

## Noncommutative geometry as a regulator

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We give a perturbative quantization of  $R^4$  space-time in the case where the commutators  $C^{\mu\nu}=[X^\mu,X^\nu]$  of the underlying algebra generators are not central. We argue that this kind of quantum space-time can be used as a regulator for quantum field theories. In particular we show, in the case of  $\phi^4$  theory, that by choosing appropriately the commutators  $C^{\mu\nu}$  we can remove all the infinities by reproducing all the counterterms. In other words, the renormalized action on  $R^4$  plus the counterterms can be rewritten as only a renormalized action on the quantum space-time  $QR^4$ . We conjecture therefore that the renormalization of quantum field theory is equivalent to the quantization of the underlying space-time  $R^4$ .

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### I. INTRODUCTION

Noncommutative geometry (NCG) [1] allows one to define the geometry of a given space in terms of its underlying algebra. It is therefore more general than ordinary differential geometry in the sense that it enables us to describe algebraically the geometry of any space whether or not it is smooth and/or differentiable. It is generally believed that NCG can be used to reformulate if not solve many problems in particle physics and general relativity such as the problem of infinities in quantum field theories (QFTs) and its possible connection to quantum gravity [2–7]. The potential for constructing new nonperturbative methods for quantum field theories using NCG is also well appreciated [2–4,8–16]. The recent major interest in NCG, however, was mainly initiated by the work of [17] on Yang-Mills theory on a noncommutative torus and its appearance as a limit of the matrix model of M theory. The relevance of NCG in string theory was further discussed in [18].

Quantum field theories on noncommutative space-time were extensively analyzed recently in the literature [19–29] and it was shown that divergences, although not completely removed, are considerably softened. The reason is that the quantization of  $R^4$  or  $R^2$  by replacing the coordinate functions  $x^\mu$  by the coordinate operators  $X^\mu$  in the sense of [6] will only modify vertices in the quantum theory and not propagators. On compact spaces, on the other hand, such as the four-sphere  $S^4$  [8], the two-sphere  $S^2$  [9], and  $CP^2$  [15] divergences are automatically canceled out when we quantize the space and that is because on compact spaces (which was not the case on noncompact spaces) quantization leads to a finite number of degrees of freedom (points).

It is hoped that noncommutative geometry will shed new light on the meaning of renormalization because it provides a very powerful tool to formulate the possible physical mechanisms underlying the renormalization process of quantum field theories. One such mechanism which was developed by Deser [38], Isham *et al.* [39], and pursued in [30,31] is Pauli's old idea that the quantization of gravity should give rise to a discrete structure of space-time which will regulate

quantum field theories. As one can immediately see the typical length scale of Pauli's lattice is of the order of Planck's scale  $\lambda_p$  which is very small compared to the weak scale and therefore corrections to the classical action will be very small compared to the actual quantum corrections. This idea, however, is still very plausible especially after the discovery made in [19] of an UV-IR mixing which could be used in a large extra-dimension-like activity to solve the above hierarchy problem.

The philosophy of this paper will be quite different. We will assume that space-time is really discrete and that the continuum picture is only an approximation [30]. The discreteness, however, is not given *a priori* but is a consequence of the requirement that the quantum field theory under consideration be finite. The noncommutativity parameter  $\theta$  is therefore expected to be a function of both the space-time and the quantum field theory and it is completely determined by the finiteness requirement. This simply means that the quantization of space-time is achieved by replacing the coordinate functions  $x^\mu$  by the coordinate operators  $X^\mu$  as in [32] but, and contrary to what was done in [6], these operators will not satisfy the centrality conditions  $[X^\mu,[X^\nu,X^\alpha]]=0$ .

This paper is organized as follows: In Sec. I we introduce the star product [33] for the case where the noncommutativity parameter  $\theta$  is not a constant. The necessary and sufficient condition under which this star product is associative turns out to be simply  $[X^\mu,[X^\nu,X^\alpha]]=0$ . The associativity requirement, however, is relaxed and allowed to be broken to the first order in this double commutator. This relaxation is necessary because one can check that we cannot generalize [6] by making the commutators  $[X^\mu,X^\nu]$  not central while simultaneously preserving associativity. The algebra  $(\mathcal{A},*)$  where  $\mathcal{A}$  is the algebra of functions on  $R^4$  is then defined.

In Sec. II we quantize perturbatively the algebra  $(\mathcal{A},*)$ . In other words we find the homomorphism  $(\mathcal{A},*)\rightarrow(\mathcal{A},\times)$  order by order in perturbation theory where  $\mathcal{A}$  is the algebra of operators generated by the coordinate operators  $X^\mu$ . The star product becomes under quantization the nonassociative operator product  $\times$  and the corresponding Moyal brackets become the commutator  $[\dots]_\times$  [34]. The difference between  $\times$  and the ordinary dot product of operators is of the order of the double commutator  $[X^\mu,[X^\nu,X^\alpha]]$ . This is basically an

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example of deformation quantization [33–36] and in particular it shows explicitly the result of [35] that the quantization prescription of Doplicher *et al.* [6] of space-time is a deformation quantization of  $R^4$ . We rederive also the space-time uncertainty relations given in [6]. In Sec. III we construct a Dirac operator on the quantum space-time  $QR^4$  and write down the action integrals of a scalar field in terms of the algebra  $(A, \times)$  as well as in terms of the algebra  $(\mathcal{A}, *)$ . The finiteness requirement is then used to fix  $\theta$  in the two-loop approximation of the  $\phi^4$  theory. Section IV contains conclusions and remarks.

## II. STAR PRODUCT

### A. Associativity

Let  $R^4$  be the space-time with the metric  $\eta_{\mu\nu} = (1, 1, 1, 1)$ . The algebra underlying the whole differential geometry of  $R^4$  is simply the associative algebra  $\mathcal{A}$  of functions  $f$  on  $R^4$ . It is generated by the coordinate functions  $x^\mu$ ,  $\mu = 0, 1, 2, 3$ . This algebra is trivially a commutative algebra under the pointwise multiplication. A review of how the algebra  $(\mathcal{A}, \cdot)$  captures all the differential geometry of  $R^4$  can be found in [2–5].

It is known that we can make the algebra  $\mathcal{A}$  noncommutative if we replace the dot product by the star product [33]. The pair  $(\mathcal{A}, *)$  is then describing a deformation  $QR^4$  of space-time which will be taken by definition to be the quantum space-time. The  $*$  product is defined for any two functions  $f(x)$  and  $g(x)$  of  $\mathcal{A}$  by [18]

$$f * g(x) = \exp\left(\frac{i}{2} C^{\mu\nu}(x) \frac{\partial}{\partial \xi^\mu} \frac{\partial}{\partial \eta^\nu}\right) f(x + \xi) g(x + \eta) \Big|_{\xi = \eta = 0}, \quad (2.1)$$

where  $C^{\mu\nu}$  form a rank-2 tensor  $C$  which in general contains a symmetric as well as an antisymmetric part [11]. It is assumed to be a function of  $x$  of the form

$$C^{\mu\nu}(x) = \chi(x) (\theta^{\mu\nu} + ia \eta^{\mu\nu}), \quad (2.2)$$

where  $\chi(x)$  is some function of  $x$ . Here  $\theta$  is the antisymmetric part and it is an  $x$ -independent tensor.  $a$  is as we will see the nonassociativity parameter and it is determined in terms of the tensor  $\theta$  as follows. The requirement that the star product (2.1) be associative can be expressed as the condition that  $I = 0$  where  $I$  is given by

$$I = (e^{ipx} * e^{ikx}) * e^{ihx} - e^{ipx} * (e^{ikx} * e^{ihx}), \quad (2.3)$$

$e^{ipx}$  are the generators of the algebra  $\mathcal{A}$  written in their bounded forms. Using the definition (2.1) we can check that

$$e^{ipx} * e^{ikx} = e^{-ipCk/2} e^{i(p+k)x} \quad (2.4)$$

and therefore Eq. (2.3) takes the form

$$\begin{aligned} I e^{-i(p+k+h)x} &= \exp\left(-\frac{1}{2} C^{\mu\nu}(x) h_\nu \frac{\partial}{\partial \xi^\mu}\right) \\ &\times \left[ \exp\left(-\frac{i}{2} p C(x + \xi) k + i(p+k)\xi\right) \right] \Big|_{\xi=0} \\ &- \exp\left(-\frac{1}{2} C^{\nu\mu}(x) p_\nu \frac{\partial}{\partial \xi^\mu}\right) \\ &\times \left[ \exp\left(-\frac{i}{2} k C(x + \xi) h + i(k+h)\xi\right) \right] \Big|_{\xi=0}. \end{aligned} \quad (2.5)$$

To see clearly what are the kind of conditions we need to ensure that the equation  $I = 0$  is an identity, we first expand both sides of Eq. (2.5) in powers of  $C$  and keep terms only up to the second order. It will then read

$$I = \frac{i}{4} \left[ C^{\mu\nu} h_\nu p \frac{\partial C}{\partial x^\mu} k - C^{\nu\mu} p_\nu k \frac{\partial C}{\partial x^\mu} h \right]. \quad (2.6)$$

As we can clearly see the associativity of the star product at this order is maintained if and only if  $C^{\mu\nu} \partial_\mu C = 0$  and  $C^{\nu\mu} \partial_\mu C = 0$ . The two consequences of these two conditions are given by the equations  $a \eta^{\mu\nu} \partial_\mu \chi = 0$  and  $\theta^{\mu\nu} \partial_\mu \chi = 0$ . The first equation is simply  $a = 0$  because the solution  $\chi = \text{const}$  will be discarded in this paper. The second equation, on the other hand, means that we can simply check that the noncommutativity matrix  $\theta$  is singular, i.e.,  $\det \theta = 0$ . We can also check that the two above conditions are necessary and sufficient to make the star product (2.1) associative at all orders because of the identities  $\theta^{\mu_1 \nu_1} \theta^{\mu_2 \nu_2} \dots \theta^{\mu_n \nu_n} \partial_{\mu_1, \mu_2, \dots, \mu_n}^n \chi = 0$ .

If we would like to avoid the singularity of the noncommutativity matrix  $\theta$ , we have then to relax the requirement of associativity. We can start by reducing the associativity of the star product (2.1) by imposing only one of the above two conditions, say,

$$\begin{aligned} C^{\mu\nu} \frac{\partial C}{\partial x^\mu} &= 0 \\ &\Rightarrow \\ C^{\mu\nu} \frac{\partial \chi}{\partial x^\mu} &= 0. \end{aligned} \quad (2.7)$$

Before we analyze this equation further, we remark that this condition on the tensor  $C$  will lead to the identities

$$C^{\mu_1 \nu_1} C^{\mu_2 \nu_2} \dots C^{\mu_n \nu_n} \partial_{\mu_1, \mu_2, \dots, \mu_n}^n C^{\alpha\beta} = 0. \quad (2.8)$$

Equation (2.7) will also lead to the equation

$$C^{\nu\mu} \frac{\partial C}{\partial x^\mu} = i \partial^\nu \chi^2 [a\theta + ia^2 \eta]. \quad (2.9)$$

In order to have a very small amount of nonassociativity in the theory we will assume that  $a$  is a very small parameter in such a way that only linear terms in  $a$  are relevant. Putting Eqs. (2.7) and (2.9) into Eq. (2.5) will then give

$$I = \frac{ia}{2} (k\theta h) O(p, k+h, \chi, \partial\chi) e^{-i\chi k\theta h/2} e^{i(p+k+h)x}, \quad (2.10)$$

where  $O$  is a function (which we will not write down explicitly) of the momenta  $p, k, h$  and of  $\chi$  and all its derivatives  $\{\partial\chi\}$ . This function  $O$  is such that it vanishes identically if  $\partial_\mu \chi = 0$ . In other words a trivial solution to the equation  $I = 0$  is  $\chi = \text{const}$  which we will discard in this paper. We would like to determine  $\chi$  from the requirement that the quantum field theory which we will eventually write down on  $\mathcal{QR}^4$  be finite. So we will leave  $\chi$  arbitrary at this stage. Clearly  $\chi$  will be model dependent and it can generally be put in the form

$$\chi(x) = \sum_{n=1} \hbar^n \chi_n(x), \quad (2.11)$$

where we do not have a tree level term because by assumption this function will be entirely determined by the different infinities of the theory which are generally of higher orders in  $\hbar$ . In other words the zero order is absent in Eq. (2.11) because QFTs are usually finite at this order.

It is instructive to solve Eq. (2.7) for  $\theta$  in terms of  $\chi$ . We assume that  $\partial_\mu \chi \neq 0$  and rewrite Eq. (2.7) in the form  $C^{\mu\nu} \partial_\mu \chi = \lambda e^\nu$  where  $\lambda$  is a small number and  $e$  is a four-vector given by  $(1, 0, 0, 0)$ . Solving Eq. (2.7) for  $\theta$  will give the following equation:

$$\frac{i \det C}{a^3 - (a/2) \sum_{\mu, \nu \neq 0} \theta_{\mu\nu} \theta^{\mu\nu}} = \lambda \frac{\chi^3}{\partial_0 \chi},$$

$$\frac{\det C}{-a^2 \theta^{0i} - ia \theta_{i\mu} \theta^{0\mu} + \theta^{jk} \sqrt{\det \theta}} = \lambda \frac{\chi^3}{\partial_i \chi}, \quad (2.12)$$

with

$$\det C = \chi^4 \left[ \det \theta + a^4 - \frac{a^2}{2} \theta_{\mu\nu} \theta^{\mu\nu} \right]. \quad (2.13)$$

$(ijk)$  are the even permutations of  $(123)$  and  $\det \theta$  is given by  $\det \theta = [\frac{1}{8} \epsilon_{\mu\nu\alpha\beta} \theta^{\mu\nu} \theta^{\alpha\beta}]^2$ . The four equations (2.12) provide four constraints on the tensor  $\theta$  which reduce at limit  $\lambda \rightarrow 0$  to one constraint given by

$$\det \theta = -a^4 + \frac{a^2}{2} \theta_{\mu\nu} \theta^{\mu\nu}. \quad (2.14)$$

This is a generalization of the quantization conditions chosen in [6]. This equation, however, can be thought of as giving the nonassociativity parameter  $a$  in terms of the noncommutativity matrix  $\theta$ . The solution is

$$a = \left[ \frac{1}{4} \theta_{\mu\nu} \theta^{\mu\nu} - \sqrt{\left( \frac{1}{4} \theta_{\mu\nu} \theta^{\mu\nu} \right)^2 - \det \theta} \right]^{1/2}. \quad (2.15)$$

As we can see from the above analysis it is necessary and sufficient to choose  $\theta$  in such a way that Eq. (2.15) is a very small number in order for the associativity of the star product (2.1) to be broken with the very small amount given by Eq. (2.10).

Using the star product (2.1) we can define the Moyal bracket of any two functions  $f(x)$  and  $g(x)$  by  $\{f(x), g(x)\} = f * g(x) - g * f(x)$  and in particular the Moyal brackets of two coordinate functions are given by

$$\{x^\mu, x^\nu\} = i\chi(x) \theta^{\mu\nu}. \quad (2.16)$$

For self-consistency these brackets should satisfy the Jacobi identity

$$\{x^\beta, \{x^\mu, x^\nu\}\} + \{x^\nu, \{x^\beta, x^\mu\}\} + \{x^\mu, \{x^\nu, x^\beta\}\} = 0, \quad (2.17)$$

but

$$\{x^\beta, \{x^\mu, x^\nu\}\} = -ia\chi(\partial^\beta \chi) \theta^{\mu\nu}. \quad (2.18)$$

Clearly at the limit of associativity ( $a \rightarrow 0$ ), Eq. (2.18) is simply zero and therefore Eq. (2.17) holds. We would like, however, to maintain Jacobi identity even for  $a \neq 0$ . We then need to impose the following constraint on  $\theta$ :

$$\theta^{\alpha\beta} \theta^{\mu\nu} + \theta^{\alpha\nu} \theta^{\beta\mu} + \theta^{\alpha\mu} \theta^{\nu\beta} = 0, \quad (2.19)$$

which will make Eq. (2.17) an identity. A class of solutions to Eq. (2.19) can be given by those antisymmetric tensors  $\theta$  such that

$$\theta^{\mu\nu} = a_\alpha^\mu a_\beta^\nu \theta_0^{\alpha\beta}, \quad (2.20)$$

where  $a_\alpha^\mu$  are arbitrary real numbers and  $\theta_0$  is an antisymmetric tensor which satisfies

$$\theta_0^{\mu\nu} \theta_0^{\alpha\beta} = (\eta^{\mu\alpha} \eta^{\nu\beta} - \eta^{\mu\beta} \eta^{\nu\alpha}). \quad (2.21)$$

Equation. (2.19) is the only constraint we need to impose on the tensors  $\theta$  in order to have both the associativity requirement in the sense of Eq. (2.10) and Jacobi identity (2.17) to be satisfied. By requiring that Eq. (2.16) should lead to a certain kind of space-time uncertainty relations we can further restrict the allowed antisymmetric tensors  $\theta$  as we will see in the next section.

## B. Algebra $(\mathcal{A}, *)$

A general element  $f(x)$  of  $\mathcal{A}$  will be defined by

$$f(x) = \int \frac{d^4 p}{(2\pi)^4} \tilde{f}(p, \chi) e^{ipx}, \quad (2.22)$$

where  $\tilde{f}$  is a smooth continuous function of the four-vector  $p$  and of the fuzziness function  $\chi$  which satisfies  $\tilde{f}^*(-p, \chi) = \tilde{f}(p, \chi)$ . It is of the general form  $\tilde{f}(p, \chi) = \tilde{f}_0(p, \chi) + a\tilde{f}_1(p, \chi)$ . The star product (2.1) can then be rewritten as

$$\begin{aligned} f * g(x) &= \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \left[ \tilde{f}(p, \chi) \tilde{g}(k, \chi) e^{-ipCk/2} \right. \\ &\quad \left. + a\tilde{f}(p, \chi) \frac{\partial \tilde{g}(k, \chi)}{\partial \chi} O(p, k, \chi, \partial \chi) \right] e^{i(p+k)x} \\ &= \int \frac{d^4 p}{(2\pi)^4} \tilde{f}^* \tilde{g}(p, \chi) e^{ipx}. \end{aligned} \quad (2.23)$$

$O(p, k, \chi, \partial \chi)$  is the function defined by Eq. (2.10). The Fourier transform  $\tilde{f}^* \tilde{g}(p, \chi) = \tilde{f}^* \tilde{g}(p, \chi)_0 + a\tilde{f}^* \tilde{g}(p, \chi)_1$  is given, on the other hand, by

$$\begin{aligned} \tilde{f}^* \tilde{g}(p, \chi) &= \int \frac{d^4 k}{(2\pi)^4} \left[ \tilde{f}(p-k, \chi) \tilde{g}(k, \chi) e^{-i(p-k)Ck/2} \right. \\ &\quad \left. + a\tilde{f}(p-k, \chi) \frac{\partial \tilde{g}(k, \chi)}{\partial \chi} O(p-k, k, \chi, \partial \chi) \right]. \end{aligned} \quad (2.24)$$

The function  $\tilde{f}(p, \chi)$  can always be expanded as  $\tilde{f}(p, \chi) = \sum_{n=0} a_n \tilde{f}_n(\chi) \tilde{f}(p)$  which suggests that Eq. (2.22) can be rewritten in the form [6]

$$f(x) = \sum_{n=0} a_n f_n(x), \quad (2.25)$$

where

$$f_n(x) = \tilde{f}_n(\chi) \int \frac{d^4 p}{(2\pi)^4} \tilde{f}_n(p) e^{ipx}. \quad (2.26)$$

$f_n(x)$  are the generators of the algebra  $(\mathcal{A}, *)$  written in a way which will allow us to see the classical limit defined by  $\chi \rightarrow 0$ . In this limit they must generate the algebra  $(\mathcal{A}, \cdot)$ . Therefore the functions  $\tilde{f}_n(\chi)$  are such that they tend to a constant when  $\chi \rightarrow 0$ . This constant can always be chosen to be 1.

### C. Change of generator basis

Finally we would like to rewrite Eq. (2.16) in a way which will be more suitable for quantization. This will involve finding a basis  $z^\mu(x)$  for which the Moyal brackets  $\{z^\mu, z^\nu\}$  are in the center of the algebra  $(\mathcal{A}, *)$ , in other words  $\{x^\alpha, \{z^\mu, z^\nu\}\} = 0$ . This is not the case for the basis  $x^\mu$  as we can see from Eq. (2.18). We then must have  $\{z^\mu, z^\nu\} = i\theta^{\mu\nu} \mathcal{C}(x)$  where  $\mathcal{C}(x)$  is any function of  $x$  which does commute (in the sense of Moyal brackets) with the elements of the algebra  $(\mathcal{A}, *)$ . To find such a basis we need first to find the central elements  $\mathcal{C}(x)$  of the algebra  $(\mathcal{A}, *)$ . To this end

we first remark that by using Eq. (2.1) the Moyal brackets of the generator  $x^\mu$  with any function  $f(x)$  are given by

$$\{x^\mu, f\} = i\chi \theta^{\mu\nu} \frac{\partial f}{\partial x^\nu}. \quad (2.27)$$

It is then clear that the only obvious solutions to the equation  $\{x^\mu, f\} = 0$  are the trivial ones, namely, the constant functions. However, choosing the central element  $\mathcal{C}(x)$  to be a constant is not good because it will lead to a singular basis at  $\chi(x) = 0$  which can be seen from the fact that the Moyal brackets  $\{z^\mu, z^\nu\}$  at  $\chi(x) = 0$  will then not vanish on the contrary to what happens to the Moyal brackets (2.16) which clearly vanish at  $\chi = 0$ . So we must find at least one central element which is not a constant function. The only clear way to find such an element is to use perturbation theory. We assume then that the quantum field theory which we will write on  $QR^4$  is relevant only up to the  $\hbar^N$  order. The function  $\chi(x)$  will then take the form

$$\chi(x) = \sum_{n=1}^N \hbar^n \chi_n(x) \quad (2.28)$$

and we would have that

$$\chi^{N+1}(x) = 0. \quad (2.29)$$

This last equation can be rewritten by using Eq. (2.27) as

$$\{x^\mu, \chi^N\} = 0; \quad (2.30)$$

in other words,  $\chi^N$  is a central element of the algebra  $\mathcal{A}$  in the  $\hbar^N$  approximation. Actually any combination of the order of  $\hbar^N$  is central as can be seen from Eqs. (2.27) and (2.29). By choosing  $\mathcal{C}(\chi) = \chi^N(x)$ , the Moyal brackets of any two coordinates  $z^\mu(x)$  and  $z^\nu(x)$  will then read

$$\{z^\mu, z^\nu\} = i\chi^N \theta^{\mu\nu}. \quad (2.31)$$

$x^\mu$  and  $z^\mu(x)$  give equivalent descriptions of the algebra  $(\mathcal{A}, *)$  and therefore the quantization of Eq. (2.16) is equivalent to the quantization of Eq. (2.31). It is obvious, however, that the quantization of Eq. (2.31) is more straightforward than the quantization of Eq. (2.16). The new basis  $z^\mu(x)$  can be found in terms of  $x^\mu$  as follows. First we note that for the purpose of finding  $z^\mu$  it is sufficient to work up to the second order in  $C$ . The star product (2.1) of any two functions  $f(x)$  and  $g(x)$  will read, up to this order,

$$\begin{aligned} f * g(x) &= f(x)g(x) + \frac{i}{2} \chi(x) (\theta^{\mu\nu} + ia\eta^{\mu\nu}) \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial x^\nu} \\ &\quad - \frac{1}{8} \chi^2(x) (\theta^{\mu\nu} + ia\eta^{\mu\nu}) (\theta^{\alpha\beta} + ia\eta^{\alpha\beta}) \\ &\quad \times \frac{\partial^2 f}{\partial x^\mu \partial x^\alpha} \frac{\partial^2 g}{\partial x^\nu \partial x^\beta}, \end{aligned} \quad (2.32)$$

and therefore the Moyal brackets of these two functions are

$$\{f, g\} = i\chi(x)\theta^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu} - \frac{ia}{2}\chi^2(x)\theta^{\alpha\beta}\eta^{\mu\nu}\frac{\partial^2 f}{\partial x^\mu\partial x^\alpha}\frac{\partial^2 g}{\partial x^\nu\partial x^\beta}. \quad (2.33)$$

In particular the Moyal brackets of the two coordinates  $z^\mu(x)$  and  $z^\nu(x)$  are given by

$$\{z^\delta, z^\sigma\} = i\chi(x)\theta^{\mu\nu}\frac{\partial z^\delta}{\partial x^\mu}\frac{\partial z^\sigma}{\partial x^\nu} - \frac{ia}{2}\chi^2(x)\theta^{\alpha\beta}\eta^{\mu\nu}\frac{\partial^2 z^\delta}{\partial x^\mu\partial x^\alpha}\frac{\partial^2 z^\sigma}{\partial x^\nu\partial x^\beta}. \quad (2.34)$$

Comparing Eqs. (2.31) and (2.34) will then give that

$$\begin{aligned} \theta^{\mu\nu}\frac{\partial z^\delta}{\partial x^\mu}\frac{\partial z^\sigma}{\partial x^\nu} &= \chi^{N-1}\theta^{\delta\sigma} \\ \Rightarrow \frac{\partial z^\mu}{\partial x^\nu} &= \chi(x)^{(N-1)/2}\eta^\mu_\nu. \end{aligned} \quad (2.35)$$

Equation (2.35) defines scaling transformations which depend on space-time points. A more thorough study of these transformations will be reported elsewhere. As we can clearly see the definition (2.35) of the new basis  $z^\mu$  in terms of  $x^\mu$  will make the quadratic term in Eq. (2.34) vanish, and for that matter all terms which are higher orders in  $C$  will also vanish. We would like now to rewrite Eq. (2.35) in a form which is better suited for quantization. To this end we make use of Eq. (2.27) for the case where  $f = z^\nu$ . We then obtain

$$\{x^\mu, z^\nu\} = i\chi(x)^{(N+1)/2}\theta^{\mu\nu}, \quad (2.36)$$

where we have used Eq. (2.35). Equation (2.36) is actually Eq. (2.35) only written in terms of Moyal brackets which under quantization will go to the commutator as we will see. For the coordinates  $z^\mu$  the Jacobi identity  $\{z^\mu, \{z^\nu, z^\alpha\}\} + \{z^\alpha, \{z^\mu, z^\nu\}\} + \{z^\nu, \{z^\alpha, z^\mu\}\} = 0$  trivially follows from Eq. (2.31).

By using Eq. (2.33) we can find that the Moyal brackets of the generator  $z^\mu$  with any function  $f$  of  $\mathcal{A}$  can be written as

$$\{z^\mu, f\} = i\chi^N\theta^{\mu\nu}\frac{\partial f}{\partial z^\nu}, \quad (2.37)$$

where we have made use of Eq. (2.35). The Moyal brackets (2.31) and (2.37) do clearly correspond to the star product

$$f * g(z) = \exp\left(\frac{i}{2}D^{\mu\nu}(z)\frac{\partial}{\partial \xi^\mu}\frac{\partial}{\partial \eta^\nu}\right)f(z + \xi)g(z + \eta)|_{\xi=\eta=0}, \quad (2.38)$$

where now  $D^{\mu\nu}(z) = \chi^N(\theta^{\mu\nu} + ia\eta^{\mu\nu})$ . This star product, however, is completely equivalent to Eq. (2.1). It is simply the star product (2.1) written in the basis  $z^\mu$ . A general element of the algebra  $(\mathcal{A}, *)$  will be written in this basis as

$$f(z) = \int \frac{d^4 p}{(2\pi)^4} \tilde{f}(p) e^{ipz}, \quad (2.39)$$

where  $\tilde{f}(p) = \tilde{f}_0(p) + a\tilde{f}_1(p)$ . The star product (2.38) will then have the form

$$\begin{aligned} f * g(z) &= \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \tilde{f}(p) \tilde{g}(k) e^{-ipDk/2} e^{i(p+k)z} \\ &= \int \frac{d^4 p}{(2\pi)^4} \tilde{f} * \tilde{g}(p) e^{ipz}, \end{aligned} \quad (2.40)$$

where  $\tilde{f} * \tilde{g}(p)$  is given by

$$\tilde{f} * \tilde{g}(p) = \int \frac{d^4 k}{(2\pi)^4} \tilde{f}(p-k) \tilde{g}(k) e^{-i(p-k)Dk/2}. \quad (2.41)$$

In this case  $\tilde{f} * \tilde{g}(p)$  is a function only of  $\chi^N$  and not of  $\chi$ . However,  $\chi^N$  is simply a constant in the  $\hbar^n$  approximation and therefore Eq. (2.40) is of the same form as Eq. (2.39).

### III. QUANTUM SPACE-TIME

#### A. Quantization

We will now show that the algebra  $(\mathcal{A}, *)$  does really describe a quantum space-time. In other words  $QR^4$  is a space-time we obtain by quantizing  $R^4$  in the following way. First of all we assume that the quantization of  $R^4$  is completely equivalent to the quantization of its underlying algebra  $(\mathcal{A}, \cdot)$  [3,4]. Then in analogy with quantum mechanics we will quantize  $(\mathcal{A}, \cdot)$  by the usual quantization prescription of replacing the coordinate functions  $x^\mu$  by the coordinate operators  $X^\mu$  so that the algebra of the functions  $(\mathcal{A}, \cdot)$  is mapped onto an algebra of operators  $(A, \times)$  [6]. If this algebra of operators  $(A, \times)$  is to describe the quantum space-time  $QR^4$ , it must be constructed in such a way that it will be homomorphic to  $(\mathcal{A}, *)$ . In other words we must construct a homomorphism  $\mathcal{X}$  from  $(A, \times)$  to  $(\mathcal{A}, *)$  which will map any element  $F(X)$  of  $A$  to the element (2.22) of  $(\mathcal{A}, *)$  in such a way that the operator product  $F(X) \times G(X)$  is mapped onto the star product (2.23). We would then have

$$F(X) \rightarrow \mathcal{X}(F(X)) = f(x) \quad (3.1)$$

together with

$$F(X) \times G(X) \rightarrow \mathcal{X}(F(X) \times G(X)) = f * g(x), \quad (3.2)$$

where  $g(x)$  is the image of the operator  $G(X)$ . In particular from Eq. (3.1) the coordinate operators  $X^\mu$  are mapped onto the coordinate functions  $x^\mu$  and from Eq. (3.2) the Moyal brackets  $\{f, g\}$  are mapped onto the commutator  $[F, G]_\times = F \times G - G \times F$  [34]. As we will see the homomorphism  $\mathcal{X}$

has no nontrivial kernels and therefore the arrows in Eqs. (3.1) and (3.2) can go the other way.

The product  $\times$  which we will call the nonassociative operator product cannot be the ordinary dot product of operators because it is clear from the definition (3.2) that  $\times$  is nonassociative whereas the dot product of operators is trivially an associative product. We can assume, however, that it will reduce at the limit of  $a \rightarrow 0$  to the ordinary dot product of operators. The difference  $\Delta$  between the nonassociative product  $\times$  and the ordinary dot product is of the order of  $a$  and it is given by

$$\Delta(F, G) = \frac{F \times G - F \cdot G}{a}, \quad (3.3)$$

where  $F \cdot G$  is defined by

$$\mathcal{X}(F(X) \cdot G(X)) = \lim_{a \rightarrow 0} f * g(x). \quad (3.4)$$

The first step in constructing this homomorphism  $\mathcal{X}$  is to impose on the coordinate operators  $X^\mu$  commutation relations which are of the same form as Eq. (2.16). We then have

$$[X^\mu, X^\nu]_\times = iR \theta^{\mu\nu}, \quad (3.5)$$

where  $R$  is an operator defined by

$$\mathcal{X}(R) = \chi(x). \quad (3.6)$$

In terms of the ordinary commutator, Eq. (3.5) will simply read

$$[X^\mu, X^\nu] = iR \theta^{\mu\nu}. \quad (3.7)$$

The contribution  $\Delta(X^\mu, X^\nu) - \Delta(X^\nu, X^\mu)$  to this commutator is identically zero because  $\Delta(X^\mu, X^\nu) = (-R/2) \eta^{\mu\nu}$ .

The operator  $R$  clearly does not commute with  $X^\mu$  because

$$[R, X^\mu]_\times = R^\mu, \quad (3.8)$$

where  $R^\mu$  are the elements of the algebra  $A$  mapped to  $\{\chi, x^\mu\}$ , i.e.,

$$\mathcal{X}(R^\mu) = \{\chi, x^\mu\} = -i\chi \theta^{\mu\nu} \partial_\nu \chi. \quad (3.9)$$

Equation (3.8) will simply mean that the Jacobi identity

$$[X^\mu, [X^\nu, X^\alpha]_\times]_\times + [X^\alpha, [X^\mu, X^\nu]_\times]_\times + [X^\nu, [X^\alpha, X^\mu]_\times]_\times = 0 \quad (3.10)$$

is not satisfied unless we choose  $\theta$  to satisfy Eq. (2.19).

In general the commutator of the generator  $X^\mu$  with any element  $F(X)$  of the algebra  $A$  is found to be

$$[X^\mu, F]_\times = \Delta F, \quad (3.11)$$

where by using Eqs. (3.1) and (3.2),  $\Delta F$  is the operator in  $A$  mapped to  $\{x^\mu, f\}$ , i.e.,

$$\mathcal{X}(\Delta F) = \{x^\mu, f\}. \quad (3.12)$$

It is clear from this equation that the central elements of the algebra  $A$  are either those operators which are mapped onto the constant functions or the operator  $O$  which is mapped onto  $\chi^N$ . The operators mapped onto the constant functions are clearly multiples of the identity operator  $\mathbf{1}$ . The operator  $O$ , on the other hand, is  $R^N$  which can be seen as follows. By using Eq. (2.32) we can prove that in the  $\hbar^N$  approximation we have that  $\chi^*(\chi^*(\chi^*(\chi \cdots (\chi^* \chi))) \cdots) = \chi^N$  where we have  $N$  factors in the product. This equation will become under quantization  $R^N + a \sum_{m=0}^{N-2} R^m \Delta(R, R^{N-m-1}) = O$ . However, by using the definition (3.3) of  $\Delta$ , one can check that in the  $\hbar^N$  approximation the second term in the expression of  $O$  is of the order of  $\hbar^{N+1}$  and therefore  $O = R^N$ . The generators  $X^\mu$  will then commute with  $R^N$ , i.e.,

$$[X^\mu, R^N]_\times = 0. \quad (3.13)$$

In general  $X^\mu$  will commute with any element of  $A$  which is of the order of  $\hbar^N$ .

The fact that  $R$  does not commute with the algebra  $A$  makes the definition (3.5) of quantum space-time not very useful when we try to construct explicitly the homomorphism  $\mathcal{X}$ . To see this more clearly we first note that general elements  $F(X)$  of the algebra  $A$  are of the form

$$F(X) = \int \frac{d^4 p}{(2\pi)^4} [\tilde{F}(p, R) e^{ipX} + e^{-ipX} \tilde{F}^+(p, R)]. \quad (3.14)$$

The nonassociative product of any two such elements  $F(X)$  and  $G(X)$  will involve four different terms because  $R$  does not commute with  $e^{(ipX)}$ . So there is no obvious way as to how to map  $F(X)$  given by Eq. (3.14) onto  $f(x)$  given by Eq. (2.22) or for that matter how to map  $F(X) \times G(X)$  onto the star product  $f * g$ .

For the purpose of quantization a better definition of quantum space-time  $QR^4$  is such that the commutators of the generators are in the center of the algebra. We need then to find a basis  $Z^\mu$  for which we have the commutators  $[Z^\mu, Z^\nu]_\times = i\theta^{\mu\nu} C$  where  $C$  is a central element of the algebra  $A$ . If  $Z^\mu$  is the operator in  $A$  mapped to the coordinate function  $z^\mu$  introduced in Eq. (2.31), then  $C$  will be simply given by  $R^N$ . We would then have

$$[Z^\mu, Z^\nu]_\times = i\theta^{\mu\nu} R^N. \quad (3.15)$$

The ordinary commutator will also be given by a similar equation  $[Z^\mu, Z^\nu] = i\theta^{\mu\nu} R^N$  because of the fact that  $\Delta(Z^\mu, Z^\nu) = (-R^N/2) \eta^{\mu\nu}$ .

The definition of the operators  $Z^\mu$  in terms of  $X^\mu$  can be given by the equation

$$[X^\mu, Z^\nu]_\times = i\theta^{\mu\nu} R_\times^{(N+1)/2}, \quad (3.16)$$

with

$$\mathcal{X}(R_\times^{(N+1)/2}) = \chi^{(N+1)/2}, \quad (3.17)$$

where we clearly have used the requirement that this equation should be mapped onto Eq. (2.36).

The coordinate operators  $Z^\mu$  are clearly unbounded and one would like to work with bounded operators. We will therefore consider instead the operators  $e^{ipZ}$  as the generators of the algebra  $A$ . A general element  $F(Z)$  of  $A$  will be defined by

$$F(Z) = \int \frac{d^4 p}{(2\pi)^4} \tilde{F}(p) e^{ipZ}. \quad (3.18)$$

$\tilde{F}$  is a smooth continuous function of the four-vector  $p$  which must satisfy  $\tilde{F}^+(-p) = \tilde{F}(p)$  in order for  $F(Z)$  to be Hermitian.

The product of any two elements  $F(Z)$  and  $G(Z)$  of  $A$  can be found to be

$$\begin{aligned} F(Z) \times G(Z) &= \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \tilde{F}(p) \tilde{G}(k) e^{-iR^N p \theta k/2} e^{i(p+k)Z} \\ &+ a \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \tilde{F}(p) \tilde{G}(k) \Delta(e^{ipZ}, e^{ikZ}), \end{aligned} \quad (3.19)$$

where we have made use of Weyl formula

$$e^{ipZ} e^{ikZ} = e^{-iR^N p \theta k/2} e^{i(p+k)Z}. \quad (3.20)$$

### B. Coherent states

Until now we did not define the homomorphism  $\mathcal{X}$  explicitly and once this is done the quantization of  $R^4$  will be completed. We claim that  $\mathcal{X}(F)$  is defined as the map taking  $F$  to its diagonal matrix element in the coherent states basis  $|x\rangle$  [11,28,29]. If we are working in the basis (3.15) instead of (3.5), then  $\mathcal{X}(F)$  is defined as the map taking  $F$  to its diagonal matrix element in the coherent states basis  $|z\rangle$ . In order to define  $\mathcal{X}$  we need first to introduce the coherent states basis  $|z\rangle$ . We start by performing a coordinates transformation to bring  $\theta$  to the standard form  $B$  given by [16]

$$B = a \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \quad (3.21)$$

where  $\sigma_2$  is the Pauli matrix.  $\theta$  and  $B$  are related by  $B = \Lambda \theta \Lambda^T$  where  $\Lambda$  is an  $SO(4)$  transformation. Equation (3.15) becomes, under this transformation,

$$[Q^\mu, Q^\nu] = iR^N B^{\mu\nu}, \quad (3.22)$$

where  $Q^\mu$  are the new coordinate operators and they are given in terms of  $Z^\mu$  by the equation  $Q^\mu = \Lambda^\mu{}_\nu Z^\nu$ . The only non-vanishing commutation relations in Eq. (3.22) are  $[Q^0, Q^1] = [Q^2, Q^3] = iaR^N$  and as we can see we have two commuting sets of conjugate variables  $(Q^0, Q^1)$  and  $(Q^2, Q^3)$ . Therefore we need to introduce only two commuting sets of creation and annihilation operators  $(a, a^\dagger)$  and  $(b, b^\dagger)$ . These creation and annihilation operators are defined by

$$\begin{aligned} a &= \frac{1}{\sqrt{2aR^N}} (Q^0 + iQ^1), \\ b &= \frac{1}{\sqrt{2aR^N}} (Q^2 + iQ^3). \end{aligned} \quad (3.23)$$

The commutation relations (3.22) in terms of these creation and annihilation operators read  $[a, a^\dagger] = [b, b^\dagger] = 1$ . A state  $|n\rangle$  ( $n \in Z_+$ ) of the harmonic oscillator  $(a, a^\dagger)$  is defined by  $a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$  and  $a |n\rangle = \sqrt{n} |n-1\rangle$ . In the same way a state  $|m\rangle$  ( $m \in Z_+$ ) of the harmonic oscillator  $(b, b^\dagger)$  is defined by  $b^\dagger |m\rangle = \sqrt{m+1} |m+1\rangle$  and  $b |m\rangle = \sqrt{m} |m-1\rangle$ . Following [37] we can then introduce the coherent states  $|q^0 q^1\rangle$  and  $|q^2 q^3\rangle$  defined by the equations

$$\begin{aligned} |q^0 q^1\rangle &= e^{-q^0^2 + q^1^2/4aR^N} \sum_{n=0}^{\infty} \frac{(q^0 + iq^1)^n}{(2aR^N)^{n/2} \sqrt{n!}} |n\rangle, \\ |q^2 q^3\rangle &= e^{-q^2^2 + q^3^2/4aR^N} \sum_{m=0}^{\infty} \frac{(q^2 + iq^3)^m}{(2aR^N)^{m/2} \sqrt{m!}} |m\rangle. \end{aligned} \quad (3.24)$$

These coherent states can also be written as

$$\begin{aligned} |q^0 q^1\rangle &= U(q^0, q^1) |0\rangle, \\ |q^2 q^3\rangle &= U(q^2, q^3) |0\rangle, \end{aligned} \quad (3.25)$$

where the operators  $U(q^0, q^1)$  and  $U(q^2, q^3)$  are given by

$$\begin{aligned} U(q^0, q^1) &= \exp\left(\frac{i}{aR^N} (q^1 Q^0 - q^0 Q^1)\right), \\ U(q^2, q^3) &= \exp\left(\frac{i}{aR^N} (q^3 Q^2 - q^2 Q^3)\right). \end{aligned} \quad (3.26)$$

These operators have the property that

$$\begin{aligned} U^{-1}(q^0, q^1) (\alpha Q^0 + \beta Q^1) U(q^0, q^1) &= \alpha(Q^0 + q^0) + \beta(Q^1 + q^1), \\ U^{-1}(q^2, q^3) (\alpha Q^2 + \beta Q^3) U(q^2, q^3) &= \alpha(Q^2 + q^2) + \beta(Q^3 + q^3), \end{aligned} \quad (3.27)$$

where  $\alpha$  and  $\beta$  are arbitrary complex numbers. This property simply means that the effect of  $U(q^0, q^1)$  or  $U(q^2, q^3)$  on the operators  $Q^0$  and  $Q^1$  or  $Q^2$  and  $Q^3$  is to translate them by the  $c$  numbers  $q^0$  and  $q^1$  or  $q^2$  and  $q^3$ , respectively. The operators  $U(q^0, q^1)$  and  $U(q^2, q^3)$  are therefore called translation operators. Finally a general coherent state of the theory is clearly given by

$$|q\rangle = |q^0 q^1\rangle |q^2 q^3\rangle = U(q^0, q^1) U(q^2, q^3) |0\rangle |0\rangle. \quad (3.28)$$

Using the above structure we can then show the identity

$$\langle q|e^{ipQ}|q\rangle=e^{ipq}e^{-aR^N p^2/4}. \quad (3.29)$$

The proof goes as follows:

$$\begin{aligned} \langle q|e^{ipQ}|q\rangle &= \langle q^0 q^1|e^{i(p_0 Q^0 + p_1 Q^1)}|q^0 q^1\rangle \\ &\times \langle q^2 q^3|e^{i(p_2 Q^2 + p_3 Q^3)}|q^2 q^3\rangle \\ &= \langle 0|e^A e^B e^{-A}|0\rangle \langle 0|e^C e^D e^{-C}|0\rangle, \end{aligned} \quad (3.30)$$

where  $A = (-i/aR^N)(q^1 Q^0 - q^0 Q^1)$ ,  $B = i(p_0 Q^0 + p_1 Q^1)$ ,  $C = (-i/aR^N)(q^3 Q^2 - q^2 Q^3)$ , and  $D = i(p_2 Q^2 + p_3 Q^3)$ . By using the Weyl formula (3.20) we can then compute that  $\exp(A)\exp(B)\exp(-A) = \exp([A, B])\exp(B)$ . However,  $[A, B] = i(p_0 q^0 + p_1 q^1)$  and therefore  $\langle 0|\exp(A)\exp(B)\exp(-A)|0\rangle = \exp i(p_0 q^0 + p_1 q^1)\langle 0|\exp(B)|0\rangle$ . Using the Weyl formula again we get that  $\exp(B) = \exp(-\xi^* a^+ + \xi a) = \exp(-|\xi|^2/2)\exp(-\xi^* a^+)\exp(\xi a)$  where  $\xi$  is given by  $\xi = (\sqrt{aR^N}/2)(ip_0 + p_1)$ . The final result is  $\langle 0|\exp(A)\exp(B)\exp(-A)|0\rangle = \exp[i(p_0 q^0 + p_1 q^1)]\exp[(-aR^N/4)(p_0^2 + p_1^2)]$ . Similar calculations will give that  $\langle 0|\exp(C)\exp(D)\exp(-C)\rangle = \exp[i(p_2 q^2 + p_3 q^3)]\exp[(-aR^N/4)(p_2^2 + p_3^2)]$ . All of this put together gives Eq. (3.29). However, the formula (3.29) is clearly valid in any other basis and not only in the basis (3.22). Rotating back to the basis (3.15) will then give

$$\langle z|e^{ipZ}|z\rangle=e^{ipz}e^{-a\chi^N p^2/4}, \quad (3.31)$$

where it is understood that  $\chi^N$  is the eigenvalue of the operator  $R^N$  on the coherent state  $|z\rangle$  defined by  $|z\rangle = U(\Lambda^{-1})|q\rangle$ . Equation (3.31) is the basic identity needed in defining the map  $\mathcal{X}$ . To show this we rewrite Eq. (3.31) in the following way:

$$\exp\left(-\frac{a\chi^N}{4}\frac{\partial}{\partial z^\mu}\frac{\partial}{\partial z_\mu}\right)(\langle z|e^{ipZ}|z\rangle)=e^{ipz}. \quad (3.32)$$

We note that at the limit of  $p \rightarrow 0$  this identity takes the form

$$\langle z|Z^\mu|z\rangle=z^\mu. \quad (3.33)$$

Equation (3.32) suggests that we define the homomorphism  $\mathcal{X}$  by

$$\begin{aligned} F(Z) \rightarrow \mathcal{X}(F(Z)) &= \exp\left(-\frac{a\chi^N}{4}\frac{\partial}{\partial z^\mu}\frac{\partial}{\partial z_\mu}\right)[\langle z|F(Z)|z\rangle] \\ &= f(z). \end{aligned} \quad (3.34)$$

Now putting Eqs. (2.39) and (3.18) into Eq. (3.34) and using Eq. (3.31) we get that  $\tilde{f}(p) = \tilde{F}(p)$  which simply means that  $\mathcal{X}$  has no non-trivial kernels [37]. The homomorphism  $\mathcal{X}$  needs also to satisfy the requirement

$$\begin{aligned} F \times G(Z) \rightarrow \mathcal{X}(F \times G(Z)) &= \exp\left(-\frac{a\chi^N}{4}\frac{\partial}{\partial z^\mu}\frac{\partial}{\partial z_\mu}\right) \\ &\times [\langle z|F \times G(Z)|z\rangle] \\ &= f * g(z), \end{aligned} \quad (3.35)$$

which can be checked by putting Eqs. (2.40) and (3.19) in this last equation and using again Eq. (3.31).

### C. Uncertainty relations

A class of solutions to the condition (2.14) which was found to be the necessary and sufficient condition for the associativity to hold approximately in the sense of Eq. (2.10) can be given by

$$\begin{aligned} -\frac{1}{a^4}\det\theta &\equiv \frac{1}{a^4}(\vec{e} \cdot \vec{b})^2 = \cosh^2\alpha, \\ -\frac{1}{2a^2}\theta_{\mu\nu}\theta^{\mu\nu} &\equiv \frac{1}{a^2}(\vec{e}^2 - \vec{b}^2) = \sinh^2\alpha, \end{aligned} \quad (3.36)$$

where  $\vec{e}$  and  $\vec{b}$  are defined by

$$\theta = \begin{pmatrix} 0 & -ie_1 & -ie_2 & -ie_3 \\ ie_1 & 0 & b_3 & -b_2 \\ ie_2 & -b_3 & 0 & b_1 \\ ie_3 & b_2 & -b_1 & 0 \end{pmatrix}, \quad (3.37)$$

and  $\alpha$  is a real number which can be taken to be a function of  $a$ . The value  $\alpha = 0$  corresponds to the case considered in [6]. From the above two equations (3.36) we can find that

$$e^2 \geq b^2 \geq a^2. \quad (3.38)$$

We would like now that the commutation relations (3.5) lead to certain space-time uncertainty relations. This will (in principle) further restrict the allowed antisymmetric tensors  $\theta$ . Using the basic identity of quantum mechanics,  $\Delta a^2 \Delta b^2 \geq \frac{1}{4}|\langle [A, B] \rangle|^2$  where  $\Delta a^2 = \langle \Delta A^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2$ , the space-time uncertainty relations are

$$(\Delta x^\mu)^2 (\Delta x^\nu)^2 \geq \frac{1}{4}|\langle [X^\mu, X^\nu] \rangle|^2 = \frac{\langle R \rangle^2}{4}|\theta^{\mu\nu}|^2$$

$\Rightarrow$

$$(\Delta x^0)^2 \sum_{i=1}^3 (\Delta x^i)^2 \geq \frac{\langle R \rangle^2}{4} \vec{e}^2$$

and

$$\sum_{1 \leq i < j \leq 3} (\Delta x^i)^2 (\Delta x^j)^2 \geq \frac{\langle R \rangle^2}{4} \vec{b}^2. \quad (3.39)$$



By using the facts  $(\Sigma \Delta x^i)^2 \geq \Sigma (\Delta x^i)^2$ ,  $(\Sigma_{i<j} \Delta x^i \Delta x^j)^2 \geq \Sigma_{i<j} (\Delta x^i \Delta x^j)^2$  and Eq. (3.38) the above uncertainty relations will take the form

$$\Delta x^0 \sum_{i=1}^{i=3} \Delta x^i \geq \frac{\lambda}{2},$$

$$\sum_{1 \leq i < j \leq 3} \Delta x^i \Delta x^j \geq \frac{\lambda}{2}, \quad (3.40)$$

where  $\lambda = a \langle R \rangle$ . These are the same uncertainty relations which were derived in [6]. We can conclude from the relations (3.40) that quantum space-time has a cellular structure. The minimal volume (the volume of one cell) is  $(\sqrt{2\pi\lambda})^4$  and therefore a finite volume  $V$  of quantum space-time contains  $V/(\sqrt{2\pi\lambda})^4$  states. An estimation of the fuzziness of space-time would determine or at least give a bound on  $\lambda$  which will restrict further the allowed tensors  $\theta$ .

#### IV. QUANTUM FIELD THEORIES ON $QR^4$

##### A. Dirac operator

Before we try to write action integrals on a given space we need always to define first the Dirac operator on it. This Dirac operator will provide the notion of derivations on this space and by constructing it we would have basically constructed Connes triplet associated to this space [1]. For  $QR^4$  this triplet consists of a representation  $\Pi(A)$  of the algebra  $A$  underlying the quantum space-time in some Hilbert space, the Dirac operator  $D$ , and the Hilbert space  $H$  on which it acts. In the last section we have already constructed the representation  $\Pi(A)$  in terms of the coherent states basis  $|x\rangle$ . The corresponding Dirac operator in the other hand will be defined by [14]

$$\int d^4x \exp\left(-\frac{a\chi}{4} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu}\right) (\langle x|[D, \Phi]_\times \times [D, \Phi]_\times |x\rangle)$$

$$= \int d^4x \partial_\mu \phi^* \partial^\mu \phi, \quad (4.1)$$

where  $\phi$  is any element of the algebra  $\mathcal{A}$  and  $\Phi$  is its corresponding operator in  $A$ . Clearly the ordinary Dirac operator  $\mathcal{D}$  on  $R^4$  given by  $\mathcal{D} = \gamma^\mu \partial_\mu$ , where  $\{\gamma^\mu\}$  is the Clifford algebra satisfying  $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ , will satisfy Eq. (4.1) in the limit  $\theta \rightarrow 0$ . In other words it will satisfy the equation  $\text{Tr}[\mathcal{D}, \Phi][\mathcal{D}, \Phi] = \int d^4x \partial_\mu \phi \partial^\mu \phi$ . It is reasonable to assume that this Clifford algebra will not get modified under quantization of space-time so that we can write  $D$  as

$$D = \gamma^\mu D_\mu + aF. \quad (4.2)$$

This assumption can be justified by the fact that the  $\gamma$ 's are not elements of the algebra  $\mathcal{A}$  and therefore quantizing the algebra will not quantize them.  $F$  in Eq. (4.2) is a connection arising from the nonassociativity of the underlying algebra  $(\mathcal{A}, *)$  and it is defined such that Eq. (4.1) takes the form

$$\int d^4x \exp\left(-\frac{a\chi}{4} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu}\right) ([D_\mu, \Phi]_\times \times [D^\mu, \Phi]_\times)$$

$$= \int d^4x \partial_\mu \phi^* \partial^\mu \phi. \quad (4.3)$$

Comparing Eqs. (4.1) and (4.3) we can find that  $F$  should satisfy the condition

$$\text{Tr}\left[\gamma^\mu [D_\mu, \Phi][F, \Phi] + \frac{i}{8} \sigma^{\mu\nu} \Delta([D_\mu, \Phi], [D_\nu, \Phi])\right]$$

$$= -\frac{1}{8} (\eta^{\mu\nu} - \gamma^\mu \gamma^\nu) \int d^4x \chi \partial_\alpha \partial^\alpha (\langle x|[D_\mu, \Phi]$$

$$\times [D_\nu, \Phi]|x\rangle), \quad (4.4)$$

where  $(i/2)\sigma^{\mu\nu} = [\gamma^\mu, \gamma^\nu]$ . A trivial solution to Eq. (4.4) is given by

$$[F, \Phi] = -\frac{i}{8} [\gamma^\alpha D_\alpha, \Phi]^{-1} \sigma^{\mu\nu} \Delta([D_\mu, \Phi],$$

$$\times [D_\nu, \Phi]) - [\gamma^\alpha D_\alpha, \Phi]^{-1} F_0,$$

where

$$\langle x|F_0|x\rangle = \frac{1}{8} (\eta^{\mu\nu} - \gamma^\mu \gamma^\nu) \chi \partial_\alpha \partial^\alpha (\langle x|[D_\mu, \Phi]$$

$$\times [D_\nu, \Phi]|x\rangle). \quad (4.5)$$

$D_\mu$  are by definition the quantum derivations on  $QR^4$  and they are given by

$$\exp\left(-\frac{a\chi}{4} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu}\right) (\langle x|[D_\mu, \Phi]_\times |x\rangle) = \partial_\mu \phi. \quad (4.6)$$

By Eqs. (3.34) and (3.35), Eq. (4.6) satisfies Eq. (4.3) trivially. To find the quantum derivations  $D_\mu$  we first have to reexpress the classical derivations in terms of the Moyal brackets and the star product introduced in Sec. I and once this is done the transition to the quantum derivations is quite straightforward. It simply consists of replacing the Moyal brackets by the commutator  $[\cdot, \cdot]_\times$  and the star product by the nonassociative operator product  $\times$  as explained in the last section. By using the Moyal brackets (2.33) any arbitrary vector field  $\mathcal{L}^\mu$  should satisfy

$$\{\mathcal{L}^\mu, \phi\} = i\chi \theta^{\alpha\beta} \partial_\alpha \mathcal{L}^\mu \partial_\beta \phi, \quad (4.7)$$

where  $\phi$  is any element of the algebra  $\mathcal{A}$ . It is clear that we have to assume that  $\mathcal{L}^\mu$  is of the order of  $\hbar^{N-1}$  in order to have only the term written in Eq. (4.7). This vector  $\mathcal{L}^\mu$ , on the other hand, will be defined by

$$\{\mathcal{L}^\mu, \phi\} = i\chi^N \theta^{\mu\beta} \partial_\beta \phi. \quad (4.8)$$

Comparing Eqs. (4.7) and (4.8) we get that

$$\begin{aligned}\partial_\alpha \mathcal{L}^\mu &= \chi^{N-1} \eta_\alpha^\mu \\ \Rightarrow \\ \mathcal{L}^\mu &= \int \chi^{N-1} dx^\mu + \mathcal{L}_0^\mu,\end{aligned}\quad (4.9)$$

where  $\mathcal{L}_0^\mu$  is an  $x$ -independent vector. In terms of Moyal brackets this last equation (4.9) will be rewritten as

$$\{x^\mu, \mathcal{L}^\nu\} = -i\chi^N \theta^{\mu\nu}. \quad (4.10)$$

Quantizing Eq. (4.8), however, will give that

$$[L^\mu, \Phi]_\times = i\theta^{\mu\nu} R^N \times [D_\nu, \Phi]_\times, \quad (4.11)$$

where we have used Eq. (4.6). This last equation can be iterated to give

$$\begin{aligned}[D_\mu, \Phi]_\times &= -iR^{-N} \theta_{\mu\nu}^{-1} [L^\nu, \Phi]_\times \\ &+ iaR^{-N} \theta_{\mu\nu}^{-1} \Delta(R^N, R^{-N} [L^\nu, \Phi]).\end{aligned}\quad (4.12)$$

This is the definition of the quantum derivations  $D_\mu$  and it is given in terms of the operators  $L^\mu$  defined by

$$\langle x | L^\mu | x \rangle = \mathcal{L}^\mu. \quad (4.13)$$

It must also satisfy

$$[X^\mu, L^\nu]_\times = -iR^N \theta^{\mu\nu}, \quad (4.14)$$

which follows from Eq. (4.10). Putting everything together the Dirac operator is then given by:

$$D = -iR^{-N} \theta_{\mu\nu}^{-1} \gamma^\mu L^\nu + a \delta D, \quad (4.15)$$

where

$$[\delta D, \Phi]_\times = [F, \Phi]_\times + iR^{-N} \theta_{\mu\nu}^{-1} \Delta(R^N, R^{-N} [L^\nu, \Phi]). \quad (4.16)$$

The Dirac operator (4.15) does act on the Hilbert space

$$H = A \otimes C^4. \quad (4.17)$$

### B. Renormalization and causality

We define scalar fields  $\hat{\Phi}$  on the quantum space-time  $QR^4$  to be elements of the algebra  $A$ ; they are given by Eq. (3.14) or (3.18). Action integrals for such fields will have the form [14]

$$\begin{aligned}S &= \int d^4x \exp\left(-\frac{a\chi}{4} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu}\right) \left[ \langle x | \frac{1}{2} [D, \hat{\Phi}]_\times \times [D, \hat{\Phi}]_\times \right. \\ &\quad \left. - \frac{m^2}{2} \hat{\Phi} \times \hat{\Phi} - \frac{g^4}{4!} (\hat{\Phi} \times \hat{\Phi}) \times (\hat{\Phi} \times \hat{\Phi}) | x \rangle \right].\end{aligned}\quad (4.18)$$

The trace is taken over the coherent states basis  $|x\rangle$ . Here  $m$  is the mass of the scalar field  $\hat{\Phi}$ , and  $g$  is the strength of the

$\hat{\Phi}^4$  interaction. These two parameters are assumed to be the physical parameters of the theory; in other words, they are finite. The field  $\hat{\Phi}$  is mapped via the homomorphism  $\mathcal{X}$  to an ordinary scalar field  $\hat{\phi}$  given by

$$\hat{\phi}(x) = \exp\left(-\frac{a\chi}{4} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu}\right) (\langle x | \hat{\Phi} | x \rangle). \quad (4.19)$$

In terms of this new field the action (4.18) will read

$$S = \int d^4x \left[ \frac{1}{2} \partial_\mu \hat{\phi}^* \partial^\mu \hat{\phi} - \frac{m^2}{2} \hat{\phi}^* \hat{\phi} - \frac{g^4}{4!} (\hat{\phi}^* \hat{\phi})^* (\hat{\phi}^* \hat{\phi}) \right]. \quad (4.20)$$

The field  $\phi$  is a general element of the algebra  $(\mathcal{A}, *)$  which is of the form (2.22). It is clearly a function of  $\chi$  which can always be put in the form

$$\begin{aligned}\hat{\phi}(x) &= \int \frac{d^4p}{(2\pi)^4} \hat{\phi}(p, \chi) e^{ipx} \\ &= \phi + \hbar \psi_1 + \hbar^2 \psi_2 + \dots + \hbar^N \psi_N.\end{aligned}\quad (4.21)$$

$\phi$  is a scalar field which is independent of  $\hbar$  and of the fuzziness functions  $\chi_1, \chi_2, \dots, \chi_N$ ; in other words it is the (commutative) classical field of the theory.  $\psi_1, \psi_2, \dots, \psi_N$ , on the other hand, are scalar fields which are also independent of  $\hbar$  but do depend on  $\chi_1, \chi_2, \dots, \chi_N$ . Their dependence on  $\chi$ 's is such that they go to zero at the limit of all  $\chi_i \rightarrow 0$ .  $\psi_1, \psi_2, \dots, \psi_N$  are assumed to be finite and therefore the noncommutative field  $\hat{\phi}$  is also finite.  $\hat{\phi}$  can then be identified with the renormalized scalar field of the theory. For simplicity we will only consider the two-loop calculation of the  $\phi^4$  theory. In this case  $N=2$  and we will have three scalar fields  $\phi, \psi_1$ , and  $\psi_2$  and two fuzzyness functions  $\chi_1$  and  $\chi_2$ . The action (4.20) in terms of these functions will read

$$S = S[\phi] + S[\phi, \psi_1, \psi_2]. \quad (4.22)$$

For the moment we will only focus on the first term in Eq. (4.22). The action  $S[\phi]$  depends only on the field  $\phi$  and it has the form

$$S[\phi] = \int d^4x \mathcal{L} + \int d^4x \Delta \mathcal{L}, \quad (4.23)$$

where

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - \frac{g^4}{4!} \phi^4 \quad (4.24)$$

and

$$\begin{aligned}
\Delta\mathcal{L} &= \int d^4x \chi \Delta\mathcal{L}_1(\phi) + \int d^4x \chi^2 \Delta\mathcal{L}_2(\phi) \\
&= \int \frac{d^4p}{(2\pi)^4} \chi(p) \Delta\mathcal{L}_1(\phi, p) + \int \frac{d^4p}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \chi(l) \\
&\quad \times \chi(p-l) \Delta\mathcal{L}_2(\phi, p). \tag{4.25}
\end{aligned}$$

In Eq. (4.25) and in all that will follow  $f(p)$  is the Fourier transform of the function  $f(x)$  and it is defined by  $f(p) = \int d^4x f(x) e^{ipx}$ .

Although  $\phi$  is the classical field, we will show below that  $\mathcal{L}$  in (4.24) generates exactly the renormalized action of the  $\phi^4$  theory, and as Eq. (4.23) suggests,  $\Delta\mathcal{L}$  will generate the corresponding counter terms.  $\chi$  will depend therefore on the classical Lagrangians  $\Delta\mathcal{L}_1$ ,  $\Delta\mathcal{L}_2$  and the usual renormalization constants  $Z_1$ ,  $Z_3$ , and  $\delta m^2$ . For consistency  $\Delta\mathcal{L}_1$  and  $\Delta\mathcal{L}_2$  should not depend on  $\chi$  which is the case as we can see from their explicit expressions

$$\begin{aligned}
\Delta\mathcal{L}_1(\phi, p) &= \int \frac{d^4k}{(2\pi)^4} \phi(k) \left[ -\frac{a}{4}(pk-k^2) \right. \\
&\quad \times (pk-k^2+m^2)\phi(p-k) \\
&\quad - \frac{3ag^4}{4!} \int \frac{d^4q}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} (ql) \\
&\quad \left. \times \phi(p-k-q-l)\phi(q)\phi(l) \right] \tag{4.26}
\end{aligned}$$

and

$$\begin{aligned}
\Delta\mathcal{L}_2(\phi, p) &= \int \frac{d^4k}{(2\pi)^4} \phi(k) \left[ \frac{1}{16}(pk-k^2+m^2) \right. \\
&\quad \times (pAk-kAk)^2\phi(p-k) \\
&\quad + \frac{1}{4} \frac{g^4}{4!} \int \frac{d^4q}{(2\pi)^4} \frac{d^4s}{(2\pi)^4} \\
&\quad \times \left( (qAs)^2 + \frac{1}{2}[(p-q-s)A(q+s)]^2 \right) \\
&\quad \left. \times \phi(p-k-q-s)\phi(q)\phi(s) \right], \tag{4.27}
\end{aligned}$$

where  $A = \theta + ia\eta$ . For the  $\phi^4$  theory it is known that in the first order of quantum theory both the mass and the coupling constant need to be renormalized. In the second order, however, we need also a field renormalization. So we would have

$$\begin{aligned}
&\int \frac{d^4p}{(2\pi)^4} \chi^{(1)}(p) \Delta\mathcal{L}_1(\phi, p) \\
&= \int d^4x \left[ -\frac{1}{2} \delta m_1^2 \phi^2 - \frac{g^4}{4!} Z_1^{(1)} \phi^4 \right] \tag{4.28}
\end{aligned}$$

and

$$\begin{aligned}
&\int \frac{d^4p}{(2\pi)^4} \chi^{(2)}(p) \Delta\mathcal{L}_1(\phi, p) \\
&\quad + \int \frac{d^4p}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \chi^{(1)}(l) \chi^{(1)}(p-l) \Delta\mathcal{L}_2(\phi, p) \\
&= \int d^4x \left[ \frac{1}{2} Z_3^{(2)} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \delta m_2^2 \phi^2 - \frac{g^4}{4!} Z_1^{(2)} \phi^4 \right]. \tag{4.29}
\end{aligned}$$

$\chi^{(1)}$  and  $\chi^{(2)}$  are given by  $\chi^{(1)} = \hbar \chi_1$  and  $\chi^{(2)} = \hbar^2 \chi_2$ . The action (4.23) will then take the form

$$S[\phi] = \int d^4x \left[ \frac{1}{2} Z_3 \partial_\mu \phi \partial^\mu \phi - \frac{m^2 + \delta m^2}{2} \phi^2 - \frac{g^4}{4!} Z_1 \phi^4 \right]. \tag{4.30}$$

$Z_3 = 1 + Z_3^{(2)}$ ,  $Z_1 = 1 + Z_1^{(1)} + Z_1^{(2)}$ , and  $\delta m^2 = \delta m_1^2 + \delta m_2^2$ . Solving Eqs. (4.28) and (4.29) for  $\chi_1$  and  $\chi_2$  will give

$$\begin{aligned}
\chi^{(1)}(p) &= \frac{\phi(p)}{\Delta\mathcal{L}_1(\phi, p)} \left[ -\frac{1}{2} \delta m_1^2 \phi(-p) \right. \\
&\quad - \frac{g^4}{4!} Z_1^{(1)} \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \phi(k)\phi(l) \\
&\quad \left. \times \phi(-p-k-l) \right] \tag{4.31}
\end{aligned}$$

and

$$\begin{aligned}
\chi^{(2)}(p) &= \frac{\phi(p)}{\Delta\mathcal{L}_1(\phi, p)} \left[ \frac{1}{2} [Z_3^{(2)} p^2 - \delta m_2^2] \phi(-p) \right. \\
&\quad - \frac{g^4}{4!} Z_1^{(2)} \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \phi(k)\phi(l) \\
&\quad \left. \times \phi(-p-k-l) \right] \\
&\quad - \frac{\Delta\mathcal{L}_2(\phi, p)}{\Delta\mathcal{L}_1(\phi, p)} \int \frac{d^4l}{(2\pi)^4} \chi^{(1)}(l) \chi^{(1)}(p-l). \tag{4.32}
\end{aligned}$$

Putting the action (4.30) back into Eq. (4.22) we get

$$S = S[\hat{\phi}] + S[\hat{\phi}, \psi_1], \tag{4.33}$$

where  $S[\hat{\phi}]$  is the action integral given by Eq. (4.30) with the substitution  $\phi \rightarrow \hat{\phi}$ , i.e.,

$$S[\hat{\phi}] = \int d^4x \left[ \frac{1}{2} Z_3 \partial_\mu \hat{\phi} \partial^\mu \hat{\phi} - \frac{m^2 + \delta m^2}{2} \hat{\phi}^2 - \frac{g^4}{4!} Z_1 \hat{\phi}^4 \right]. \tag{4.34}$$

This is exactly the standard renormalized action of the  $\phi^4$  theory with all of its counter terms.  $S[\hat{\phi}, \psi_1] = \int d^4x \mathcal{L}(\hat{\phi}, \psi_1)$ , on the other hand, is given by

$$\begin{aligned} \mathcal{L}[\hat{\phi}, \psi_1] = & -\frac{a\hbar^2\chi_1}{2}\partial_\mu\partial_\nu\hat{\phi}\partial^\mu\partial^\nu\psi_1 + \frac{a\hbar^2m^2\chi_1}{2}\partial_\mu\hat{\phi}\partial^\mu\psi_1 \\ & + \frac{a\hbar^2g^4\chi_1}{4}\hat{\phi}^2\partial_\mu\hat{\phi}\partial^\mu\psi_1 + \frac{a\hbar^2g^4\chi_1}{4}\hat{\phi}\psi_1\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} \\ & + \hbar\delta m_1^2\hat{\phi}\psi_1 + \frac{\hbar g^4Z_1^{(1)}}{6}\hat{\phi}^3\psi_1. \end{aligned} \quad (4.35)$$

As we can immediately remark, this action does not contain the field  $\psi_2$  given in Eq. (4.21). So at this order of perturbations,  $\psi_2$  is an arbitrary finite scalar field.

Clearly the finiteness requirement given by equations (4.28) and (4.29) reduces considerably the noncausality of the field  $\hat{\phi}$ . However, as we can see from Eqs. (4.33), (4.34), and (4.35) this scalar field is still highly noncausal in the sense that its conjugate momentum, which follows from the action (4.33), is not given by the ordinary expression  $Z_3\partial_0\hat{\phi}$ . It does contain extra corrections coming from Eq. (4.35). Nevertheless, one can construct a causal field  $\hat{\phi}_c$  from  $\hat{\phi}$  as follows. The action integral of  $\hat{\phi}_c$  will be by definition given by

$$S[\hat{\phi}_c] = \int d^4x \left[ \frac{1}{2}Z_3\partial_\mu\hat{\phi}_c\partial^\mu\hat{\phi}_c - \frac{m^2 + \delta m^2}{2}\hat{\phi}_c^2 - \frac{g^4}{4!}Z_1\hat{\phi}_c^4 \right], \quad (4.36)$$

and it should be equal to Eq. (4.33), i.e.,  $S[\hat{\phi}_c] = S$ . The field  $\hat{\phi}$ , on the other hand, will be defined by

$$\hat{\phi}_c = \hat{\phi} + \hbar^2\psi'_2. \quad (4.37)$$

From Eqs. (4.33) and (4.36) the field  $\psi'_2$  should satisfy

$$\hbar^2\psi'_2 = -\frac{\mathcal{L}(\hat{\phi}, \psi_1)}{[\partial^2 + m^2 + (g^4/6)\hat{\phi}^2]\hat{\phi}}. \quad (4.38)$$

The field  $\hat{\phi}_c$  is causal but not necessarily finite. The field  $\hat{\phi}$ , on the other hand, is finite but not causal. Clearly the finite and causal scalar field theory which we can construct on  $QR^4$  is such that  $\hat{\phi}_c = \hat{\phi}$ . The solution to this condition is clearly  $\psi'_2 = 0$  which can be reexpressed as a constraint on the field  $\psi_1$ :

$$\mathcal{L}(\hat{\phi}, \psi_1) = 0. \quad (4.39)$$

The only consistent solution to this equation is the trivial one:  $\psi_1 = 0$ . It is the only solution we can check which is compatible with the field  $\hat{\phi}$  being finite. The class of fields  $\hat{\phi}$  given by Eqs. (4.21) and (4.39) are the only both causal and finite scalar fields which we can write down on  $QR^4$ . The corresponding action integral is given by Eq. (4.20) or, equivalently, Eq. (4.34).

It is very instructive to perform the following consistency check on our results. First we remark that the commutation relations (2.16) combined with the solutions (4.31) and (4.32) lead directly to the conclusion that the coordinates  $x^\mu$  diverge, which definitely needs to be avoided in order to keep the finiteness of the field  $\hat{\phi}$ . The solution to this problem is to assume that the antisymmetric tensor  $\theta$  scales in such a way that the coordinates  $x^\mu$  remain well defined. We then write  $\theta = Z_\theta\theta_F$ , and compute the commutation relations (2.16) which will then take the form  $\{x^\mu, x^\nu\} = iZ_\theta[a_1Z_1^2 + a_2\delta m^4 + a_3Z_1 + a_4\delta m^2 + a_5Z_1\delta m^2 + a_6Z_3 + a_7]\theta_F^{\mu\nu}$ .  $a_i = a_i(x)$ ,  $i = 1, 7$ , are finite (computable) functions on  $R^4$ . The minimal prescription that will keep this commutator from diverging is that  $Z_\theta = 1/(Z_1^2\delta m^4Z_3)$ . In other words  $\theta$  directly measures the infinities of the field theory. We will leave the discussion on how really small  $\theta$  is to a future communication.

The action (4.18) or (4.20) may be viewed as a classical action describing a  $\phi^4$  theory of a (classical) noncommutative field  $\hat{\phi}$  living on a quantum space-time  $QR^4$ . The noncommutativity effects of this theory were shown to be exactly the quantum effects of an ordinary quantum  $\phi^4$  theory of a quantum (commutative) field  $\hat{\phi}$  living on  $R^4$ . This map between the noncommutative classical field theory and the commutative quantum field theory is consistent by construction because the two limits, the classical limit  $\eta \rightarrow 0$  and the commutative limit  $\chi \rightarrow 0$ , are identical.

## V. CONCLUSION

We showed that the scalar field action on  $R^4$  plus its counterterms can be rewritten only as a renormalized action on  $QR^4$  with no counterterms. This leads us to believe that renormalization of quantum field theory is in general equivalent to the process of quantizing the underlying space-time.

Finding phenomenological consequences of NCG such as the correction to the Coulomb potential due to the noncommutativity of space-time will be very interesting because it will allow us to put bounds on the nature of space-time at the very short distances. Results will be reported elsewhere.

Trying to include gravity as the source of the regularization and not merely as another term in the action is also under investigation. We would like the commutation relation (2.15) or (3.3) to be a consequence of quantum gravity. A large extra dimensionlike activity will be then used to make quantum gravity corrections of the same order as the quantum corrections. This will clearly involve going to higher dimensions.

The connection of the quantum space-time constructed in this paper to ordinary lattices is also very important to such matters as confinement and asymptotic freedom.

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