Out of equilibrium thermal field theories: Finite time after switching on the interaction and Wigner transforms of projected functions

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We study out of equilibrium thermal field theories with switching on the interaction occurring at finite time using the Wigner transforms of two-point functions. For two-point functions we define the concept of a projected function: it is zero if any of the times refers to the time before switching on the interaction; otherwise it depends only on the relative coordinates. This definition includes bare propagators, one-loop self-energies, etc. For the infinite-average-time limit of the Wigner transforms of projected functions we define the analyticity assumptions: (1) The function of energy is analytic above (below) the real axis. (2) The function goes to zero as the absolute value of energy approaches infinity in the upper (lower) semiplane. Without use of the gradient expansion, we obtain the convolution product of projected functions. We sum the Schwinger-Dyson series in closed form. In the calculation of the Keldysh component (both resummed and single self-energy insertion approximation) contributions appear which are not the Fourier transforms of projected functions, signaling the limitations of the method. In the Feynman diagrams there is no explicit energy conservation at vertices; there is an overall energy-smearing factor taking care of the uncertainty relations. The relation between the theories with the Keldysh time path and with the finite time path enables one to rederive the results, such as the cancellation of pinching, collinear, and infrared singularities, hard thermal loop resummation, etc.

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I. INTRODUCTION

Out of equilibrium thermal field theory [1,2] has recently attracted considerable interest [3–15]. In many applications one considers the properties of the Green functions of almost equilibrated systems, at infinite time after switching on the interaction. A recent approach based on first principles has been successful in demonstrating the cancellation of collinear [16,17] and pinching singularities [18–23], the extension of the hard thermal loop (HTL) approximation [24–26] to out of equilibrium [27,28], and applications to heavy-ion collisions [29–31]. A weak point of the approach was that most of these results were obtained under the assumption that the variation of slow Wigner variables could be ignored or, in other words, that these results were valid only in the lowest order of the gradient expansion [32–34].

For some problems, e.g., heavy-ion collisions, both limitations are undesired. One would like to consider large deviations from equilibrium. One cannot wait infinitely long as these systems go apart after a very short time, probably without reaching the stage of equilibrium. In nuclear collisions, short-time-scale features have been studied in a number of papers [35–40].

In the theories with switching on the interaction infinitely long before the time of interest, one tries to get some information by extrapolation to early times. However, in doing so, this information is either deformed or lost. Indeed, relaxation phenomena include many processes that are expected to terminate as one goes sufficiently far from the switching-on time of the interaction. Thus one can expect that the full theory also describes early times. In many papers, problems with a finite switching-on time [9], especially those related to the inflatory phase of the Universe (see Ref. [41] for further references), have been studied by Wigner transforming in relative distance and studying the dependence on relative time directly [42]. In such an approach, one relies heavily on differential equations and numerical methods.

In an attempt to remove the weak points in both cases, we suggest the application of the method of Wigner transform (but now also in relative time) to the case of switching on the interaction at finite time.The time-integration path C (see Fig. 1) is now closing the part of the real axis between t_i (the switching-on time) and t_f (the switching-off time); in the rest of the paper the switching-off time is pushed to infinity. If we push the switching-on time to minus infinity, the connection to the Keldysh integration path is established. It turns out that this connection is highly nontrivial.

To understand the limitations coming from the finite switching-on time, we start with a generic two-point function $G(x,y)$. The quantities x and y are four-vector variables with the time components in the range $t_i<x_0,\ y_0<\infty$ (here t_i is the time at which we switch on the interaction; it is usually set to $-\infty$, but we set it to $t_i=0$). We define the Wigner variables s (relative space-time, relative variable) and X (average space-time, slow variable) as usual:

$$X=\frac{x+y}{2},\quad s=x-y,$$
$$G(x,y)=G\left(X+\frac{s}{2},X-\frac{s}{2}\right). \tag{1.1}$$

The lower limit on $x_0,\ y_0$ implies the following conditions on X_0 and s_0: $0<X_0,\ -2X_0<s_0<2X_0$. To define the pro-

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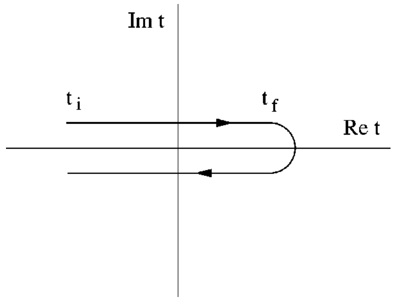

FIG. 1. Finite switching-on time integration path.

jected function (truncated, ‘‘mutilated function’’ [43], PF in further text), we add two additional properties: the function does not change with $\vec{X}$ (homogeneity assumption), it is a function of $(s_0,\vec{s})$ within the interval $-2X_0<s_0<2X_0$ and is identical to zero outside the interval:

$$F\left(X+\frac{s}{2},X-\frac{s}{2}\right)=\Theta(X_0)\Theta(2X_0-s_0)\Theta(2X_0+s_0)\overline{F}(s_0,\vec{s}), \tag{1.2}$$

$$\overline{F}(s_0,\vec{s})=\lim_{X_0\to\infty}F\left(X+\frac{s}{2},X-\frac{s}{2}\right).$$

Analogously, the Wigner transform of the projected function (WTPF) is obtained from the Wigner transform of the function defined on the infinite carrier of s_0 $(-\infty<s_0<\infty)$ with the help of the projection operators P_{X_0}, which are the Fourier transforms of the above given Θ’s:

$$F_{X_0}(p_0,\vec{p})=\int_{-\infty}^{\infty}dp_0'P_{X_0}(p_0,p_0')F_\infty(p_0',\vec{p}),$$

$$P_{X_0}(p_0,p_0')=\frac{1}{\pi}\Theta(X_0)\frac{\sin[2X_0(p_0-p_0')]}{p_0-p_0'},$$

$$\lim_{X_0\to\infty}P_{X_0}(p_0,p_0')=\delta(p_0-p_0'), \tag{1.3}$$

where the subscript ‘‘∞,’’ as it appears in F_∞ in (1.3) and in other expressions of this paper, is the short notation for the ‘‘$\lim_{X_0\to\infty}$.’’ As an illustration, the Wigner transform of the convolution product of two-point functions

$$C=A*B\Leftrightarrow C(x,y)=\int dzA(x,z)B(z,y) \tag{1.4}$$

is given by the gradient expansion (note that we have assumed the homogeneity in space coordinates, which excludes any dependence on $\vec{X}$):

$$C_{X_0}(p_0,\vec{p})=e^{-i\diamond}A_{X_0}(p_0,\vec{p})B_{X_0}(p_0,\vec{p}),$$

$$\diamond=\frac{1}{2}(\partial^A_{X_0}\partial^B_{p_0}-\partial^A_{p_0}\partial^B_{X_0}). \tag{1.5}$$

This general expression should be contrasted with our result valid for A and B being projected functions (see Sec. V):

$$C_{X_0}(p_0,\vec{p})=\int dp_{01}dp_{02}P_{X_0}\left(p_0,\frac{p_{01}+p_{02}}{2}\right)\times\frac{1}{2\pi}\frac{ie^{-iX_0(p_{01}-p_{02}+i\epsilon)}}{p_{01}-p_{02}+i\epsilon}\times A_\infty(p_{01},\vec{p})B_\infty(p_{02},\vec{p}). \tag{1.6}$$

The analytic properties of the WTPF in the $X_0\to\infty$ limit as a function of complex energy are very important for further analysis. We define the following properties: (1) the function of p_0 is analytic above (below) the real axis, (2) the function goes to zero as $|p_0|$ approaches infinity in the upper (lower) semiplane. The choice above (below) and upper (lower) refers to R (A) components (note here that we do not require such properties for the Keldysh components). It is easy to recognize that the properties (1) and (2) are just the definition of the retarded (advanced) function. However, it is nontrivial, and not always true, that the functions with the R (A) index satisfy them.

Under the assumption that A or B satisfies (1) and (2) (A as advanced or B as retarded) Eq. (1.6) can be integrated even further. We obtain

$$C_{X_0}(p_0,\vec{p})=\int dp_0'P_{X_0}(p_0,p_0')A_\infty(p_0',\vec{p})B_\infty(p_0',\vec{p}). \tag{1.7}$$

The convolution product of two two-point functions which are WTPF’s and satisfy (1) and (2) is also a WTPF. This product is then expressed through the projection operator acting on a simple product of two WTPF’s given at $X_0^A=X_0^B=\infty$. In the $X_0\to\infty$ limit the convolution product of two WTPF’s which satisfy (1) and (2) is equal to the lowest-order contribution in the gradient expansion. The result makes sense since moving $X_0\to\infty$ is equivalent to $t_i\to-\infty$ at fixed X_0 in the standard approach.

We find that some quantities, obtained at low orders in the perturbative expansion, e.g., bare propagators, one-loop self-energies, belong to WTPF’s. This enables us to sum the Schwinger-Dyson series with the propagators and self-energies as both these quantities are WTPF. Under the conditions (1) and (2), the retarded, advanced, and Keldysh components at finite X_0 are obtained by a simple action (smearing) of the projection operator onto the corresponding quantities obtained at $X_0=\infty$, and the convolution product is a simple multiplication:

$$\mathcal{G}_{R,X_0}(p_0,\vec{p})=\int dp_{01}P_{X_0}(p_0,p_{01})\mathcal{G}_{R\infty}(p_{01},\vec{p}),$$

$$\mathcal{G}_{R,\infty}(p_{01},\vec{p})=\frac{G_{R,\infty}(p_{01},\vec{p})}{1-i\Sigma_{R,\infty}(p_{01},\vec{p})G_{R,\infty}(p_{01},\vec{p})}, \tag{1.8}$$

and similarly for the advanced component. The calculation of the Keldysh component requires a more elaborate treatment: one reduces the multiplication to a double (a single in some cases) convolution product; the result contains terms that are non-WTPF terms. Also, the one-loop contribution to the Keldysh component is corrected by non-WTPF terms.

Now, it is $\mathcal{G}_{R,\infty}(p_0,\vec{p})$ as given in Eq. (1.8) (and similar expressions for the advanced and Keldysh components and the single self-energy insertion approximation to $\mathcal{G}_{R(A,K),\infty}$) to which previous results on the cancellation of pinching singularities [18,19,21,23] and the HTL resummation [27,28] (and also on the cancellation of collinear [16,17] and infrared singularities if the properties (1) and (2) hold at the two-loop level) apply.

From our study one can deduce a general rearrangement of the perturbation expansion at the non-Keldysh integration path: the contributions look like the zeroth order of the gradient expansion with the slow coordinate (X_0) pushed to $+\infty$, but the use of the PF manifests itself as the appearance of the $(\Sigma_j q_{0j}+i\epsilon)^{-1}$ factor instead of $-\pi\delta(\Sigma_j q_{0j})$ at each vertex, and as an overall projection (smearing) operator instead of the exact conservation of energy.

Our study suggests that the results obtained by using the Keldysh integration path ($t_i\rightarrow-\infty$) could be related to the results of our approach (t_i finite). This relation is possible at low orders of the perturbation expansion, i.e., as long as the expressions involved are the projected functions not breaking assumptions (1) and (2). Technically, the amplitudes are related by (1.3), where ''F_{X_0}'' is the contribution in our approach (t_i finite) and ''F_∞'' is substituted by the corresponding lowest-order contribution in the gradient expansion in the theories with $t_i\rightarrow-\infty$.

Finally, we note that our method is helpful in problems related to the time evolution of the system. Additional problems related to the gradient expansion in space components, appearing together with space inhomogeneity, will not benefit from our method.

The paper is organized as follows. In Sec. II we give a general setup of out of equilibrium thermal field theory. In Sec. III we define finite-time Wigner transforms, define the projection operators, and introduce the notion of projected functions. We define analyticity assumptions (1) and (2). In Sec. IV we define a few important examples of projected functions, namely, bare propagators and one-loop self-energies, and find that they satisfy the analyticity assumptions (1) and (2). In Sec. V we analyze the properties of the product of two and n two-point functions. In Sec. VI these properties are used to study the product of two pole contributions, and to discuss the reduction of the inverse bare propagator to the space of projected functions and the equations of motion. We sum the Schwinger-Dyson series for retarded, advanced, and Keldysh components of the propagator. We discuss the appearance of pinching singularities in our scheme. Section VII is devoted to the modifications of Feynman rules in coordinate and momentum space. It is indicated that, in the absence of breakdown of assumptions (1) and (2), all energy denominators appearing at vertices can be replaced by delta functions. Section VIII is a summary of the results and ideas described in the paper.

II. OUT OF EQUILIBRIUM SETUP

We start by assuming that the system has been prepared at some initial time $t_i=0$ (to avoid inessential complications, we assume that the zero-temperature renormalization has already been performed). At t_i the interaction is switched on and at time t_f it is switched off (we shall take the limit $t_f\rightarrow\infty$). For $t_i<t_1,t_2<t_f$, the system evolves under the evolution operator [8]

$$U(t_2,t_1)=T_c\left[\exp\left(i\int_c d^4x'\,\mathcal{L}_I(x')\right)\right], \tag{2.1}$$

where c is the integration contour connecting t_1 and t_2 in the complex time plane and T_c is the contour ordering operator. We provide T_c with an extra property: for all times not belonging to the contour it gives zero.

The Heisenberg field operator $\phi(x)$ is obtained from the free field $\phi_{in}(x)$ in the interaction picture as

$$\phi(x)=U(t_i,t)\,\phi_{in}(x)\,U(t,t_i), \tag{2.2}$$

$$\phi(x)=T_C\left[\phi_{in}(x)\exp\left(i\int_C d^4x'\,\mathcal{L}_I(x')\right)\right], \tag{2.3}$$

where all fields on the right-hand side are in the interaction picture, and C is the integration contour of Fig. 1 (with the switching-off time pushed to infinity, $t_f\rightarrow+\infty$). In the Heisenberg picture, the average values of the operators are obtained as

$$\langle O(t)\rangle=\mathrm{Tr}\,\rho O(t), \tag{2.4}$$

where ρ is the density operator admitting the Wick decomposition. Especially, we define the two-point Green function as

$$G^{(C)}(x,x')=-i\langle T_C\phi(x)\phi(x')\rangle. \tag{2.5}$$

With the help of (2.1) it can be written as

$$G^{(C)}(x,x')=-i\left\langle T_C\left(\exp\left[i\int_C d^4x''\mathcal{L}_I(x'')\right]\times\phi_{in}(x)\,\phi_{in}(x')\right)\right\rangle. \tag{2.6}$$

In Eq. (2.6) it is implicitly assumed that the interaction Lagrangian does not depend on time explicitly. Indeed, direct time dependence through time-dependent perturbation, or through the background field which depends directly on time, $\psi(x,t)=\phi(t)+\overline{\psi}(x,t)$, would break our scheme through the appearance of two-point functions which are not projected functions.

We assume the single-particle density operator to be stationary with respect to the free Hamiltonian $H_0=\Sigma_j H_j$ (for an alternative choice of the initial density, see Ref. [44]):

$$\rho=\sum_n |\psi_n\rangle p_n\langle\psi_n|=\frac{1}{Z}\exp\left(-\sum_j \beta_j H_j\right), \tag{2.7}$$

where the ''temperature'' function [9] β_j (the ''temperature'' of the jth degree of freedom) is adopted to obtain the given initial-state particle distribution. Now, one has

$$\sum_j \beta_j H_j=\int d^4p\,\beta(p_0)p_0\Theta(p_0)\,\delta(p_0^2-\vec{p}^2-m^2)a_p^+a_p, \tag{2.8}$$

and obtains the distribution function $f(p_0)$ as a function of $\beta(p_0)$,

$$f_\beta(p_0)=\frac{1}{p_0}\,\mathrm{Tr}\,p_0\rho=\frac{1}{\exp\beta(p_0)p_0\mp 1}, \tag{2.9}$$

or the inverse relation

$$\beta_f(p_0)=\frac{\log\left(\frac{1}{f(p_0)}\pm 1\right)}{p_0}. \tag{2.10}$$

The free fields are expanded in creation and annihilation operators as

$$\phi(x)=\int\frac{d^3p}{(2\pi)^3 2\omega_p}(a_p e^{-ipx}+a_p^+e^{ipx}), \tag{2.11}$$

with $p_0=\omega_p=(\vec{p}^2+m^2)^{1/2}$:

$$\langle a_p^+a_{p'}\rangle=(2\pi)^3 2\omega_p f(\omega_p)\,\delta(\vec{p}-\vec{p}\,'),$$

$$\langle a_p a_{p'}^+\rangle=(2\pi)^3 2\omega_p(1\pm f(\omega_p))\delta(\vec{p}-\vec{p}\,'), \tag{2.12}$$

where $f(\omega_p)$ is the given initial distribution.

For completeness, in Eqs. (2.8)–(2.12) we include free fermions (lower case) in addition to free bosons (upper case). In the case of spin-1/2, spin-1, or higher-spin particles, additional spinor or tensor indices must appear. For simplicity, we do not show them explicitly.

The noninteracting contour Green function is given as

$$G_{in}^{(C)}(x,x')=-i\langle T_C\phi_{in}(x)\phi_{in}(x')\rangle. \tag{2.13}$$

Depending on whether the times x_0 and x_0' belong to the upper (''1'') or lower (''2'') part of the path C, the function $G_{in}^{(C)}(x,x')$ splits into the components $G_{\mu,\nu,in}(x,x')$, $\mu,\nu=1,2$. For the times $x_0<0$ or $x_0'<0$, the Green function is equal to zero owing to our definition of T_C.

III. PROJECTED FUNCTIONS

Let us start with the two-point function $G(x,y)$. The quantities x and y are four-vector variables with time components in the range $t_i<x_0,\ y_0<\infty$ (here t_i is the time at which we switch on the interaction; it is usually set to $-\infty$, but we set it to $t_i=0$). We define the Wigner variables as usual:

$$X=\frac{x+y}{2},\quad s=x-y,$$

$$G(x,y)=G\left(X+\frac{s}{2},X-\frac{s}{2}\right). \tag{3.1}$$

The lower limit on x_0, y_0 implies $0<X_0$, $-2X_0<s_0<2X_0$. The values of the function G for the (X,s) not satisfying these conditions are physically irrelevant. Our definiton of time ordering operator (see Sec. II) sets them to zero, so that we can rewrite Eq. (3.1) as

$$G(x,y)=\Theta(X_0)\Theta(2X_0-s_0)\Theta(2X_0+s_0)\overline{G}\left(X+\frac{s}{2},X-\frac{s}{2}\right). \tag{3.2}$$

Note here that the function $\overline{G}$ defined by Eq. (3.2) in general depends on X_0. At the points (X,s) which do not belong to the carrier of projection operator, the values of $\overline{G}$ are arbitrary. This freedom is used to define projected functions. The two-point function can be expressed in terms of the Wigner transform (i.e., Fourier transform with respect to s_0,s_i):

$$G\left(X+\frac{s}{2},X-\frac{s}{2}\right)=(2\pi)^{-4}\int d^4p e^{-i(p_0s_0-\vec{p}\vec{s})}G(p_0,\vec{p};X). \tag{3.3}$$

Here

$$\begin{aligned}G(p_0,\vec{p};X)&=\int_{-2X_0}^{2X_0}ds_0\int d^3s e^{i(p_0s_0-\vec{p}\vec{s})}G\left(X+\frac{s}{2},X-\frac{s}{2}\right)\\&=\int_{-\infty}^{\infty}\int ds_0\int d^3s e^{i(p_0s_0-\vec{p}\vec{s})}\Theta(X_0)\\&\quad\times\Theta(2X_0-s_0)\Theta(2X_0+s_0)\\&\quad\times\overline{G}\left(X+\frac{s}{2},X-\frac{s}{2}\right).\end{aligned} \tag{3.4}$$

We adopt a simplifying assumption of the homogeneity in space coordinates. This assumption excludes any dependence on $\vec{X}$ and we drop it as an argument of the function.

The product of Θ functions is a projection operator with a simple Fourier transform

$$\begin{aligned}P_{X_0}(p_0,p_0')&=\frac{1}{2\pi}\Theta(X_0)\int_{-2X_0}^{2X_0}ds_0e^{is_0(p_0-p_0')}\\&=\frac{1}{\pi}\Theta(X_0)\frac{\sin[2X_0(p_0-p_0')]}{p_0-p_0'},\end{aligned} \tag{3.5}$$

and

$$e^{-is_0p_0'}\Theta(X_0)\Theta(2X_0+s_0)\Theta(2X_0-s_0)$$
$$=\int dp_0 e^{-is_0p_0}P_{X_0}(p_0,p_0'). \tag{3.6}$$

It is important to note that

$$\lim_{X_0\to\infty} P_{X_0}(p_0,p_0')=\lim_{X_0\to\infty}\frac{1}{\pi}\frac{\sin[2X_0(p_0-p_0')]}{p_0-p_0'}=\delta(p_0-p_0'). \tag{3.7}$$

There is a hierarchy of the P_{X_0} projectors:

$$P_{X_{0,M}}(p_0,p_0'')=\int dp_0' P_{X_0}(p_0,p_0')P_{X_0'}(p_0',p_0''),$$
$$X_{0,M}=\min(X_0,X_0'). \tag{3.8}$$

In this paper, the projected function is a very special two-point function $F(x,y)=F(X+s/2,X-s/2)$: it does not depend on $\vec{X}$, it is a function of $(s_0,\vec{s})$ within the interval $-2X_0<s_0<2X_0$ and identical to zero outside:

$$F\left(X+\frac{s}{2},X-\frac{s}{2}\right)=\Theta(X_0)\Theta(2X_0-s_0)\Theta(2X_0+s_0)\overline{F}(s_0,\vec{s}). \tag{3.9}$$

Function $\overline{F}$ is related to the limit $X_0\to\infty$:

$$\lim_{X_0\to\infty} F\left(X+\frac{s}{2},X-\frac{s}{2}\right)=\overline{F}(s_0,\vec{s}). \tag{3.10}$$

An important property of the projected function is that the whole X_0 dependence is introduced by the projection operator

$$F_{X_0}(p_0,\vec{p})=[P_{X_0}F_\infty](p_0,\vec{p})$$
$$=\int_{-\infty}^{\infty} dp_0' P_{X_0}(p_0,p_0')F_\infty(p_0',\vec{p}). \tag{3.11}$$

Important examples of projected functions are retarded, advanced, and Keldysh components of free propagators. Further examples will emerge in the next sections.

For further analysis, the analytic properties of the $X_0\to\infty$ limit of the WTPF as a function of complex energy are very important. We define the following properties: (1) the function of p_0 is analytic above (below) the real axis, (2) the function goes to zero as $|p_o|$ approaches infinity in the upper (lower) semiplane. The choice above (below) and upper (lower) refers to R (A) components.

One should note that the properties of the projected functions are tightly related to the abrupt cutoff at t_i. Any smoothing of the cutoff would also change these properties.

IV. EXAMPLES OF PROJECTED FUNCTIONS

A. Poles in the energy plane

We start with the simplest projected functions: simple poles in the energy plane. The pole contribution to the Green function is

$$\mathcal{G}_{\infty,pole}(p_0)=\frac{a}{p_0-\overline{p_0}}. \tag{4.1}$$

For $\mathrm{Im}\,\overline{p_0}<0$, it satisfies assumptions (1) and (2) as a retarded component, but not as an advanced component (and for $\mathrm{Im}\,\overline{p_0}>0$, just the opposite).

It can be projected to finite X_0:

$$\mathcal{G}_{X_0,pole}(p_0)=a\frac{1-e^{-2iX_0(p_0-\overline{p_0})sgn(Im\,\overline{p_0})}}{p_0-\overline{p_0}}. \tag{4.2}$$

For any finite X_0, the function $\mathcal{G}_{X_0,pole}(p_0)$ is regular at $\overline{p_0}$. For large X_0, Eq. (4.2) exhibits the exponential decay $e^{-2X_0|\mathrm{Im}\,\overline{p_0}|}$ independently of the sign of $\mathrm{Im}\,\overline{p_0}$.

It can be transformed back to the variables (X_0,s_0):

$$\mathcal{G}_{pole}\left(X_0+\frac{s_0}{2},X_0-\frac{s_0}{2}\right)=iae^{-i\overline{p_0}s_0}(\Theta(s_0\,\mathrm{sgn}(\mathrm{Im}\,\overline{p_0}))$$
$$-\Theta(-s_0\,\mathrm{sgn}(\mathrm{Im}\,\overline{p_0})-2X_0)). \tag{4.3}$$

Evidently, this contribution is a projected function. For $\mathrm{Im}\,\overline{p_0}<0$, it is different from zero only at $0<s_0<2X_0$; i.e., it is a retarded function, and for $\mathrm{Im}\,\overline{p_0}>0$, it is an advanced function.

B. Propagator

We start with Eqs. (2.11), (2.12), and (2.13). The transition to the R/A basis is straightforward. Careful calculation gives for the retarded component $(0<x_0,\ 0<y_0)$

$$G_R(x,y)=-G_{1,1}+G_{1,2}=\int d^4p\,\frac{-i}{p^2-m^2+2i\epsilon p_0}e^{-ip(x-y)}, \tag{4.4}$$

and for the Keldysh component

$$G_K(x,y)=G_{1,1}+G_{2,2}$$
$$=\int d^4p\,2\pi\delta(p^2-m^2)(1\pm 2f(\omega_p))e^{-ip(x-y)}. \tag{4.5}$$

As our G_R and G_K depend only on $s=x-y$ and vanish at times before switching on the interaction, they are projected functions. The Wigner transform over the infinite x_0-y_0 interval gives as usual [4,12,15] [note, however, that we avoid [23] using the nonanalytic function $\epsilon(p_0)$ in the expression for $G_{K,\infty}$]

$$G_{R,\infty}(p)=\frac{-i}{p^2-m^2+2i\epsilon p_0},$$

$$G_{K,\infty}(p)=-[1\pm 2f(\omega_p)]\omega_p^{-1}[p_0G_{R,\infty}(p)-p_0G_{A,\infty}(p)]. \tag{4.6}$$

The ϵ parameter, which regulates these expressions, should be kept uniformly finite during the calculations, and the limit $\epsilon\to 0$ should be taken last of all [8]. This especially means that $\lim_{X_0\to\infty}\exp(-X_0\epsilon)=0$ and the terms containing this factor vanish in the $X_0\to\infty$ limit.

The finite Wigner transform $[x_0>0, y_0>0, X_0=(x_0+y_0)/2]$ is obtained by smearing

$$G_{R,X_0}(p)=[P_{X_0}G_{R,\infty}](p)=-G^*{}_{A,X_0}(p),$$
$$G_{K,X_0}(p)=[P_{X_0}G_{K,\infty}](p). \tag{4.7}$$

It is easy to verify that neither the spinor nor the tensor factor changes our conclusion (4.7). One can even integrate expression (4.7). For a scalar particle, one obtains

$$G^0_{R,X_0}(p)=\frac{-i}{p_0^2-\vec{p}^2-m^2+2i\epsilon p_0}\left[1-\left(\cos 2X_0\omega_p - i\frac{p_0}{\omega_p}\sin 2X_0\omega_p\right)e^{2iX_0(p_0+i\epsilon)}\right]$$
$$=G^0_{R,\infty}(p)\left[1-\left(\cos 2X_0\omega_p - i\frac{p_0}{\omega_p}\sin 2X_0\omega_p\right)\times e^{2iX_0(p_0+i\epsilon)}\right]. \tag{4.8}$$

It is important to observe that, at any finite X_0, the above expression is not singular at $p_0=\pm\omega_p$.

Evidently, for $X_0\to\infty$, the first term in $G_{R,X_0}(p)$ gives $G_{R,\infty}$, while the other two ''oscillate out.'' For the Keldysh component, one needs

$$(p_0G^0_R)_{X_0}(p)=p_0G^0_{R,\infty}(p)\left[1-\left(\cos 2X_0\omega_p - i\frac{\omega_p}{p_0}\sin 2X_0\omega_p\right)e^{2iX_0(p_0+i\epsilon)}\right]. \tag{4.9}$$

Then one can use the analogy to (4.6):

$$G^0_{K,X_0}(p)=-[1\pm 2f(\omega_p)]\omega_p^{-1}[(p_0G^0_R)_{X_0}(p)-(p_0G^0_A)_{X_0}(p)]. \tag{4.10}$$

For a spinor particle, one obtains

$$G^{1/2}_{R,X_0}(p)=iG^0_{R,\infty}(p)\gamma^0 e^{2iX_0(p_0+i\epsilon)}\left(\omega_p-\frac{p_0^2}{\omega_p}\right)\sin 2X_0\omega_p + G^o_{R,X_0}(p)(\gamma^\mu p_\mu+m). \tag{4.11}$$

Similarly, for a vector particle (for simplicity, we choose the Feynman gauge):

$$G^1_{\mu,\nu,R,X_0}(p)=g_{\mu,\nu}G^0_{R,X_0}(p). \tag{4.12}$$

We note here that the explicit expressions (4.8)–(4.12) will not be necessary for further discussion.

C. One-loop self-energy

To discuss the amputated one-loop self-energy, we start with (the underlying theory includes bosons and fermions with three-point vertices, but spin and internal symmetry indices are suppressed for simplicity of presentation)

$$\Sigma(x,y)\propto g^2S(x,y)D(x,y)$$
$$\propto g^2\int d^4p\,d^4p'\,e^{-ips}P_{X_0}(p_0,p_0')S_\infty(p_0',\vec{p})\times\int d^4q\,d^4q'\,e^{-iqs}P_{X_0}(q_0,q_0')D_\infty(q_0',\vec{q}). \tag{4.13}$$

The Wigner transform (with respect to $s=x-y$) is

$$\Sigma_{X_0}(p_{01},\vec{p}_1)\propto g^2\int_{-2X_0}^{2X_0}ds_0\int dp_0\,d^3p\,dq_0\times e^{-i(p_0+q_0)s_0}\Theta(2X_0-s_0)\Theta(2X_0+s_0)\times S_\infty(p_0',\vec{p})D_\infty(q_0',\vec{p}_1-\vec{p})$$
$$\propto g^2\int dp_0'\,d^3p\,dq_0'\,P_{X_0}(p_{01},p_0'+q_0')\times S_\infty(p_0',\vec{p})D_\infty(q_0',\vec{p}_1-\vec{p}), \tag{4.14}$$

where as an intermediary step we have used the representation of the bare propagators (4.7) and the representation of the projectors (3.5) and (3.6). Finally, one reads Eq. (4.14) in the R/A basis $[\Sigma_{R(A)}=-(\Sigma_{1,1}+\Sigma_{2,1(1,2)}),\ \Sigma_K=\Sigma_{1,1}+\Sigma_{2,2}]$, as

$$\Sigma_{R(A),X_0}(p)=[P_{X_0}\Sigma_{R(A),\infty}](p),\quad \Sigma_{K,X_0}(p)=[P_{X_0}\Sigma_{K,\infty}](p). \tag{4.15}$$

To calculate $\Sigma_{R,\infty}$ and $\Sigma_{K,\infty}$ we start with Eqs. (2.23)–(2.25) in [23]. After taking into account that the product of the retarded with the advanced function, with the same time variables, vanishes, one obtains

$$\Sigma_{R,\infty}(q)=\frac{ig^2}{2}\int\frac{d^4k}{(2\pi)^4}[h(k_0,\omega_k)+h(q_0-k_0,\omega_{q-k})]\times D_{R,\infty}(k)S_{R,\infty}(q-k)F, \tag{4.16}$$

where D and S are bare scalar propagators, $h(k_0,\omega_k)=-k_0\omega_k^{-1}[1\pm 2f(\omega_p)]$, and the factor $F=F(k_0,|\vec{k}|,q_0,|\vec{q}|,\vec{k}\vec{q},\ldots)$ includes the information about spin and internal degrees of freedom ($F=1$ if all particles are scalars). Now, to verify that the one-loop self-energy $\Sigma_{R,\infty}$ satisfies assumptions (1) and (2), one observes that the vacuum contribution satisfies them (for exceptions, see, e.g., Refs. [33,45]), while the contributions to $\Sigma_{R,\infty}$ from various k_0 points are linear and additive in distribution functions.

For finite ϵ, this contribution possesses singularities only below the real axis in the complex q_0 plane, and vanishes as $|q_0|\rightarrow\infty$ in the upper semiplane. However, there is no guarantee that the imaginary part of $\Sigma_{R,\infty}$ is negative.

Using the same method one calculates the Keldysh component. Although assumptions (1) and (2) are not imposed on Σ_K, one can decompose this component into two pieces $\Sigma_K=-\Sigma_{K,R}+\Sigma_{K.A}$:

$$\Sigma_{K,R(A),\infty}(q)=\mp\frac{ig^2}{2}\int\frac{d^4k}{(2\pi)^4}[1+h(k_0,\omega_k)h(q_0-k_0,\omega_{q-k})]D_{R(A),\infty}(k)S_{R(A),\infty}(q-k)F, \tag{4.17}$$

where $\Sigma_{K,R(A)}$ satisfies assumptions (1) and (2) in the way the retarded (advanced) function does. General analytic properties of the expressions of the type (4.16) and (4.17) are well known: there is a discontinuity (cut) along the real axis, starting at thresholds for real processes and extending to $\pm\infty$.

V. CONVOLUTION PRODUCT OF TWO TWO-POINT FUNCTIONS

Let us now consider the convolution product of two Green functions:

$$C=A*B\Leftrightarrow C(x,y)=\int dzA(x,z)B(z,y). \tag{5.1}$$

In terms of Wigner transforms

$$C(p_0,\vec{p};X)=\int_{-2X_0}^{2X_0}ds_0\int d^3s\int d^4z e^{i(p_0s_0-\vec{p}\vec{s})}\frac{1}{(2\pi)^4}\times\int d^4p_1e^{-i(p_{01}s_{01}-\vec{p}_1\vec{s}_1)}A(p_{01},\vec{p}_1;X_1)\frac{1}{(2\pi)^4}\times\int d^4p_2e^{-i(p_{02}s_{02}-\vec{p}_2\vec{s}_2)}B(p_{02},\vec{p}_2;X_2),$$

$$X_1=X+\frac{s_2}{2},\quad X_2=X-\frac{s_1}{2},\ \ s_1=x-z,\ \ s_2=z-y. \tag{5.2}$$

The assumed translational invariance helps us easily integrate the space components of momenta and coordinates. To do so, we substitute $d^3\vec{s}d^3\vec{z}$ by $d^3\vec{s}_1d^3\vec{s}_2$ (Jacobian $J=1$)

$$\vec{s}=\vec{s}_1+\vec{s}_2,\ \ \vec{z}=\frac{-\vec{s}_1+\vec{s}_2}{2}+\vec{X}. \tag{5.3}$$

The momenta should be equal ($\vec{p}=\vec{p}_1=\vec{p}_2$) and one obtains (note that the dependence on X's is reduced to the dependence on X_0's; further in the text it is indicated as an index),

$$C_{X_0}(p_0,\vec{p})=\frac{1}{(2\pi)^2}\int_{-2X_0}^{2X_0}ds_0\int dz_0\int dp_{01}\int dp_{02}\times e^{i(p_0s_0-p_{01}s_{01}-p_{02}s_{02})}A_{X_{01}}(p_{01},\vec{p})B_{X_{02}}(p_{02},\vec{p}). \tag{5.4}$$

For energy integrals, we proceed in a somewhat different way. We shrink our choice of the functions $A(x,z)$ and $B(z,y)$ to the projected functions. Then we can use the connection to the Wigner transforms on the infinite carrier:

$$C_{X_0}(p_0,\vec{p})=\frac{1}{(2\pi)^2}\int_{-2X_0}^{2X_0}ds_0\int dz_0\int dp_{01}\int dp_{02}\times e^{i(p_0s_0-p_{01}s_{01}-p_{02}s_{02})}\int dp'_{01}P_{X_{01}}(p'_{01},p_{01})\times A_\infty(p'_{01},\vec{p})\int dp'_{02}P_{X_{02}}(p'_{02},p_{02})B_\infty(p'_{02},\vec{p}). \tag{5.5}$$

The integration $dp_{01}dp_{02}$ is easily performed with the help of Eq. (3.6) and one obtains

$$C_{X_0}(p_0,\vec{p})=\frac{1}{(2\pi)^2}\int_{-2X_0}^{2X_0}ds_0\int dz_0\int dp_{01}\int dp_{02}\times e^{i(p_0s_0-p_{01}s_{01}-p_{02}s_{02})}\Theta(2X_{01}+s_{01})\times\Theta(2X_{01}-s_{01})\Theta(2X_{02}+s_{02})\Theta(2X_{02}-s_{02})\times A_\infty(p_{01},\vec{p})B_\infty(p_{02},\vec{p}). \tag{5.6}$$

The product of Θ functions is transformed into $\Theta(2X_0+s_0)\Theta(2X_0-s_0)\Theta(z_0)$. Then

$$C_{X_0}(p_0,\vec{p})=\int\int dp_{01}dp_{02}\delta(p_0,p_{01},p_{02})\times A_\infty(p_{01},\vec{p})B_\infty(p_{02},\vec{p}). \tag{5.7}$$

Here

$$\delta(p_0,p_{01},p_{02})=\frac{1}{(2\pi)^2}\int_{-2X_0}^{2X_0}ds_0\int_0^\infty dz_0e^{i(p_0s_0-p_{01}s_{01}-p_{02}s_{02})}=\frac{1}{(2\pi)^2}\int_{-2X_0}^{2X_0}ds_0\int_0^\infty dz_0\times e^{i[s_0(p_0-p_{01}+p_{02}/2)+(z_0-X_0)(p_{01}-p_{02}+i\epsilon)]}=P_{X_0}\left(p_0,\frac{p_{01}+p_{02}}{2}\right)\frac{1}{2\pi}\frac{i}{p_{01}-p_{02}+i\epsilon}\times e^{-iX_0(p_{01}-p_{02}+i\epsilon)}, \tag{5.8}$$

where we have used

$$\int_0^\infty dz_0 e^{iz\alpha} = \frac{i}{\alpha + i\epsilon}. \tag{5.9}$$

We can write the final expression as

$$C_{X_0}(p_0,\vec{p}) = \int dp_{01}dp_{02}P_{X_0}\left(p_0,\frac{p_{01}+p_{02}}{2}\right) \times \frac{1}{2\pi}\frac{ie^{-iX_0(p_{01}-p_{02}+i\epsilon)}}{p_{01}-p_{02}+i\epsilon} \times A_\infty(p_{01},\vec{p})B_\infty(p_{02},\vec{p}). \tag{5.10}$$

Expression (5.10) is the key for finite-time thermal field theory.

If A is an operator satisfying assumptions (1) and (2) for advanced components, we can integrate expression (5.10) even further. After closing the p_{01} integration contour in the lower semiplane, one obtains [if B is an operator satisfying (1) and (2) for retarded components, one can achieve the same result by closing the p_{02} integration contour in the upper semiplane]

$$C_{X_0}(p_0,\vec{p}) = \int dp_{01}P_{X_0}(p_0,p_{01})A_\infty(p_{01},\vec{p})B_\infty(p_{01},\vec{p}). \tag{5.11}$$

This is an extraordinary result: the convolution product of two WTPF's is a WTPF under conditions (1) and (2).

As expected, in the $X_0=\infty$ limit, Eq. (5.11) becomes a simple product

$$\lim_{X_0\to\infty} C_{X_0}(p_0,\vec{p}) = A_\infty(p_0,\vec{p})B_\infty(p_0,\vec{p}). \tag{5.12}$$

At finite X_0, Eq. (5.11) exhibits a smearing of energy (as much as it is necessary to preserve the uncertainty relations).

Convolution product of *n* projected functions

The product of n two-point functions is obtained by repeating the above procedure:

$$C_{X_0}(p_0,\vec{p}) = \int \prod_{j=1}^{n-1}(dp_{0,j})dp_{0,n}P_{X_0}(p_0,(p_{0,1}+p_{0,n})/2) \times \prod_{j=1}^{n-1}\left(A_{j,\infty}(p_{0,j},\vec{p})\frac{1}{2\pi}\frac{i}{p_{0,j}-p_{0,j+1}+i\epsilon}\right) \times e^{-iX_0[p_{0,1}-p_{0,n}+i(n-1)\epsilon]}A_{n,\infty}(p_{0,n},\vec{p}). \tag{5.13}$$

For the intermediate products in Eq. (5.13) to hold we must require that at least $n-1$ of the functions in the product satisfy assumptions (1) and (2). Furthermore, the order of these functions is important: the retarded functions should be on the right, the advanced on the left, and the function that is neither advanced nor retarded in the middle. However, this is not the order in which the components appear in the Schwinger-Dyson equation.

If the above requirement is fulfilled, one obtains (index R for the retarded component, a similar expression for the advanced component)

$$I_j(p_{0,j-1},p_{0,j+1},\vec{p}) = \int dp_{0,j}\frac{1}{2\pi}\frac{i}{p_{0,j-1}-p_{0,j}+i\epsilon} \times A_{R,j,\infty}(p_{0,j},\vec{p})\frac{1}{2\pi}\frac{i}{p_{0,j}-p_{0,j+1}+i\epsilon} = A_{R,j,\infty}(p_{0,j-1},\vec{p})\frac{1}{2\pi} \times \frac{i}{p_{0,j-1}-p_{0,j+1}+i\epsilon}. \tag{5.14}$$

Then one finds

$$C_{X_0}(p_0,\vec{p}) = \int dp_{0,1}P_{X_0}(p_0,p_{0,1})\prod_{j=1}^{n}A_{j,\infty}(p_{0,1},\vec{p}). \tag{5.15}$$

VI. EXAMPLES OF CONVOLUTION PRODUCTS

A. Convolution products of pole contributions

We assume two pole contributions as shown in Eq. (4.1): $A_{\infty,pole,i}=a_i/(p_0-\overline{p}_{0,i})$, $i=1,2$. The product $C=A_1*A_2$ is simple in the cases in which both contributions are retarded functions, or both are advanced, or A_1 is an advanced and A_2 a retarded function. Then one can simply use Eq. (5.11) to obtain

$$C_{X_0}(p_0) = \int dp_{01}P_{X_0}(p_0,p_{01})\frac{a_1}{p_0-\overline{p}_{0,1}}\frac{a_2}{p_0-\overline{p}_{0,2}}. \tag{6.1}$$

The case in which A_1 is a retarded and A_2 an advanced function (i.e., $\mathrm{Im}\,\overline{p}_{01}<0$, $\mathrm{Im}\,\overline{p}_{02}>0$) requires additional care. After substituting them into Eq. (5.10), we choose new variables $P_0=(p_{01}+p_{02})/2$ and $\Delta_0=p_{01}-p_{02}$, and integrate over Δ_0 to obtain

$$C_{X_0}(p_0) = \int dP_0 P_{X_0}(p_0,P_0)\frac{a_1}{P_0-\overline{p}_{0,1}}\frac{a_2}{P_0-\overline{p}_{0,2}} + \int dP_0 P_{X_0}(p_0,P_0)\frac{2a_1a_2}{\overline{p}_{01}+\overline{p}_{02}-2P_0} \times\left(\frac{e^{iX_0[2(P_0-\overline{p}_{01})-i\epsilon]}}{2(P_0-\overline{p}_{01})-i\epsilon}+\frac{e^{-iX_0[2(P_0-\overline{p}_{02})+i\epsilon]}}{2(P_0-\overline{p}_{02})+i\epsilon}\right). \tag{6.2}$$

The first term is formally identical to Eq. (6.1). The second term consists of two non-WTPF pieces.

As we shall see in the subsection discussing pinching singularities (Sec. VI D), non-WTPF contributions appear also in the convolution product of the type $G_R^* \Sigma_K * G_A$ or, more generally, in the convolution products containing retarded components positioned on the left from the advanced components $(\ldots * A_R * \ldots * B_A * \ldots)$. The non-WTPF terms depend directly on X_0; they are carrying the nontrivial information about the time evolution (i.e., about the dependence on X_0). However, the non-WTPF terms cannot be convoluted further using Eqs. (5.10) or (5.11). This fact indicates the natural limits of the applicability of the methods developed in this paper.

B. Inverse propagator and the equations of motion

To define the inverse propagator, we use the results of Sec. (IV B). We define the restriction of G_R^{-1} on the subspace of projected functions as

$$G_{R,X_0}^{-1}(p_0,\vec{p}) = \int dp_0' P_{X_0}(p_0,p_0') G_{R,\infty}^{-1}(p_0',\vec{p}),$$
$$G_{R,\infty}^{-1}(p_0',\vec{p}) = i(p^2 - m^2 + 2i\epsilon p_0). \tag{6.3}$$

This integral does not converge in the absolute sense, thus we cannot calculate the dependence of G_R^{-1} on X_0. Nevertheless, we can apply it from the left to some class of functions. For example, we can apply it formally to G_{R,X_0}: $G_R^{-1} * G_R = 1$, or written out more explicitly

$$\int dp_0' P_{X_0}(p_0,p_0') i(p^2 - m^2 + 2ip_0\epsilon) G_{R,\infty}(p_0',\vec{p}) = 1. \tag{6.4}$$

This equality is obtained using a simple integration over p_{02} in the expression of the type (5.10). We cannot verify the second identity $G_R * G_R^{-1} = 1$ directly owing to the divergence of the integrals, but we can apply it to the projected function C

$$G_R * G_R^{-1} * C = C, \tag{6.5}$$

under the only requirement that $C_\infty(p_0,\vec{p})$ should satisfy assumptions (1) and (2) in the way a retarded function does and vanish rapidly enough at $p_0 \to \infty$ to make the integral over $G_{R,\infty}^{-1}(p_0) C_\infty(p_0)$ convergent.

Equation (6.4) is the equation of motion for G_R. In the $X_0 \to \infty$ limit, it reduces to the well-known equation for G_R. For the Keldysh component of the propagator, the equation of motion is given by

$$G_R^{-1} * G_K = G_R - 1 * (hG_R - hG_A) = 0, \tag{6.6}$$

where we have ignored the terms $O(\epsilon)$. Owing to the presence of the product $G_R^{-1} * hG_A$, this equation cannot be verified directly (the integrals diverge). Instead (analogously to the case of the product $G_R * G_R^{-1}$), one multiplies it from the left by the function C, which vanishes rapidly at $p_o \to \infty$ and satisfies assumptions (1) and (2) in the way an advanced function does.

C. Resummed Schwinger-Dyson series

We write the Schwinger-Dyson equations in the form

$$\mathcal{G}_R = G_R + iG_R * \Sigma_R * \mathcal{G}_R, \quad \mathcal{G}_A = G_A + iG_A * \Sigma_A * \mathcal{G}_A, \tag{6.7}$$
$$\mathcal{G}_K = G_K + iG_R * \Sigma_K * \mathcal{G}_A + iG_K * \Sigma_A * \mathcal{G}_A + iG_R * \Sigma_R * \mathcal{G}_K.$$

The formal solution (where all products are convolution products and the operators are kept in the proper order) is

$$\mathcal{G}_R = G_R * (1 - i\Sigma_R * G_R)^{-1}, \quad \mathcal{G}_A = G_A * (1 - i\Sigma_A * G_A)^{-1}, \tag{6.8}$$

$$\mathcal{G}_K = \mathcal{G}_R * (h(p_0,\omega_p)(G_A^{-1} - G_R^{-1}) + i\Sigma_K) * \mathcal{G}_A. \tag{6.9}$$

To use the formal solution of the Schwinger-Dyson series, we assume that the functions $G_{R(A)}$, $\mathcal{G}_{R(A)}$, and $\Sigma_{R(A)}$ satisfy requirements (1) and (2) for the retarded components in the upper and for the advanced in the lower semiplane.

This assumption deserves a few comments: For the retarded (advanced) bare propagators, our assumption is valid. If the retarded component is real between the cuts on the part of the real axis, the Schwartz theorem tells us that assumptions (1) and (2) valid in the upper semiplane are also valid in the lower semiplane of the first Riemann sheet.

At equilibrium, perturbation theory yields the full propagator as a set of Fourier coefficients. The analytic continuation in the energy plane is not unique. This freedom is used to choose an analytic continuation that satisfies requirements (1) and (2) defined in Sec. III. The positivity property of the spectral density then implies that the propagator has neither zeros nor poles off the real axis [8]. A further implication is that the exact self-energy $\Sigma_R(p_0,\vec{p})$ at equilibrium also satisfies the properties (1) and (2). This is not guaranteed for approximate expressions for self-energy.

In the formal solution of the retarded propagator, the factors G_R and Σ_R alternate regularly. This fact can improve the convergence properties of some integrals.

Now it is easy to write down the resummed Schwinger-Dyson series for the retarded (advanced) propagator (with any exact self-energy obtained by the perturbation expansion that satisfies our assumptions). In terms of the corresponding propagator calculated at $X_0 = \infty$

$$\mathcal{G}_{R(A),X_0}(p_0,\vec{p}) = \int dp_{01} P_{X_0}(p_0,p_{01}) \mathcal{G}_{R(A),\infty}(p_{01},\vec{p}), \tag{6.10}$$

where

$$\mathcal{G}_{R(A),\infty}(p_{01},\vec{p}) = \frac{G_{R(A),\infty}(p_{01},\vec{p})}{1 - i\Sigma_{R(A),\infty}(p_{01},\vec{p}) G_{R(A),\infty}(p_{01},\vec{p})}. \tag{6.11}$$

Starting from Eq. (6.11) one obtains the HTL-resummed [27,28] retarded (advanced) component of the propagator without use of the gradient expansion.

Some more work is necessary to calculate the Keldysh component. Now, in addition to individual terms, the sum $\mathcal{G}_{R,\infty}(p_0,\vec{p})$ should also satisfy (1) and (2), i.e., the imaginary part of $\Sigma_{R,\infty}$ should be negative.

However, there is a possibility that $\mathrm{Im}\,\Sigma_R$ is positive in some kinematical region. Then the resummed Schwinger-Dyson equation for a retarded component can create the pole in the upper semiplane. However, this case is very questionable: one sums infinitely many retarded functions (i.e., the functions which vanish at $t<t'$) and obtains the function which is not retarded (i.e., nonzero at $t<t'$). Such cases are usually classified as pathology [45,33]. At this point one should cautiously consider the use of the ''physical'' gauge [46], in order to prevent eventual gauge artifacts.

Some indication that, in some cases, $\mathcal{G}_{R,\infty}(p_0,\vec{p})$ does satisfy assumptions (1) and (2) comes from the HTL limit. Indeed, at equilibrium, the HTL limit of $\mathcal{G}_{R,\infty}(p_{01},\vec{p})$ must satisfy (1) and (2), as it is easy to verify. As the properties of density functions enter only through the thermal mass and the position of isolated poles, the same must be true of any distribution allowing the HTL approximation.

Owing to the fact that the Keldysh component of self-energy does not satisfy the analyticity assumptions (1) and (2), we can only try to integrate expression (6.9) using approximate and numerical methods.

However, it is possible that Σ_K can be decomposed into two pieces satisfying assumptions (1) and (2) as retarded and as advanced functions, respectively: $\Sigma_K=-\Sigma_{K,R}+\Sigma_{K,A}$. For example, this happens in the case of one-loop self-energy. Then Eq. (6.9) becomes

$$\mathcal{G}_K=\mathcal{G}_R*[h(p_0,\omega_p)G_A^{-1}+i\Sigma_{K,A}]*\mathcal{G}_A-i\mathcal{G}_R*[h(p_0,\omega_p)G_R^{-1}+i\Sigma_{K,R}]*\mathcal{G}_A. \tag{6.12}$$

Owing to the fact that the functions $\mathcal{G}_R$ and $\mathcal{G}_A$ are not singular in the point $p_0=\pm\omega_p$, the terms containing G_R^{-1} and G_A^{-1} cancel mutually. As one of the remaining convolutions includes factors of the same type (RR or AA), we are left with a single convolution multiplication. This convolution contains neither the advanced first factor nor the retarded second factor; thus, in general, it cannot be worked out in a simple way, and it will contain non-WTPF contributions. However, it may be performed at least numerically.

The appearance of the non-WTPF contributions signals stepping out of the space of projected functions. Indeed, the calculation of the more complex diagrams, containing subdiagrams resummed into $\mathcal{G}_K$, will not enjoy advantages of the presented calculus.

Finally, we note here that the calculation starting with Eqs. (6.7) and ending with Eqs. (6.10)–(6.12) cannot be performed with the true (i.e., calculated, in some miraculous way, to all orders) Σ_K, Σ_R, and Σ_A. Indeed, we have anticipated that the true Σ_K, Σ_R, and Σ_A contain non-WTPF terms, and thus one cannot use instead of G_R^{-1} and G_A^{-1} their restrictions to the subspace of projected functions. Using the gradient expansion (under large-X_0 assumption) one obtains familiar equations of motion for the Green functions of interacting fields [2,3,5,34]. However, the advantage of the presented calculus will be observed in the properties of collision integrals, where one can expect considerable simplifications and the possibility of evaluating contributions of more complex diagrams. A more complete discussion of collision integrals is out of the scope of the present paper, and we hope to publish it elsewhere.

D. Pinching singularities

The pinchlike contribution to the Keldysh component of the resummed propagator is expressed as [23] (we treat only the scalar case)

$$G_{Kp}=iG_R*(-\overline{\Sigma}_{K,R}+\overline{\Sigma}_{K,A})*G_A, \tag{6.13}$$

where we have introduced the short notation $\overline{\Sigma}_{K,R(A)}=h(p_0,\omega_p)\Sigma_{R(A)}+\Sigma_{K,R(A)}$.

Similarly as in the case of resummed contributions, we can perform convolution between alike components (RR or AA). Then one can integrate the terms containing Σ_R and $\Sigma_{K,R}$ with respect to p_{02}, and the terms containing Σ_A and $\Sigma_{K,A}$ with respect to p_{01}. The result is intriguing:

$$\begin{aligned}G_{Kp,X_0}(p_0,\vec{p})=&-\int dp_{01}P_{X_0}(p_0,p_{01})\frac{1}{p_{01}^2-\omega_p^2+2i\epsilon p_{01}}i\\&\times\overline{\Sigma}_K\frac{1}{p_{01}^2-\omega_p^2-2i\epsilon p_{01}}-\frac{1}{2\omega_p}\sum_{\lambda=-1}^{1}\lambda\\&\times\Bigg(\int dp_{01}\frac{e^{2iX_0(p_0-p_{01})}-e^{-2iX_0(p_0-\lambda\omega_p)}}{i\pi(2p_0-p_{01}-\lambda\omega_p)}\\&\times\overline{\Sigma}_{K,R}(p_{01}+i\epsilon,\vec{p})\frac{1}{p_{01}^2-\omega_p^2+2i\epsilon p_{01}}\\&+\int dp_{01}\frac{e^{-2iX_0(p_0-p_{01})}-e^{2iX_0(p_0-\lambda\omega_p)}}{i\pi(2p_0-p_{01}-\lambda\omega_p)}\\&\times\overline{\Sigma}_{K,A}(p_{01}-i\epsilon,\vec{p})\frac{1}{p_{01}^2-\omega_p^2-2i\epsilon p_{01}}\Bigg).\end{aligned} \tag{6.14}$$

The first term in Eq. (6.14) is a projected function (WTPF) that becomes the usual pinchlike term in the $X_0\to\infty$ limit. It is this contribution to which the conclusions [23] about the cancellation of pinching singularities apply. However, the other terms are of non-WTPF nature; contrary to the case of the product of simple pole terms, the discontinuity along the real axis appearing in the functions $\Sigma_{R(A)}$ and $\Sigma_{K,R(A)}$ now prevents the vanishing of these terms.

A full discussion on pinching singularities in the finite-time-after-switching formulation requires more efforts and we hope to publish it elsewhere.

VII. MODIFICATIONS OF THE FEYNMAN RULES

In this section, in the framework of the generic field theory with bosons and fermions, we discuss the changes of Feynman rules that are due to the ''finite time'' assumption. We further analyze the diagrams with respect to the question of energy nonconservation. Indeed, we find that this feature appears together with the non-WTPF contributions.

The calculations performed so far already contain all of the modifications of the Feynman rules required by the finite-t_i assumption. In coordinate space, the only modification is that the bare propagators [Eqs. (4.4) and (4.5)] are limited by $0<x_0$ and $0<y_0$; thus they are projected functions. In energy-momentum space, the above change reflects in the change of propagators, vertices, and the overall factor.

To transform to energy-momentum space, we choose some vertex j, arrange the orientation so that all lines i become outgoing, and use the propagators represented by Eqs. (4.4), (4.5), and (4.6) (the p_i momentum is joined to the line i). Exponentials attached to x_j are easily integrated with the help of Eq. (5.9):

$$\frac{1}{2\pi}\int_0^\infty dx_j e^{-ix_j(\Sigma_i p_i - i\epsilon)} = \frac{i}{2\pi\left(-\sum_i p_i + i\epsilon\right)}. \tag{7.1}$$

After performing this integration, instead of the bare propagators we obtain their $X_0\to\infty$ limits [Eq. (4.6)], which are the familiar propagators of the usual $(t_i\to-\infty)$ theory. At the vertices the usual energy-conserving $\delta(\Sigma_i p_{0i})$ is substituted by $i(2\pi)^{-1}(-\Sigma_i p_{0i}+i\epsilon)^{-1}$.

Under the momentum integrals there is a leftover factor at the vertices j_A (by subscript A we indicate that j_A are vertices with amputated legs):

$$e^{-i\sum_{j_A} x_{j_A}(\sum_{i_{j_A}} \lambda_{i_{j_A}} p_{i_{j_A}})}, \tag{7.2}$$

where $\lambda=\pm$ depends on whether the corresponding momentum is outgoing of or incoming to the vertex j_A, and i_{j_A} is running through the nonamputated lines.

The overall factor in the case of two-point functions is treated in a simple way: introduce a slow Wigner variable as the average over the times of boundary vertices, and the relative time [Eq. (3.1)]. Finally, one can Fourier transform over the relative time. There emerges an overall energy-smearing factor $P_{X_0}(p_0,p_0')$ for two-point functions and similarly for n-point functions. In the case of n-point functions, the choice of variables is large and might not be unique; namely, depending on the diagram calculated, one chooses the most appropriate set of variables. The overall factor takes care of uncertainty relations: the larger the elapsed ''time'' X_0, the smaller the energy smearing.

In the vertex factor the energy is not explicitly conserved. This energy nonconservation is, through the uncertainty relations, related to the finiteness of X_0. In the limit of infinite X_0, energy conservation is recovered. Here we want to argue that for some choice of propagators entering the vertex, the energy is conserved explicitly. To see this conservation, assume, for a moment, that at least one of the unspecified propagators (D, G, and S) related to the chosen vertex, say D_∞, is a retarded function, D_R. In this case, one can integrate over q_0, close the integration path from above (owing to $e^{iX_0(p_{o,j}+i\epsilon)}$, closing from below is out of question), and collect the contributions from singularities. If there are no singularities [and we know that conditions (1) and (2) are valid for bare propagators], one just obtains the energy-conservation condition $\delta(\Sigma_i p_{0i})$. The same is achieved with the outgoing momenta and advanced components of the propagator with closing the integration path from below. Now we are going to show that, indeed, one of these possibilities is realized at each vertex.

Each individual denominator $(\Sigma_i p_{0i}-i\epsilon)^{-1}$ (the lines are all oriented out) can be easily integrated. To demonstrate this, we have to sum over the indices of the corresponding vertex. We rename the basis (i,j), $i,j=1,2$ into $[\mu,\nu]$, where $\mu,\nu=-1$ correspond to $i,j=2$, and $\mu,\nu=1$ correspond to $i,j=1$. Then we find the relations of the type (we assume a three-field vertex, but the proof extends easily to any number of fields)

$$D_{[\mu,\nu]}=\frac{1}{2}(D_K-\mu D_R-\nu D_A). \tag{7.3}$$

The sum over the indices in the chosen vertex (S, D, G propagators of the outgoing lines; the factor μ for the negative coupling of the vertex to which the index-2 ends of the propagators are attached) is

$$\begin{aligned}\sum_\mu \mu S_{[\mu,\lambda]}D_{[\mu,\rho]}G_{[\mu,\nu]}=\frac{1}{4}[&S_R D_R G_R+(S_K+\lambda S_A)\\ &\times(D_K+\rho D_A)G_R+(S_K+\lambda S_A)\\ &\times D_R(G_K+\nu G_A)+S_R(D_K+\rho D_A)\\ &\times(G_K+\nu G_A)].\end{aligned} \tag{7.4}$$

Expression (7.4) contains only terms including at least one retarded propagator: S_R, or D_R, or G_R. Thus one can integrate the terms separately and find that the factor $(\Sigma_i p_{0i}-i\epsilon)^{-1}$ is effectively replaced by $i\pi\delta(\Sigma_i p_{0i})$.

As there is nothing special at this vertex (the indices λ,ρ,ν remain unspecified), one may conclude that this is a general feature. Nevertheless, one should do it very cautiously, step by step, while problems may appear at some degree of complexity. Then, as seen in Eqs. (5.10) and (5.13), we find a new element in addition to the energy denominator $(-p_{0j}+p_{0j+1}-i\epsilon)^{-1}$. One obtains the extra factor $e^{-iX_0(p_{0j}-p_{0j+1}-i\epsilon)}$. With the help of this factor, even the contributions from the poles of the retarded component in the upper semiplane will decay exponentially with the time X_0.

However, the diagrams with resummed self-energy subdiagrams are particularly sensitive. In this case, one is strongly advised to undertake an intermediate step: to Fourier transform the two-point function with respect to the relative time, to investigate the analytic structure, and then to perform the multiplication of two-point functions. Owing to the cuts in $\mathcal{G}_{R(A)}$ and to the non-WTPF contributions to $\mathcal{G}_K$, it is likely that this is just the point after which we have to live without the advantages of projected functions.

VIII. SUMMARY

We consider out of equilibrium thermal field theories with switching on the interaction occurring at finite time ($t_i=0$). We study Wigner transforms (also in the relative time s_0) of two-point functions. To develop a calculation scheme based on first principles, we define a very useful concept of projected functions: a two-point function with the property that it is zero for $x_o<t_i$ and for $y_0<t_i$; for $t_i<x_0$ and $t_i<y_0$, the function depends only on x_0-y_0. We find that many important functions are of this type: bare propagators, one-loop self-energies, resummed Schwinger-Dyson series with one-loop self-energies for the case of retarded and advanced components of the propagator, etc. The properties of the Wigner transforms in the $X_0\rightarrow\infty$ limit are particularly simple if they satisfy these analyticity assumptions: (1) The function of p_0 is analytic above the real axis (for a retarded component, but below it for an advanced component). (2) The function goes to zero as $|p_0|$ approaches infinity in the upper (lower) semiplane. We find that these assumptions are very natural at low orders of the perturbation expansion. The convolution product of projected functions is remarkably simple, much simpler than what one would expect from the gradient expansion.

The Schwinger-Dyson series, with bare propagators and self-energies being projected functions satisfying assumptions (1) and (2), is resummed in closed form without the need for the gradient expansion. The calculation of the resummed Keldysh component is simplified to a double (and under a certain analyticity assumption, to a single) convolution product. This contribution signals the stepping out of the comfortable space of projected functions.

The Feynman diagram technique is reformulated: there is no explicit energy conservation at vertices, there is an overall energy-smearing factor taking care of the finite elapsed time (X_0) and the uncertainty relations.

The relation between the amplitudes (valid at low orders of the perturbation expansion) of the theory with switching on the interaction in the remote past and the theory with finite switching-on time, enables one to rederive the results such as cancellation of pinching singularities, cancellation of collinear and infrared singularities, HTL resummation, etc. Previously, these results were considered applicable only to lowest-order contributions in the gradient expansion.

The question arises whether higher-order contributions also remain within the space of projected functions satisfying assumptions (1) and (2). The answer depends on the eventual positivity of $\mathrm{Im}\,\Sigma_R$, explicitly time-dependent perturbation, and the appearance of the one-loop approximated or resummed Keldysh component. The positive $\mathrm{Im}\,\Sigma_R$ can create the pole in the upper semiplane in the resummed Schwinger-Dyson series. However, this case is very questionable: one sums infinitely many retarded functions (i.e., functions which vanish for $t<t'$) and obtains the function which is not retarded (i.e., nonzero at $t<t'$). Such cases are usually classified as pathology [33,45]. The way of breaking the scheme explicitly is to introduce direct time dependence through time-dependent perturbation, or through the background field which depends directly on time: $\psi(x,t)=\phi(t)+\overline{\psi}(x,t)$. In this way, one obtains the two-point functions which are not projected functions. A natural step out of the space of projected functions occurs in the calculation of the resummed Keldysh component of the propagator. The appearance of the non-WTPT contributions signals that the calculation of the more complex diagrams containing subdiagrams resummed into $\mathcal{G}_K$ will not enjoy advantages of the presented calculus.

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[1] J. Schwinger, J. Math. Phys. **2**, 407 (1961).
[2] L. V. Keldysh, Zh. Éksp. Teor. Fiz. **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)].
[3] L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (Benjamin, New York, 1962).
[4] A. J. Niemi, Phys. Lett. B **203**, 425 (1988).
[5] P. Danielewicz, Ann. Phys. (N.Y.) **152**, 239 (1984).
[6] K.-C. Chou, Z.-B. Su, B.-L. Hao, and L. Yu, Phys. Rep. **118**, 1 (1985).
[7] J. Rammer and H. Smith, Rev. Math. Phys. **58**, 323 (1986).
[8] N. P. Landsman and Ch. G. van Weert, Phys. Rep. **145**, 141 (1987).
[9] E. Calzetta and B. L. Hu, Phys. Rev. D **37**, 2878 (1988).
[10] O. Eboli, R. Jackiw, and S.-Y. Pi, Phys. Rev. D **37**, 3557 (1988).
[11] E. A. Remler, Ann. Phys. (N.Y.) **202**, 351 (1990).
[12] M. Le Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996).
[13] D. A. Brown and P. Danielewicz, Phys. Rev. D **58**, 094003 (1998).
[14] C. Greiner and S. Leupold, Report No. UGI-98-12.
[15] J-P. Blaizot, E. Iancu, and J-Y. Ollitrault, in *Quark-Gluon Plasma II*, edited by R. C. Hwa (World Scientific, Singapore, 1995).
[16] M. Le Bellac and H. Mabilat, Z. Phys. C **75**, 137 (1997).
[17] A. Niégawa, Eur. Phys. J. C **5**, 345 (1998).
[18] T. Altherr and D. Seibert, Phys. Lett. B **333**, 149 (1994).
[19] T. Altherr, Phys. Lett. B **341**, 325 (1995).
[20] P. F. Bedaque, Phys. Lett. B **344**, 23 (1995).
[21] A. Niégawa, Phys. Lett. B **416**, 137 (1998).
[22] C. Greiner and S. Leupold, Eur. Phys. J. C **8**, 517 (1999).
[23] I. Dadić, Phys. Rev. D **59**, 125012 (1999).

[24] R. D. Pisarski, Phys. Rev. Lett. **63**, 1129 (1989).
[25] E. Braaten and R. D. Pisarski, Nucl. Phys. **B337**, 569 (1990).
[26] J. Frenkel and J. C. Taylor, Nucl. Phys. **B334**, 199 (1990).
[27] M. E. Carrington, H. Defu, and M. H. Thoma, Eur. Phys. J. C **7**, 347 (1999).
[28] H. Defu and M. U. Heinz, Eur. Phys. J. C **7**, 101 (1999).
[29] R. Baier, M. Dirks, and K. Redlich, Phys. Rev. D **55**, 4344 (1997).
[30] R. Baier, M. Dirks, K. Redlich, and D. Schiff, Phys. Rev. D **56**, 2548 (1997).
[31] R. Baier, M. Dirks, and K. Redlich, Acta Phys. Pol. B **28**, 2873 (1997).
[32] D. C. Langreth and J. W. Wilkins, Phys. Rev. B **6**, 3189 (1972).
[33] P. A. Henning, Phys. Rep. **253**, 263 (1995).
[34] J.-P. Blaizot and E. Iancu, Nucl. Phys. **B557**, 183 (1999).
[35] P. Grangé, H. A. Weidenmüller, and G. Wolschin, Ann. Phys. (N.Y.) **136**, 190 (1981).
[36] M. Tohyama, Phys. Lett. **163B**, 14 (1985).
[37] P.-G. Reinhard, H. L. Yadav, and C. Toepffer, Nucl. Phys. **A458**, 301 (1986).
[38] M. Tohyama, Phys. Rev. C **36**, 187 (1987).
[39] C. Greiner, K. Wagner, and P.-G. Reinhard, Phys. Rev. C **49**, 1693 (1994).
[40] K. Morawetz and H. S. Köhler, Eur. Phys. J. A **4**, 291 (1999).
[41] D. Boyanovsky, H. J. de Vega, R. Holman, S. P. Kumar, R. D. Pisarski, and J. Salgado, hep-ph/9810209.
[42] D. Boyanovsky, H. J. de Vega, and S. -Y. Wang, Phys. Rev. D **61**, 065006 (2000).
[43] R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain* (American Mathematical Society, New York, 1934).
[44] J. Baacke, K. Heitmann, and C. Pätzold, Phys. Rev. D **57**, 6398 (1998).
[45] P. A. Henning, Nucl. Phys. **A546**, 653 (1992).
[46] P. V. Landshoff and A. Rebhan, Nucl. Phys. **B393**, 607 (1993).