}\partial_{i_1}X^{\mu_1}\cdots\partial_{i_{p+1}}X^{\mu_{p+1}}\Gamma_{\mu_1\cdots\mu_{p+1}}, \tag{5.3}$$

where X^μ denote the bosonic coordinates on the brane world volume, Γ_μ are the Dirac gamma matrices in target space, and $\Gamma_{\mu_1\cdots\mu_n}=\Gamma_{\mu_1}\cdots\Gamma_{\mu_n}$. We need to look at supersymmetry transformations which preserve this gauge choice. These are the transformations

$$\delta\Theta=\frac{1}{2}(1-\Gamma)\epsilon, \tag{5.4}$$

for an infinitesimal spinor parameters ϵ. To preserve the supersymmetries of the background, we require in addition that ϵ be in fact a Killing spinor of the background. The corresponding supercharge is given by

$$Q=\frac{1}{2}\bar{\Theta}(1-\Gamma). \tag{5.5}$$

so that $\delta\Theta=\{Q\cdot\epsilon,\Theta\}$.

In the static gauge we are using, the expression for Γ may be seen to be (using expressions given in [10])

$$\Gamma=\frac{1}{r^{p+1}}\Bigg[\frac{H}{\lambda}\Gamma_0+P(\sqrt{1-r^2}\sin\phi\Gamma_{p+2}+\sqrt{1-r^2}\cos\phi\Gamma_{p+1})+(j-r^{p+1})\times\left(\frac{\cos\phi}{\sqrt{1-r^2}}\Gamma_{p+2}-\frac{\sin\phi}{\sqrt{1-r^2}}\Gamma_{p+1}\right)\Bigg]\Gamma_{p\cdots1}, \tag{5.6}$$

where we have now used a coordinate system on the sphere S^{p+2} which has angles $\theta_1,\ldots,\theta_{p+2}$. The angles $\theta_1,\ldots,\theta_p$ are on the S^p on which the p-brane is wrapped, while the relationships between θ_{p+1}, θ_{p+2} with r,ϕ are given by

$$\sqrt{1-r^2}\cos\phi=\cos\theta_{p+2},\qquad \sqrt{-r^2}\sin\phi=\sin\theta_{p+2}\cos\theta_{p+1}. \tag{5.7}$$

Using an analysis similar to that in [9] and [10], it is straightforward to see from Eqs. (5.6) and (5.5) that the condition that half of the supercharges vanish implies $P=p_r/\lambda=0$.

What does this condition imply in phase space? The coordinates (r,ϕ) and the momenta (p_r,p_ϕ) satisfy standard Poisson brackets. In particular,

$$[r,\phi]_{\mathrm{PB}}=0. \tag{5.8}$$

However, we have shown that BPS motions have $p_r=0$. Thus we should regard this as a constraint in phase space,

$$\psi_1=p_r=0. \tag{5.9}$$

as a weak condition. Taking Poisson brackets with the Hamiltonian yields a secondary constraint

$$\psi_2 = \frac{dV}{dr} = 0. \tag{5.10}$$

Actually, this can also be seen as following from the symmetry reduction condition $Q=0$. There are no further constraints. This is because a direct computation yields

$$[H,\psi_2]_{\rm PB} = -\frac{\lambda^2 p_r}{H g_{rr}(r)}\frac{d^2V}{dr^2}. \tag{5.11}$$

which vanishes on the constraint surface because of Eq. (5.9). Finally, the two constraints ψ_1 and ψ_2 form a second class system with the Poisson brackets (PB's)

$$[\psi_1,\psi_2]_{\rm PB} = -\frac{d^2V}{dr^2}. \tag{5.12}$$

so that the matrix of PB's of the constraints is

$$C = -i\frac{d^2V}{dr^2}\sigma_2, \tag{5.13}$$

where σ_2 is the Pauli matrix

To analyze the dynamics of these restricted set of motions, we need to look at the brackets of unconstrained variables on the reduced phase space. Alternatively, we should look at Dirac brackets. These may be computed in a straightforward manner, and the result is

$$[r,\phi]_{\rm DB} = [r,\phi]_{\rm PB} - [r,\psi_1]_{\rm PB}(C^{-1})^{12}[\psi_2,\phi]_{\rm PB} = -\frac{1}{\lambda}\frac{\partial^2 V/\partial r \partial j}{\partial^2 V/\partial r^2}. \tag{5.14}$$

On the constraint surface this is evaluated as

$$[r,\phi]_{\rm DB} = \frac{R^{p-1}}{N}\frac{r^{2-p}}{p-1}, \tag{5.15}$$

which is nonzero. In a quantum theory we should replace these Dirac brackets by a commutator and one would have noncommuting coordinates on the sphere. The noncommutativity is proportional to $1/N$ as expected. Furthermore, this is divergent for $p=1$ and reverses sign for $p=0$. However, these are the two cases where is no true bound for the angular momentum.

Alternatively, one can consider the quantum theory directly. Now the condition $p_r=0$ should be imposed on the space of states. If P denotes the projection operator on these states, the relevant dynamical quantities in this subspace of states are PrP and $P\phi P$. These will not commute even though r and ϕ do.

The origin of noncommutativity in our problem is similar to the way noncommutativity arises in the quantum Hall effect when one restricts to the lowest Landau level. Here again the restriction to the lowest Landau level may be viewed at the classical level as constraints which set the velocities to zero as weak conditions. The Dirac brackets for the coordinates are then nontrivial. One justification of this procedure for restricting the analysis to the lowest Landau level is given by taking the zero-mass limit. In the present case we have a much more elegant reason for such a reduction, namely, supersymmetry.

Interestingly, the above commutation relation turns out to be identical to the one obtained by the following heuristic argument in [7] where it is also shown that for $p=2$ these are the same as the commutators which define a fuzzy S^4. We have seen that for motions with $P=0$ the size of the brane is related to the angular momentum j by the relation $j=r^{p-1}$. We can then consider r^{p-1} as the canonical conjugate to ϕ. This leads to a commutation relation between r and ϕ, which is the same obtained by replacing the Dirac brackets (5.15) by a commutator. However, this heuristic argument does not throw light on the origin of noncommutativity, which lies in the fact that we are working on a subspace of states. Most importantly, as we have argued, the reduction to a noncommutative space can be understood as a Hamiltonian reduction based on supersymmetry.

VI. CONCLUSIONS

Our analysis has provided an important consistency check on the giant graviton picture; viz, BPS states have bounded angular momenta, while there are non-BPS states which can have arbitrary angular momenta. This is consistent with the stringy exclusion principle. We have shown that at the classical level such states have the same dispersion relation as that of a graviton; the brane tension is canceled by the Lorentz force due to the field strength to which the brane couples. Furthermore, the very existence of the BPS bound required precise coefficients in front the DBI and Chern Simons terms—these incorporate flux quantization as well as the details of the geometry.

In [9] and [10] it has been shown that these BPS states are in fact those which preserve half of the supersymmetries of the background. We showed that one can impose these supersymmetry conditions only when $p_r=0$ or equivalently the size of the brane is fixed during its motion. We can then impose the condition that half of the supercharges vanish as a strong condition, which would then imply this restriction of the motion of the branes which which lead to bounded angular momenta for $p\geqslant 2$. The same restriction also led us to the fact that for such motions the two transverse coordinates on the sphere can be regarded as noncommuting. It is important to realize that there is nothing noncommuting at the fundamental level. This arises purely because we are considering motion on a constrained surface on phase space. This constrained surface can be understood as given by the condition that on it half of the supercharges vanish. In other words, space can be regarded as noncommutative if and only if we restrict ourselves to the subspace of BPS states. We believe that this fact can have implications for the suggestion that noncommutativity could be the origin of the stringy exclusion principle.

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