Bimetric gravity and ''dark matter''

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We present a bimetric theory of gravity containing a length scale of galactic size. For distances less than this scale the theory satisfies the standard tests of general relativity. For distances greater than this scale the theory yields an effective gravitational constant much larger than the locally observed value of Newton's constant. The transition from one regime to the other through the galactic scale can explain the observed rotation curves of galaxies and hence the effects normally attributed to the presence of dark matter. Phenomena on an extragalactic scale such as galactic clusters and the expansion of the universe are controlled by the enhanced gravitational coupling. This provides an explanation of the missing matter normally invoked to account for the observed value of Hubble's constant in relation to observed matter.

I. INTRODUCTION

The presence of dark matter in the universe has been invoked to explain a number of phenomena: the rotation curves of galaxies [1–4], the apparent mass of galactic clusters [5,6] and the observed expansion of the universe. Many attempts to detect this conjectured matter are under way but at the moment there has been no direct observation of it. In this paper we present a model with a modified version of gravity that can accommodate these points in a relatively straightforward way. It does so without the necessity of postulating dark matter. The theory, which is geometrical in character, is perfectly covariant, locally Lorentz invariant and satisfies the equivalence principle. It also satisfies all the tests of general relativity on the scales at which they have been carried out.

The theory is a bi-metric theory. Such theories have a long history [7–9] and have recently been used as a way of realizing variable speed of light (VSL) theories [10–19]. An earlier incomplete version of the theory presented here was motivated in this way [20]. Bi-metric theories ran into difficulties when, as then formulated, they appeared to be inconsistent with the tests of general relativity [21,22]. However the theory proposed here does meet all those tests [21–25]. Of course VSL is built into bi-metric theories. This will be important in applying the model to early universe studies. However, in this paper we concentrate on those implications of our theory for the gravitational interaction of matter that provide an explanation of the current state of the universe without recourse to dark matter.

Our proposal is to introduce into the space-time manifold two vierbein bundles. Each bundle supports its own metric. One is associated with matter and the other with (underlying) gravity. The matter vierbein can be strained and scaled relative to the gravitational vierbein. The dynamics of the theory includes this straining and scaling as dynamical degrees of freedom. The justification for introducing these new effects is ultimately in the results that emerge. However, if dark matter is not present, some modification of dynamics, as in the modified nonrelativistic dynamics (MOND) scenario [26–28], or gravity is essential. We choose to modify gravity in a way that permits the introduction of a galactic scale, something that is impossible with standard general relativity.

The equations of motion are obtained from an action comprising three parts: a gravitational term I_G of the standard curvature form, a matter term I_M of the standard form based on the matter metric, and a linking action I_L that depends on the variables that determine the relationship between the two vierbein bundles. The full action, I, is the sum of all three terms:

$$I = I_G + I_L + I_M. \tag{1}$$

Each term has its own gravitational coupling constant with the dimensions of Newton's constant, G_N. In developing the theory we show how these different constants are related to one another and to Newton's constant.

II. GENERAL STRUCTURE

The geometrical character of the theory is clearly revealed by its formulation in the vierbein formalism. The local Lorentz invariance in both the gravitational and matter vierbein frame is explicit throughout as a gauge invariance. We introduce a vierbein bundle appropriate to gravity, $\{e_{\mu a}\}$, with the associated metric

$$g_{\mu\nu} = e_{\mu a} e_\nu{}^a, \tag{2}$$

where the raising and lowering of a indices is carried out with the standard Lorentz metric $\eta_{ab} = \{1, -1, -1, -1\}$. The inverse vierbein is $\{e^{a\mu}\}$ so that

$$e_{\mu a} e^{a\nu} = \delta_\mu^\nu, \quad e^{a\mu} e_{\mu b} = \delta_b^a, \tag{3}$$

and

$$g^{\mu\nu} = e^{a\mu} e_a{}^\nu. \tag{4}$$

The vierbein associated with matter is $\{\bar{e}_{\mu\bar{a}}\}$ and the raising and lowering of $\bar{a}$ indices is by means of the Lorentz metric, $\eta_{\bar{a}\bar{b}} = \{1, -1, -1, -1\}$. The associated metric is

$$\bar{g}_{\mu\nu} = \bar{e}_{\mu\bar{a}} \bar{e}_\nu{}^{\bar{a}}. \tag{5}$$

The two vierbein bundles are related by a local linear transformation

$$\bar{e}_{\mu\bar{a}}=e_{\mu a}M^{a}{}_{\bar{a}}e^{\phi}, \tag{6}$$

where the matrix M is an element of $SL(4,R)$ and the local scaling is introduced through the factor e^{ϕ}. The determinants associated with the volume elements of the bundles are J and $\bar{J}$ where

$$J=\det\{e_{\mu}{}^{a}\} \quad \text{and} \quad \bar{J}=\det\{\bar{e}_{\mu}{}^{\bar{a}}\}. \tag{7}$$

We have therefore $\bar{J}=Je^{4\phi}$. We denote the inverse matrix by $M^{\bar{a}}{}_{a}$ so that

$$M^{a}{}_{\bar{a}}M^{\bar{a}}{}_{b}=\delta^{a}_{b}, \quad M^{\bar{a}}{}_{a}M^{a}{}_{\bar{b}}=\delta^{\bar{a}}_{\bar{b}}. \tag{8}$$

The vierbein connections and associated coordinate connections are defined so that the vierbeins are covariantly constant in the appropriate way:

$$D_{\mu}e_{\nu a}=\partial_{\mu}e_{\nu a}+\omega_{\mu a}{}^{b}e_{\nu b}-\Gamma^{\lambda}_{\mu\nu}e_{\lambda a}=0 \tag{9}$$

and

$$\bar{D}_{\mu}\bar{e}_{\nu\bar{a}}=\partial_{\mu}\bar{e}_{\nu\bar{a}}+\bar{\omega}_{\mu\bar{a}}{}^{\bar{b}}\bar{e}_{\nu\bar{b}}-\bar{\Gamma}^{\lambda}_{\mu\nu}\bar{e}_{\lambda\bar{a}}=0. \tag{10}$$

The requirement that η_{ab} and $\eta_{\bar{a}\bar{b}}$ be covariantly constant implies that $\omega_{\mu ab}=-\omega_{\mu ba}$ and $\bar{\omega}_{\mu\bar{a}\bar{b}}=-\bar{\omega}_{\mu\bar{b}\bar{a}}$.

It is convenient to define a covariant derivative of M that includes both the right and left vierbein connections:

$$D_{\mu}M^{a}{}_{\bar{a}}=\partial_{\mu}M^{a}{}_{\bar{a}}+\omega_{\mu}{}^{a}{}_{b}M^{b}{}_{\bar{a}}-M^{a}{}_{\bar{b}}\bar{\omega}_{\mu}{}^{\bar{b}}{}_{\bar{a}}. \tag{11}$$

However, further differentiation requires the coordinate connection, $\Gamma^{\lambda}_{\mu\nu}$. The covariant derivative $\bar{D}_{\mu}$ can be defined and extended in a similar way. Its effect on M is the same as that of D_{μ} but a second differentiation must use the coordinate connection $\bar{\Gamma}^{\lambda}_{\mu\nu}$.

Gravitational curvature tensors appropriate to each of the bundles are defined so that

$$[D^{L}_{\mu},D^{L}_{\nu}]V_{a}=R_{ab\mu\nu}V^{b} \tag{12}$$

and

$$[D^{R}_{\mu},D^{R}_{\nu}]V_{\bar{a}}=\bar{R}_{\bar{a}\bar{b}\mu\nu}V^{\bar{b}}, \tag{13}$$

where D^{L}_{μ} includes the left vierbein connection field, $\omega_{\mu ab}$ but not the coordinate connection, $\Gamma^{\lambda}_{\mu\nu}$. Similarly D^{R}_{μ} includes the only the right vierbein connection, $\bar{\omega}_{\mu\bar{a}\bar{b}}$. We have then

$$R_{ab\mu\nu}=\partial_{\mu}\omega_{\nu ab}-\partial_{\nu}\omega_{\mu ab}+\omega_{\mu a}{}^{c}\omega_{\nu cb}-\omega_{\nu a}{}^{c}\omega_{\mu cb}, \tag{14}$$

with a similar definition for $\bar{R}_{\bar{a}\bar{b}\mu\nu}$. It follows that

$$[D_{\mu},D_{\nu}]M^{a}{}_{\bar{a}}=R^{a}{}_{b\mu\nu}M^{b}{}_{\bar{a}}-M^{a}{}_{\bar{b}}\bar{R}^{\bar{b}}{}_{\bar{a}\mu\nu}-2C^{\lambda}_{\mu\nu}D_{\lambda}M^{a}{}_{\bar{a}}, \tag{15}$$

where

$$C^{\lambda}_{\mu\nu}=\frac{1}{2}(\Gamma^{\lambda}_{\mu\nu}-\Gamma^{\lambda}_{\nu\mu}). \tag{16}$$

The quantity $C^{\lambda}_{\mu\nu}$ is the torsion tensor in the coordinate basis. It is not in general zero.

III. GRAVITATIONAL ACTION

The gravitational action has the standard form

$$I_{G}=-\frac{1}{16\pi G}\int d^{4}xJR, \tag{17}$$

where

$$R=e^{a\mu}e^{b\nu}R_{ab\mu\nu} \tag{18}$$

and G is a coupling with the dimensions of Newton's constant, G_{N}. The vierbeins and the connections are treated as independent variables. If we vary the former, then

$$\delta I_{G}=\frac{1}{8\pi G}\int d^{4}xJ\,\delta e_{\sigma c}\left(e^{c\mu}R^{\sigma}{}_{\mu}-\frac{1}{2}e^{c\sigma}R\right). \tag{19}$$

The variation of the vierbein connection yields

$$\delta I_{G}=-\frac{1}{16\pi G}\int d^{4}xJe^{a\mu}e^{b\nu}(D^{L}_{\mu}\delta\omega_{\nu ab}-D^{L}_{\nu}\delta\omega_{\mu ab}). \tag{20}$$

Switching to the full covariant derivative we get

$$\delta I_{G}=-\frac{1}{16\pi G}\int d^{4}xJe^{a\mu}e^{b\nu}(D_{\mu}\delta\omega_{\nu ab}$$
$$-D_{\nu}\delta\omega_{\mu ab}+2C^{\lambda}_{\mu\nu}\delta\omega_{\lambda ab}). \tag{21}$$

Using the result

$$JD_{\mu}V^{\mu}=\partial_{\mu}(JV^{\mu})+2JC^{\lambda}_{\lambda\mu}V^{\mu}, \tag{22}$$

we can integrate by parts and obtain finally

$$\delta I_{G}=-\frac{1}{8\pi G}\int d^{4}xJe^{a\mu}e^{b\nu}(C^{\lambda}_{\lambda\mu}\delta^{\sigma}_{\nu}-C^{\lambda}_{\lambda\nu}\delta^{\sigma}_{\mu}+C^{\sigma}_{\mu\nu})\delta\omega_{\sigma ab}. \tag{23}$$

If there is no other interaction in the theory, we can deduce from the vanishing of these variations that $C^{\lambda}_{\mu\nu}=0$ and

$$R^{\sigma}{}_{\mu}-\frac{1}{2}\delta^{\sigma}_{\mu}R=0,$$

the standard equations for matterless gravity.

IV. LINKING ACTION

Proposals for constructing an action for the linear transformation relating the bundles have involved parametrizing it in terms of a vector or scalar field [15–19]. Our proposal treats all the degrees of freedom inherent in the transformation. This is crucial for the structure of the theory.

Because scaling commutes with the other elements of the group of linear transformations, it can be treated separately. The Lagrangian for ϕ can be chosen to be proportional to $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. The remaining degrees of freedom are represented by the matrix M which lies in the non-linear manifold $SL(4,R)$. A natural way of constructing a Lagrangian for such a theory is to invoke the mechanism of the non-linear sigma model and express it as a quadratic form in the derivatives $(D_\mu M)M^{-1}$ with the appropriate structure. The use of the covariant derivative guarantees the local Lorentz gauge invariance relative to both bundles and the presence of M^{-1} guarantees that the derivative is a proper element of the tangent space to the manifold $SL(4,R)$. We take as our action

$$I_L = \frac{1}{16\pi F}\int d^4x J g^{\mu\nu}\,\mathrm{Tr}(j_\mu j_\nu)$$

$$+ \frac{1}{16\pi F'}\int d^4x J g^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi), \qquad (24)$$

where F and F' are new gravitational constants with the same physical dimensions as G. The matrix valued current j_μ is given by

$$j_\mu = (D_\mu M)M^{-1} \qquad (25)$$

or, more explicitly,

$$j_\mu^{ab} = (D_\mu M^a_{\ \bar b})M^{\bar b b}. \qquad (26)$$

It is also convenient to define an alternative version of the current, appropriate to the barred vierbein bundle, $\bar j_\mu^{\bar a\bar b}$, as

$$\bar j_\mu^{\bar a\bar b} = (M^{-1}D_\mu M)^{\bar a\bar b} = M^{\bar a}_{\ a}D_\mu M^{a\bar b}. \qquad (27)$$

We also include in I_L ''mass'' terms of the form

$$-\frac{1}{16\pi F}\int d^4x J\frac{m^2}{4}(M^a_{\ \bar a}M_a^{\ \bar a}+M_{\bar a}^{\ a}M^{\bar a}_{\ a}-\gamma)$$

$$-\frac{1}{16\pi F'}\int d^4x J m^2\phi^2. \qquad (28)$$

For simplicity we have chosen the mass parameter m to be the same in both these additional terms. For the choice $\gamma = 8$, the action vanishes when M represents a Lorentz transformation. Departures from this value introduce a cosmological constant term in the action. By construction these additional terms clearly maintain local Lorentz invariance in both vierbein bundles. The mass terms are crucial for the effectiveness of the theory because we identify the galactic distance scale with m^{-1}. For effects on a scale much less than this therefore m will be viewed as a small parameter that can be neglected in certain circumstances.

We treat the vierbein, which enters through $g^{\mu\nu}$, the matrix, M, and the connections $\omega_{\mu ab}$ and $\bar\omega_{\mu\bar a\bar b}$ as independent variables. The result for the linking action from the vierbein variation is

$$\delta I_L = -\frac{1}{8\pi F}\int d^4x J\delta e_{\sigma c}\left(e^{c\nu}g^{\sigma\mu}-\frac{1}{2}e^{c\sigma}g^{\mu\nu}\right)\mathrm{Tr}(j_\mu j_\nu)$$

$$-\frac{1}{8\pi F}\int d^4x J\delta e_{\sigma c}e^{c\sigma}\frac{m^2}{8}(M^a_{\ \bar a}M_a^{\ \bar a}+M_{\bar a}^{\ a}M^{\bar a}_{\ a}-\gamma)$$

$$-\frac{1}{8\pi F'}\int d^4x J\delta e_{\sigma c}\left(e^{c\nu}g^{\sigma\mu}-\frac{1}{2}e^{c\sigma}g^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi$$

$$-\frac{1}{8\pi F'}\int d^4x J\delta e_{\sigma c}e^{c\sigma}\frac{m^2}{2}\phi^2. \qquad (29)$$

From the left vierbein connection we have

$$\delta I_L = \frac{1}{8\pi F}\int d^4x J\delta\omega_{\mu ab}j^{\mu ba} \qquad (30)$$

and, from the right vierbein connection,

$$\delta I_L = -\frac{1}{8\pi F}\int d^4x J\delta\bar\omega_{\mu\bar a\bar b}\bar j^{\mu\bar b\bar a}. \qquad (31)$$

On varying the matrix M we obtain

$$\delta I_L = -\frac{1}{8\pi F}\int d^4x J(\delta MM^{-1})^{ab}\left[D_\mu j^\mu_{ba}-2C^\lambda_{\lambda\mu}j^\mu_{ba}\right.$$

$$\left.+\frac{m^2}{4}(M_a^{\ \bar c}M_{b\bar c}-M^{\bar c}_{\ a}M_{\bar c b})\right], \qquad (32)$$

where the square brackets $[\cdots]$ indicate the traceless version of the quantity contained within them. The quantity δMM^{-1}, being an arbitrary element of the $SL(4,R)$ Lie algebra, is sufficiently general to identify the other factor in the integrand. Finally, on varying ϕ, we have

$$\delta I_L = -\frac{1}{8\pi F'}\int d^4x J\delta\phi[g^{\mu\nu}(D_\nu\partial_\mu\phi$$

$$-2C^\lambda_{\lambda\nu}\partial_\mu\phi)+m^2\phi]. \qquad (33)$$

V. MATTER ACTION

We assume that matter is propagated in the vierbein background $\{\bar e_{\mu\bar a}\}$. This seems a consistent approach since it implies that matter behaves in a conventional way in relation to the gravitational field it experiences. In particular the equivalence principle is satisfied. However, the theory does change the relationship of this observed gravitational field to the distribution of matter density. We have

$$\delta I_M = -\frac{1}{2}\int d^4x\bar J\delta\bar g_{\mu\nu}\bar T^{\mu\nu}, \qquad (34)$$

where $\bar T^{\mu\nu} = \bar T^{\nu\mu}$ is the symmetric energy momentum tensor for matter. Since

$$\bar g_{\mu\nu} = \bar e_{\mu\bar a}\bar e_\nu^{\ \bar a}, \qquad (35)$$

it follows that

$$\delta I_M = -\int d^4x \bar{J} \delta \bar{e}_{\mu\bar{a}} \bar{e}_\nu{}^{\bar{a}} \bar{T}^{\mu\nu}. \tag{36}$$

However,

$$\delta \bar{e}_{\mu\bar{a}} = \delta e_{\mu a} M^a{}_{\bar{a}} e^\phi + e_{\mu a} \delta M^a{}_{\bar{a}} e^\phi + \delta \phi \bar{e}_{\mu\bar{a}}, \tag{37}$$

so the variation of the matter action takes the form

$$\delta I_M = -\int d^4x J e^{4\phi} [\delta e_{\sigma c} T^{\sigma c} + \mathrm{Tr}(\delta M M^{-1} U) + \delta\phi \bar{T}], \tag{38}$$

where $\bar{T} = \bar{g}_{\mu\nu} \bar{T}^{\mu\nu}$,

$$U^{ba} = e_\mu{}^a M^b{}_{\bar{a}} \bar{e}_\nu{}^{\bar{a}} \bar{T}^{\mu\nu} - \frac{1}{4} \eta^{ab} \bar{T}, \tag{39}$$

and

$$T^{\sigma c} = M^c{}_{\bar{a}} \bar{e}_\nu{}^{\bar{a}} \bar{T}^{\sigma\nu}, \tag{40}$$

$$T^\sigma{}_\lambda = e_{\lambda c} T^{\sigma c} = \bar{e}_{\lambda\bar{a}} \bar{e}_\nu{}^{\bar{a}} \bar{T}^{\sigma\nu} = \bar{T}^{\sigma\nu} \bar{g}_{\lambda\nu}. \tag{41}$$

If we adopt the convention that barred quantities, that is those appropriate to the gravitational background of the matter, have spatial indices raised and lowered with the barred metric, we can define

$$\bar{T}^\sigma{}_\lambda = \bar{T}^{\sigma\nu} \bar{g}_{\lambda\nu}. \tag{42}$$

Hence we get the simple seeming result

$$T^\sigma{}_\lambda = \bar{T}^\sigma{}_\lambda. \tag{43}$$

However, it is important to recall that

$$T^{\sigma\tau} = T^\sigma{}_\lambda g^{\lambda\tau} \neq \bar{T}^{\sigma\tau}. \tag{44}$$

In fact $T^{\sigma\tau}$ is not necessarily symmetric. This does not cause any difficulty in the theory because, as we will show below, the matter energy momentum tensor $\bar{T}^{\mu\nu}$, which is symmetric, obeys the appropriate conservation equation. Note that $T = T^\mu{}_\mu = \bar{T}$.

VI. EQUATIONS OF MOTION

We obtain the equations of motion by requiring that the variation of the total action be zero. The result is

$$\frac{1}{8\pi G} \left(R^\sigma{}_\mu - \frac{1}{2} \delta^\sigma_\mu R \right) - \frac{1}{8\pi F} \left(\mathrm{Tr}(j^\sigma j_\mu) - \frac{1}{2} \delta^\sigma_\mu \mathrm{Tr}(j^\lambda j_\lambda) \right)$$

$$- \frac{1}{8\pi F} \frac{m^2}{8} \delta^\sigma_\mu (M^a{}_{\bar{a}} M_a{}^{\bar{a}} + M^{\bar{a}}{}_a M_{\bar{a}}{}^a - \gamma)$$

$$- \frac{1}{8\pi F'} \left((g^{\sigma\nu} \partial_\nu \phi \partial_\mu \phi) - \frac{1}{2} \delta^\sigma_\mu (g^{\lambda\nu} \partial_\lambda \phi \partial_\nu \phi) \right)$$

$$- \frac{1}{8\pi F'} \frac{m^2}{2} \delta^\sigma_\mu \phi^2 = e^{4\phi} \bar{T}^\sigma{}_\rho. \tag{45}$$

Equation (45) is a generalization of the standard equation of general relativity. The new straining degrees of freedom are controlled by the following equation:

$$\frac{1}{8\pi F} \left(D_\mu j^\mu_{ba} - 2C^\lambda_{\lambda\mu} j^\mu_{ba} + \frac{m^2}{4} [M_a{}^{\bar{c}} M_{b\bar{c}} - M^{\bar{c}}{}_a M_{\bar{c}b}] \right)$$

$$+ e^{4\phi} U^{ba} = 0. \tag{46}$$

The generalization of the "no torsion" rule in general relativity is represented by the two equations

$$j^{\sigma[b,a]} = \frac{F}{G} e^{a\mu} e^{b\nu} (C^\lambda_{\lambda\mu} \delta^\sigma_\nu - C^\lambda_{\lambda\nu} \delta^\sigma_\mu + C^\sigma_{\mu\nu}). \tag{47}$$

$$\bar{j}^{[\bar{b},\bar{a}]}_\mu = 0. \tag{48}$$

Finally the local expansion is controlled by

$$\frac{1}{8\pi F'} [g^{\mu\nu}(D_\nu \partial_\mu \phi - 2C^\lambda_{\lambda\nu} \partial_\mu \phi) + m^2 \phi] + e^{4\phi} \bar{T} = 0. \tag{49}$$

Equation (45) implies that

$$\frac{1}{8\pi G} R^\sigma{}_\mu - \frac{1}{8\pi F} \mathrm{Tr}(j^\sigma j_\mu) - \frac{1}{8\pi F'} g^{\sigma\nu} \partial_\nu \phi \partial_\mu \phi$$

$$+ \frac{1}{8\pi F} \frac{m^2}{8} \delta^\sigma_\mu (M^a{}_{\bar{a}} M_a{}^{\bar{a}} + M^{\bar{a}}{}_a M_{\bar{a}}{}^a - \gamma)$$

$$+ \frac{1}{8\pi F'} \frac{m^2}{2} \delta^\sigma_\mu \phi^2 = e^{4\phi} \left(\bar{T}^\sigma{}_\mu - \frac{1}{2} \delta^\sigma_\mu \bar{T} \right). \tag{50}$$

Although rather complex, these equations are surprisingly susceptible of analysis as we show below.

VII. BIANCHI IDENTITY

Just as in standard general relativity it is necessary to check that the theory satisfies the integrability conditions associated with the Bianchi identity. In the presence of torsion this is changed to the following:

$$R_{\lambda\tau\mu\nu;\sigma} + R_{\lambda\tau\nu\sigma;\mu} + R_{\lambda\tau\sigma\mu;\nu} = -2(R_{\lambda\tau\rho\mu} C^\rho_{\nu\sigma} + R_{\lambda\tau\rho\nu} C^\rho_{\sigma\mu}$$

$$+ R_{\lambda\tau\rho\sigma} C^\rho_{\mu\nu}), \tag{51}$$

where $;\mu$ indicates the covariant derivative D_μ. In the contracted version it becomes

$$\left(R^\mu{}_\sigma - \frac{1}{2} \delta^\mu_\sigma R \right)_{;\mu} = R^{\mu\nu}{}_{\rho\sigma} C^\rho_{\mu\nu} + 2R^\mu{}_\rho C^\rho_{\sigma\mu}. \tag{52}$$

If we take the covariant divergence of the left side of Eq. (45) and make use of the equations of motion together with the identity

$$j^{ab}_\mu = e^{a\nu}(\bar{\Gamma}^\lambda_{\mu\nu} - \Gamma^\lambda_{\mu\nu}) e_\lambda{}^b - \eta^{ab} \partial_\mu \phi, \tag{53}$$

we obtain the result

$$\bar{D}_\sigma \bar{T}^\sigma{}_\lambda = 2\bar{C}^\sigma{}_{\sigma\lambda}\bar{T} - 2\bar{C}^\tau{}_{\sigma\lambda}\bar{T}^\sigma{}_\tau. \tag{54}$$

It is easily checked that this is equivalent to the standard conservation law

$$D_\sigma^{(\bar{g})}\bar{T}^\sigma{}_\lambda = 0, \tag{55}$$

where $D_\sigma^{(\bar{g})}$ is the covariant derivative formed from the metric connection arising from $\bar{g}_{\mu\nu}$.

VIII. CONNECTION STRUCTURE

In standard general relativity the assumption that matter couples to gravity only through the metric means that torsion plays no role in the theory. In the theory presented here we make the same assumption about matter. However, because of the extra complexity of the theoretical structure, torsion is in general not zero. Nevertheless, the equations of motion do permit the connection structure to be elucidated in a straightforward way.

The starting point of the analysis is Eq. (48) which reveals that the antisymmetric part of $\bar{j}_{\mu\bar{a}\bar{b}}$ vanishes. It allows us to express $\bar{\omega}_{\mu\bar{a}\bar{b}}$ in terms of the other dynamical variables and hence eliminate it from the equations of motion.

We now express $j_{\mu ab}$ in terms of the symmetric part of $\bar{j}_{\mu\bar{a}\bar{b}}$ as follows:

$$j_{\mu ab} = M_a{}^{\bar{a}}\bar{j}_{\mu\bar{a}\bar{b}}M^{\bar{b}}{}_b = M_a{}^{\bar{a}}\bar{j}_{\mu\{\bar{a},\bar{b}\}}M^{\bar{b}}{}_b. \tag{56}$$

We can write this more explicitly in the form

$$j_{\mu ab} = \frac{1}{2}M_a{}^{\bar{a}}(M_{\bar{a}}^{-c}\partial_\mu M_{c\bar{b}} + M_{\bar{a}}^{-c}\omega_{\mu c}{}^d M_{d\bar{b}} + M_{\bar{b}}^{-c}\partial_\mu M_{c\bar{a}}$$

$$+ M_{\bar{b}}^{-c}\omega_{\mu c}{}^d M_{d\bar{a}})M^{\bar{b}}{}_b, \tag{57}$$

which shows that $j_{\mu ab}$ is a linear function of $\omega_{\mu ab}$.

The torsion tensor is also a linear function of $\omega_{\mu ab}$. The covariant constancy of $e_{\mu a}$ implies that

$$\Gamma^\lambda{}_{\mu\nu} = e^{a\lambda}(\partial_\mu e_{\nu a} + \omega_{\mu a}{}^b e_{\nu b}). \tag{58}$$

Hence

$$C^\lambda{}_{\mu\nu} = \frac{1}{2}e^{a\lambda}(\partial_\mu e_{\nu a} + \omega_{\mu a}{}^b e_{\nu b} - \partial_\nu e_{\mu a} - \omega_{\nu a}{}^b e_{\mu b}). \tag{59}$$

If we define

$$C_{\lambda\mu\nu} = g_{\lambda\sigma}C^\sigma{}_{\mu\nu} \tag{60}$$

and

$$\omega_{\mu\lambda\nu} = e_\lambda{}^a e_\nu{}^b \omega_{\mu ab}, \tag{61}$$

then we obtain $\omega_{\mu ab}$ as a linear function of the torsion:

$$\omega_{\mu\lambda\nu} = C_{\lambda\mu\nu} - C_{\nu\mu\lambda} + C_{\mu\lambda\nu} + \hat{\omega}_{\mu\lambda\nu}, \tag{62}$$

where $\hat{\omega}_{\mu\lambda\nu}$ is the metric version of $\omega_{\mu\lambda\nu}$ and is given by

$$\hat{\omega}_{\mu\lambda\nu} = \frac{1}{2}(e_\nu{}^a\partial_\mu e_{\lambda a} - e_\lambda{}^a\partial_\mu e_{\nu a} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\nu\mu}). \tag{63}$$

We note that Eq. (47) relates the torsion tensor $C^\lambda{}_{\mu\nu}$ linearly to $j_{\mu ab}$:

$$C_{\sigma\mu\nu} + C^\lambda{}_{\lambda\mu}g_{\nu\sigma} - C^\lambda{}_{\lambda\nu}g_{\mu\sigma} = -X_{\sigma\mu\nu}, \tag{64}$$

where

$$X_{\sigma\mu\nu} = \frac{G}{F}j_{\sigma[\mu,\nu]} \tag{65}$$

and

$$j_{\sigma\mu\nu} = e_\mu{}^a e_\nu{}^b j_{\sigma ab}. \tag{66}$$

It follows that

$$C_{\sigma\mu\nu} = -X_{\sigma\mu\nu} - \frac{1}{2}X^\lambda{}_{\lambda\mu}g_{\nu\sigma} + \frac{1}{2}X^\lambda{}_{\lambda\nu}g_{\mu\sigma}. \tag{67}$$

Finally we have the linear relation

$$\omega_{\mu\lambda\nu} = -X_{\lambda\mu\nu} + X_{\nu\mu\lambda} - X_{\mu\lambda\nu} - X^\tau{}_{\tau\lambda}g_{\mu\nu} + X^\tau{}_{\tau\nu}g_{\mu\lambda} + \hat{\omega}_{\mu\lambda\nu}, \tag{68}$$

which determines $\omega_{\mu ab}$ in terms of the other dynamical variables.

For future reference and to show that the above equation can take a simple form in special cases we compute $j_{\lambda ab}$ when the matrix M takes a diagonal form, namely

$$M^a{}_{\bar{a}} = \Lambda_a \delta^a_{\bar{a}} \quad \text{and} \quad M^{\bar{a}}{}_a = \Lambda_a^{-1}\delta^{\bar{a}}_a. \tag{69}$$

We enforce an obvious correspondence between the values of the symbols a and $\bar{a}$ to give meaning to the δ symbols. We can compute $j_{\mu ab}$ from Eq. (57) to obtain

$$j_{\lambda ab} = \frac{\partial_\lambda \Lambda_a}{\Lambda_a}\eta_{ab} + \frac{1}{2}\omega_{\lambda ab}\left(1 - \frac{\Lambda_a^2}{\Lambda_b^2}\right), \tag{70}$$

with the simple result

$$j_{\lambda[a,b]} = \frac{1}{4}\omega_{\lambda ab}\left(2 - \frac{\Lambda_a^2}{\Lambda_b^2} - \frac{\Lambda_b^2}{\Lambda_a^2}\right). \tag{71}$$

Used in conjunction with Eq. (68) this leads to an easy evaluation of the vierbein connection.

IX. WEAK FIELD LIMIT

In order to apply it to planetary motion, galaxies and galactic clusters it is appropriate to examine the theory in the limit of weak gravitational fields. Since we are considering relatively local objects in an effectively flat background, we will choose the parameter $\gamma = 8$ in order to set the cosmological constant to zero.

The weak field limit has the form

$$e_{\mu a}=e^{(0)}_{\mu a}+h_{\mu a}, \tag{72}$$

where

$$e^{(0)}_{\mu a}e^{(0)}{}_\nu{}^a=\eta_{\mu\nu}. \tag{73}$$

Similarly we can set

$$\bar{e}_{\mu\bar{a}}=\bar{e}^{(0)}_{\mu\bar{a}}+\bar{h}_{\mu\bar{a}}, \tag{74}$$

where

$$\bar{e}^{(0)}_{\mu\bar{a}}\bar{e}^{(0)}{}_\nu{}^{\bar{a}}=\eta_{\mu\nu}. \tag{75}$$

The connections $\omega_{\mu ab}$, $\bar{\omega}_{\mu\bar{a}\bar{b}}$ and the scaling field ϕ are first order quantities. The matrix M has the form

$$M^a{}_{\bar{a}}=M^{(0)a}{}_{\bar{a}}+m^a{}_{\bar{a}}, \tag{76}$$

where $M^{(0)}$ is a constant Lorentz transformation. We have then

$$M^{(0)}{}^{\bar{a}}{}_a=M^{(0)}{}_a{}^{\bar{a}}. \tag{77}$$

We can use $M^{(0)}$ and its inverse and $e^{(0)}$ and $\bar{e}^{(0)}$ to convert superfixes and suffixes between the various bases. For example we have

$$m^a{}_b=m^a{}_{\bar{a}}M^{(0)}{}^{\bar{a}}{}_b. \tag{78}$$

The requirement that $\det M=1$ implies that

$$m^a{}_a=m^{\bar{a}}{}_{\bar{a}}=m^\mu{}_\mu=0. \tag{79}$$

The relationship between e and $\bar{e}$ implies that

$$\bar{h}_{\mu\bar{a}}=h_{\mu\bar{a}}+m_{\mu\bar{a}}+\phi e^{(0)}_{\mu\bar{a}} \tag{80}$$

or

$$\bar{h}_{\mu\nu}=h_{\mu\nu}+m_{\mu\nu}+\phi\eta_{\mu\nu}, \tag{81}$$

together with corresponding equations in other bases.

To lowest order,

$$R_{ab\mu\nu}=\partial_\mu\omega_{\nu ab}-\partial_\nu\omega_{\mu ab}. \tag{82}$$

Hence

$$R^\sigma{}_\mu=e^{(0)a\sigma}e^{(0)b\nu}R_{ab\mu\nu}, \tag{83}$$

so that

$$R^\sigma{}_\mu=\partial_\mu\omega_\nu{}^{\sigma\nu}-\partial_\nu\omega_\mu{}^{\sigma\nu}, \tag{84}$$

and

$$R=2\partial_\mu\omega_\nu{}^{\mu\nu}. \tag{85}$$

Again, in the lowest order approximation,

$$j_{\mu ab}=\partial_\mu m_{ab}+\omega_{\mu ab}-\bar{\omega}_{\mu ab}. \tag{86}$$

If we convert to the coordinate basis, we have

$$j_{\mu\lambda\tau}=\partial_\mu m_{\lambda\tau}+\omega_{\mu\lambda\tau}-\bar{\omega}_{\mu\lambda\tau}. \tag{87}$$

We can also evaluate $\bar{j}_\mu$. In this lowest approximation it coincides with j_μ. From Eq. (47) we have

$$\bar{j}_{\mu[\lambda,\tau]}=j_{\mu[\lambda,\tau]}=0. \tag{88}$$

That is

$$\partial_\sigma m_{[\lambda,\tau]}+\omega_{\sigma\lambda\tau}-\bar{\omega}_{\sigma\lambda\tau}=0. \tag{89}$$

From Eq. (47) we see that the torsion in the gravitational vierbein bundle vanishes. Explicitly we have

$$C^\lambda_{\mu\nu}=\frac{1}{2}(\partial_\mu h_\nu{}^\lambda-\partial_\nu h_\mu{}^\lambda+\omega_\mu{}^\lambda{}_\nu-\omega_\nu{}^\lambda{}_\mu)=0. \tag{90}$$

Equation (46) yields

$$\eta^{\mu\sigma}\partial_\mu(\partial_\sigma m_{\lambda\tau}+\omega_{\sigma\lambda\tau}-\bar{\omega}_{\sigma\lambda\tau})+m^2 m_{\{\lambda,\tau\}}=-8\pi F U_{\lambda\tau}, \tag{91}$$

where

$$U_{\lambda\tau}=\bar{T}_{\lambda\tau}-\frac{1}{4}\eta_{\lambda\tau}\bar{T}, \tag{92}$$

and we have assumed that $\bar{T}_{\mu\nu}$ and hence $U_{\mu\nu}$ are first order quantities. Making use of Eq. (89) we obtain the result

$$\eta^{\mu\sigma}\partial_\mu\partial_\sigma m_{\{\lambda,\tau\}}+m^2 m_{\{\lambda,\tau\}}=-8\pi F\left(\bar{T}_{\lambda\tau}-\frac{1}{4}\eta_{\lambda\tau}\bar{T}\right). \tag{93}$$

Because we set $\gamma=8$ in Eq. (45), we obtain

$$R_{\sigma\lambda}-\frac{1}{2}\eta_{\sigma\lambda}R=8\pi G\bar{T}_{\sigma\lambda}. \tag{94}$$

In the present approximation $m_{[\mu,\nu]}$ can be removed by gauge transformations of the form

$$\omega_{\sigma\lambda\tau}\rightarrow\omega_{\sigma\lambda\tau}+\partial_\sigma\phi_{\lambda\tau} \quad \text{and} \quad \bar{\omega}_{\sigma\lambda\tau}\rightarrow\bar{\omega}_{\sigma\lambda\tau}+\partial_\sigma\bar{\phi}_{\lambda\tau}. \tag{95}$$

We can assume therefore that in this approximation $m_{[\mu,\nu]}$ vanishes. Therefore $m_{\mu\nu}$ may be assumed symmetric. It satisfies

$$(\partial^2+m^2)m_{\mu\nu}=-8\pi F\left(\bar{T}_{\mu\nu}-\frac{1}{4}\eta_{\mu\nu}\bar{T}\right). \tag{96}$$

The gauge invariance referred to above means also that we are free to choose $h_{\mu\nu}$ to be symmetric with the result that $\bar{h}_{\mu\nu}$ is also symmetric. Under these circumstances we can solve Eq. (90) to yield

$$\omega_{\nu\lambda\mu} = \partial_{\mu} h_{\nu\lambda} - \partial_{\lambda} h_{\nu\mu}. \tag{97}$$

Equation (45) now implies that

$$\partial_{\mu}\partial_{\nu}h^{\nu}{}_{\sigma} + \partial_{\sigma}\partial_{\nu}h^{\nu}{}_{\mu} - \partial^{2}h_{\mu\sigma} - \partial_{\mu}\partial_{\sigma}h^{\nu}{}_{\nu}$$

$$- \eta_{\sigma\mu}(\partial_{\nu}\partial_{\tau}h^{\nu\tau} - \partial^{2}h^{\nu}{}_{\nu}) = 8\pi G\bar{T}_{\sigma\mu}. \tag{98}$$

We now refine our coordinate system by choosing the harmonic gauge

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0. \tag{99}$$

In the lowest approximation it yields

$$\partial_{\mu} h^{\mu}{}_{\lambda} = \frac{1}{2} \partial_{\lambda} h^{\mu}{}_{\mu}. \tag{100}$$

The equation of motion then becomes

$$\partial^{2}\left(h_{\mu\sigma} - \frac{1}{2}\eta_{\mu\sigma}h^{\tau}{}_{\tau}\right) = -8\pi G\bar{T}_{\mu\sigma} \tag{101}$$

or

$$\partial^{2}h_{\mu\sigma} = -8\pi G\left(\bar{T}_{\mu\sigma} - \frac{1}{2}\eta_{\mu\sigma}\bar{T}\right). \tag{102}$$

The equation for ϕ is

$$(\partial^{2} + m^{2})\phi = -8\pi F'\bar{T}. \tag{103}$$

To demonstrate how the theory matches up to the standard tests of general relativity we consider its implications for phenomena on a scale such as the solar system that is much smaller than the galactic scale. For such applications we can set $m=0$. We will return to the problem with the galactic scale parameter later.

By combining the above equations we find for the massless case

$$\partial^{2}\bar{h}_{\mu\nu} = -8\pi(G+F)\bar{T}_{\mu\nu} + 8\pi\left(\frac{1}{2}G + \frac{1}{4}F - F'\right)\eta_{\mu\nu}\bar{T}. \tag{104}$$

By making use of the result $\partial_{\sigma}\bar{T}^{\sigma}{}_{\nu} = 0$, we can show also that

$$\partial^{2}[\partial_{\mu}\bar{h}^{\mu}{}_{\nu} - \xi\partial_{\nu}\bar{h}^{\mu}{}_{\mu}] = 0, \tag{105}$$

where

$$\xi = \frac{\frac{1}{2}G + \frac{1}{4}F - F'}{G - 4F'}. \tag{106}$$

Assuming then that the relevant fields are sourced by the energy momentum tensor we can maintain the condition

$$\partial_{\mu}\bar{h}^{\mu}{}_{\nu} - \xi\partial_{\nu}\bar{h}^{\mu}{}_{\mu} = 0. \tag{107}$$

For arbitrary F' this gauge condition on $\bar{h}_{\mu\nu}$ is different from that on $h_{\mu\nu}$. However, the choice

$$F' = -\frac{1}{4}F \tag{108}$$

is of special interest. We have then $\xi = \frac{1}{2}$. As a result $\bar{h}_{\mu\nu}$ satisfies the same harmonic condition as $h_{\mu\nu}$:

$$\partial_{\mu}\bar{h}^{\mu}{}_{\nu} - \frac{1}{2}\partial_{\nu}\bar{h}^{\mu}{}_{\mu} = 0. \tag{109}$$

The wave equation for $h_{\mu\nu}$ also takes a significant form and becomes

$$\partial^{2}\bar{h}_{\mu\nu} = -8\pi(G+F)\left(\bar{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{T}\right). \tag{110}$$

It follows that for this choice of F' weak field gravity is related to the matter distribution exactly as in general relativity if we set Newton's constant $G_{N} = G + F$. Hereafter we will assume that F' has this special value and that Newton's constant is obtained in this way from the theory.

For example in a static situation where $\partial^{2} = -\nabla^{2}$, $\bar{T}_{00} = \bar{T} = \rho$ (the density of matter) we have

$$\nabla^{2}\bar{h}_{00} = 4\pi G_{N}\rho. \tag{111}$$

Of course we can interpret $\bar{h}_{00}$ as the gravitational potential experienced by a material particle. In addition the spatial part of the metric satisfies

$$\nabla^{2}\bar{h}_{ij} = 4\pi G_{N}\rho\delta_{ij}. \tag{112}$$

This is precisely the form for the spatial metric to yield Einstein's prediction for the deflection of light and to satisfy the time-of-flight measurements of radio signals [17,18].

The remaining solar system scale test is the precession of the orbit of Mercury. This requires a higher order correction than the Newtonian approximation of the weak field limit. For reasons of space we do not present this calculation here but we have checked the consistency of the theory on this point by examining the asymptotic behavior of the Schwarzschild-like solution. The result is that the refinement of the Newtonian potential that produces the precession is correctly given by the theory.

It is clear from the wave equation for $\bar{h}_{\mu\nu}$, Eq. (110), and the associated gauge condition, Eq. (109), that the gravitational waves emitted by a time dependent matter distribution will be exactly the same as predicted by general relativity (GR). The detection of these waves by ordinary matter will also be entirely conventional. It is reasonable to conclude that the observations of the slowing of a binary quasar and its conformity with the predictions of GR will be reproduced in our theory [23–25].

There are circumstances in which the bi-metric theory could show differences with general relativity. These would occur were there to be a form of matter that coupled directly to the metric $g_{\mu\nu}$. Such matter would act as a source for and

react to the unbarred metric. In the weak field limit it would therefore behave as if $G_N = G$ and if mixed with other matter would imply that the equivalence principle did not hold. Part of our theory of ordinary matter and gravity is that such anomalous matter is not present. This is the assumption we use throughout this paper.

X. ''DARK MATTER'' AND THE GALACTIC SCALE

In order to apply our theory to objects of galactic or extra-galactic size we restore the ''mass'' parameter m. Our hypothesis is that m^{-1} is a length of the order of 30 kpc. Assigning such a value to m^{-1} achieves the purpose that the theory exhibits three regions of length scale, namely (i) $0-1$ kpc in which conventional Newtonian gravity holds sway and for which the theory of the previous section is relevant, (ii) $1-100$ kpc, a transition region appropriate to galactic dynamics, and (iii) above 100 kpc in which Newtonian gravity with an enhanced coupling appears.

To analyze the gravitational effect of the galactic scale we set

$$\bar{h}_{\mu\nu} = h_{\mu\nu} + h'_{\mu\nu}, \qquad (113)$$

where

$$h'_{\mu\nu} = m_{\mu\nu} + \phi\eta_{\mu\nu}, \qquad (114)$$

while $h_{\mu\nu}$ satisfies Eq. (102) and $h'_{\mu\nu}$ satisfies

$$(\partial^2 + m^2)h'_{\mu\nu} = -8\pi F\left(\bar{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{T}\right). \qquad (115)$$

Therefore while a highly localized matter distribution of mass M at the origin yields

$$h_{00} = -GM\frac{1}{r} \qquad (116)$$

it gives rise to

$$h'_{00} = -FM\frac{e^{-mr}}{r} \qquad (117)$$

for the remaining part of the metric. We have then

$$\bar{h}_{00} = -\frac{GM}{r}\left(1 + \frac{F}{G}e^{-mr}\right). \qquad (118)$$

It follows that the effective gravitational coupling for $mr \ll 1$ is $G_N = G + F$, while at distances beyond the galactic scale for which $mr \gg 1$, the effective coupling is G. Our hypothesis is that $G_N < G$ so that gravity is weaker at short distances than at long distances. This is the basis of our explanation for ''dark matter.'' For this to be true we clearly must have $F < 0$. We set

$$F = -\epsilon G, \qquad (119)$$

with $\epsilon > 0$, so the above gravitational potential becomes

$$\bar{h}_{00} = -\frac{G_N M}{(1-\epsilon)}\frac{(1-\epsilon e^{-mr})}{r}. \qquad (120)$$

More generally for a matter distribution $\rho(\mathbf{r})$ we have

$$\bar{h}_{00}(\mathbf{r}) = -\int d^3\mathbf{r}'\rho(\mathbf{r}')\frac{G_N}{(1-\epsilon)}\frac{(1-\epsilon e^{-m|\mathbf{r}-\mathbf{r}'|})}{|\mathbf{r}-\mathbf{r}'|}. \qquad (121)$$

We believe that this choice of a negative sign for F is in fact the ''natural'' choice and stabilizes the theory against tumbling of one vierbein relative to the other [21]. A simplified analysis supporting this view is presented in the Appendix. Tumbling can turn a time direction in one vierbein into a space direction in the other if it is too severe. Such a situation would create difficulties for causal structure in the theory.

A. Galactic clusters

For objects much greater than galactic size such as galactic clusters, there is ample evidence that the observed matter is insufficient to account for the gravitational potential inferred from applications of the virial theorem to the motion of galaxies in the cluster [5,6]. Currently the picture of such clusters is that $5-10\%$ of the mass is galactic in origin with a further contribution from hot gas. The bulk of the gravitational potential is accounted for by dark matter. It is also significant that a detailed analysis of clusters suggests that the dark matter distribution follows that of the visible matter [6]. The clusters appear to be condensed versions of the local background.

The explanation of this effect is straightforward in our model. The appearance of dark matter is simply the consequence of an enhanced gravitational coupling of visible matter. The parameter ϵ represents the dark fraction of apparent matter. On the basis of the above observations we should expect $\epsilon \simeq 0.9 - 0.95$. Note that because our extragalactic dynamics is still Newtonian, although with an enhanced coupling, it is still possible to apply analyses of galactic clusters that rely on the virial theorem for an inverse square law of force between galaxies [5].

The inverse square law is also crucial for the argument relating the motion of the local group to optical flux due to galaxies [32]. The close alignment of the motion of the local group through the cosmic microwave background (CMB) with the net optical flux confirms that fluctuations in the distribution of visible and dark matter are closely correlated. Disparities in the visible and dark matter distributions would tend to destroy this alignment. In our theory of course, the distribution of dark matter is identical to that of visible matter of which it is merely a reflection. There is no room for any bias between visible and dark matter fluctuations.

We will see later that the re-interpretation of dark matter as an enhancement of the Newtonian constant at large scales extends also to the dynamics of the expanding universe.

This picture of strong large-scale gravity with weak short-scale gravity has already been proposed as the FLAG scenario by Sanders [29,30]. There are distinctions between this

model and our proposal. As emphasized by Sanders the FLAG model is an intuitive extension of the vector meson anti-gravity suggestion of Scherk [31]. Our model is a complete geometrical theory with a well-defined dynamical starting point. Some of the predictions of the FLAG model are not acceptable. In particular the gravitational deflection of light is determined by strong gravity and so is an order of magnitude greater than the Einstein deflection even on the scale of the solar system. In our model, as we have shown above, the deflection, along with all other short-scale phenomena, is consistent with the predictions of general relativity.

B. Galactic rotation curves

On a galactic scale we are concerned with the rotation curves of galaxies. The apparent asymptotic flatness of many of these curves is usually taken as the most direct evidence for dark matter [1–4]. Our explanation rests on the modification of the gravitational interaction described above. It turns out that by choosing a value for ϵ in the range suggested by large scale dynamics it is possible, using a disk model, appropriate for certain spiral galaxies, to compute rotation curves that exhibit the features of observed rotation curves.

As an example we construct a model galaxy with a thin disk of surface density

$$\sigma(r) = \sigma_d e^{-a_d r}. \tag{122}$$

The mass of the disk is $m_d = 2\pi\sigma_d/a_d^2$. The gravitational potential in the plane of the galaxy is

$$\psi(r) = -\frac{G_N \sigma_d}{1-\epsilon} \int d^2\mathbf{r}' e^{-a_d r'} \frac{1}{|\mathbf{r}-\mathbf{r}'|} (1 - \epsilon e^{-m|\mathbf{r}-\mathbf{r}'|}), \tag{123}$$

and the tangential rotational velocity, v, is given by

$$v^2 = \mathbf{r}\cdot\nabla\psi(r)$$

$$= \frac{G_N \sigma_d a_d}{1-\epsilon} \int d^2\mathbf{r}' e^{-a_d r'} \mathbf{r}\cdot\hat{\mathbf{r}}' \frac{1}{|\mathbf{r}-\mathbf{r}'|} (1 - \epsilon e^{-m|\mathbf{r}-\mathbf{r}'|}). \tag{124}$$

The galaxy NGC 3198 has a well-measured rotation curve and it is accepted that a simple exponential disk model gives a good account of its luminosity distribution [4]. We assume a mass distribution with the same exponential shape. In Fig. 1 we show the resulting rotation curve where we have chosen the galaxy parameters to be $a_d = 0.38$ kpc^{-1} and $m_d = 2.9 \times 10^{10} M_\odot$; these are quite close to the values of a previous dark matter analysis [4]. The theory parameters are chosen to be $\epsilon = 0.937$ and $m = 0.035$ kpc^{-1} or $m^{-1} = 28.6$ kpc. We stress that these parameters are not a best fit but merely the result of eyeball exploration. However, it is not easy to reduce the value of ϵ value by much and obtain a convincing shape. It is encouraging that ϵ does lie in the range we anticipated from our discussion of extragalactic structure.
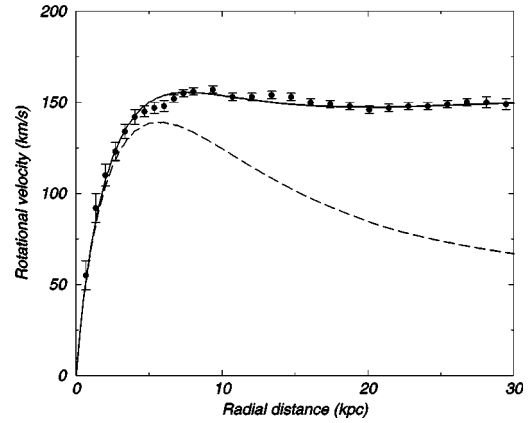

FIG. 1. Data for NGC3198 compared to exponential disk model with enhanced gravity (solid line) and the Newtonian (no enhancement) curve for the same mass (dashed line).

Figure 1 also shows the standard Newtonian curve for the assumed mass distribution. The difference between the two curves is normally attributed to dark matter. Here we achieve the same effect by means of our modified gravity theory. Figure 2 shows the extrapolation of the curve to larger radius. There is clearly a beginning of a falloff around 90–100 kpc although there is still a substantial rotational velocity out to 400 kpc. The rotation curve becomes Newtonian but with an enhanced gravitational coupling

$$v \simeq \sqrt{\frac{GM_d}{r}}. \tag{125}$$

Of course the challenge to our theory is to fit all galactic rotation curves simultaneously with common values for m and ϵ and plausible masses and mass distributions for the galaxies. This is no easy task since many galaxies have a more complicated structure than NGC3198 and detailed modeling will be required to determine the adequacy of our
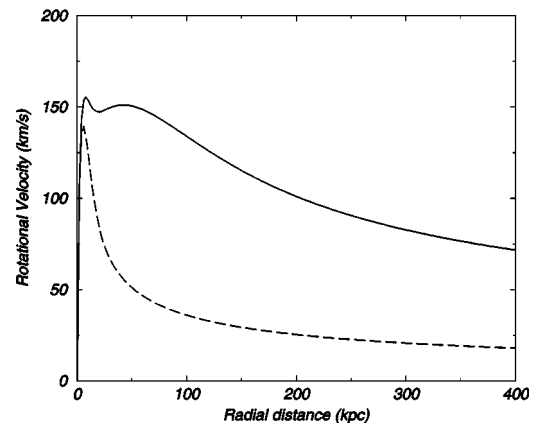

FIG. 2. The extended rotation curve for the exponential disk model with enhanced gravity (solid line) and the Newtonian (no enhancement) curve for the same mass (dashed line).

theoretical predictions. We intend to pursue this task in the future. However, one prediction of the theory does not require such a detailed attack. We predict that *all* galactic rotation curves will fall away in the range 100–200 kpc from the galactic center. Measurements in this range and beyond would be a direct test of our ideas.

Similar success in fitting the rotation curve of NGC3198 was achieved by Sanders within his FLAG model [29,30]. Indeed in parametrizing a modified Newtonian law for matter the two models are equivalent. For this reason Sanders also anticipated many of the points made above.

Our proposal of a galactic scale determining the shapes of rotation curves must be reconciled with the construction of a proposed universal scaling curve for galaxies [33]. This approach takes the optical radius of the galaxy as the significant length scale and uses a statistical approach to obtain well-defined rotation curves for classes of galaxies, thus revealing systematic relationships among them. Whether this galactic phenomenology can be accommodated within our proposal remains to be seen. There is freedom within our model to parametrize mass distributions and this flexibility gives some hope that the detailed modeling required to test our theory will provide an equally good description of galactic rotation curves. More importantly, however, if a common scale for the underlying structure of all galaxies were to be established, it would hold out the prospect of better distance measurements and hence the possibility of measuring the Hubble constant with more certainty.

C. Expanding universe

Finally we wish to show that when the matter density is small as in the present epoch, the expansion of the universe is controlled by the standard equations of general relativity but with Newton's constant, G_N, replaced by the enhanced constant G. In other words the dark matter component required to relate the visible matter to the observed Hubble constant is again supplied by the enhancement mechanism. Of course a complete reconciliation of the equations with observation requires a contribution to the energy density from the cosmological constant. This necessity can be catered for also in our theory.

To apply our theory we take the usual starting point that the spatial sections of the universe are isotropic and homogeneous. In the present approach we meet these requirements by assuming that there exist basis vierbein fields, $E_{\mu a}$, such that $E_{ta}E_{tb}\eta^{ab}=1$ and $E_{ta}E_{\mu b}\eta^{ab}=0$ for $\mu \neq t$. We choose our coordinates so that E_{ta} is constant in space-time and $E_{\mu a}$ depends only on coordinates on the spatial section when $\mu \neq t$. We finally fix the nature of the space-time model by choosing structure constants f_{bca} so that

$$\partial_\mu E_{\nu a} - \partial_\nu E_{\mu a} = -f_{bca}E_\mu{}^b E_\nu{}^c. \qquad (126)$$

Of course f_{bca} vanishes if any of the suffixes is 0 (timelike). For a spatially flat universe all structure constants have the value zero. For a universe of positive spatial curvature the spatial constants $f_{jki}=2\epsilon_{jki}$, where 2 is a convenient normalization, and for a negatively curved universe $f_{jki}=\delta_{ij}n_k - \delta_{ik}n_j$ where n_k is an arbitrary unit three-vector.

We construct the expanding universe by choosing the gravitational vierbein to have the form

$$e_{\mu a}=A_a E_{\mu a}, \qquad (127)$$

where

$$A_0=1 \quad \text{and} \quad A_i=A(t), \qquad (128)$$

A similar structure is maintained for the matter vierbein by requiring the transformation matrix M to be diagonal. That is (we equivalence the labels a and $\bar{a}$ in the obvious way),

$$\bar{e}_{\mu\bar{a}}=e^\phi \Lambda_a e_{\mu a}=\bar{A}_a E_{\mu a} \quad \text{and} \quad \bar{A}_a=\Lambda_a A_a e^\phi, \qquad (129)$$

where $\Lambda_0=\Lambda(t)$ and $\Lambda_i=\Lambda_S(t)$. The requirement that $\det M=1$ implies that $\Lambda_S=\Lambda^{-1/3}$.

Using the above information and Eq. (63) we find that

$$\hat{\omega}_{i0j}=-\hat{\omega}_{ij0}=\frac{\dot{A}}{A}\delta_{ij} \qquad (130)$$

and

$$\hat{\omega}_{ijk}=-\frac{1}{2A}(f_{ijk}+f_{kij}+f_{kji}). \qquad (131)$$

All other components vanish. A similar structure is found for ω_{abc}. From Eq. (71) we obtain

$$\omega_{i0j}=-\omega_{ij0}=\frac{\dot{A}}{AE}\delta_{ij}, \qquad (132)$$

where

$$E=1-\frac{G}{4F}\left(2-\frac{\Lambda^2}{\Lambda_S^2}-\frac{\Lambda_S^2}{\Lambda^2}\right) \qquad (133)$$

and

$$\omega_{ijk}=\hat{\omega}_{ijk}. \qquad (134)$$

From Eq. (14) we find, for the relevant components of the curvature tensor,

$$R_{00}=-\frac{3}{A}\partial_t\left(\frac{\dot{A}}{E}\right) \qquad (135)$$

and

$$R_{ij}=\left[\frac{1}{A}\partial_t\left(\frac{\dot{A}}{E}\right)+2\left(\frac{\dot{A}}{AE}\right)^2\right]\delta_{ij}-\frac{1}{4A^2}(f_{kjl}+f_{ljk}+f_{lkj})$$

$$\times(f_{lik}+f_{kil}+f_{kli})+\frac{1}{2A^2}(f_{jik}+f_{kji}+f_{kij})f_{lkl}. \qquad (136)$$

If we evaluate this expression for the three curvature cases, we find

$$R_{ij}=\left[\frac{1}{A}\partial_t\left(\frac{\dot{A}}{E}\right)+2\left(\frac{\dot{A}}{AE}\right)^2+\frac{2k}{A^2}\right]\delta_{ij}, \qquad (137)$$

where conventionally $k=0,\pm1$ according as the spatial curvature is zero, positive or negative. We also have, from Eq. (70),

$$j_{tab}j_t{}^{ba}=\frac{4}{3}\left(\frac{\acute\Lambda}{\Lambda}\right)^2 \qquad (138)$$

and

$$j_{iab}j_j{}^{ba}=\frac{1}{2}\left(\frac{\acute A}{AE}\right)^2\left(2-\frac{\Lambda^2}{\Lambda_S^2}-\frac{\Lambda_S^2}{\Lambda^2}\right)\delta_{ij}. \qquad (139)$$

If we denote the matter energy density and pressure by ρ and p respectively, then from Eq. (45) we obtain

$$-\frac{3}{A}\partial_t\left(\frac{\acute A}{E}\right)-\frac{4G}{3F}\left(\frac{\acute\Lambda}{\Lambda}\right)^2-\frac{G}{F'}\dot\phi^2+\frac{G}{F}\frac{m^2}{8}(\Lambda^2+\Lambda^{-2}+3\Lambda_S^2$$

$$+3\Lambda_S^{-2}-\gamma)+\frac{G}{F'}\frac{m^2}{2}\phi^2=4\pi Ge^{4\phi}(\rho+3p) \qquad (140)$$

and

$$\frac{1}{A}\partial_t\left(\frac{\acute A}{E}\right)+2E\left(\frac{\acute A}{AE}\right)^2+\frac{2k}{A^2}-\frac{G}{F}\frac{m^2}{8}(\Lambda^2+\Lambda^{-2}+3\Lambda_S^2$$

$$+3\Lambda_S^{-2}-\gamma)-\frac{G}{F'}\frac{m^2}{2}\phi^2=4\pi Ge^{4\phi}(\rho-p). \qquad (141)$$

From Eq. (46) we find

$$\partial_t\left(\frac{\acute\Lambda}{\Lambda}\right)+3\frac{\acute A}{A}\frac{\acute\Lambda}{\Lambda}-\frac{3}{2}\left(\frac{\acute A}{AE}\right)^2\left(\frac{\Lambda^2}{\Lambda_S^2}-\frac{\Lambda_S^2}{\Lambda}\right)$$

$$+\frac{3m^2}{16}(\Lambda^2-\Lambda^{-2}-\Lambda_S^2+\Lambda_S^{-2})=-6\pi Fe^{4\phi}(\rho+p). \qquad (142)$$

Finally from Eq. (49) we obtain

$$\ddot\phi+\frac{3\acute A}{A}\dot\phi+m^2\phi=-8\pi F'e^{4\phi}(\rho-3p). \qquad (143)$$

Equations (140) and (141) can be combined to eliminate the second derivative in t:

$$E\left(\frac{\acute A}{AE}\right)^2+\frac{k}{A^2}-\frac{2}{9}\frac{G}{F}\left(\frac{\acute\Lambda}{\Lambda}\right)^2-\frac{1}{6}\frac{G}{F'}\dot\phi^2-\frac{G}{F}\frac{m^2}{24}(\Lambda^2+\Lambda^{-2}$$

$$+3\Lambda_S^2+3\Lambda_S^{-2}-\gamma)-\frac{G}{F'}\frac{m^2}{6}\phi^2=\frac{8\pi G}{3}e^{4\phi}\rho. \qquad (144)$$

If we multiply this equation by A^2 and differentiate with respect to t, we can use the above equations of motion to deduce that

$$\dot\rho=\left(\frac{\acute\Lambda}{\Lambda}-3\frac{\acute A}{A}-3\dot\phi\right)(\rho+p). \qquad (145)$$

This is easily re-expressed as

$$\partial_t\left(\rho\frac{e^{3\phi}A^3}{\Lambda}\right)+p\,\partial_t\left(\frac{e^{3\phi}A^3}{\Lambda}\right) \qquad (146)$$

or

$$\partial_t(\rho\bar A^3)+p\,\partial_t(\bar A^3), \qquad (147)$$

where $\bar A=e^\phi A\Lambda^{-1/3}$ is the cosmic radius parameter appropriate to matter. Equation (147) is therefore the standard equation for the conservation of the energy-momentum tensor in this special case, as we should have expected.

The above equations are rather complicated but can be simplified if we ask how they might be applied to our expanding universe in which the pressure vanishes and the matter density is low and getting lower. In the absence of matter Eq. (145) has a solution for which $\phi\equiv0$. For weak density we assume that there is a solution for which $\phi\simeq O(\rho)$. We have then, as a leading approximation,

$$\ddot\phi+\frac{3\acute A}{A}\dot\phi+m^2\phi=-8\pi F'\rho. \qquad (148)$$

The form of this equation suggests that, ignoring transients, the solution is to a good approximation

$$\phi=-\frac{8\pi F'}{m^2}\rho=-\frac{2\pi\epsilon G}{m^2}\rho. \qquad (149)$$

We expect then that ϕ will remain $O(\rho)$. If we set $\Lambda=e^\xi$ and omit all terms $O(\xi^2)$, then $E\simeq1$ and Eq. (143) becomes

$$\ddot\xi+3\frac{\acute A}{A}\dot\xi+\left[m^2-8\left(\frac{\acute A}{A}\right)^2\right]\xi=-6\pi F\rho. \qquad (150)$$

Again we expect there to be a solution $\xi=O(\rho)$. If now we neglect all terms $O(\rho^2)$, we obtain the equations

$$\left(\frac{\acute A}{A}\right)^2+\frac{k}{A^2}=\frac{8\pi G}{3}(\rho+\rho_{CC}), \qquad (151)$$

where the energy density associated with the cosmological constant, ρ_{CC}, is given by

$$\rho_{CC}=\frac{m^2}{8\pi F}\left(1-\frac{\gamma}{8}\right)=\frac{m^2(1-\epsilon)}{8\pi G_N\epsilon}\left(\frac{\gamma}{8}-1\right) \qquad (152)$$

and

$$\partial_t(\rho A^3)=0. \qquad (153)$$

These are the standard equations for the expanding universe but with Newton's constant G_N replaced by the enhanced constant G. To lowest order in ρ the parameter t is also the proper time of comoving matter in its own metric since the

two metrics coincide under these circumstances. It follows that Hubble's constant $H_0 = \dot{A}/A$ and the current deceleration parameter is $q_0 = -\ddot{A}A/\dot{A}^2$ and that the critical density for a flat universe is

$$\rho_c = \frac{3H_0^2}{8\pi G} = \frac{3H_0^2}{8\pi G_N}(1-\epsilon). \qquad (154)$$

So ρ_c in our theory is between 1/10th and 1/20th the value appropriate to standard general relativity. This brings it within range of visible matter [34]. In fact recent measurements of distant supernovas [35] lead to an estimate of q_0 that suggests $\rho = 0.3 \times \rho_c$ and $\rho_{CC} = 0.7 \times \rho_c$ (assuming a flat universe [36]) with the further implication that in the present theory, visible matter may support the current expansion of the universe albeit with the help of a cosmological constant.

XI. CONCLUSIONS

In this paper we have constructed a modified theory of gravity that fits all the standard tests of general relativity that can be made on the scale of the solar system including the bending of light by the sun, time delay measurements and the precession of Mercury's orbit. The theory is a bi-metric theory of a novel kind with a very geometrical structure. It has the flexibility to permit the introduction of a galactic length-scale of roughly 30 kpc. Gravitational effects of matter at distances below the galactic scale, in the solar system for example, are of a conventional kind and of a strength determined by Newton's constant, G_N. This outcome is achieved in the theory as the result of a competition between two gravitational effects, a strong attraction and a repulsion that is nearly as strong. Over the range of the galactic scale the theory allows the repulsion to fade out exponentially leaving the much stronger underlying gravity to show through. The gravitational effects of matter on this large scale are also conventional in character but of a strength determined by the much stronger underlying gravitational constant $G \simeq 10-20 \times G_N$. These results are consistent with observations of galactic clusters and also apply to the expansion of the universe as a whole. As a result the critical density for a flat universe is between 1/10th and 1/20th of the value calculated in general relativity. This makes it more plausible that visible baryonic matter can support the expansion of the universe without dark matter, though supplemented by a cosmological constant.

The theory provides a natural explanation for the tendency of dark matter to follow the distribution of visible matter since the former is simply an amplification of the latter. Dark matter does not have a separate dynamics of its own. On this basis the apparent dark matter is not in any way biased in its fluctuation structure relative to visible matter.

We showed that at least for the galaxy NGC3198 the theory can reproduce the rotation curve out to 30 kpc (the end of the measured range) equally as well as dark matter models. The theory also predicts that the rotation curve will fall away in the 100–200 kpc range and eventually tend to a standard inverse square root of distance behavior but with an apparent mass 10–20 times the expected mass of the galaxy

($\sim 3 \times 10^{10} M_\odot$ for NGC3198) if the conventional Newtonian constant is used for the estimate. Our theory however assigns the expected mass but uses the enhanced value for the gravitational constant appropriate at extra-galactic distances. This phenomenon should appear for all galaxies. It is therefore important for the theory to test rotation curves at distances of 100–200 kpc from the galactic center. It is also important to test the effectiveness of the theory in a range of galaxies and to arrive at a consistent picture with a common set of gravitational parameters. This task requires a good understanding of the structure of individual galaxies and accurate knowledge of their distance. However, we note that the establishment of a fixed scale associated with all galaxies would be of considerable help in establishing galactic distances.

In addition to the incomplete phenomenological analyses discussed above there are many issues yet to be explored in our bi-metric theory. Of particular importance are the development of density fluctuations, dynamics and formation of galaxies using the modified gravitational law, and evolution of the early universe. In this last context the ability of bi-metric theories to support anomalous propagation of signals (VSL phenomena) will be of great importance. The existence within the theory of black holes is also a topic that it would be very interesting to resolve. On intuitive grounds it seems reasonable to suppose that black holes will exist in our theory but the nature of the horizon may be more complicated than in general relativity because of its double light-cone structure.

APPENDIX

Because of the indefinite character of the local Lorentz metric, it turns out that the degrees of freedom in the linking field M can lie in a number of distinct sectors. We can illustrate this idea by considering a configuration for which M is non-trivial only in the (0,1) plane. By confining attention to this plane we can treat M as a two-dimensional matrix of the form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad (A1)$$

where $ad - bc = 1$. The quantity

$$f = a^2 + d^2 - b^2 - c^2 \qquad (A2)$$

is invariant under separate left and right (two-dimensional) Lorentz transformations of M. We can therefore set

$$M = LSL', \qquad (A3)$$

where L and L' are independent Lorentz transformations and S can be given a special form that depends on the value of f as follows:

For $f \geq 2$ the special form for S is

$$S = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad (A4)$$

where $-\infty < \lambda < \infty$ and $f = \lambda^2 + \lambda^{-2}$. For $-2 \leq fg \leq 2$ S has the form

$$S = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad (A5)$$

where $0 \leq \theta \leq \pi/2$ and $f = 2\cos^2\theta - 2\sin^2\theta = 2\cos 2\theta$. The remaining possibility is $f \leq -2$ when S is given by

$$S = \begin{pmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{pmatrix}. \qquad (A6)$$

where $f = -(\lambda^2 + \lambda^{-2}) \leq -2$.

The three ranges together cover all possibilities for the value of f. We therefore can conclude that the matrix M will be gauge equivalent to the appropriate special form S provided that form is chosen according to the value of f.

If f lies in the first range, then

$$j_\mu \simeq \partial_\mu \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{\partial_\mu \lambda}{\lambda} \right), \qquad (A7)$$

and the linking action will have the form

$$I_L \simeq -\frac{2}{16\pi\epsilon G} \int d^4 x J g^{\mu\nu} \left(\frac{\partial_\mu \lambda}{\lambda} \right) \left(\frac{\partial_\nu \lambda}{\lambda} \right). \qquad (A8)$$

If f lies in the second range, then

$$j_\mu \simeq \partial_\mu \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \partial_\mu \theta, \qquad (A9)$$

and the linking action will have the form

$$I_L \simeq \frac{2}{16\pi\epsilon G} \int d^4 x J g^{\mu\nu} \partial_\mu \theta \partial_\nu \theta. \qquad (A10)$$

The third range can be dealt with similarly.

A key point is the change sign in the action between the two ranges. The second range represents tumbling of the matter vierbein relative to the gravitational vierbein. This is an undesirable effect since it can lead to an interchange of space and time. However, the relevant sign for the action suggests that a positive energy is generated by motion of this kind with the result that it is likely to be restrained. In the first region the kinetic energy term is negative, suggesting an instability for this kind of straining between the veirbeins. This is not obviously a bad thing. An effect of this kind can be discerned in the equations for the expansion of an Friedman-Robertson-Walker (FRW-) type universe discussed above. More complicated possibilities of a similar kind exist for the four dimensional case.

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