Stars and black holes in varying speed of light theories

João Magueijo
Theoretical Physics, The Blackett Laboratory, Imperial College, Prince Consort Road, London, SW7 2BZ, United Kingdom
(Received 31 July 2000; published 18 January 2001)

We investigate spherically symmetric solutions to a recently proposed covariant and locally Lorentz-invariant varying speed of light theory. We find the metrics and variations in c associated with the counterpart of black holes, the outside of a star, and stellar collapse. The remarkable novelty is that c goes to zero or infinity (depending on parameter signs) at the horizon. We show how this implies that, with appropriate parameters, observers are prevented from entering the horizon. Concomitantly stellar collapse may end in a ''Schwarzchild radius'' remnant. We then find formulas for gravitational light deflection, gravitational redshift, radar echo delay, and the precession of the perihelion of Mercury, highlighting how these may differ distinctly from their Einstein counterparts but still evade experimental constraints. The main tell-tale signature of this theory is the prediction of the observation of a different value for the fine structure constant, α, in spectral lines formed in the surface of stars. We close by mentioning a variety of new classical and quantum effects near stars, such as aging gradients and particle production.

DOI: 10.1103/PhysRevD.63.043502 PACS number(s): 98.80.Hw

I. INTRODUCTION

The possibility that the speed of light c might vary has recently attracted considerable attention [1–21]. Most notably, in a cosmological setting, temporal variations in c have been shown to solve the so-called cosmological puzzles — the horizon, flatness, and Lambda problems of big-bang cosmology. At a more conceptual level it is clear that varying speed of light (VSL) theories require extreme departures from the standard framework of physics, since they contradict the leading postulate behind relativity and Lorentz invariance. A number of alternative implementations for VSL have been discussed, involving either hard [2] or soft [1] breaking of Lorentz invariance.

In a recent paper [19] it was shown that contrary to popular belief it is possible to set up covariant and locally Lorentz invariant VSL theories, as long as these concepts are subject to very minimal generalizations. As a matter of fact the necessary generalizations glean from the usual definitions all that is operationally meaningful, in the sense that the aspects they preserve are exactly those which can be the outcome of experiment. Such a formulation arguably provides the most conservative VSL theory one may set up. It is found that in such theories the local value of c is determined via a differential equation, containing as source terms the cosmological constant and the matter Lagrangian.

Naturally in such theories c varies not only in time (over cosmological time scales) but also in space, once the inhomogeneity of the Universe is taken into account [22]. In the simplest case one should investigate such a phenomenon by seeking static and spherically symmetric solutions. Such is the purpose of this paper. We investigate VSL solutions representing the counterpart of black holes, the exterior of a star, and stellar collapse. It should be stressed that it is such solutions, not the cosmological ones, that bear relevance to many experimental tests (a point entirely missed by [20]).

In Sec. II we start by reviewing the key aspects of the theory proposed in [19]. Then in Sec. III we consider static spherically symmetric solutions, both in isotropic and radial coordinates. We find the limit under which the Schwarzchild solution is still a solution of our theory, and note that c goes to zero or infinity at the horizon. We also find the most general solution, which is similar to the solution found in [24]; however the relationship between the various parameters in [24] is new. In all of these solutions we find that c must go to zero or infinity at the horizon. This is not accidental, and in Sec. IV we sketch a proof showing why this is generally the case.

The last result has two very significant implications. The first is discussed in Sec. V, and corresponds to the naive expectation that if c goes to zero fast enough at the horizon then no observer can actually reach it. Indeed c still acts as a local speed limit. This insight proves to be true, even when a number of complications are taken into account. First the field c may also act as a gravitational field, pushing free-falling particles off geodesics, accelerating or braking them. Secondly, as c changes so do all fine structure constants, and also the time rates of the interactions they promote. One should attach the definition of time to these rates, and examine the problem of an observer falling into a black hole from the point of view of the number of ticks of such ''interaction time.'' We find that when all this is taken into account, there is still a large region of parameter space for which reaching the horizon requires infinite free-falling time. VSL black holes are therefore not covered by an ''horizon'' but instead, the horizon represents an edge of space-time to be put on the same level as the asymptotic spatial infinity. This has the implication that the singularity may be excised from the manifold — we conjecture that perhaps one can get rid of all singularities in a similar way in VSL theories.

Another interesting result is described in Sec. VI: stellar collapse may take infinite interaction time when viewed by an observer on the surface of the collapsing star. This follows directly from the above considerations, and implies that the end point of stellar collapse must be a Schwarzchild remnant. As this is formed the speed of light goes to zero for all points inside the star, thereby freezing all processes and preventing the formation of a singularity.

The final part of this paper, contained in Sec. VII, is devoted to possible experimental tests for this theory based upon solar system gravitational physics. We concentrate on the classical tests of GR (general gelativity), leaving to a future publication the analysis of more recent (but more complex) experiments, such as the binary pulsar PSR 1913 + 16 [23]. We examine the effects of VSL upon the orbits of planets, gravitational light deflection, and the radar echo time-delay. The real novelty is, however, the effects upon the spectral lines formed at the surface of stars, for which our theory predicts a fine structure different from laboratory measurements. We find that it is possible to reproduce all the standard GR tests, and still have a non-negligible spectral effect. The application of techniques similar to the ones developed by Webb *et al.* [25] should put this theory to the test.

We conclude with a brief qualitative discussion of an assortment of exotic new phenomena expected in the vicinity of very massive stars in VSL theories.

II. SUMMARY OF THE THEORY

We first summarize the covariant and locally Lorentz invariant VSL theory proposed in [19]. In this theory the speed of light plays 3 distinct roles (corresponding to independent aspects of the theory) parametrized by numbers q, κ, a, b, and β.

At its most innocuous, VSL is nothing but a theory predicting changing fine structure constants $\alpha_i = g_i^2/(\hbar c)$ (in which i labels the various interactions, and g_i are charges), with fixed ratios α_i/α_j. Choosing units such that changes are attributed primarily to c is useful simply because they lead to a simpler picture. A fixed-c dual theory may be obtained by a change of units, but the ensuing local dynamics is then rather contrived. Also, important global features may be missed in fixed-c units (e.g. the trans-eternal regions, or the black hole edges discussed in [19]). In [19] we then required that the matter Lagrangian should not depend on c; this fact alone fixes the scaling with c of all Lagrangian parameters up to the $\hbar(c)$ dependence. In particular particle rest energies scale like $E_0 \propto \hbar c$, and all gauge charges like $g_i \propto \hbar c$. Taking $\hbar c \propto c^q$ we then have $\alpha_i \propto g_i \propto \hbar c \propto c^q$. In summary, c's first role is to parametrize changes in all ''constants'' in minimal changing α theories for which the Lagrangian itself is required to remain invariant.

One must then endow c with its own action, and note that c appears in the gravitational Lagrangian as part of a conversion factor between curvature and energy density. As pointed out in [19] the definition of c in terms of a field, its dynamics, and its coupling to gravity and matter may be defined in many different ways. In the simplest $\psi = \log(c/c_0)$, and

$$S = \int d^4x \sqrt{-g} \left(e^{a\psi}(R + \mathcal{L}_\psi) + \frac{16\pi G}{c_0^4} e^{b\psi} \mathcal{L}_m \right) \tag{1}$$

where we shall not impose $a - b = 4$ (and we have set $\Lambda = 0$). The simplest dynamics for ψ derives from

$$\mathcal{L}_\psi = -\kappa(\psi) \nabla_\mu \psi \nabla^\mu \psi. \tag{2}$$

Hence the second aspect of c, as a dynamical field and coupling constant, is parameterized by a, b and κ. In [19] it was shown how this aspect of the theory allows for analogies with some string theories to be made [26].

Thirdly, the theory proposed in [19] is covariant and locally Lorentz invariant, in a generalized sense which accommodates a varying c. The generalization is trivial; and essentially amounts to the use of an x^0 coordinate in all differential geometry formulas. The only significant difference is that if c varies, local measurements of space dx and time dt do not generally lead to closed forms (i.e. $d^2 t \neq 0$ or $d^2 x \neq 0$), leading to a fiber bundle structure where usually one finds a tangent bundle. However they admit integrating factors, so that $dt\,\psi^\beta$ and $dx\,\psi^{\beta-1}$ are closed forms. Hence c appears in a third role, as a conversion factor between space and time, and as an integrating factor defining the change of units which would convert the theory into a fixed c standard covariant and locally Lorentz invariant theory. The parameter β needs not be related to any other parameters, but we considered the cases $\beta = 3 - q/2$ and $\beta = 1 - q/2$.

The equations for such a theory are

$$G_{\mu\nu} = \frac{8\pi G}{c_0^4 e^{(a-b)\psi}} T_{\mu\nu} + \kappa \left(\nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} g_{\mu\nu} \nabla_\delta \psi \nabla^\delta \psi \right) + e^{-a\psi} (\nabla_\mu \nabla_\nu e^{a\psi} - g_{\mu\nu} \Box e^{a\psi}) \tag{3}$$

and

$$\Box \psi + a \nabla_\mu \psi \nabla^\mu \psi = \frac{8\pi G}{c_0^4 e^{(a-b)\psi} (2\kappa + 3a^2)} (aT - 2b\mathcal{L}_m). \tag{4}$$

A change of units rephrases these theories as Brans-Dicke theories [24] only when $b + q = 0$ and $\beta = 1 - q/2$. However there is a formal analogy between action (1) and Brans-Dicke theory in the Jordan frame, established with the following identifications:

$$\phi_{bd} = e^{a\psi} \tag{5}$$

$$\omega_{bd} = \frac{\kappa}{a^2} \tag{6}$$

$$T^{bd}_{\mu\nu} = e^{b\psi} T_{\mu\nu}. \tag{7}$$

The analogy is always valid in vacuum, but breaks down when $b \neq 0$ inside matter distributions. Indeed $T^{bd}_{\mu\nu}$ then depends on ϕ_{bd} *in the Jordan frame*. Bearing this in mind, we shall make use of this analogy for reading off solutions from [24]. However careful rederivation will be required to account for novelties induced by $b \neq 0$.

A further analogy with scalar-tensor theories arises from the conformal equivalence of various $\{a, b, \kappa\}$ theories. Conformal transformations do not change the speed of light, mapping VSL theories into VSL theories; but the gravitational action is modified leading to different values for a, b, and κ. This may be used to simplify the dynamics, in particular reducing it to Brans-Dicke dynamics. The relevant transformations are spelled out in Appendix B, where a set of results is derived which may then be used to provide alternative derivations for many results in the main body of this paper.

However, as stressed in Appendix B, the frame realizing Brans-Dicke dynamics can only be achieved with very restricted forms of matter. In particular, one must require that $\mathcal{L}_m$ be homogeneous in the metric; clearly far from true in general. In the particular case in which we only consider classical point particles the Lagrangian takes the form

$$S=-\frac{E_0}{2\alpha}\int d\lambda[-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^\alpha \qquad (8)$$

in which α can *a priori* be any number. In metric theories of gravity the value of α is irrelevant, because u^2 (with $u=\dot{x}$) is a constant. One usually takes $\alpha=1/2$, so that the action becomes the length of the world-line. The value of α is however physically relevant if $b\neq 0$ [19], and the results in this paper do depend on α. Arguments for $\alpha=1$ were put forward in [19], and we shall adopt this assumption in the main body of this paper. This implies that for classical point particles $\mathcal{L}_m=-\rho/2$, with ρ the energy density. Hence $\mathcal{L}_m$ is homogeneous degree 1 in the metric.

For general forms of matter, minimal coupling, that is, the requirement that $\mathcal{L}_m$ does not depend on c, is not conformally invariant. Therefore a conformal frame (and so a set of a and b) is picked for its simplicity in describing nongravitational physics (a point clearly made in [31]). This renders the construction described in Appendix B a useful mathematical tool, but with limited physical meaning, except when the generality of $\mathcal{L}_m$ can be swept under the carpet.

III. VACUUM SPHERICALLY SYMMETRIC SOLUTIONS

Let us consider static spherically symmetric (SSS) solutions. We shall work with both radial coordinates:

$$ds^2=-Bd\xi^2+Adr^2+r^2d\Omega^2 \qquad (9)$$

(where $d\Omega^2=d\theta^2+\sin^2\theta d\phi^2$) and with isotropic coordinates:

$$ds^2=-Fd\xi^2+G(d\rho^2+\rho^2d\Omega^2). \qquad (10)$$

Recall that as in [19] the usual tools of differential geometry are unaffected under the condition that an x^0-type of coordinate is used, here denoted by $d\xi=cdt$.

A. The VSL Schwarzchild solution

The simplest SSS solution to VSL is the Schwarzchild solution:

$$ds^2=-\left(1-\frac{2Gm}{c_\infty^2 r}\right)d\xi^2+\frac{dr^2}{1-\frac{2Gm}{c_\infty^2 r}}+r^2d\Omega. \qquad (11)$$

This is valid whenever the field ψ does not gravitate, e.g. in the bimetric theory discussed in the Appendix of [19] (a case developed further in Appendix A). This is also true in the theory described above in the limit $\kappa,a\rightarrow 0$. We may then have κ/a and κ/b finite, or κ/a and b finite if $\kappa/a\gg b$. As we will see the latter case distinguishes itself by predicting non-geodesic motion (but no corrections to the metric) while the former predicts geodesic motion.

The horizon is at $r_h=2Gm/(c_\infty^2 r)$, and the mass m is identified by comparing g_{00} and the weak field solution to this theory. Later we shall see that m need not be the Keplerian mass, if $b\neq 0$ (the case in which planets do not follow geodesics).

Integrating Eq. (4) with metric (11) leads to the exact solution

$$\psi=\frac{b-a}{2\kappa}\log\left(1-\frac{2Gm}{c_\infty^2 r}\right) \qquad (12)$$

in which the factor $(b-a)/(2\kappa)$ can be found using the weak field limit. Hence

$$c=c_\infty\left(1-\frac{2Gm}{c_\infty^2 r}\right)^{(b-a)/2\kappa}. \qquad (13)$$

We see that the speed of light goes to either zero or infinity at the horizon depending on the couplings, a property we shall prove in general in Sec. IV.

Physically the effect of the coupling parameters' signs and relative magnitudes is as follows. Let $\kappa>0$ so that the energy in the VSL field ψ is positive (but negligible, since $\kappa\rightarrow 0$). The field ψ is then driven by direct couplings to matter and to gravity, with strengths proportional to the couplings b and a respectively [cf. Eq. (54) of [19]]. If both b and a are positive the first coupling drives c to decrease close to matter concentrations, the second to increase. If $b=a$ (such as in the case of the dilaton coupling at tree level, as discussed in [19]) the speed of light does not change near matter concentrations. If $b>a$ light slows down close to massive bodies; if $b<a$ it speeds up. In either case, we found that near a black hole's horizon something extreme must happen: c must go to either zero or infinity. The fact that something extreme must happen is due to the structure of space-time, and can be linked to the usual proofs of the no-hair theorem as we shall see. The choice between the two options is made by the relative strengths of the a and b couplings, and follows whatever trend in c is already present in the weak field region.

The solution we have just found will be extremely useful in clarifying the meaning of more complicated solutions. It preserves the simplicity of the Schwarzchild solution while allowing for a variety of non-gravitational VSL effects to be present.

B. Brans-Dicke type of solutions

Given the formal analogy in vacuum between VSL theories and Brans-Dicke theories, we may use [24] to write the following exact solution:

$$ds^2 = -F^{2/\lambda} d\xi^2 + \left(1+\frac{\rho_0}{\rho}\right)^4 F^{2(\lambda-C-1)/\lambda}(d\rho^2+\rho^2 d\Omega^2) \tag{14}$$

$$c = c_0 F^{C/a\lambda} \tag{15}$$

with

$$F = \frac{1-\rho_0/\rho}{1+\rho_0/\rho} \tag{16}$$

$$\lambda^2 = (C+1)^2 - C(1-\kappa C/(2a^2)). \tag{17}$$

However, the weak field limit imposes a relation between $\{C,\lambda,\rho_0\}$ and $\{a,b,\kappa,m\}$ which goes beyond the identifications (5)–(7). This is due to the fact that when ''Brans-Dicke'' language is adopted for VSL theories the matter Lagrangian now depends on ϕ_{bd}, when $b\neq 0$ [cf. Eq. (7)].

Mimicking the weak field calculation presented in [24], we find that Eqs. (3) and (4) lead to

$$\psi = \frac{a-b}{3a^2+2\kappa}\frac{2m}{r} \tag{18}$$

$$-g_{00} = 1 - \frac{4m}{r}\frac{2a^2+\kappa-\frac{ba}{2}}{3a^2+2\kappa} \tag{19}$$

in which recall we have assumed $\mathcal{L}_m = -\rho/2$ [cf. Eq. (8) and its following discussion]. Defining a Poisson mass

$$M = 2m\frac{2a^2+\kappa-\frac{ba}{2}}{3a^2+2\kappa} \tag{20}$$

we then have

$$\psi = \frac{a-b}{2a^2+\kappa-ba/2}\frac{M}{r} \tag{21}$$

$$-g_{00} = 1 - \frac{2M}{r}. \tag{22}$$

We note once more that M need not be the Keplerian mass.

If we now expand Eqs. (14) and (15) we obtain

$$\psi = -\frac{CM}{ar} \tag{23}$$

$$-g_{00} = 1 - \frac{2M}{r} \tag{24}$$

with $M = 2\rho_0/\lambda$. Comparing with Eqs. (21) and (22) we gather

$$C = -\frac{a^2-ba}{2a^2+\kappa-ab/2} \tag{25}$$

with λ to be obtained from Eq. (17). We stress that a direct substitution of Eq. (6) in the Brans-Dicke result [24] misses the terms in b.

The metric (14) may be cast into an Eddington-Robertson expansion [27]:

$$ds^2 = -\left(1-2\frac{M}{\rho}+2\beta\frac{M^2}{\rho}\right)d\xi^2 + \left(1+2\gamma\frac{M}{\rho}\right)(d\rho^2+\rho^2 d\Omega^2) \tag{26}$$

with the parametrized post-Newtonian (PPN) parameters $\beta = 1$ and

$$\gamma = C+1 = \frac{a^2+\kappa+ab/2}{2a^2+\kappa-ab/2}. \tag{27}$$

The Schwarzchild limit may be obtained by letting $a,\kappa\to 0$, keeping κ/a and b finite but with $\kappa/a \gg b$. Then $\gamma\approx 1$, $M\approx m$, and the metric reduces to Schwarzchild. However the variation in c is non-negligible even in this regime:

$$c = c_\infty\left(1-\frac{b-a}{\kappa}\frac{Gm}{c_\infty^2 r}\right). \tag{28}$$

Also deviations from geodesic motion, due to $b\neq 0$, may be non-negligible. Hence it is possible to introduce two types of new VSL effects without modifying the metric, a feature which we shall use to solve a variety of problems.

IV. THE SPEED OF LIGHT MUST GO TO ZERO OR INFINITY AT THE HORIZON

The fact that in the examples above c goes to either zero or infinity at the black hole's horizon is far from accidental. It may be generally proved by adapting techniques used in proving the no-hair theorem [32]. Here we sketch how such a general proof might proceed, taking the particular case of a scalar c (as opposed to a complex c undergoing spontaneous symmetry breaking as discussed in [19], or a c derived from a spinorial field).

Let us consider a static, vacuum, not necessarily spherically symmetric solution which is asymptotically flat and contains an horizon. Let the metric take the form

$$ds^2 = -L d\xi^2 + h_{ij}dx^i dx^j \tag{29}$$

with L and h_{ij} time-independent. Let us discuss the problem in terms of $\phi_{bd} = e^{a\psi} \geq 0$, which must satisfy

$$\frac{1}{\sqrt{Lh}}(\sqrt{Lh}h^{ij}\phi^{bd}_{,i})_{,j} = 0. \tag{30}$$

We first multiply this expression by $\sqrt{Lh}$ and integrate over the region Ω bounded by the horizon (where L must go to zero) and infinity. Integrating by parts reveals

$$\int_{\Omega} dx^3 \sqrt{Lh} h^{ij} \phi^{bd}_{,i} \phi^{bd}_{,j} - \int_{\partial\Omega} \sqrt{Lh} \phi^{bd} h^{ij} \phi^{bd}_{,i} dS_j = 0. \tag{31}$$

The piece of the surface integral corresponding to infinity is zero, by virtue of asymptotic flatness.

At this point VSL differs from relativity. In the usual GR proof one then shows that the integral over the horizon must also be zero since $L \to 0$ there. The only escape route is if ϕ_{bd} or its gradient blow up at the horizon. This is precluded by the requirement that the scalar field energy density be finite. Hence the surface integral is zero, and since the volume integral is semi-positive definite it must be zero, so that the identity is satisfied. This implies that $\phi_{bd} = 0$ everywhere outside the horizon.

Clearly the last part of the argument may break down in VSL, because the ψ gravitation may be negligible. Hence its divergence at the horizon need not produce a singularity. This is the case in the parameter region which produces a Schwarzchild solution. More generally we may define a region in the space $\{a, b, \kappa\}$ for which this type of behavior occurs.

It may also happen that the ψ divergence at the ''horizon''causes a singularity. For instance [28,29] have shown that this happens for $(1+C)/\lambda < 2$. However such a singular horizon is not a problematic ''naked singularity'' in VSL theories because, as we shall see in the next section, for some regions of the theory's couplings information cannot flow out of (or into) the singular surface. Hence a singular horizon need not have the pathological connotations it has in general relativity.

Whatever happens to the metric at the ''horizon,'' a nontrivial solution for ϕ_{bd} always requires that the surface integral in Eq. (31) diverges at the horizon. This implies that c must go either to zero or infinity at the horizon.

The generalization of this argument to stationary solutions, to more general fields (i.e. when c is derived from a bosonic invariant associated with a fermionic field ψ), or in the presence of an electromagnetic field, leads to the same conclusion.

A word on terminology is in order. We are loosely using the word horizon to describe what can in fact be a naked singularity. However, in either case VSL theories predict that such a surface cannot be reached, as we shall show in the next section. Perhaps the wording ''black hole edge'' would be more appropriate, since such a surface becomes part of the spatial infinity of the space-time. However we shall use the expression horizon in what follows for simplicity.

V. THE INACCESSIBILITY OF SINGULARITIES

This theorem has the interesting implication that, at least for suitable couplings, the horizon, as well as the region inside it, are not physically accessible. Naively one might expect this to happen if c goes to zero sufficiently fast at the horizon. Indeed c still acts as a local speed limit, and so $c \to 0$ seems to imply that nothing can enter the horizon. However two extra complications come into the problem: free-falling particles do not generally follow geodesics, and interaction rates change (due to changing α_i) near the black hole. We shall use Schwarzchild VSL black holes as an illustration.

A. Free fall into VSL black holes

As pointed out in [19], $b \neq 0$ VSL theories satisfy a weak form of the equivalence principle (and do not conflict with the Eötvos experiment); however they predict non-geodesic motion. Indeed the action for a point particle, with $b \neq 0$, is given by

$$S = -\frac{E_0}{2} \int d\lambda e^{b\psi} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \tag{32}$$

in which the ''x^0-affine'' parameter is given by $d\lambda = c d\tau$, where $d\tau$ is the actual affine parameter (proper time in the VSL units, for a time-like particle). Hence, if $b \neq 0$, particles do not follow lines of extremal length, but instead minimize the functional (32). Varying Eq. (32) shows that source terms appear in the geodesic equation, specifically

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = -b \left(\dot{x}^\mu \dot{x}^\nu - \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta g^{\mu\nu} \right) \psi_{,\nu}. \tag{33}$$

Consider now radial geodesics ($\dot{\theta} = \dot{\phi} = 0$) in the Schwarzchild metric, so that

$$\mathcal{L} = e^{b\psi} \left(-B\dot{\xi}^2 + \frac{\dot{r}^2}{B} \right) \tag{34}$$

$$\frac{c}{c_0} = e^{\psi} = B^{b/2\kappa} \tag{35}$$

$$B = 1 - \frac{2Gm}{c_\infty^2 r} \tag{36}$$

(we have assumed the usual limit, with $b \gg a$). There are two conserved quantities:

$$E = e^{b\psi} B \dot{\xi} \tag{37}$$

$$\mathcal{L} = -\epsilon \tag{38}$$

(with $\epsilon = 1$ for time-like particles) from which we derive

$$\dot{r} = \sqrt{E^2 B^{-b^2/\kappa} - B^{1 - b^2/2\kappa}}. \tag{39}$$

If the speed of light does not change, we have

$$\tau = \int_{r_i}^{r_h} \frac{dr}{c_0 \sqrt{E^2 - B}} \tag{40}$$

[where r_i and $r_h = 2Gm/(c_\infty^2)$ label the starting point and the horizon], and so the proper time taken for a free falling observer to reach the horizon converges. However, as is well known, such a process takes infinite coordinate time:

$$t=\int_{r_i}^{r_h}\frac{E}{c_0 B}dr=\infty. \quad (41)$$

If c changes, the proper time required to reach the horizon is now

$$\tau=\int_{r_i}^{r_h}\frac{dr}{c\sqrt{E^2B^{-b^2/\kappa}-B^{1-b^2/2\kappa}}}. \quad (42)$$

Let us first assume that $b\ll 1$ but b/κ is non-negligible (so that $b^2/\kappa\ll 1$). Then this differs from the fixed c case in that "$v\propto c$," as naively expected. Hence the horizon is unreachable if c goes to zero faster than $r-2m$, that is if $b/(2\kappa)\geq 1$. When b^2/κ is non-negligible, the field ψ also acts as an extra gravitational force, accelerating or braking free-falling particles. In the general case τ diverges if

$$\frac{b}{2\kappa}(1-b)\geq 1 \quad (43)$$

[with $1+b^2/(2\kappa)>0$ and $\kappa>0$].

In general (that is without assuming a Schwarzchild solution) there are regions of parameter space for which the horizon may be regarded as a boundary of space-time, since it is located at infinite affine distance from any point in its exterior.

B. Interaction clocks in the vicinity of black holes

However one should bear in mind an extra complication, already discussed in [19]. Interaction paces also change near the black hole, since all fine structure constants change. Strong decays are faster than weak ones because $\alpha_s\gg\alpha_w$. Similarly, as the strength of all interactions varies near the black hole, so will the time rates of all the processes they promote.

Somewhat philosophically it was pointed out in [19] that our sensation of time flow derives precisely from change, and this is imparted by interactions and their rates. Hence we introduced the concept of an "interaction clock," a device ticking to the time scales set by the α_i (the fact that the ratios between all α_i are constant removes any ambiguity). The tick of such a clock is given by $\tau_0(\alpha_i)=\tau_0(c)$ [19], with

$$\tau_0=\frac{\hbar}{\alpha^2 Q}\propto\frac{1}{c^{2q+1}} \quad (44)$$

in which Q is the energy scale of the process producing the tick τ_0. One such construction is a muon clock. Let us produce a large number of non-relativistic muons. When half of them have decayed the clock ticks, and produces another large number of muons. Such a clock would tick to a rate [35]

$$\tau_\mu=\frac{96\pi^3\hbar}{E_\mu\alpha_w^2\left(\frac{m_\mu}{m_W}\right)} \quad (45)$$

where m_μ and m_W are the muon and the W masses, and $\alpha_w=g_w/(\hbar c)$ is the weak fine structure constant. Another example is an atomic clock, the period of which is given by

$$\tau_e=\frac{\hbar}{\alpha_e^2 E_e} \quad (46)$$

where $E_e=m_e c^2$ is the electron rest mass. Since $E_e\propto c^q$ (like all other relativistic energies) we have that $\tau_e\propto 1/c^{2q+1}$. These are two realizations of interaction clocks; if all else fails remember that τ_0 is the pace at which we age [34].

A better formulation of the question of whether an observer may or may not reach the horizon is then: how many τ_0 ticks are required? For a Schwarzchild solution this means computing the dimensionless number:

$$\mathcal{N}=\int_{r_i}^{r_h}\frac{d\tau}{\tau_0}=\int_{r_i}^{r_h}\frac{dr}{\tau_0 c\sqrt{E^2B^{-b^2/\kappa}-B^{1-(b^2/2\kappa)}}} \quad (47)$$

which diverges if

$$-\frac{b}{2\kappa}[2q+b]\geq 1. \quad (48)$$

This condition defines the parameter space for which the horizon should be counted as part of the spatial infinity of the black hole.

C. Are there VSL singularities?

This result is extremely interesting. Our solution has a singularity at $r=0$ (in some cases for the general solution there is in fact a naked singularity at $r=r_h$). However this singularity is physically inaccessible; not just in the sense that information cannot flow from it into the asymptotically flat region, but also in the sense that no observer starting from the asymptotically flat region can actually reach it. The singularity lies in a disconnected piece of the manifold, which should simply be excised as unphysical.

It is tempting to conjecture that all singularities are subject to the same constraint, in which case we seem to have eliminated the singularity problem, by means of a stronger version of the cosmic censorship principle.

VI. COLLAPSING STARS AND THEIR REMNANTS

We now discuss stellar collapse making use of the Oppenheimer-Snyder solution, in which a spherical dust ball collapses. This is the correct solution in the limit $a,\kappa,b\rightarrow 0$, keeping κ/a and κ/b finite. In this case the metric is Schwarzchild and motion is geodesics. Other cases are more complicated (see [30] for an investigation in the context of Brans-Dicke theory).

The Oppenheimer-Snyder solution makes use of Birkoff's theorem to match a Schwarzchild outside solution, to a Friedmann closed solution in collapsing stage (in general we note that the solutions derived in Sec. III apply to the outside of a static star for the same reason). The inside metric is then

$$ds^2 = -d\zeta^2 + R^2(\zeta)[d\chi^2 + \sin^2\chi d\Omega^2]. \tag{49}$$

Here ζ is the proper x^0 of free-falling observers, χ is the radial coordinate of a 3-sphere, and R is the expansion factor. The latter satisfies standard Friedmann equations (which are valid in the regime under study [36]) for a dust Universe with density ρ. One can show that there is no jump in the curvature provided that

$$m = \frac{4}{3}\pi\rho R_0^3 \tag{50}$$

$$R_0 = \sin\chi_0 R(\zeta) \tag{51}$$

in which χ_0 is the radial coordinate indexing the surface of the star (which follows a geodesic). The internal value for the speed of light is given by

$$\psi = \frac{b-a}{\kappa}\log\left(1 - \frac{8\pi G\rho\sin^2\chi_0 R^2}{3c_\infty^2}\right). \tag{52}$$

Even though the Oppenheimer-Snyder solution may be adapted to our circumstances, the physics of collapse is entirely different. The arguments applied in the previous section to free falling observers are also valid for observers on the surface of the star. In standard relativity collapse takes infinite coordinate time, but finite proper time for an observer on the surface of the star. In VSL theories the proper time, as felt by interaction clocks on the surface of a collapsing star, is infinite (for the parameter region identified in the last section). As the surface of the star approaches its Schwarzchild radius, all processes freeze-out. We are left with a Schwarzchild remnant, the surface of which is part of spatial infinity. The star itself has left the manifold. It's still black, but it is not a hole; rather its surface is an edge of space.

When this happens there is also a divergence for the number of ticks for any process for observers inside the star, since c inside the star must also go to zero or infinity. Hence the singularity is never formed, a fact which in any case has little physical relevance. The inside of the star is pickled for eternity as the Schwarzchild remnant is formed.

VII. GRAVITATIONAL PHYSICS AROUND STARS

We now turn to the study of gravitational phenomena in the vicinity of VSL stars. A more detailed study within the framework of the PPN formalism [23] is warranted, but shall not be attempted here.

In summary we find the following. There are three classes of effects: upon planetary orbits (e.g. the precession of the perihelion of Mercury), upon light (e.g. gravitational light bending, or the radar echo time-delay), and upon the fine structure of absorption lines. These are caused, in different combinations, by three distinct facts which we can switch on and off independently: corrections to the Schwarzchild metric, violations of energy conservation, and spatial variations in α.

If there are only corrections to the Schwarzchild metric we obtain corrections to the GR result for the planetary and light trajectories similar to those found in Brans-Dicke theory. These corrections are embodied in the PPN parameter γ computed above. However there is a limit in which we recover the Schwarzchild metric (and $\gamma=1$) but in which there are significant violations of energy conservation. In this limit we recover the GR results for light properties, but we find non-negligible corrections to planetary orbits. Finally it is possible to switch off these two effects, and so recover the classical tests of GR, and still produce significant changes in α and consequently in the fine structure of spectra in light emitted at the surface of stars. It is also possible to switch off the latter, and keep either of the former two effects.

A. The precession of the perihelion of Mercury

We start by deriving the orbits of point particles, considering first the Schwarzchild metric. We are therefore in the limit $a,\kappa\to 0$, but we shall assume that b is finite and $\kappa/a \gg b$ so that we may exhibit deviations from geodesic motion. Setting $\theta=\pi/2$, $\dot\theta=0$, the Lagrangian is [cf. Eq. (32)]

$$\mathcal{L} = e^{b\psi}\left(-B\dot\xi^2 + \frac{\dot r^2}{B} + r^2\dot\phi^2\right). \tag{53}$$

There are three conserved quantities:

$$E = e^{b\psi}B\dot\xi \tag{54}$$

$$J = r^2 e^{b\psi}\dot\phi \tag{55}$$

$$\mathcal{L} = -\epsilon \tag{56}$$

where $\epsilon = 0,1$ for light and particles respectively. It follows that

$$\dot r^2 = E^2 e^{-2b\psi} - \epsilon e^{-b\psi}B - \frac{J^2}{r^2}e^{-2b\psi}B. \tag{57}$$

Using the standard transformations

$$u = \frac{G}{rc_\infty} \tag{58}$$

$$\frac{d}{d\lambda} = \dot\phi\frac{d}{d\phi} \tag{59}$$

and differentiating we get

$$u'' + u = 3mu^2 + \frac{m\epsilon}{J^2}\left(1 + \frac{b^2}{2\kappa}\right)(1-2mu)^{b^2/2\kappa} \tag{60}$$

in which we have used Eq. (13). Expanding the VSL contribution (terms arising from $b^2/\kappa \neq 0$) up to first order in mu leads to

$$u'' + u = 3mu^2 + \frac{m\epsilon}{J^2}\left(1 + \frac{b^2}{2\kappa}\right) - \frac{m^2\epsilon}{J^2}\frac{b^2}{\kappa}\left(1 + \frac{b^2}{2\kappa}\right)u \tag{61}$$

to be compared with the Newtonian result

$$u''+u=\frac{m\epsilon}{J^2} \tag{62}$$

and the GR result

$$u''+u=3mu^2+\frac{m\epsilon}{J^2}. \tag{63}$$

The Newtonian solutions are elliptical orbits:

$$u_0=\frac{m}{J^2}(1+e\cos\phi) \tag{64}$$

where e is the eccentricity. The GR term $3mu^2$ causes a precession of the perihelion by

$$\Delta\phi=\frac{6\pi m^2}{J^2} \tag{65}$$

per revolution. In the case of Mercury this amounts to about 43″ per century.

VSL causes two extra effects, even in the limit where the metric remains Schwarzchild. First it causes a shift in the Keplerian mass, that is, the Newtonian formula still applies but with mass

$$\mathcal{M}=m\left(1+\frac{b^2}{2\kappa}\right). \tag{66}$$

This can be guessed by comparing the relevant term in Eq. (61) with the Newtonian expression (62). A derivation of Kepler's third law, with a more rigorous derivation of Eq. (66) may be found in Appendix C. Secondly, the last term in Eq. (61) induces a shift in the frequency, causing a precession per revolution of

$$\Delta\phi=-\frac{4\pi m^2}{J^2}\frac{b^2}{2\kappa}\left(1+\frac{b^2}{2\kappa}\right). \tag{67}$$

We see that, as announced above, even in the limit in which the metric remains Schwarzchild, VSL may induce significant corrections to the orbit of Mercury.

It may make more sense to rewrite $\Delta\phi$ in terms of $\mathcal{M}$, since this is the mass measured using Kepler's third law. Then, to first order in b^2/κ, the joint GR and VSL effect is

$$\Delta\phi=\frac{6\pi\mathcal{M}^2}{J^2}\left(1-\frac{4}{3}\frac{b^2}{\kappa}\right). \tag{68}$$

In the case of Mercury, in addition to the usual GR effect there is a precession of about 57″ times *minus* b^2/κ, purely due to violations of energy conservation.

The general case is more difficult to compute. We use the Eddington-Robertson form of the metric in radial coordinates:

$$ds^2=-Bd\xi^2+Adr^2+r^2d\Omega^2 \tag{69}$$

$$B=1-2\frac{M}{r}+2(1-\gamma)\frac{M^2}{r^2} \tag{70}$$

$$A=1+2\gamma\frac{M}{r} \tag{71}$$

with M and γ given by Eqs. (20) and (27). Using the same techniques as above we arrive at

$$u'^2=\frac{E^2}{ABJ^2}-\frac{u^2}{A}-\frac{\epsilon e^{b\psi}}{AJ^2}. \tag{72}$$

When $e^{b\psi}=1$ this expression leads to standard results (see [27]). Hence we should add to these results any corrections induced by the new terms associated with the $e^{b\psi}$ factor. To find the new terms we need $e^{b\psi}$ up to second order in Mu. Noting that

$$\rho=r[1-(1+C)u] \tag{73}$$

and expanding Eq. (15) we find

$$e^{b\psi}=1-\frac{bC}{a}Mu+\frac{bC}{a}\left(\frac{bC}{2a}-1-C\right)(Mu)^2. \tag{74}$$

Hence the new terms in Eq. (72) are

$$u'^2=\cdots+\frac{bC}{a}\frac{Mu}{J^2}-\frac{M^2u^2}{J^2}\frac{bC}{a}\left(2\gamma+\frac{bC}{2a}-1-C\right) \tag{75}$$

where the ellipsis denotes terms present in the fixed c calculation for PPN metrics. This leads to

$$u''=\cdots+\frac{bC}{2a}\frac{M}{J^2}-\frac{M^2u}{J^2}\frac{bC}{a}\left(2\gamma+\frac{bC}{2a}-1-C\right). \tag{76}$$

Again the Keplerian mass receives a shift

$$\mathcal{M}=M\left(1+\frac{bC}{a}\right). \tag{77}$$

As for the perihelion precession we should now add to the standard formula

$$\Delta\phi_0=\frac{6\pi M^2}{J^2}\frac{1+2\gamma}{3} \tag{78}$$

[with γ given by Eq. (27)] the extra term

$$\Delta\phi_1=-\frac{2\pi M^2}{J^2}\frac{bC}{a}\left(2\gamma+\frac{bC}{2a}-1-C\right). \tag{79}$$

This result reduces to Eq. (67) in the limit $a,\kappa\rightarrow 0$, and $\kappa/a \gg b$. An expression containing only physically meaningful quantities can then be obtained by rewriting these formulas in terms of $\mathcal{M}$ by means of Eq. (77).

B. Gravitational light deflection

Considering now light trajectories, we should set $\epsilon=0$ in Eq. (72). This cancels out the term in $e^{b\psi}$ and so VSL induces no effects on light trajectories other than those induced by distortions to the Schwarzschild metric. Hence if $a,\kappa \rightarrow 0$, and $\kappa/a \gg b$ we predict the same result as GR for gravitational light bending:

$$\Delta\phi = \frac{4Gm}{r_0 c_\infty^2} \tag{80}$$

where r_0 is the impact parameter. In the case of a light ray grazing the Sun $\Delta\phi = 1.75''$. The general case is

$$\Delta\phi = \frac{4GM}{r_0 c_\infty^2}\frac{1+\gamma}{2} \tag{81}$$

with γ given by Eq. (27).

It would seem at first that in the limit in which the metric remains Schwarzschild there are no corrections to GR for light bending, but the formula for the perihelion of Mercury precession may be modified. This is a distinctive feature of VSL, distinguishing it from Brans-Dicke theories, and can be traced to violations of energy conservation in the Jordan frame in these theories. In practice however the situation is very different. The masses m or M are not directly accessible; the mass of the Sun being estimated via Kepler's law. The result is a Keplerian mass $\mathcal{M}$ given by either Eq. (66) or Eq. (77). Hence, even though VSL corrections of order b^2/κ only affect time-like orbits, these corrections filter through to formulas for light trajectories, because these must be expressed in terms of Keplerian masses. The relevant result is obtained by substituting Eq. (66) or Eq. (77) in Eq. (80) or Eq. (81).

This situation is a good object lesson against harsh applications of conformal transformations. As spelled out in Appendix B, if we ignore the most general type of $\mathcal{L}_m$ it is possible to map the dynamics of our theory into Brans-Dicke dynamics. This explains why our formulas for planets (which are not conformally invariant) differ from Brans-Dicke results, but the same does not happen to light (which is conformally invariant). However, such a direct application of a conformal transformation would miss the interconnection between conformally invariant and non-invariant results which we have just pointed out.

C. Radar echo time delay

Naively one might expect a different result for radar echo time delays in VSL theories. Indeed if light traveled slower or faster near the Sun, the echo time-delay should be larger or smaller. As we shall see this is not true in our theory, a feature due to the fact that we have not broken local Lorentz invariance. As pointed out in [19] this manifests itself in the absence of a global time coordinate, the differential structure associated with time forming a fiber bundle rather than a tangent bundle. Hence non-local calculations involving time should be done with the coordinate ξ, the conversion to time to be done locally. As a consequence whatever happens to c locally along the path of the radar wave does not affect the final result.

We start by deriving results valid *if* we were to break local Lorentz invariance. Let r_0 be the point of closest approach to the Sun. Then the time taken for the radar signal to move up to distance r, in the absence of gravitational effects, is

$$\Delta t = \int dr \frac{r}{c(r)[r^2 - r_0^2]^{1/2}}. \tag{82}$$

With a variation in c analogous to Eq. (13) we would get

$$\Delta t = \frac{[r^2 - r_0^2]^{1/2}}{c_\infty} + \frac{b-a}{\kappa}\frac{Gm}{c_\infty^3}\log\left(\frac{r+[r^2-r_0^2]^{1/2}}{r_0}\right). \tag{83}$$

Hence to the usual gravitational time-delay, we would have to add a delay (if $\alpha > 0$) due to a lower value for c close to the Sun. Comparing Eq. (83) with the usual PPN formula [27] we find that this effect, due to explicit violations of Lorentz invariance, simulates a PPN parameter $\gamma = (b - a)/\kappa$.

Nothing like that happens in a locally Lorentz invariant VSL theory. From

$$\dot{r}^2 = \frac{E^2 e^{-2b\psi}}{AB} - \epsilon e^{-b\psi}B - \frac{J^2}{r^2}e^{-2b\psi}B \tag{84}$$

we obtain, after setting $\epsilon = 0$ and making use of

$$\dot{r} = \frac{dr}{d\xi}\frac{E}{e^{b\psi}B} \tag{85}$$

the expression

$$\left(\frac{dr}{d\xi}\right)^2 = \frac{1 - \frac{J^2 B}{E^2}}{A/B} \tag{86}$$

in which all factors in $e^{b\psi}$ have cancelled out. This leads to the standard expression [27]

$$\Delta\xi = [r^2 - r_0^2]^{1/2} + (1+\gamma)\frac{GM}{c_\infty^2}\log\left(\frac{r+[r^2-r_0^2]^{1/2}}{r_0}\right) + \frac{GM}{c_\infty^2}\left(\frac{r-r_0}{r+r_0}\right)^{1/2}. \tag{87}$$

One needs now to transform ξ into time, but that is done on the Earth, where $c \approx c_\infty$. Hence $\Delta t = \Delta\xi/c_\infty$, leading to the same result as in Brans-Dicke theories.

The only novelty is again that M is not the Keplerian mass, if b^2/κ is non-negligible. Once more we find that even though VSL is equivalent to Brans-Dicke in light experiments, the fact that masses are estimated using time-like objects induces corrections in formulas for light. In the present case we should use Eq. (77) to replace M with $\mathcal{M}$ in Eq. (87).

D. Spectral lines

Naturally the hallmark and real novelty of VSL is a changing electromagnetic fine structure constant. This should affect the fine structure of absorption lines created on the surface of stars, and be detectable using techniques similar to Webb *et al.* [25]. As we shall see the larger the potential difference, the stronger the effect, so perhaps dwarfs, or even neutron stars might be better candidates for this experiment.

We first consider the effect upon spectra in the non-relativistic regime. We find that all spectral lines are proportional to the Rydberg energy, given by $E_R = m_e e^4/\hbar^2 = E_e\alpha^2$, where $E_e = m_e c^2$ is the electron's rest energy. Hence spectral lines have wavelengths proportional to $\lambda = \hbar c/E_R \propto 1/\alpha^2 \propto c^{-2q}$. Considering that photons in free flight have a constant wavelength (see Sec. V A of [19]) we conclude that when we compare spectral lines coming from the surface of a star with those measured on an Earth laboratory, we find an extra ''redshift'' effect, due to VSL, of magnitude

$$\frac{\Delta\lambda}{\lambda} = -2q\frac{\Delta c}{c} = \frac{2bq}{\kappa}\frac{Gm}{c_\infty^2 r} \tag{88}$$

where the last identity is valid only in the limit $a,\kappa \rightarrow 0$, and $\kappa/a \gg b$. We therefore conclude that VSL theories have a PPN parameter $\alpha_{PPN} = 2bq/\kappa$ [23]. Pound-Rebka-Snider experiments are capable of constraining this parameter, but not by more than $|\alpha_{PPN}| < 10^{-3}$ (see Fig. 14.3 of [23]). As will be shown in [36] the combination bq/κ is of order $\Delta\alpha/\alpha$ at cosmological redshifts or order 1. Hence the observations made by Webb *et al.* [25], when interpreted with VSL, imply violations of the weak equivalence principle at the level $\alpha_{PPN} \sim 10^{-5}$, consistent with current experimental tests. In particular, measurements of non-relativistic spectral lines formed on the surface of the Sun do not constrain α_{PPN} by more than $|\alpha_{PPN}| < 10^{-2}$. More compact objects, such as dwarfs or pulsars, display a stronger VSL redshift effect, but the effect is degenerate with respect to Doppler shifts induced by their unknown velocities with respect to us. For such objects one has to go to look into the fine structure in order to measure, without degeneracy, the possible effects upon spectra lines of varying constants.

Considering now the relativistic fine structure of spectral lines, we find that they directly measure the tell-tale signature of VSL, since they are directly related to $\alpha = e^2/(\hbar c)$ (not to be confused with α_{PPN}). For small deviations we have

$$\frac{\Delta\alpha}{\alpha} = q\frac{\Delta c}{c} = q\psi = -\frac{q(\gamma-1)}{a}\frac{GM}{c_\infty^2 r} \tag{89}$$

where we have used Eq. (15) (recall that $\alpha \propto c^q$). It is interesting to note that Eq. (89) may be large even choosing parameters which render the metric Schwarzchild, and non-geodesic effects associated with $b \neq 0$ negligible. In the limit $a,\kappa \rightarrow 0$, and $\kappa/a \gg b$ (so that $\gamma \approx 1$) we have

$$\frac{\Delta\alpha}{\alpha} = -\frac{bq}{\kappa}\frac{Gm}{c_\infty^2 r}. \tag{90}$$

This may be non-negligible even with negligible b^2/κ (so that no corrections to the GR result are present in the perihelion of Mercury). The prefactor bq/κ may be inferred from cosmological observations [36] and can at most be of order 10^{-4}. Hence we need an object sufficiently compact, such as active galactic nuclei (AGN), a pulsar or a white dwarf, for the effect to be non-negligible. Furthermore we need the ''chemistry'' of such an object to be sufficiently simple, so that line blending does not become problematic.[1]

Generally (i.e. for any matter configurations) the larger the gravitational potential differences, the stronger the effect. Indeed, for static configurations, both $\Delta\alpha/\alpha$ and the gravitational potential satisfy Poisson equations, with source terms related by a multiplicative constant. Hence the local value of α should map the gravitational potential, and one would need to have big variations in the gravitational potential to observe corresponding spatial variations in α. It would be interesting to use this to infer α maps from *N*-body simulations, so as to deduce possible observational signatures of VSL on cluster and supercluster scale.

VIII. THEORETICAL AND OBSERVATIONAL OUTLOOK

We have provided ample evidence for how VSL stars and ''black holes'' may be rather exotic indeed. We have used the covariant and locally Lorentz invariant formulation proposed in [19], and stress that the results derived are by no means generic to all VSL theories. Indeed in Appendix A we showed how bimetric VSL black holes may differ distinctly from the ones considered here. In this regard it would be of great interest to derive the properties of black holes in the bimetric theory of Clayton and Moffat [9–11] and Drummond [12]. Another variation upon the theme are VSL theories which explicitly break local Lorentz invariance, such as the one proposed by Albrecht and Magueijo [2], and for which black hole solutions remain elusive. In Sec. VII C we derived a distinctive effect to be expected in such theories (a different radio echo time-delay) which is not present in locally Lorentz invariant VSL theories. Hence the exotic results derived in this paper are generic to the type of theories proposed in [19], but by no means to all VSL theories.

Yet, even within the framework of the VSL theories proposed in [19], a large number of new effects still remain to be explored. We close this paper first by highlighting a few obvious areas of interest which should prompt further theoretical work, and then describing observational prospects.

An important omission in this paper is quantum effects, which we have ignored. However it was shown in [19] that a varying-*c* induces quantum particle creation (a point noted before, in other VSL theories, by [15]). That being the case, VSL black holes might be sources of radiation in a process

[1] I would like to thank Lance Miller and Graça Rocha for tutoring me on the details of stellar spectral lines.

complementary to Hawking's radiation. The exact details of such a process remain to be worked out. Also the interaction between a changing c and standard Hawking radiation is far from obvious. These phenomena are currently being investigated.

Further quantum effects arise from the fact that all gauge field strengths becoming zero or infinite will no doubt reshape the low-energy aspect of any quantum field theory. Indeed, the scaling arguments mentioned in Sec. V B should break down when the line $\alpha=1$ is crossed. Therein nonperturbative interactions will become perturbative, or vice versa, a process which may have dramatic implications. For instance, the vacuum of a given theory may change. The impact upon phenomena like confinement may be massive.

There are also other interesting classical effects beyond those described in this paper. All the arguments developed in this paper concerned free-falling point particles. One may wonder what happens to free-falling extended objects. As is well known, they will feel gravity by means of tidal forces. Should $b\neq 0$ they will also feel inertial forces, corresponding to their acceleration (or braking) by the field ψ. Furthermore there will also be effects induced by the gradients in c. Let us consider a body moving along a negative gradient of c (and assume $b=0$). Given that $v\propto c$, such a body would get squashed along the direction of motion. In general a stress proportional to $\mathbf{v}\cdot\boldsymbol{\nabla}\psi$ will be felt.

Another finite size effect involves the time rates associated with "interaction clocks" derived in Sec. V B. For a point particle falling into a black hole a slowing down of this rate means merely the slowing down of its progression towards the horizon. However, for an extended object there will also be an aging gradient, closely mapping the c gradient, in addition to the stresses mentioned above. These issues, as well as the quantum effects described above, will be the subject of a future publication.

Besides these interesting topics for future theoretical work, there is the obvious hurdle of experiment. We saw that the theory produces effects very similar to Brans-Dicke theory, plus additional effects, namely departures from geodesic motion for non-null particles, and distorted fine structure in spectral lines in stellar light. If $b=0$ the classical tests of GR impose the constraint [23]

$$|\gamma-1|<10^{-3}. \tag{91}$$

If we adopt the Schwarzchild limit (in which case $\gamma=1$) this constraint becomes

$$\frac{b^2}{|\kappa|}<10^{-3}. \tag{92}$$

In between these two limits a rather complex combination of a, b, and κ is constrained to the same order of magnitude.

Should there be any departures from GR results in these classical experiments, however, VSL would be an interesting competitor to Brans-Dicke theory, since it predicts corrections to light and planetary formulas distinct from Brans-Dicke theory. More interesting still is that, unlike Brans-Dicke theory, the theory does not become trivial in the limit in which the classical tests of GR are reproduced ($\omega_{BD}\gg 1$ for Brans-Dicke, $a,\kappa\ll 1$, $\kappa/a\gg b$, and $b^2/\kappa\ll 1$ for VSL). In this limit the theory still predicts a shift in α observable in the fine structure of spectra from stars or other compact objects. This effect makes VSL an interesting experimental target.

ACKNOWLEDGMENTS

I would like to thank Andy Albrecht, John Barrow, Kim Baskerville, Malcolm Perry, Kelly Stelle, and John Webb, for discussion or comments. I am grateful to the Isaac Newton Institute for support and hospitality while part of this work was done.

APPENDIX A: BLACK HOLES IN BIMETRIC THEORIES

As a curiosity we now show an example of an alternative VSL theory which evades the theorem described in Sec. IV. We show how this happens, using as an example the theory described in the Appendix of [19]. In this theory there are two metrics, g coupling to gravitation and matter, and h coupling to the field c only. The action is

$$S=S_1+S_2$$

$$S_1=\int d^4x\sqrt{-g}\left(R+\frac{16\pi G}{c_0^4 e^{4\psi}}\mathcal{L}_m\right)$$

$$S_2=\int d^4x\sqrt{-h}(H-\kappa h^{\mu\nu}\partial_\mu\psi\partial_\nu\psi) \tag{A1}$$

where $g_{\mu\nu}$ and $h_{\mu\nu}$ lead to two Einstein tensors $G_{\mu\nu}$ and $H_{\mu\nu}$. Varying with respect to g, ψ, and h leads to equations

$$G_{\mu\nu}=\frac{8\pi G}{c_0^4 e^{4\psi}}T_{\mu\nu} \tag{A2}$$

$$\Box_h\psi=\frac{32\pi G}{c_0^4 e^{4\psi}\kappa}\sqrt{\frac{g}{h}}\mathcal{L}_m \tag{A3}$$

$$H_{\mu\nu}=\kappa\left(\nabla_\mu\psi\nabla_\nu\psi-\frac{1}{2}h_{\mu\nu}\nabla_\alpha\psi\nabla^\alpha\psi\right). \tag{A4}$$

Let us now consider SSS solutions to this theory. It is immediately obvious that $g_{\mu\nu}$ is the Schwarzchild solution, with mass m. The solutions for c and h_μ can be obtained by applying to this theory an argument similar to the one followed in Sec. III B. Solutions (14) and (15) are still valid, since we are in vacuum. However the weak field limit now produces

$$-h_{00}=1-\frac{2Gm}{c_\infty r}\frac{-ab}{2\kappa} \tag{A5}$$

$$\psi=-\frac{b}{\kappa}\frac{Gm}{c_\infty r} \tag{A6}$$

in which we have $b=-4$ and $a\to 0$. Hence, comparing with the asymptotic forms of Eqs. (14) and (15), we find

$$C=-2 \tag{A7}$$

$$\lambda a=\sqrt{2\kappa}. \tag{A8}$$

This leads to the result for $h_{\mu\nu}$ and c:

$$ds^2=-d\xi^2+\left(1-\left(\frac{\rho_0}{\rho)}\right)^2\right)^2(d\rho^2+\rho^2 d\Omega^2) \tag{A9}$$

$$c=c_0\left(\frac{1-\rho_0/\rho}{1+\rho_0/\rho}\right)^{-\sqrt{2/\kappa}} \tag{A10}$$

in which the ''horizon'' is at

$$\rho_0=\frac{\sqrt{2}m}{\kappa}. \tag{A11}$$

The horizon of $g_{\mu\nu}$ and that of $h_{\mu\nu}$ (which is where c goes to infinity) therefore do not need to be at the same place.

APPENDIX B: CONFORMAL DUALS

Here we examine the effect of conformal transformations on VSL theories. These are to be distinguished from changes of units which render c constant, leading to fixed c duals, as studied in [19]. Conformal transformations take the form

$$d\hat{t}=dt\Omega \tag{B1}$$

$$d\hat{x}=dx\Omega \tag{B2}$$

$$\hat{g}_{\mu\nu}=g_{\mu\nu} \tag{B3}$$

$$d\hat{E}=dE\Omega^{-1} \tag{B4}$$

or equivalently

$$d\hat{t}=dt \tag{B5}$$

$$d\hat{x}=dx \tag{B6}$$

$$\hat{g}_{\mu\nu}=\Omega^2 g_{\mu\nu} \tag{B7}$$

$$d\hat{E}=dE\Omega^{-1}. \tag{B8}$$

These transformations do not change the value of c, and so map VSL theories into VSL theories; but the gravitational action is modified leading to different values for a, b, and κ. The point we wish to make is that the degeneracy of conformally related theories is usually broken by the presence of matter. Indeed minimal coupling (the requirement that $\mathcal{L}_m$ does not depend on c) is not conformally invariant, and so a conformal frame (and so a set of a and b) is picked for its simplicity in describing non-gravitational physics (a point clearly made in [31]). Another example of a case where a preferred ''physical'' conformal frame is present was given in [33].

If we can ignore generic matter fields, however, conformal transformation may be a useful mathematical trick. Of particular interest is the ''Jordan'' or ''geodesic'' frame, in which $b=0$, and ψ does not couple to $\mathcal{L}_m$. In such a frame there is energy conservation, and particles follow geodesics. Two other frames of interest are the Einstein frame $(a=0)$ and the string frame $(a=b)$.

Consider then an action of the form

$$S=\int d^4x\sqrt{-g}\Big(\phi^\alpha\hat{R}-\frac{\omega}{\phi^\beta}\nabla_\mu\phi\nabla^\mu\phi-V(\phi)+\frac{16\pi G_0}{c_0^4}f(\phi)\mathcal{L}_m\Big) \tag{B9}$$

in which, in our case, $f(\phi)=\phi^{b/a}$. Under a conformal transformation $\hat{g}_{\mu\nu}=\Omega^2 g_{\mu\nu}$, the transformed action is

$$\hat{S}=\int d^4x\sqrt{-\hat{g}}\Big\{\Omega^{-2}\phi^\alpha\hat{R}+6\phi^\alpha\Omega^{-4}\hat{\nabla}_\mu\Omega\hat{\nabla}^\mu\Omega-6\alpha\phi^{\alpha-1}\Omega^{-3}\hat{\nabla}_\mu\phi\hat{\nabla}^\mu\Omega-\omega\phi^{-\beta}\Omega^{-2}\hat{\nabla}_\mu\phi\hat{\nabla}^\mu\phi-V(\phi)\Omega^{-4}+\frac{16\pi G_0}{c_0^4}f(\phi)\hat{\mathcal{L}}_m(\hat{g}_{\mu\nu}\Omega^{-2})\Big\}. \tag{B10}$$

If a portion of $\mathcal{L}_m$ is homogeneous degree α in the metric, it is possible to transform away any coupling between ϕ and $\mathcal{L}_m$ by setting $\Omega^2=\phi^n$ with $n=b/(a\alpha)$. Note that this is only possible if $\mathcal{L}_m$ is homogeneous in the metric, something which is not generally true (for instance kinetic terms are first order in the metric whereas interaction terms are zeroth order). If we stick to classical particles, α is the power of u to be used in the Lagrangian

$$S=-\frac{E_0}{2\alpha}\int d\lambda[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^\alpha. \tag{B11}$$

In standard GR this does not matter, but here it is crucial. In [19] we have argued for $\alpha=1$, but this need not be the case ($\alpha=1/2$ is the value usually used in the literature, so that the action becomes the length of the world-line). We could even consider the case in which different types of classical matter had different α, another good example of a situation in which the geodesic frame would not exist (as indeed $\mathcal{L}_m$ would then not be homogeneous in the metric).

Action (B9) then becomes

$$\hat{S}=\int d^4x\sqrt{-\hat{g}}\Big\{\phi^{1-n}\hat{R}-\frac{\omega+3n(1-n/2)}{\phi^{1+n}}\hat{\nabla}_\mu\phi\hat{\nabla}^\mu\phi-V(\phi)\phi^{-2n}+\frac{16\pi G_0}{c_0^4}f(\phi)\hat{\mathcal{L}}_m(\hat{g}_{\mu\nu}\Omega^{-2})\Big\}. \tag{B12}$$

Setting

$$\chi=\phi^{1-n} \tag{B13}$$

and $V=0$ we finally recover the Brans-Dicke action with

$$\hat{S}=\int d^4x\sqrt{-\hat{g}}\left\{\chi\hat{R}-\frac{\hat{\omega}}{\chi}\hat{\nabla}_\mu\chi\hat{\nabla}^\mu\chi+\frac{16\pi G_0}{c_0^4}\hat{\mathcal{L}}_m(\hat{g}_{\mu\nu})\right\} \tag{B14}$$

with

$$\hat{\omega}=\frac{\omega+3n(1-n/2)}{(1-n)^2}. \tag{B15}$$

By means of this transformation it is now possible to confirm most of the results derived in this paper. For instance, Eq. (27) may be derived from the usual Brans-Dicke result (with terms in b included). On the contrary the careless application of this tool to the prediction of the precession of the perihelion of Mercury and gravitational light deflection may be very misleading. One might expect light properties to remain unaffected by this transformation. While this is true on the surface, it is not in reality. Formulas for the light deflection contain the Keplerian mass, which is affected by conformal transformations. This point is made clear in Sec. VII B.

APPENDIX C: KEPLERIAN ORBITS IN VSL THEORIES

The Keplerian mass is estimated from Kepler's third law, which here we simplify to circular orbits. Then planets at distance R have periods T such that R^3/T^2 is a constant, proportional to the mass of the Sun. Kepler's law is used to estimate the mass of the Sun, and therefore any corrections it receives filter through to all formulas involving the mass of the Sun.

We consider first a VSL Schwarzchild metric, so that [cf. Eq. (72)]

$$u'^2=\frac{E^2}{J^2}-u^2B-\frac{e^{b\psi}B}{J^2}. \tag{C1}$$

Following [27] we now set to zero both u' and also its derivative with respect to u (the latter required for stability of the orbit). This leads to

$$E^2=B(J^2u^2+e^{b\psi}) \tag{C2}$$

$$J^2=\frac{[1+b^2/(2\kappa)]B^{b^2/2\kappa}B'}{u^2(2Bu-B')}. \tag{C3}$$

From Eqs. (54) and (55) we have

$$\frac{d\phi}{d\xi}=\frac{Ju^2B}{E}\approx(m[1+b^2/(2\kappa)]u^3)^{1/2} \tag{C4}$$

in which the last approximation reflects the fact that for all planets used to estimate the mass of the Sun $mu\ll 1$. Hence, with $\omega=2\pi/T$, we have

$$\omega^2R^3=\mathcal{M}=m\left(1+\frac{b^2}{2\kappa}\right). \tag{C5}$$

A similar exercise using the general form of the equations of motion confirms Eq. (77).

[1] J. Moffat, Int. J. Mod. Phys. D **2**, 351 (1993); Found. Phys. **23**, 411 (1993).
[2] A. Albrecht and J. Magueijo, Phys. Rev. D **59**, 043516 (1999).
[3] J.D. Barrow, Phys. Rev. D **59**, 043515 (1999).
[4] J.D. Barrow and J. Magueijo, Phys. Lett. B **443**, 104 (1998).
[5] J.D. Barrow and J. Magueijo, Phys. Lett. B **447**, 246 (1999).
[6] J.D. Barrow and J. Magueijo, Class. Quantum Grav. **16**, 1435 (1999).
[7] J.D. Barrow and J. Magueijo, Astrophys. J. Lett. **532**, L87 (2000).
[8] J. Moffat, astro-ph/9811390.
[9] M.A. Clayton and J.W. Moffat, Phys. Lett. B **460**, 263 (1999).
[10] M.A. Clayton and J.W. Moffat, Phys. Lett. B **477**, 269 (2000).
[11] M.A. Clayton and J.W. Moffat, gr-qc/0003070.
[12] I. Drummond, gr-qc/9908058.
[13] P.P. Avelino and C.J.A.P. Martins, Phys. Lett. B **459**, 468 (1999).
[14] P.P. Avelino, C.J.A.P. Martins, and G. Rocha, Phys. Lett. B **483**, 210 (2000).
[15] T. Harko and M.K. Mak, Gen. Relativ. Gravit. **31**, 849 (1999); Class. Quantum Grav. **16**, 2741 (1999).
[16] E. Kiritsis, J. High Energy Phys. **10**, 010 (1999).
[17] S. Alexander, hep-th/9912037.
[18] B. Basset *et al.*, Phys. Rev. D **62**, 103518 (2000).
[19] J. Magueijo, Phys. Rev. D **62**, 103521 (2000).
[20] S. Landau, P. Sisterna, and H. Vecetich, astro-ph/0007108.
[21] T. Jacobson and D. Mattingly, gr-qc/0007031.
[22] J. Barrow and C. O'Toole, astro-ph/9904116.
[23] C. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, England, 1993).
[24] C. Brans and R. Dicke, Phys. Rev. **124**, 925 (1961).
[25] J.K. Webb, V.V. Flambaum, C.W. Churchill, M.J. Drinkwater, and J.D. Barrow, Phys. Rev. Lett. **82**, 884 (1999).
[26] T. Damour and A.M. Polyakov, Nucl. Phys. **B423**, 532 (1994); Gen. Relativ. Gravit. **26**, 1171 (1994).
[27] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[28] M. Campanelli and C.O. Lousto, Int. J. Mod. Phys. D **2**, 451 (1993).
[29] A.G. Agnese and M. La Camera, Phys. Rev. D **51**, 2011 (1995).
[30] M.A. Scheel, S.L. Shapiro, and S.A. Teukolsky, Phys. Rev. D

51, 4208 (1995); **51**, 4236 (1995).
[31] I. Quiros, gr-qc/9904004; Phys. Rev. D **61**, 124026 (2000).
[32] J. Bekenstein, Phys. Rev. D **5**, 1239 (1972); **5**, 2403 (1972).
[33] T. Damour, G. Gibbons, and C. Gundlach, Phys. Rev. Lett. **64**, 123 (1990).
[34] J.D. Barrow and F.J. Tipler, *The Anthropic Cosmological Principle* (Oxford University Press, New York, 1986).
[35] F. Mandl and G. Shaw, *Quantum Field Theory* (Wiley, New York, 1984).
[36] J. Barrow and J. Magueijo, ''Cosmological scenarios arising from covariant and locally Lorentz-invariant varying speed of light theories'' (in preparation).