Fractional fermion number in a (1+1)-dimensional Dirac equation with a scalar Coulomb field

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An elementary example of fermion number fractionalization is given. The model considered is a massive Dirac fermion moving on the positive halfline in an external scalar Coulomb field. The theory is symmetric under charge conjugation. We demonstrate the existence of a nondegenerate, normalizable, and self-charge-conjugate zero energy solution in the theory. The vacuum thus has a fermion number $\pm 1/2$ depending on whether the lowest positive energy state without the external scalar field was filled or empty.

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The occurrence of a fractional fermion number of the vacuum is certainly one of the most interesting discoveries in the theory of quantum fields. It is a manifestation of the effects of Dirac's negative energy sea [1,2]. It was first observed by Jackiw and Rebbi [3] that, in a charge conjugation symmetric theory of one-dimensional massless Dirac fermions interacting with a solitonic background field (the kink), the vacuum acquires a fractional charge $\pm 1/2$. In this case the existence of a normalizable zero energy solution of the Dirac equation is rather essential, and the fermion number fractionalization is closely related to the spectral asymmetry of the theory considered. Later, Goldstone and Wilczek [4] showed that for a general Dirac Hamiltonian without conjugation symmetry, the fermion number can be a nontrivial continuous function of the various parameters in the theory. Soon after, examples of fermion number fractionalization in various space-time dimensions were also considered [5,6]. Mathematically, the vacuum fermion number N_V is related to the η invariant, η_H, [7] of the pertinent Dirac Hamiltonian H defined in a classical background field

$$N_V = -\frac{1}{2}\eta_H \equiv -\frac{1}{2}\left\{\sum_{E_n\geqslant 0} 1 - \sum_{E_n<0} 1\right\}, \qquad (1)$$

where E_n are the energy eigenvalues of H. Experimentally, fermion number fractionalization was verified in certain one-dimensional polymers such as polyacetylene [8]. It has also found a place in such subjects as the fractional quantum Hall effect [9] and the chiral bag model of nucleons [10]. Most of the cases of fractional fermion numbers considered in the literature involved certain nontrivial solitonic background fields.

In this Brief Report we would like to provide an elementary example in which a fractional fermion number could be induced from the interaction of a massive fermion with a scalar Coulomb potential.

Let us consider a $(1+1)$-dimensional Dirac Hamiltonian with special potentials of the kind

$$H = -i\alpha_x \frac{d}{dx} + \beta(m+U(x)) + V(x), \qquad (2)$$

where m is the mass of the fermion, α_x and β are the Dirac matrices, and U and V are scalar and vector potentials, respectively. Such a model is of interest in both the theory of the nuclear shell model [10,11] and the model [with $V(x) = 0$] of the self-compatible field of a quark system [12]. In the presence of a vector potential, the Dirac Hamiltonian does not exhibit a charge conjugation symmetry since a charge coupling treats particles and antiparticles differently. So the existence of zero mode does not necessarily imply a fractional fermion number. On the other hand, owing to charge-conjugation symmetry, a theory with only a scalar potential U generally has no spectral asymmetry, unless there exists a zero energy solution [10].

We now consider a theory described by the Hamiltonian, Eq. (2), on the positive halfline, i.e., $x\geqslant 0$. We take $V(x) = 0$ and $U(x) = -q/x\,(q>0)$. The Dirac matrices can be represented in terms of the Pauli matrices as

$$\alpha_x = \sigma_2, \quad \beta = \sigma_1. \qquad (3)$$

In this representation, the Dirac equation $H\psi = E\psi$ for the two-component wave function

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \qquad (4)$$

takes the form

$$\frac{d\psi_1}{dx} + \left(m - \frac{q}{x}\right)\psi_1 = E\psi_2, \qquad (5)$$

$$-\frac{d\psi_2}{dx} + \left(m - \frac{q}{x}\right)\psi_2 = E\psi_1. \qquad (6)$$

Here ψ_1 and ψ_2 are, respectively, the upper and lower component of the Dirac wave function, and E is the eigenenergy. The boundary conditions satisfied by the wave function are $\psi_{1,2}(x) = 0$ as $x\to 0,\infty$. The system is symmetric under charge conjugation. In fact, in the representation, Eq. (3), the charge-conjugation operator is given by σ_3. If Ψ is a solution of Eqs. (5) and (6) with energy E, then $\Psi^c \equiv \sigma_3\Psi^*$ is also a solution, but with energy $-E$.

Now it is easy to see that the Dirac equation has a solution with zero energy which is normalizable, nondegenerate, and self-charge-conjugate. The wave function of this state is

$$\psi_0(x) \sim \begin{pmatrix} x^q e^{-mx} \\ 0 \end{pmatrix}. \tag{7}$$

We must have the lower component of Eq. (7) set to zero since the general solution of Eq. (6), which is $x^{-q}e^{mx}$ for $E=0$, is divergent at both $x=0$ and $x=\infty$. This ground state is self-charge-conjugate, since $\sigma_3\Psi_0=\Psi_0$. From Eq. (1), we conclude that the vacuum in the presence of the external scalar field, defined with the zero energy state empty, has a fermion number $N_V=-1/2$. Clearly, there is a degenerate lowest-energy configuration with the zero energy state filled, yielding fermion number $N_V=1/2$.

Finally, we note here that the solutions of Eqs. (5) and (6) may be expressed in terms of the Whittaker functions, and the discrete energy spectrum is given by

$$E^2 = m^2\left[1-\frac{q^2}{(q+n)^2}\right]. \tag{8}$$

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