Geodesic deviation in Kaluza-Klein theories

Richard Kerner,[1,*] Jérôme Martin,[2,†] Salvatore Mignemi,[3,‡] and Jan-Willem van Holten[4,§]
[1]*LGCR, Université Paris-VI, Tour 22, Boîte 142, 4, Place Jussieu, 75005 Paris, France*
[2]*Institut d'Astrophysique de Paris, 98 boulevard Arago, 75014 Paris, France*
[3]*Dipartimento di Matematica, Università di Cagliari, viale Merello 92, 09123 Cagliari, Italy*
[4]*NIKHEF, P.O. Box 41882, 1009 DB Amsterdam, The Netherlands*
(Received 3 October 2000; published 27 December 2000)

We study in detail the equations of the geodesic deviation in multidimensional theories of Kaluza-Klein type. We show that their 4-dimensional space-time projections are identical with the equations obtained by direct variation of the usual geodesic equation in the presence of the Lorentz force, provided that the fifth component of the deviation vector satisfies an extra constraint derived here.

DOI: 10.1103/PhysRevD.63.027502 PACS number(s): 04.50.+h

It is well known that the equation satisfied by a world-line $x^\mu(\tau)$ of a massive charged particle in a 4-dimensional Riemannian space-time in the presence of an electromagnetic field is given by

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}+\frac{q}{m}F_\lambda{}^\mu\frac{dx^\lambda}{d\tau}=0,\quad \mu,\nu,\ldots=0,\ldots,3. \tag{1}$$

In this expression, m and q are the mass and the charge, respectively, of a test particle, τ is the 4-dimensional proper time and the electromagnetic field $F_{\mu\nu}$ is defined by $F_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu$ where $A_\mu(x^\kappa)$ is the vector potential. The $\Gamma^\mu_{\nu\lambda}$ are the 4-dimensional Christoffel symbols. The second term of the above equation represents the inertial force whereas the last term is the Lorentz force.

The study of geodesics in multidimensional theories of Kaluza-Klein type has been performed by many authors [1–5] in quite an exhaustive manner. It has been known since the very beginning [6,7] that, in the simplest version of the theory (without scalar field), the geodesic equation in 5 dimensions coincides with the geodesic equation in 4 dimensions with an extra term which can be identified with the Lorentz force. Indeed, in 5 dimensions, the geodesic equation is given by

$$\frac{d^2x^A}{ds^2}+\left\{\begin{matrix}A\\BC\end{matrix}\right\}\frac{dx^B}{ds}\frac{dx^C}{ds}=0,\quad A,B,\ldots=1,\ldots,5, \tag{2}$$

where s is the 5-dimensional interval length, and the brackets $\{^A_{BC}\}$ are the 5-dimensional Christoffel symbols. Projecting this equation under the assumption that all the quantities are independent of the fifth coordinate, $\partial_5=0$, it is easy to establish, using the formulas displayed in the Appendix, that one obtains

$$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds}+\left(\frac{dx^5}{ds}+A_\mu\frac{dx^\mu}{ds}\right)F_\lambda{}^\mu\frac{dx^\lambda}{ds}=0,$$

$$\frac{d}{ds}\left(\frac{dx^5}{ds}+A_\mu\frac{dx^\mu}{ds}\right)=0. \tag{3}$$

The second equation, telling us that the quantity $Q\equiv dx^5/ds+A_\mu(dx^\mu/ds)$ is constant along the 5-dimensional geodesics parametrized by s, reflects the fact that the backgrounds have been chosen such that x^5 is a cyclic coordinate. Since we have the following relation between the squares of the intervals in 5 and 4 dimensions:

$$\begin{aligned}ds^2&=(g_{\mu\nu}+A_\mu A_\nu)\,dx^\mu dx^\nu+2\,A_\mu dx^\mu dx^5+(dx^5)^2\\&=d\tau^2+\left(\frac{dx^5}{ds}+A_\mu\frac{dx^\mu}{ds}\right)\left(\frac{dx^5}{ds}+A_\nu\frac{dx^\nu}{ds}\right)ds^2,\end{aligned} \tag{4}$$

which amounts to $ds^2(1-Q^2)=d\tau^2$, we see that the 4-dimensional equation (1) is recovered provided that we make the following identification:

$$\frac{q}{m}=\frac{Q}{\sqrt{1-Q^2}}\Leftrightarrow Q=\frac{q/m}{\sqrt{1+(q/m)^2}}. \tag{5}$$

This means that we suppose that $Q^2<1$, so that ds is timelike whenever $d\tau$ is timelike, and vice versa. A more general situation is discussed in detail in [8]. The non-Abelian generalization has been considered in [1] and [2]. Since the multidimensional theories of Kaluza-Klein type are constructed as copies of Einstein's general relativity theory in more than 4 dimensions, all the usual mathematical corollaries remain valid. For example, it is possible to cancel the Christoffel symbols along a given geodesic curve by an appropriate choice of coordinates, which amounts to the annulation of forces acting on a test particle moving along that geodesic line. In 4 dimensions, the cancellation along the worldline of the 5-dimensional Christoffel symbols may be interpreted as the simultaneous compensation of gravitational and the Lorentz forces by an appropriate acceleration field.

It is also a well known fact that such cancellation can be performed along only one given geodesic line at once, but

*Email address: rk@ccr.jussieu.fr
†Email address: jmartin@iap.fr
‡Email address: smignemi@vaxca1.unica.it
§Email address: v.holten@nikhef.nl

not necessarily along all its neighbors. This fact becomes obvious when one looks at the *geodesic deviation* equation

$$\frac{\mathrm{D}^2(\delta x^\mu)}{\mathrm{D}\tau^2} = {}^4R^\mu{}_{\rho\nu\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\delta x^\lambda. \tag{6}$$

where δx^λ is an infinitesimal ''geodesic deviation'' vector, and $\mathrm{D}/\mathrm{D}\tau$ denotes the pull-back of covariant derivatives along the time-like geodesics. For a massive *and* charged test particle in the presence of both gravitational and electromagnetic fields ($R^\mu{}_{\nu\lambda\rho}\neq 0$ and $F_{\mu\nu}\neq 0$), it is not difficult to derive the generalized world-line deviation equation by taking direct variation of the world-line equation (1):

$$\frac{\mathrm{D}^2(\delta x^\mu)}{\mathrm{D}\tau^2} = {}^4R^\mu{}_{\rho\nu\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\delta x^\lambda + \frac{q}{m}\left[(\nabla_\rho F^\mu{}_\nu)\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\delta x^\rho + F^\mu{}_\nu\frac{\mathrm{D}(\delta x^\nu)}{\mathrm{D}\tau}\right]. \tag{7}$$

Here the Riemann tensor appears explicitly, making it automatically impossible to cancel its influence by any local or global coordinate or gauge transformations. This is why the study of the geodesic deviation in multidimensional theories, which reads

$$\frac{\mathrm{D}^2(\delta x^A)}{\mathrm{D}s^2} = R^A{}_{BCE}\frac{\mathrm{d}x^B}{\mathrm{d}s}\frac{\mathrm{d}x^C}{\mathrm{d}s}\delta x^E, \tag{8}$$

is of particular interest. Indeed, when explicited in the form that splits up the 4-dimensional space-time and the *D*-dimensional internal space, *a priori*, new terms show up, containing quadratic expressions of the type $F^\mu{}_\nu F_{\mu\lambda}$ appearing in the 5-dimensional Riemann tensor (cf. Appendix), that cannot be foreseen or derived from a purely 4-dimensional point of view, even if one tries to introduce the interaction of charges with gauge and scalar fields. It will fix in a canonical way the terms describing the purely gravitational influence

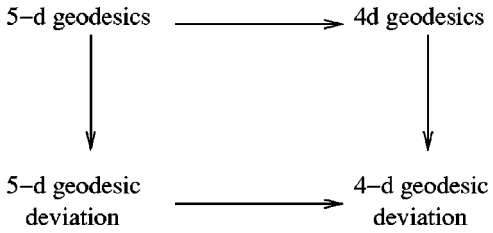

FIG. 1. Kaluza-Klein reduction of geodesic deviations.

of those fields, which by their energy density must influence the trajectories of chargeless massive particles, too, provoking tidal effects which should deform the initially parallel geodesic lines.

Hence the question arises whether calculation of the geodesic deviation after reduction to 4 dimensions yields the same result as projecting the 5-dimensional geodesic deviations to 4 dimensions, rendering the diagram of Fig. 1 commutative. This is of course an important issue since any difference could be used to discriminate a purely 4-dimensional theory from a Kaluza-Klein approach.

The purpose of this Brief Report is to address this question. To our knowledge, the analysis of this problem cannot be found in the existing literature, and we believe that the present study will close this gap, and in addition, will shed some new light on the interplay between the gauge fields and gravitation, and on the interpretation of the equivalence principle in multidimensional theories as well.

We consider here the version of the 5-dimensional Kaluza-Klein theory in which the scalar field φ is put equal to 1 from the beginning, which makes the field equations arising from the variational principle in 5 dimensions strictly equivalent to the Einstein-Maxwell system. We shall perform all the calculations in a holonomous coordinate system in order to make the interpretation of the geodesic equations and the affine parameters as straightforward as possible. The space-time components obey the equation

$$\frac{\mathrm{D}^2(\delta x^\mu)}{\mathrm{D}s^2} = \left[{}^4R^\mu{}_{\nu\rho\kappa} - \frac{3}{4}F^\mu{}_\rho F_{\nu\kappa}\right]\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\frac{\mathrm{d}x^\rho}{\mathrm{d}s}\delta x^\kappa - \frac{Q}{2}[\nabla_\kappa F_\rho{}^\mu + \nabla^\mu F_{\rho\kappa}]\frac{\mathrm{d}x^\rho}{\mathrm{d}s}\delta x^\kappa - \frac{1}{2}(A_\kappa\delta x^\kappa + \delta x^5)(\nabla_\rho F^\mu{}_\nu)\frac{\mathrm{d}x^\rho}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} - \frac{Q^2}{4}F^\mu{}_\lambda F_\rho{}^\lambda\delta x^\rho + \frac{Q}{4}(A_\kappa\delta x^\kappa + \delta x^5)F^\mu{}_\lambda F_\rho{}^\lambda\frac{\mathrm{d}x^\rho}{\mathrm{d}s}. \tag{9}$$

Equation (9) still is not explicit enough to be solved as function of the space-time variables. The derivatives with respect to 5-dimensional interval $\mathrm{d}s$ should be replaced by the derivatives with respect to the 4-dimensional proper time $\mathrm{d}\tau$. Next, the second-order covariant derivative appearing on the right-hand side and containing the 5-dimensional connection coefficients and their partial derivatives, has to be expressed in terms of ordinary derivatives and 4-dimensional Christoffel symbols along with the gauge-invariant quantities containing the Faraday tensor $F_{\mu\nu}$. The relation between the covariant derivations with respect to the parameters $\mathrm{d}s$ (the 5-dimensional line element) and $\mathrm{d}\tau$ (particle's proper time in 4 space-time dimensions) is quite complicated when it comes to the second-order covariant derivatives. This is why we omit all the intermediary calculations, giving here the final result. The second covariant derivatives are related as follows for the 4-dimensional components of a given 5-dimensional vector u^A:

$$\left(\frac{ds}{d\tau}\right)^2 \frac{D^2u^\mu}{Ds^2} = \frac{D^2u^\mu}{D\tau^2} + \frac{1}{2}\frac{Q}{\sqrt{1-Q^2}}\nabla_\rho F_\nu{}^\mu \frac{dx^\rho}{d\tau}u^\nu - \frac{1}{2}(A_\kappa u^\kappa + u^5)\nabla_\rho F_\nu{}^\mu \frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau} + \frac{Q}{\sqrt{1-Q^2}}F_\nu{}^\mu \frac{Du^\nu}{D\tau}$$
$$+ \frac{1}{4}\frac{Q^2}{(1-Q^2)}F^\mu_\nu F^\nu_\rho u^\rho - \frac{1}{4}\frac{Q}{\sqrt{1-Q^2}}(A_\kappa u^\kappa + u^5)F^\mu_\nu F^\nu_\rho \frac{dx^\rho}{d\tau} - \frac{3}{4}F^\mu{}_\lambda F_{\kappa\rho}\frac{dx^\lambda}{d\tau}\frac{dx^\kappa}{d\tau}u^\rho$$
$$+ F_\lambda{}^\mu \frac{dx^\lambda}{d\tau}\left[\frac{d}{d\tau}(A_\kappa u^\kappa + u^5) + F_{\kappa\rho}\frac{dx^\kappa}{d\tau}u^\rho\right]. \tag{10}$$

Combining Eqs. (9) and (10) and using homogeneous Maxwell's equations, $\nabla_\lambda F^\mu_\nu + \nabla_\nu F_\lambda{}^\mu + \nabla^\mu F_{\nu\lambda} = 0$ and the identification of the physical charge-to-mass ratio (5), one obtains

$$\frac{D^2(\delta x^\mu)}{D\tau^2} = {}^4R^\mu{}_{\rho\nu\lambda}\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\lambda + \frac{q}{m}\left[(\nabla_\rho F^\mu{}_\nu)\frac{dx^\nu}{d\tau}\delta x^\rho + F^\mu{}_\nu \frac{D(\delta x^\nu)}{D\tau}\right] + F^\mu{}_\lambda \frac{dx^\lambda}{d\tau}\left[\frac{d}{d\tau}(A_\kappa \delta x^\kappa + \delta x^5) + F_{\kappa\rho}\frac{dx^\kappa}{d\tau}\delta x^\rho\right]. \tag{11}$$

This equation would coincide with the usual 4-dimensional deviation Eq. (7) if it were not for the last term, which contains the standard Lorentz force multiplied by the expression in square brackets, linear in the infinitesimal deviation vector. However, it is easily recognized that the last term just represents the deviation of the 5th component of the momentum Q, which by Eq. (3) is conserved, and which we identified with the charge through Eq. (5):

$$\delta Q = \left[\frac{d}{ds}(\delta x^5 + A_\lambda \delta x^\lambda) + F_{\lambda\rho}\frac{dx^\rho}{ds}\delta x^\lambda\right]. \tag{12}$$

It is noteworthy that the same result can be obtained by using the simpler, but non-covariant form of the geodesic deviation equation (8):

$$\frac{d^2(\delta x^A)}{ds^2} + 2\left\{\begin{matrix} A \\ BC \end{matrix}\right\}\frac{dx^B}{ds}\frac{d(\delta x^C)}{ds} + \left(\partial_D\left\{\begin{matrix} A \\ BC \end{matrix}\right\}\right)\frac{dx^B}{ds}\frac{dx^C}{ds}\delta x^D = 0, \tag{13}$$

which makes the calculations much less tedious. The fifth component of the previous equation leads to

$$\frac{d}{ds}\left[\frac{d}{ds}(\delta x^5 + A_\lambda \delta x^\lambda) + F_{\lambda\rho}\frac{dx^\rho}{ds}\delta x^\lambda\right] = 0, \tag{14}$$

which means that δQ is indeed a constant. From a purely mathematical point of view, this constant can take on any real value, depending on the arbitrary choice of *initial conditions*, which include the initial values of *ten* variables, $\delta x^A(0)$ and $[d(\delta x^B)/ds](0)$, as it is the case for any system of five ordinary differential equations of second order. The fact that δQ is a constant means that not all the initial data can be independent. As a matter of fact, the first derivative of the fifth component of the deviation, $[d(\delta x^5)/ds](0)$, is an imposed function of the four-dimensional initial deviations, namely

$$\frac{d(\delta x^5)}{ds}(0) = \delta Q - \left[\frac{d}{ds}(A_\lambda \delta x^\lambda) + F_{\lambda\rho}\frac{dx^\rho}{ds}\delta x^\lambda\right](0). \tag{15}$$

Requiring that $\delta Q = 0$ is the condition which must be imposed if we want to maintain a one-to-one correspondence between the geodesic deviation equation in 5-dimensional Kaluza-Klein space and the usual deviation equation in presence of the electromagnetic field in 4 dimensions.

Now, returning to Eq. (11), we get the final result that can be stated very simply as follows:

The space-time projection of the five-dimensional Kaluza-Klein geodesic deviation equation yields for fixed Q the usual four-dimensional world-line deviation equation in the presence of both gravitational and electromagnetic fields, for particles of the same q/m; geodesic deviations between five-dimensional world lines with different values of Q describe the four-dimensional deviation of world-lines for particles with different values of q/m.

The only influence of the electromagnetic fields on chargeless particles comes through the term linear in the 4-dimensional Riemann tensor, which is a solution of the coupled Einstein-Maxwell equations (see Ref. [11]).

Although the final answer to the problem of the projection of the geodesic deviation equation from the 5-dimensional Kaluza-Klein metric space onto its 4-dimensional space-time basis is very simple and does not bring any surprise, it is worth checking (the above calculations have never been published elsewhere, at least to our knowledge), and does not seem to be totally trivial. It can be interpreted as a strong *equivalence principle* generalized to the 5-dimensional theory incorporating electromagnetism into geometry.

Our result can be easily generalized to the non-Abelian case [1,2,9,10], where the conservation of charge Q is replaced by a condition on the rotation of the charge isovector in the Lie algebra space in which it takes its value. There is no guarantee that the higher-order deviation equations, obtained with a similar technique of new independent variation of Einstein-Maxwell equations do also project properly, although such a statement seems very plausible.

However, the introduction of the *dilaton field*, i.e. supposing that the radius of the compactified 5-th dimension depends on the space-time position x^μ, may bring new effects leading to certain anomalies in the deviation equation and its projection onto the usual space-time. One can apply a similar technique of probing the deviations to the equations of motions of the *p-branes* embedded in multi-dimensional spaces, which represent a natural generalization of geodesic curves in Kaluza-Klein theories.

These developments should become the object of another independent study.

For J.W.v.H. this work is part of the research program of the Foundation for Fundamental Research of Matter (FOM).

APPENDIX

The 5-dimensional metric tensor of the theory reads $(A,B,\ldots=1,2,\ldots,5)$

$$\gamma_{AB}=\begin{pmatrix} g_{\mu\nu}+A_\mu A_\nu & A_\nu \\ A_\mu & 1 \end{pmatrix}, \quad \text{with} \quad ds^2=\gamma_{AB}\,dx^A dx^B, \tag{A1}$$

where $g_{\mu\nu}=g_{\mu\nu}(x^\lambda)$ and $A_\mu=A_\mu(x^\lambda)$, which means that we consider that both $g_{\mu\nu}$ and A_μ do not depend on the fifth coordinate x^5. Here are the Christoffel symbols of the metric (A1):

$$\begin{Bmatrix} \mu \\ \nu\lambda \end{Bmatrix}=\Gamma^\mu_{\nu\lambda}+\frac{1}{2}(A_\lambda F_\nu{}^\mu+A_\nu F_\lambda{}^\mu),$$

$$\begin{Bmatrix} 5 \\ \mu\nu \end{Bmatrix}=\frac{1}{2}(\nabla_\mu A_\nu+\nabla_\nu A_\mu)-\frac{1}{2}A^\rho(A_\nu F_{\mu\rho}+A_\mu F_{\nu\rho}), \tag{A2}$$

$$\begin{Bmatrix} \mu \\ 5\nu \end{Bmatrix}=\begin{Bmatrix} \mu \\ \nu 5 \end{Bmatrix}=\frac{1}{2}F_\nu{}^\mu, \quad \begin{Bmatrix} 5 \\ 5\mu \end{Bmatrix}=\begin{Bmatrix} 5 \\ \mu 5 \end{Bmatrix}=-\frac{1}{2}A^\kappa F_{\mu\kappa},$$

$$\begin{Bmatrix} \mu \\ 55 \end{Bmatrix}=0, \quad \begin{Bmatrix} 5 \\ 55 \end{Bmatrix}=0. \tag{A3}$$

With this in mind, we can proceed with the computation of the components of 5-dimensional Riemann tensor in a holonomous system. Using the convention

$$R^A{}_{BCD}=\partial_C\begin{Bmatrix} A \\ DB \end{Bmatrix}-\partial_D\begin{Bmatrix} A \\ CB \end{Bmatrix}+\begin{Bmatrix} A \\ CE \end{Bmatrix}\begin{Bmatrix} E \\ DB \end{Bmatrix}-\begin{Bmatrix} A \\ DE \end{Bmatrix}\begin{Bmatrix} E \\ CB \end{Bmatrix}, \tag{A4}$$

and after some calculus, using also the Bianchi identities satisfied by the tensor $F_{\mu\nu}$, we get

$$R^\rho{}_{\lambda\mu\nu}={}^{(4)}R^\rho{}_{\lambda\mu\nu}+\tfrac{1}{4}(F_\mu{}^\rho F_{\lambda\nu}-F_\nu{}^\rho F_{\lambda\mu}+2F_\lambda{}^\rho F_{\mu\nu})-\tfrac{1}{2}A_\lambda\nabla^\rho F_{\mu\nu}+\tfrac{1}{2}(A_\nu\nabla_\mu F_\lambda{}^\rho-A_\mu\nabla_\nu F_\lambda{}^\rho)+\tfrac{1}{4}A_\lambda F_\sigma{}^\rho(A_\mu F_\nu{}^\sigma-A_\nu F_\mu{}^\sigma), \tag{A5}$$

$$R^5{}_{\mu\nu\lambda}=-{}^{(4)}R^\rho{}_{\mu\nu\lambda}A_\rho+\tfrac{1}{2}\nabla_\mu F_{\nu\lambda}+\tfrac{1}{4}F_\mu{}^\rho(A_\lambda F_{\rho\nu}-A_\nu F_{\rho\lambda})-\tfrac{1}{2}A_\mu A_\rho\nabla^\rho F_{\lambda\nu}+\tfrac{1}{2}A_\rho(A_\nu\nabla_\lambda F_\mu{}^\rho-A_\lambda\nabla_\nu F_\mu{}^\rho)+\tfrac{1}{4}A_\mu A^\sigma F_{\rho\sigma}(A_\lambda F_\nu{}^\rho-A_\nu F_\lambda{}^\rho)+\tfrac{1}{4}A^\rho(F_{\lambda\rho}F_{\mu\nu}+F_{\nu\rho}F_{\lambda\mu}+2F_{\mu\rho}F_{\lambda\nu}), \tag{A6}$$

$$R^\rho{}_{5\mu\nu}=-\tfrac{1}{2}\nabla^\rho F_{\mu\nu}+\tfrac{1}{4}F_\sigma{}^\rho(A_\mu F_\nu{}^\sigma-A_\nu F_\mu{}^\sigma), \tag{A7}$$

$$R^\rho{}_{\mu 5\nu}=-\tfrac{1}{2}\nabla_\nu F_\mu{}^\rho+\tfrac{1}{4}A_\mu F_\sigma{}^\rho F_\nu{}^\sigma, \tag{A8}$$

$$R^5{}_{5\mu\nu}=\tfrac{1}{4}F_{\rho\sigma}A^\sigma(F_\mu{}^\rho A_\nu-F_\nu{}^\rho A_\mu)+\tfrac{1}{2}A_\rho\nabla^\rho F_{\mu\nu}, \tag{A9}$$

$$R^5{}_{\mu 5\nu}=\tfrac{1}{4}F_\mu{}^\rho F_{\nu\rho}+\tfrac{1}{2}A_\rho\nabla_\nu F_\mu{}^\rho-\tfrac{1}{4}A_\mu A^\lambda F_\nu{}^\sigma F_{\sigma\lambda}, \tag{A10}$$

$$R^\rho{}_{5\lambda 5}=-\tfrac{1}{4}F_\sigma{}^\rho F_\lambda{}^\sigma, \tag{A11}$$

$$R^5{}_{5\lambda 5}=\tfrac{1}{4}A_\rho F_\sigma{}^\rho F_\lambda{}^\sigma. \tag{A12}$$

Here the Greek indices $\mu,\nu,\ldots$ are raised and lowered by means of the 4-dimensional metric tensors $g^{\mu\lambda}$ and $g_{\lambda\rho}$.

[1] R. Kerner, Ann. Inst. Henri Poincare, Sect. A **9**, 147 (1968).
[2] C. Wong, Nuovo Cimento A **65**, 689 (1970).
[3] H. Liu and P.S. Wesson, Class. Quantum Grav. **14**, 1651 (1997).
[4] H. Arodź, Phys. Lett. **116B**, 251 (1982).
[5] H. Arodź, A. Sitarz, and P. Wegrzyn, Acta Phys. Pol. B **20**, 921 (1989).
[6] Th. Kaluza, Sitzungsber. K. Preuss. Akad. Wiss. **K1**, 966 (1921).
[7] O. Klein, Z. Phys. **37**, 895 (1926); **46**, 188 (1927).
[8] E. Leibowitz and N. Rosen, Gen. Relativ. Gravit. **4**, 449 (1973).
[9] R. Kerner, Ann. Inst. Henri Poincare, Sect. A **24**, 437 (1981).
[10] A.B. Balakin and F.G. Suslikov, C. R. Acad. Sci., Ser. IIb: Mec., Phys., Chim., Astron. **324**, 619 (1997).
[11] A. Balakin, J.-W. van Holten, and R. Kerner, ''Motions and world-line deviations in Einstein-Maxwell theory,'' Report No. LGCR-2000/08/2 and NIKHEF/00-023, gr-qc/0009016.