Rotational modes of relativistic stars: Analytic results

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We study the *r* modes and rotational ''hybrid'' modes (inertial modes) of relativistic stars. As in Newtonian gravity, the spectrum of low-frequency rotational modes is highly sensitive to the stellar equation of state. If the star and its perturbations obey the same one-parameter equation of state (as with barotropic stars), there exist *no pure r modes at all*—no modes whose limit, for a star with zero angular velocity, is an axial-parity perturbation. Rotating stars of this kind similarly have no pure *g* modes, no modes whose spherical limit is a perturbation with polar parity and vanishing perturbed pressure and density. In spherical stars of this kind, the *r* modes and *g* modes form a degenerate zero-frequency subspace. We find that rotation splits the degeneracy to *zeroth* order in the star's angular velocity Ω, and the resulting modes are generically hybrids, whose limit as $\Omega \rightarrow 0$ is a stationary current with both axial and polar parts. Because each mode has definite parity, its axial and polar parts have alternating values of l. We show that each mode belongs to one of two classes, axial-led or polar-led, depending on whether the spherical harmonic with the lowest value of l that contributes to its velocity field is axial or polar. Newtonian barotropic stars retain a vestigial set of purely axial modes (those with $l=m$); however, for relativistic barotropic stars, we show that these modes must also be replaced by axial-led hybrids. We compute the post-Newtonian corrections to the $l=m$ modes for uniform density stars. On the other hand, if the star is nonbarotropic (that is, if the perturbed star obeys an equation of state that differs from that of the unperturbed star), the *r* modes alone span the degenerate zero-frequency subspace of the spherical star. In Newtonian stars, this degeneracy is split only by the order-Ω^2 rotational corrections. However, when relativistic effects are included, the degeneracy is again broken at zeroth order. We compute the *r* modes of a nonbarotropic, uniform density model to first post-Newtonian order.

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I. INTRODUCTION

The discovery that the *r* modes in rotating stars are generically unstable due to the emission of gravitational waves [1,2] has attracted a large amount of attention in the last two years. The current models suggest that the *r*-mode instability may cause a newly born neutron star to spin down to a fraction of the Kepler frequency (which provides the limit of dynamical stability) in the first few months of its existence [3,4]. Since a considerable amount of gravitational radiation is generated in the process, the *r* modes provide a promising source for the generation of gravitational-wave interferometers that are currently under construction [5]. It is also speculated that the instability associated with the *r* modes may be relevant for older neutron stars in accreting systems [6,7].

Since the instability was first discovered and its potential astrophysical relevance was appreciated, there have been many attempts to improve on the detailed physics incorporated in the models. This effort leads to difficult questions regarding, for example, neutron star superfluidity [8], the interplay between the magnetic field and fluid pulsations [9–11], and the formation of a solid crust as a young neutron star cools [12–18]. These and several other issues must be addressed before the true astrophysical relevance of the *r* modes can be assessed. Our understanding of the *r*-mode instability, however, is based almost entirely on Newtonian calculations, and it is important to compute these modes in a relativistic context, where instability growth times may differ significantly from Newtonian-based estimates. (The closely related instability of the *f* modes of rapidly rotating stars is sharply strengthened by relativistic effects; see [19] for a review.)[1]

The purpose of the present investigation is to understand how general relativity affects the properties of the *r* modes. In order to address this issue, we first need to discuss the general nature of the modes of rotating stars.

The spherical symmetry of a nonrotating star implies that

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[1]There are as yet no fully relativistic calculations of other pulsation modes (like the *f* mode) of rapidly rotating stars, except in the Cowling approximation [20].

its perturbations can be divided into two classes, polar or axial, according to their behavior under parity. Where polar tensor fields on a two-sphere can be constructed from the scalars Y_l^m and their gradients ∇Y_l^m (and the metric on a two-sphere), axial fields involve the pseudovector $\hat{r}\times\nabla Y_l^m$, and their behavior under parity is opposite to that of Y_l^m. That is, axial perturbations of odd l are invariant under parity, and axial perturbations with even l change sign. Because a rotating star is also invariant under parity, its perturbations may also be divided into distinct parity eigenstates. If a mode varies continuously along a sequence of equilibrium configurations that starts with a spherical star and continues along a path of increasing rotation, the mode will be called axial if it is axial for the spherical star. Its parity cannot change along the sequence, but l is well defined only for modes of spherical configuration.

It is useful to subdivide stellar pulsation modes according to the physics dominating their behavior. This classification was first developed by Cowling [21] for the polar perturbations of Newtonian polytropic models. The p modes of spherical models are polar-parity modes having pressure as their dominant restoring force. They typically have large pressure and density perturbations and high frequencies (higher than a few kilohertz for neutron stars). The g modes are polar-parity modes that are chiefly restored by gravity. They typically have very small pressure and density perturbations and low frequencies. Indeed, for spherical barotropic stars, which are marginally stable to convection, the g modes are all zero frequency and have vanishing perturbed pressure and density.[2] Similarly, all axial-parity perturbations of nonrotating perfect fluid models have zero frequency. The perturbed pressure and density as well as the radial component of the fluid velocity all vanish for axial perturbations; being rotational scalars, they must have polar parity. Thus, the axial perturbations of a spherical star are simply stationary horizontal fluid currents. This Newtonian picture of stellar pulsation is readily generalized to the relativistic case. The only difference is that the various modes will now generate gravitational waves. This means that they are no longer ''normal modes,'' but satisfy outgoing-wave boundary conditions at spatial infinity. Furthermore, one can identify an additional class of such outgoing modes in relativistic stars. Like the modes of black holes, these modes are essentially associated with the dynamical spacetime geometry and have been termed w modes or gravitational-wave modes [22]. For a general discussion of the oscillations of relativistic stars, we refer the reader to the recent review article by Kokkotas and Schmidt [23].

In general, the classification of modes is relevant also for rotating stars, even though the character of the various modes may be significantly affected by rotation. In particular, rotation imparts a finite frequency to the zero-frequency perturbations of spherical stars. Because these modes are restored by the Coriolis force, their frequencies are proportional to the star's angular velocity Ω. In fluid mechanics, such modes are generally known as inertial modes [24,25,14]. In nonbarotropic stars these rotationally restored modes all have axial parity (the polar g modes are nondegenerate already in a spherical nonbarotropic star because of internal entropy or composition gradients). In astrophysics, these modes were first studied in Newtonian gravity by Papaloizou and Pringle [26], who called them r modes because of their similarity to the Rossby waves of terrestrial meteorology. In barotropic stars, however, the space of zero-frequency modes of the spherical model includes the polar g modes in addition to the axial r modes. This large degenerate subspace of zero-frequency modes is split by rotation to zeroth order in the angular velocity, and the rotationally restored (inertial) modes of barotropic stars are generically hybrids whose spherical limits are mixtures of axial and polar perturbations. This has been shown in Newtonian gravity by Lockitch and Friedman [27] (see also [28,29]). In order to distinguish between the two classes of inertial modes, we refer to modes which become purely axial in the spherical limit as r modes, while modes that limit to a mixed parity state are called rotational hybrid modes. This is a natural nomenclature given the standard distinction between axial and polar modes in relativistic studies of spherical stars.

Attempts to study the r modes of rotating relativistic stars were not made until rather recently [1,30–34]. In fact, the present investigation is the first study of this problem that puts all its different facets in the proper context. In particular, we prove that (apart from a set of stationary dipole modes) rotating relativistic barotropic stars have *no* pure r modes (modes whose limit for a spherical star is purely axial). This is in contrast with barotropic Newtonian stars which retain a vestigial set of purely axial modes (those having spherical harmonic indices $l=m$). Instead, the Newtonian r modes with $l=m\geq 2$ acquire relativistic corrections with both axial and polar parity to become discrete hybrid modes of the corresponding relativistic models. We compute these corrections for slowly rotating barotropic stars to first post-Newtonian order.

For nonbarotropic relativistic stars the situation is somewhat different. In the slow-motion approximation in which they have so far been studied, nonbarotropic stars have, remarkably, a *continuous* spectrum. Kojima [30] has shown that purely axial modes would be described by a single, second-order ordinary differential equation (ODE) for the modes' radial behavior. He then argues that the continuous spectrum is implied by the fact that the eigenvalue problem is singular (the coefficient of the highest derivative term of the equation vanishes at some value of the radial coordinate). This claim has been made mathematically precise by Beyer

[2]The lowest p mode for each value of l and m is termed an f mode or fundamental mode; it may also be regarded as a g mode, in that it is present in uniform-density models, but it has finite frequency in barotropic stars. We use the term ''barotropic'' here to denote a star for which the perturbed and unperturbed configurations satisfy the same one-parameter equation of state. In earlier work, we had used ''isentropic'' instead, because isentropic stars with no composition gradient have this property, and ''barotropic'' is not always used to include perturbations. Here, however, because the departure of neutron-star matter from a one-parameter equation of state is dominated by a composition gradient, not an entropy gradient, ''isentropic'' is inaccurate.

and Kokkotas [31]. As the latter authors point out, the continuous spectrum may be an artifact of the vanishing of the imaginary part of the frequency in the slow-rotation limit. (Or, more broadly, it may be an artifact of the slow rotation approximation itself.) In this paper we show that, in addition to the continuous spectrum, certain discrete modes also exist as solutions to Kojima's equation. These modes are the relativistic analogue to the Newtonian r modes in nonbarotropic stars. We compute these modes for slowly rotating nonbarotropic stars.

In a complementary study of the relativistic r modes, Kojima and Hosonuma [34] have recently derived the order-Ω^2 rotational corrections to Kojima's equation. Working in the time domain, they derive a set of evolution equations for an axial perturbation and its lowest order polar and axial corrections. Direct numerical evolution of these equations (with appropriate initial data) would provide a useful comparison with our results on the modes of nonbarotropic relativistic stars.

When does one need to take into account the departure of a neutron star from barotropy in computing rotational modes? Because of bulk viscosity, a gravitational-wave-driven instability is unlikely to set in above about 10^{10} K. This is well below the Fermi temperature of the star's baryons, and the departure from barotropy appears to be dominated by composition gradients in the crust and interior. These have been discussed in the context of g modes of spherical stars by Finn [35] and by Reisenegger and Goldreich [36,37] and for rotating stars by Lai [38]. Because the time scale of perturbations is too slow to allow the beta and inverse beta decays that would allow a displaced fluid element to adjust its composition to that of the surrounding star, $\Delta \log p/\Delta \log \rho$ is greater than $d\log p/d\log\rho$ by a factor of $1+\frac{1}{2}x$, where $x=n_p/n\approx 6\times 10^{-3}\rho/\rho_{\rm nuclear}$ is the local ratio of protons to baryons. This leads in the star's interior to g-mode frequencies limited by the Brunt-Väisälä frequency,

$$g\left(\frac{3\rho}{10P}x\right)^{1/2}\sim(500\,{\rm s}^{-1})\left(\frac{\rho}{\rho_{\rm nuclear}}\right)^{1/2}$$

(with g the local acceleration of gravity); when a crust is present, crustal g modes have comparable frequencies. The g-mode frequencies of spherical stars are then of order 100–200 Hz; when this is smaller than the frequencies of the rotationally restored modes of the barotropic models, one expects the barotropic approximation to be valid.

The plan of the paper is as follows. We begin, in Sec. II, with a brief review of the Eulerian and Lagrangian perturbation formalisms, both of which are used in the paper. In Sec. III, we consider the time-independent perturbations of spherical relativistic stars and prove that the subspace of nonradial zero-frequency modes is spanned by the r and g modes in barotropic models, but by the r modes alone in nonbarotropic models. Because of this difference, the character of the mode spectrum in rotating barotropic models differs considerably from that of nonbarotropic models. In Sec. IV A, we consider rotating nonbarotropic stars and argue that the problem of finding their r modes is well defined. In Sec. IV B, we consider the barotropic case and derive a set of perturbation equations whose structure parallels the corresponding Newtonian equations of Lockitch and Friedman [27].[3] This similarity between the Newtonian and relativistic equations leads to an identical structure of the mode spectrum and to a parallel theorem that every nonradial mode is either an axial-led or polar-led hybrid (the result has so far been proved only for slowly rotating relativistic stars). We consider the relativistic r-mode solutions of barotropic stars, finding that the zero-frequency dipole ($l=1$) solutions are the only purely axial solutions allowed. In other words, there are no nonstationary modes in barotropic relativistic stars whose limit as $\Omega\to 0$ is a pure axial perturbation. In particular, Newtonian r modes having $l=m\geqslant 2$ do not exist in barotropic relativistic stars and must be replaced by axial-led hybrid modes. This section concludes with a discussion of the boundary conditions appropriate to the relativistic modes (Sec. IV C). Finally, in Sec. V we construct the post-Newtonian corrections to the well-known Newtonian r modes in uniform-density stars, both barotropic and nonbarotropic. Some of the detailed equations, as well as the proof of the theorem regarding the barotropic mode spectrum, are presented in Appendixes A–C. We use geometrized units ($G=c=1$) throughout the paper.

II. EULERIAN AND LAGRANGIAN PERTURBATIONS

In general relativity, a complete description of a self-gravitating perfect fluid configuration is provided by a spacetime with metric $g_{\alpha\beta}$, sourced by an energy-momentum tensor,

$$T_{\alpha\beta}=(\epsilon+p)u_\alpha u_\beta+pg_{\alpha\beta}, \tag{2.1}$$

where the fluid four-velocity u^α is a unit timelike vector field,

$$u^\alpha u_\alpha=-1, \tag{2.2}$$

and ϵ and p are, respectively, the total energy density and pressure of the fluid as measured by an observer moving with four-velocity u^α. The metric and fluid variables satisfy an equation of state,

$$\epsilon=\epsilon(p,s), \tag{2.3}$$

with s the entropy per baryon, as well as the Einstein equation

$$G_{\alpha\beta}=8\pi T_{\alpha\beta}. \tag{2.4}$$

An equilibrium stellar model is a stationary solution $(g_{\alpha\beta},u^\alpha,\epsilon,p)$ to these equations. In this paper we will consider only equilibrium models obeying a one-parameter equation of state,

$$\epsilon=\epsilon(p), \tag{2.5}$$

[3] We will refer to equations from Ref. [27] by the equation number with the prefix ''LF.'' For example, Eq. (LF, 25) will mean Eq. (25) from Ref. [27].

because this accurately models the equilibrium configuration of a neutron star.

Adiabatic perturbations of such a star may be studied using either the Eulerian or the Lagrangian perturbation formalism [39,40]. An Eulerian perturbation may be described in terms of a smooth family $[\overline{g}_{\alpha\beta}(\lambda),\overline{u}^{\alpha}(\lambda),\overline{\epsilon}(\lambda),\overline{p}(\lambda)]$ of solutions to the exact equations (2.2)–(2.4) that coincides with the equilibrium solution at $\lambda=0$:

$$[\overline{g}_{\alpha\beta}(0),\overline{u}^{\alpha}(0),\overline{\epsilon}(0),\overline{p}(0)]=(g_{\alpha\beta},u^{\alpha},\epsilon,p).$$

Then the Eulerian change δQ in a quantity Q may be defined (to linear order in λ) as

$$\delta Q\equiv\left.\frac{dQ}{d\lambda}\right|_{\lambda=0}. \tag{2.6}$$

Thus a Eulerian perturbation is simply a change $(h_{\alpha\beta},\delta u^{\alpha},\delta\epsilon,\delta p)$ in the equilibrium configuration at a particular point in spacetime (where we have written the change in the metric as $h_{\alpha\beta}\equiv\delta g_{\alpha\beta}$). These must satisfy the perturbed Einstein equation $\delta G_{\alpha}^{\beta}=8\pi\delta T_{\alpha}^{\beta}$, together with an equation of state relating $\delta\epsilon$ and δp that may, in general, differ from that of the equilibrium configuration [see Eq. (2.13) below].

In the Lagrangian perturbation formalism [39,40], on the other hand, perturbed quantities are expressed in terms of the Eulerian change in the metric, $h_{\alpha\beta}$, and a Lagrangian displacement vector ξ^{α}, which connects fluid elements in the equilibrium star to the corresponding elements in the perturbed star. The Lagrangian change ΔQ in a quantity Q is related to its Eulerian change δQ by

$$\Delta Q=\delta Q+\pounds_{\xi}Q, \tag{2.7}$$

with $\pounds_{\xi}$ the Lie derivative along ξ^{α}.

The identities

$$\Delta g_{\alpha\beta}=h_{\alpha\beta}+2\nabla_{(\alpha}\xi_{\beta)}, \tag{2.8}$$

$$\Delta\varepsilon_{\alpha\beta\gamma\delta}=\tfrac{1}{2}\,\varepsilon_{\alpha\beta\gamma\delta}g^{\mu\nu}\Delta g_{\mu\nu} \tag{2.9}$$

then allow one to express the fluid perturbation in terms of $h_{\alpha\beta}$ and ξ^{α},

$$\Delta u^{\alpha}=\tfrac{1}{2}\,u^{\alpha}u^{\beta}u^{\gamma}\Delta g_{\beta\gamma}, \tag{2.10}$$

$$\frac{\Delta p}{\Gamma_1 p}=\frac{\Delta\epsilon}{\epsilon+p}=\frac{\Delta n}{n}=-\frac{1}{2}q^{\alpha\beta}\Delta g_{\alpha\beta}, \tag{2.11}$$

where Γ_1 is the adiabatic index, n is the baryon density, and $q^{\alpha\beta}\equiv g^{\alpha\beta}+u^{\alpha}u^{\beta}$. Using Eqs. (2.7)–(2.11), it is straightforward to express the corresponding Eulerian changes also in terms of $h_{\alpha\beta}$ and ξ^{α}, e.g.,

$$\delta u^{\alpha}=q^{\alpha}_{\beta}\pounds_{u}\xi^{\beta}+\tfrac{1}{2}\,u^{\alpha}u^{\beta}u^{\gamma}h_{\beta\gamma}. \tag{2.12}$$

For an adiabatic perturbation of an equilibrium model obeying a one-parameter equation of state, Eqs. (2.7) and (2.11) imply that the Eulerian changes in the pressure and energy density are related by

$$\frac{\delta p}{\Gamma_1 p}=\frac{\delta\epsilon}{(\epsilon+p)}+\xi^{\alpha}A_{\alpha}, \tag{2.13}$$

where we have introduced the Schwarzschild discriminant

$$A_{\alpha}=\frac{1}{(\epsilon+p)}\nabla_{\alpha}\epsilon-\frac{1}{\Gamma_1 p}\nabla_{\alpha}p, \tag{2.14}$$

which governs convective stability in the star. In general, the adiabatic index Γ_1 need not be the function

$$\Gamma\equiv\frac{(\epsilon+p)}{p}\frac{dp}{d\epsilon} \tag{2.15}$$

associated with the equilibrium equation of state. In terms of this function, we have

$$A_{\alpha}=\left(\frac{1}{\Gamma}-\frac{1}{\Gamma_1}\right)\frac{1}{p}\nabla_{\alpha}p. \tag{2.16}$$

We will call a model barotropic if the perturbed configuration satisfies the same one-parameter equation of state as the unperturbed configuration. In this case, $\Gamma_1\equiv\Gamma$ and the Schwarzschild discriminant vanishes identically. Such stars are marginally stable to convection. In this paper we study low-frequency pulsation modes of slowly rotating relativistic stars. We consider both barotropic and nonbarotropic models.

III. STATIONARY PERTURBATIONS OF SPHERICAL STARS

The equilibrium of a spherical perfect fluid star is described by a static, spherically symmetric spacetime with metric $g_{\alpha\beta}$ of the form

$$ds^2=-e^{2\nu(r)}dt^2+e^{2\lambda(r)}dr^2+r^2d\theta^2+r^2\sin^2\theta\,d\varphi^2 \tag{3.1}$$

and the energy-momentum tensor (2.1) with the fluid four-velocity given by

$$u^{\alpha}=e^{-\nu}t^{\alpha}. \tag{3.2}$$

Here $t^{\alpha}=(\partial_t)^{\alpha}$ is the timelike Killing vector of the spacetime.

For barotropic stars, the pressure and energy density are related by an equation of state of form

$$p=p(\epsilon). \tag{3.3}$$

In addition to this, the various quantities must satisfy the Einstein equation $G_{\alpha\beta}=8\pi T_{\alpha\beta}$, which leads to the standard Tolman-Oppenheimer-Volkov (TOV) equations

$$\frac{dp}{dr}=-\frac{(\epsilon+p)(M+4\pi r^3p)}{r(r-2M)}, \tag{3.4}$$

$$\frac{dM}{dr}=4\pi r^2\epsilon, \tag{3.5}$$

and

$$\frac{d\nu}{dr} = -\frac{1}{(\epsilon+p)}\frac{dp}{dr}, \tag{3.6}$$

where

$$M(r) \equiv \tfrac{1}{2} r(1-e^{-2\lambda}). \tag{3.7}$$

Our main focus in this study is on the low-frequency oscillations, corresponding to rotationally restored modes (r modes and other inertial modes) of slowly rotating stars. As in Newtonian theory, we expect these modes to limit to stationary perturbations of a spherical star as the rotation rate goes to zero. In other words, we are interested in the space of zero-frequency modes: the linearized, time-independent perturbations of the static equilibrium. As in the Newtonian case [27], we find that this zero-frequency subspace is spanned by two classes of perturbations. To identify these classes explicitly, we must examine the equations governing the perturbed configuration.

Using the Eulerian formalism, we express the perturbed configuration in terms of the set $(h_{\alpha\beta}, \delta u^{\alpha}, \delta\epsilon, \delta p)$, satisfying the perturbed Einstein equation $\delta G_{\alpha}^{\beta} = 8\pi \delta T_{\alpha}^{\beta}$, together with an equation of state relating $\delta\epsilon$ and δp.

The perturbed Einstein tensor is given by

$$\delta G_{\alpha}^{\beta} = -\tfrac{1}{2}\{\nabla_{\gamma}\nabla^{\gamma}h_{\alpha}^{\beta} - \nabla_{\gamma}\nabla^{\beta}h_{\alpha}^{\gamma} - \nabla^{\gamma}\nabla_{\alpha}h_{\gamma}^{\beta} + \nabla_{\alpha}\nabla^{\beta}h + 2R_{\alpha}^{\gamma}h_{\gamma}^{\beta} + (\nabla^{\gamma}\nabla^{\delta}h_{\gamma\delta} - \nabla_{\gamma}\nabla^{\gamma}h - R^{\gamma\delta}h_{\gamma\delta})\delta_{\alpha}^{\beta}\}, \tag{3.8}$$

where $h \equiv g^{\alpha\beta}h_{\alpha\beta}$, ∇_{α} is the covariant derivative associated with the equilibrium metric, and

$$R_{\alpha}^{\beta} = 8\pi(T_{\alpha}^{\beta} - \tfrac{1}{2}T\delta_{\alpha}^{\beta}) = 8\pi[(\epsilon+p)u_{\alpha}u^{\beta} + \tfrac{1}{2}(\epsilon-p)\delta_{\alpha}^{\beta}] \tag{3.9}$$

is the equilibrium Ricci tensor. The perturbed energy-momentum tensor is given by

$$\delta T_{\alpha}^{\beta} = (\delta\epsilon + \delta p)u_{\alpha}u^{\beta} + \delta p\,\delta_{\alpha}^{\beta} + (\epsilon+p)\delta u_{\alpha}u^{\beta} + (\epsilon+p)u_{\alpha}\delta u^{\beta}. \tag{3.10}$$

Following Thorne and Campolattaro [41], we expand our perturbed variables in scalar, vector, and tensor spherical harmonics. The perturbed energy density and pressure are scalars and therefore must have polar parity

$$\delta\epsilon = \delta\epsilon(r)Y_l^m, \tag{3.11}$$

$$\delta p = \delta p(r)Y_l^m. \tag{3.12}$$

The perturbed four-velocity for a polar-parity mode can be written

$$\delta u_P^{\alpha} = \left\{\frac{1}{2}H_0(r)Y_l^m t^{\alpha} + \frac{1}{r}W(r)Y_l^m r^{\alpha} + V(r)\nabla^{\alpha}Y_l^m\right\}e^{-\nu} \tag{3.13}$$

[where r^{α} is the coordinate vector field $(\partial_r)^{\alpha}$], while that of an axial-parity mode can be written

$$\delta u_A^{\alpha} = -U(r)e^{(\lambda-\nu)}\epsilon^{\alpha\beta\gamma\delta}\nabla_{\beta}Y_l^m u_{\gamma}\nabla_{\delta}r. \tag{3.14}$$

(We have chosen the exact form of these expressions for later convenience.)

To simplify the form of the metric perturbation we will again follow Thorne and Campolattaro [41] and work in the Regge-Wheeler [42] gauge. The metric perturbation for a polar-parity mode can then be written

$$h_{\mu\nu}^{P} = \begin{bmatrix} H_0(r)e^{2\nu} & H_2(r) & 0 & 0 \\ \text{symm} & H_2(r)e^{2\lambda} & 0 & 0 \\ 0 & 0 & r^2K(r) & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta K(r) \end{bmatrix} Y_l^m, \tag{3.15}$$

while that of an axial-parity mode can be written

$$h_{\mu\nu}^{A} = \begin{bmatrix} 0 & 0 & h_0(r)\left(\dfrac{-1}{\sin\theta}\right)\partial_{\varphi}Y_l^m & h_0(r)\sin\theta\,\partial_{\theta}Y_l^m \\ 0 & 0 & h_1(r)\left(\dfrac{-1}{\sin\theta}\right)\partial_{\varphi}Y_l^m & h_1(r)\sin\theta\,\partial_{\theta}Y_l^m \\ \text{symm} & \text{symm} & 0 & 0 \\ \text{symm} & \text{symm} & 0 & 0 \end{bmatrix}. \tag{3.16}$$

The Regge-Wheeler gauge is unique for perturbations having spherical harmonic index $l \geq 2$. However, when $l = 1$ or $l = 0$, there remain additional gauge degrees of freedom.[4] In addition, the components of the perturbed Einstein equation acquire a slightly different form in each of the three cases. (Campolattaro and Thorne [43] discuss the difference between the $l \geq 2$ and $l = 1$ cases.)

We have derived the components of the perturbed Einstein equation using the Maple tensor package[5] by substituting expressions (3.11)–(3.16) into Eqs. (3.8) and (3.10) [making liberal use of the equilibrium equations (3.4) through (3.7) to simplify the expressions]. The resulting set of equations for the case $l \geq 2$ are equivalent to those presented by Thorne and Campolattaro [41] upon specializing their equations to the case of stationary perturbations and making the necessary changes of notation.[6] Similarly, the set of equations for the case $l = 1$ is equivalent to that presented by Campolattaro and Thorne [43]. For completeness, the equations governing stationary perturbations of spherical stars are given in Appendix A.

Decomposition of the zero-frequency subspace

By inspection of the three sets of perturbation equations given in Appendix A, it is evident that they decouple into two independent classes. We find that any solution

$$(H_0, H_1, H_2, K, h_0, W, V, U, \delta\epsilon, \delta p) \tag{3.17}$$

to the equations governing the time-independent perturbations of a static, spherical star is a superposition of (i) a solution

$$(0, H_1, 0, 0, h_0, W, V, U, 0, 0) \tag{3.18}$$

to Eqs. (A6)–(A8) or (A21) and (ii) a solution

$$(H_0, 0, H_2, K, 0, 0, 0, 0, \delta\epsilon, \delta p) \tag{3.19}$$

to Eqs. (A1)–(A5), (A11)–(A14), or (A18)–(A20).

For solutions of type (ii), the vanishing of the (tr), $(t\theta)$, and $(t\varphi)$ components of the perturbed metric in our coordinate system implies that these solutions are static. If one assumes the linearization stability[7] of these solutions, i.e., that any solution to the static perturbation equations is tangent to a family of exact static solutions, then the theorem that any static self-gravitating perfect fluid is spherical implies that any solution of type (ii) is simply a neighboring spherical equilibrium.

Thus, under the assumption of linearization stability, we have shown that all stationary nonradial $(l > 0)$ perturbations of a spherical star have

$$H_0 = H_2 = K = \delta\epsilon = \delta p = 0$$

and satisfy Eqs. (A6)–(A8). That is,

$$0 = H_1 + \frac{16\pi(\epsilon + p)}{l(l+1)} e^{2\lambda} r W, \tag{3.20}$$

$$0 = e^{-(\nu-\lambda)}[e^{(\nu-\lambda)} H_1]' + 16\pi(\epsilon + p) e^{2\lambda} V, \tag{3.21}$$

$$h_0'' - (\nu' + \lambda') h_0' + \left[\frac{(2 - l^2 - l)}{r^2} e^{2\lambda} - \frac{2}{r}(\nu' + \lambda') - \frac{2}{r^2} \right] h_0 = \frac{4}{r}(\nu' + \lambda') U, \tag{3.22}$$

where a prime denotes a derivative with respect to r. Observe that if we use Eq. (3.20) to eliminate $H_1(r)$ from Eq. (3.21), we obtain

$$V = \frac{e^{-(\nu+\lambda)}}{l(l+1)(\epsilon + p)} [(\epsilon + p) e^{\nu+\lambda} r W]'. \tag{3.23}$$

This equation is clearly the generalization to relativistic stars of the conservation of mass equation in Newtonian gravity, Eq. (LF, 13). The other two equations relate the two dynamical degrees of freedom of the spacetime metric to the perturbation of the fluid four-velocity and vanish in the Newtonian limit.

These perturbations must be regular everywhere and satisfy the boundary condition that the Lagrangian change in the pressure vanish at the surface of the star, $r = R$. We show in Sec. IV C below that this boundary condition requires only

$$W(R) = 0, \tag{3.24}$$

leaving $W(r)$ and $U(r)$ otherwise undetermined. If $W(r)$ and $U(r)$ are specified, then the functions $H_1(r)$, $h_0(r)$, and $V(r)$ are determined by the above equations. The solutions for the metric variables are subject to matching conditions to the solutions in the exterior spacetime, which must also be regular at infinity: see Sec. IV C.

Finally, we consider the equation of state of the perturbed star. We have seen that for an adiabatic oscillation of a star obeying a one-parameter equation of state, Eq. (2.11), implies that the perturbed pressure and energy density are related by

[4]Letting e_{AB} be the metric on a two-sphere with ϵ_{AB} and D_A the associated volume element and covariant derivative, respectively, one finds the following: When $l \geq 2$, the polar tensors $D_A D_B Y_l^m$ and $e_{AB} Y_l^m$ are linearly independent, but when $l = 1$, they coincide. In addition, the axial tensor $\epsilon_{(A}{}^{B} D_{C)} D_B Y_l^m$ vanishes identically for $l = 1$ and, of course, $D_A Y_l^m$ vanishes for $l = 0$.

[5]www.maplesoft.com

[6]In particular, their fluid variables (denoted by the subscript TC) are related to ours as follows: $U_{TC}(r,t) = Ut$, $W_{TC}(r,t) = -re^{\lambda} Wt$, $V_{TC}(r,t) = Vt$, and $\delta\epsilon/(\epsilon + p) = \delta p/\Gamma_1 p = -(K + \frac{1}{2} H_0)$. Their equilibrium metric has the opposite signature and differs in the definitions of the metric potentials $\nu_{TC} = \frac{1}{2}\nu$ and $\lambda_{TC} = \frac{1}{2}\lambda$.

[7]We are aware of a proof of this linearization stability property under assumptions on the equation of state that are satisfied by uniform density stars, but would not allow polytropes [44].

$$\frac{\delta p}{\Gamma_1 p} = \frac{\delta\epsilon}{(\epsilon+p)} + \xi^r\left[\frac{\epsilon'}{(\epsilon+p)} - \frac{p'}{\Gamma_1 p}\right] \tag{3.25}$$

for some adiabatic index $\Gamma_1(r)$.

The Lagrangian displacement vector ξ^α is related to our perturbation variables by Eq. (2.12):

$$q^\alpha_\beta \pounds_u \xi^\beta = \delta u^\alpha - \tfrac{1}{2} u^\alpha u^\beta u^\gamma h_{\beta\gamma}. \tag{3.26}$$

Thus we have

$$e^{-\nu}\partial_t \xi^r = \delta u^r \tag{3.27}$$

or [taking the initial displacement (at $t=0$) to be zero]

$$\xi^r = t e^{\nu} \delta u^r. \tag{3.28}$$

For the class of perturbations under consideration, we have seen that $\delta p = \delta\epsilon = 0$. Thus Eqs. (3.25) and (3.28) require that

$$\delta u^r\left[\frac{\epsilon'}{(\epsilon+p)} - \frac{p'}{\Gamma_1 p}\right] = 0. \tag{3.29}$$

For axial-parity perturbations this equation is automatically satisfied, since δu^α_A has no r component: cf. Eq. (3.14). In other words, a spherical star always admits a set of axial zero-frequency modes (the r modes).

For polar-parity perturbations, $\delta u^r_P = e^{-\nu} W(r)/r \neq 0$, and Eq. (3.29) will be satisfied if and only if

$$\Gamma_1(r) \equiv \Gamma(r) = \frac{(\epsilon+p)}{p}\frac{dp}{d\epsilon}. \tag{3.30}$$

Thus we see that a spherical star admits a class of polar zero-frequency modes (the g modes) if and only if the star is barotropic; that is, if and only if the perturbed star obeys the same one-parameter equation of state as the equilibrium star.

That all axial-parity fluid perturbations of a spherical relativistic star are time independent was first shown by Thorne and Campolattaro [41]. The time-independent g modes in spherical, barotropic, relativistic stars were found by Thorne [45].

To summarize: We have shown that a spherical star always admits a class of zero-frequency r modes (stationary fluid currents with axial parity), but admits zero-frequency g modes (stationary fluid currents with polar parity) if and only if the star is barotropic. Conversely, the zero-frequency subspace of nonradial perturbations of a nonbarotropic spherical star is spanned by the r modes only, while the zero-frequency subspace of non-radial perturbations of a spherical barotropic star is spanned by the r and g modes—that is, by convective fluid motions having both axial and polar parity and with vanishing perturbed pressure and density. Being stationary, these r and g modes do not couple to gravitational radiation, although the r modes do induce a nontrivial metric perturbation $(h_{t\theta}, h_{t\varphi} \neq 0)$ in the spacetime exterior to the star (frame dragging). One would expect this large subspace of modes, which is degenerate at zero frequency, to be split by rotation, as it is in Newtonian stars. Our aim is to investigate this issue in more detail, and we will begin by considering perturbations of slowly rotating relativistic stars.

IV. PERTURBATIONS OF SLOWLY ROTATING STARS

The equilibrium of a perfect fluid star that is rotating slowly with uniform angular velocity Ω is described [46,47] by a stationary, axisymmetric spacetime with metric $g_{\alpha\beta}$ of the form

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2 - 2\omega(r) r^2 \sin^2\theta\, dt\, d\varphi \tag{4.1}$$

(accurate to order Ω). The energy-momentum tensor follows from Eq. (2.1) and the fluid four-velocity to order Ω:

$$u^\alpha = e^{-\nu}(t^\alpha + \Omega\varphi^\alpha). \tag{4.2}$$

Here $t^\alpha = (\partial_t)^\alpha$ and $\varphi^\alpha = (\partial_\varphi)^\alpha$ are, respectively, the timelike and rotational Killing vectors of the spacetime.

That the star is rotating slowly corresponds to the assumption that Ω is small compared to the Kepler velocity, $\Omega_K \sim \sqrt{M/R^3}$, the angular velocity at which the star is dynamically unstable to mass shedding at its equator. In particular, we neglect all quantities of order Ω^2 or higher. To order Ω the star retains its spherical shape, because the centrifugal deformation of its figure is an order-Ω^2 effect [46]. This means that Eqs. (3.3)–(3.7) governing a spherical star remain relevant also for a slowly rotating equilibrium configuration. In addition we need to solve an equation [46] that determines the new metric function $\omega(r)$ in terms of the spherical metric functions $\nu(r)$ and $\lambda(r)$:

$$\frac{e^{(\nu+\lambda)}}{r^4}\frac{d}{dr}\left(r^4 e^{-(\nu+\lambda)}\frac{d\bar{\omega}}{dr}\right) - \frac{4}{r}\left(\frac{d\nu}{dr} + \frac{d\lambda}{dr}\right)\bar{\omega} = 0, \tag{4.3}$$

where

$$\bar{\omega}(r) \equiv \Omega - \omega(r). \tag{4.4}$$

This new metric variable is a quantity of order Ω that governs the dragging of inertial frames induced by the rotation of the star [46]. Apart from the frame-dragging effect, however, the spacetime is unchanged from the spherical configuration. Outside the star, Eq. (4.3) has the solution

$$\bar{\omega} = \Omega - \frac{2J}{r^3}, \tag{4.5}$$

where J is the angular momentum of the spacetime. This relation can be used to provide boundary conditions for $\bar{\omega}$ (and its derivative) at the surface of the star in terms of Ω and J. Specifically, the solution to Eq. (4.3) is normalized by requiring that

$$\bar{\omega}(R) + \tfrac{1}{3} R \bar{\omega}'(R) = \Omega, \tag{4.6}$$

where R is the radius of the star.

Note that $\overline{\omega_c}$ and $\overline{\omega}$ satisfy the inequalities $0<\overline{\omega_c}\leqslant\overline{\omega}\leqslant\Omega$ (where an index c denotes the value at the center of the star). This means that $0\leqslant\omega\leqslant\Omega-\overline{\omega_c}$ and that Ω, ω, and $\overline{\omega}$ are positive for all values of r. Defining a rescaled variable $\tilde{\omega}=\overline{\omega}/\Omega$, we have $\tilde{\omega}_c=\overline{\omega_c}/\Omega\leqslant\tilde{\omega}\leqslant 1$. Then, to linear order in Ω, a single integration of Eq. (4.3) suffices to determine the frame dragging for all Ω and a specific stellar model (a given equation of state and, say, the central density).

We now consider the nonradial perturbations of these slowly rotating equilibrium models to linear order in Ω. Since the equilibrium spacetime is stationary and axisymmetric, we may decompose our perturbations into modes of the form[8] $e^{i(\sigma t+m\varphi)}$. The perturbation equations have been written down in the Eulerian formalism by Kojima [48], but we will find it convenient to work also in the Lagrangian formalism. We therefore begin by expanding the perturbed density and pressure, the displacement vector ξ^α, and the metric perturbations $h_{\alpha\beta}$ in tensor spherical harmonics.

The Eulerian change in the density and pressure may be written as

$$\delta\epsilon=\sum_{l=m}^{\infty}\delta\epsilon_l(r)Y_l^m e^{i\sigma t} \tag{4.7}$$

and

$$\delta p=\sum_{l=m}^{\infty}\delta p_l(r)Y_l^m e^{i\sigma t}, \tag{4.8}$$

respectively.

The Lagrangian displacement vector can be written

$$\xi^\alpha\equiv\frac{1}{i\kappa\Omega}\sum_{l=m}^{\infty}\left\{\frac{1}{r}W_l(r)Y_l^m r^\alpha+V_l(r)\nabla^\alpha Y_l^m - iU_l(r)P_\mu^\alpha\epsilon^{\mu\beta\gamma\delta}\nabla_\beta Y_l^m\nabla_\gamma t\nabla_\delta r\right\}e^{i\sigma t}, \tag{4.9}$$

where we have defined

$$P_\mu^\alpha\equiv e^{(\nu+\lambda)}(\delta_\mu^\alpha-t_\mu\nabla^\alpha t) \tag{4.10}$$

and introduced the comoving frequency

$$\kappa\Omega\equiv\sigma+m\Omega. \tag{4.11}$$

The exact form of expression (4.9) has been chosen for later convenience. In particular, we have chosen a gauge in which $\xi_t\equiv 0$. Note also the chosen relative phase between the terms in Eq. (4.9) with polar parity (those with coefficients W_l and V_l) and the terms with axial parity (those with coefficients U_l).

Working in the Regge-Wheeler gauge, we express our metric perturbation as

$$h_{\mu\nu}=e^{i\sigma t}\sum_{l=m}^{\infty}\begin{bmatrix} H_{0,l}e^{2\nu}Y_l^m & H_{1,l}(r)Y_l^m & h_{0,l}(r)\left(\frac{m}{\sin\theta}\right)Y_l^m & ih_{0,l}(r)\sin\theta\partial_\theta Y_l^m \\ H_{1,l}(r)Y_l^m & H_{2,l}(r)e^{2\lambda}Y_l^m & h_{1,l}(r)\left(\frac{m}{\sin\theta}\right)Y_l^m & ih_{1,l}(r)\sin\theta\partial_\theta Y_l^m \\ \text{symm} & \text{symm} & r^2K_l(r)Y_l^m & 0 \\ \text{symm} & \text{symm} & 0 & r^2\sin^2\theta K_l(r)Y_l^m \end{bmatrix}. \tag{4.12}$$

Again, note the choice of phase between the polar-parity components (those with coefficients $H_{0,l}$, $H_{1,l}$, $H_{2,l}$, and K_l) and the axial-parity components (those with coefficients $h_{0,l}$ and $h_{1,l}$).

The use of the Lagrangian formalism introduces additional gauge freedom into the problem of stellar perturbations. This freedom is associated with a class of trivial displacements that leave all physical quantities invariant [49,50]. One eliminates this gauge freedom by restricting attention to the ''canonical'' displacements—those that conserve vorticity in constant entropy surfaces [51,39]. This conservation law, known as Ertel's theorem, is essentially the curl of the perturbed Euler equation and in general relativity has the form [39]

$$\Delta\left\{\pounds_u\omega_{\alpha\beta}-\frac{2}{n^2}\nabla_{[\alpha}n\nabla_{\beta]}p\right\}=0 \tag{4.13}$$

or

$$\pounds_u\Delta\omega_{\alpha\beta}=\frac{2}{n^2}\{\nabla_{[\alpha}\Delta n\nabla_{\beta]}p+\nabla_{[\alpha}n\nabla_{\beta]}\Delta p\}, \tag{4.14}$$

where

[8]We will always choose $m\geqslant 0$ since the complex conjugate of an $m<0$ mode with real frequency σ is an $m>0$ mode with frequency $-\sigma$. Note that σ is the frequency measured by an inertial observer at infinity.

$$\omega_{\alpha\beta} \equiv 2\nabla_{[\alpha}\left(\frac{\epsilon+p}{n}u_{\beta]}\right) \tag{4.15}$$

is the relativistic vorticity. For our slowly rotating equilibrium star, Eq. (4.14) can be written using Eq. (2.11) and (2.16) as

$$i\kappa\Omega e^{-\nu}\Delta\omega_{\alpha\beta} = \frac{2}{n}A_r\nabla_{[\alpha}r\nabla_{\beta]}\Delta p, \tag{4.16}$$

since $A_\alpha = A_r\nabla_\alpha r$: cf. Eq. (2.16). Note that the three spatial components of the perturbed vorticity are not independent, being related by the identity

$$\nabla_{[\alpha}\Delta\omega_{\beta\gamma]} = 0. \tag{4.17}$$

We seek those modes that in the limit $\Omega \to 0$ belong to the zero-frequency subspace considered in the previous section. We have shown that such modes must have axial parity in nonbarotropic stars, but may be either polar or axial in the barotropic case. We will, therefore, require that our perturbation variables obey an ordering in powers of Ω that reflects this spherical limit:

$$U_l, h_{0,l} \sim O(1),$$

$$W_l, V_l, H_{1,l} \sim \begin{cases} O(1) & \text{barotropic stars} \\ O(\Omega^2) & \text{nonbarotropic stars,} \end{cases}$$

$$H_{0,l}, H_{2,l}, K_l, h_{1,l}\delta\epsilon_l, \delta p_l, \sigma \sim O(\Omega). \tag{4.18}$$

In addition to the new equations that arise at order Ω, the zeroth-order quantities must obey the zeroth-order perturbation equations (3.20), (3.22), and (3.23), for all l. The degeneracy of the zero-frequency modes will be split at *zeroth* order if there is a subset of the $O(\Omega)$ equations that involves only the $O(1)$ variables. While this occurs in Newtonian gravity only for barotropic stars [27], in general relativity it occurs also for nonbarotropic stars.

A. Nonbarotropic case

In a search for the relativistic r modes, Kojima [30] has recently applied his general perturbation equations [48] to the case of a mode whose spherical limit is purely axial. Accordingly, he assumes an ordering of his perturbation variables in powers of Ω that agrees with our nonbarotropic ordering (although he does not distinguish between the barotropic and nonbarotropic cases). Kojima then finds that the zeroth-order equation (3.22) is joined at order Ω by an additional pair of equations, which can be written

$$l(l+1)\left\{ i(\sigma+m\omega)e^{-2\nu}\left[h_{0,l}' - 2\frac{h_{0,l}}{r}\right] + \frac{(l-1)(l+2)h_{1,l}}{r^2}\right\} - 2im\omega' e^{-2\nu}h_{0,l} = 0 \tag{4.19}$$

and

$$l(l+1)\left\{ i(\sigma+m\omega)e^{-2\nu}h_{0,l} - e^{-2\lambda}h_{1,l}' - \left[\frac{2M}{r^2} + 4\pi(p-\epsilon)r\right]h_{1,l}\right\}$$
$$+ im[16\pi(p+\epsilon)r^2\overline{\omega}e^{-2\nu}h_{0,l} - 2r\omega' e^{-2\nu-2\lambda}h_{0,l} + \omega' r^2 e^{-2\nu-2\lambda}h_{0,l}'] - 16\pi m\overline{\omega}(p+\epsilon)r^2 e^{-2\nu}U_l = 0. \tag{4.20}$$

These two equations can be combined to give a second relation between the zeroth-order variables $h_{0,l}$ and U_l:

$$\left[\sigma + m\Omega - \frac{2m\overline{\omega}}{l(l+1)}\right]U_l + (\sigma+m\Omega)h_{0,l} = 0. \tag{4.21}$$

Kojima derived this equation from the perturbed Einstein equation, but as we will see in Sec. IV B, the equation can be written down immediately in the Lagrangian formalism as one of the spatial components of Eq. (4.16), $\Delta\omega_{\theta\varphi} = 0$ [cf. Eq. (4.53)]. Inspection of the Eulerian equations (as, for example, given by Kojima [48]) appears to suggest that there is no other equation in addition to Eqs. (3.22) and (4.21) that involves only the zeroth-order axial variables $h_{0,l}$ and U_l. However, after a closer study we find that for barotropic stars there is, in fact, a third such equation, implying that the system is overdetermined. While the existence of this third equation is obscured by the Eulerian formalism, it arises naturally in the Lagrangian framework as the other independent spatial component of Eq. (4.16). In nonbarotropic stars, this equation couples the $O(1)$ variables occurring on the left-hand side (LHS) to the $O(\Omega)$ variables (such as the perturbed pressure and density) appearing on the RHS. However, for barotropic stars the RHS vanishes identically (since $A_r \equiv 0$) and the third equation relating only the $O(1)$ variables emerges [cf. Eq. (4.54) or (4.55)].

Hence, for barotropic stars, the assumption that the mode is purely axial as $\Omega \to 0$ leads to an overdetermined system of equations. The appropriate spherical limit is therefore one that also includes the polar variables W_l, V_l, and $H_{1,l}$, as in Eq. (4.18). For nonbarotropic stars, on the other hand, the r-mode assumption appears to be consistent. Combining Eq. (4.21) with Eq. (3.22) gives Kojima's "master" equation for $h_{0,l}$:

$$\left[\sigma + m\Omega - \frac{2m\bar{\omega}}{l(l+1)}\right]\left\{ e^{\nu-\lambda}\frac{d}{dr}\left[e^{-\nu-\lambda}\frac{dh_{0,l}}{dr}\right] - \left[\frac{l(l+1)}{r^2} - \frac{4M}{r^3} + 8\pi(p+\epsilon)\right]h_{0,l}\right\} + 16\pi(p+\epsilon)(\sigma+m\Omega)h_{0,l} = 0. \tag{4.22}$$

Kojima used this equation to argue that the r-mode spectrum of a relativistic star is continuous. The conclusion that the equation supports a continuous spectrum was shown with more mathematical rigour by Beyer and Kokkotas [31]. Basically, the continuous spectrum arises because Eq. (4.22) is a singular eigenvalue problem; the combination $\sigma + m\Omega - 2m\bar{\omega}/l(l+1)$ may have a zero in the interval $r \in [0,\infty]$. It is interesting to ask whether the presence of a continuous part of the spectrum is physical or whether it is an artifact of the approximations we have introduced. That the latter may be the case can be argued for in the following way. To leading order in the slow-rotation expansion, the mode frequency σ is a real-valued quantity, but at higher orders it must have also an imaginary part (corresponding to dissipation due to gravitational wave emission). If we were to consider Eq. (4.22) for complex frequencies, the problem will be regular and there will likely be no continuous spectrum. The possible presence of a continuous spectrum is an interesting issue that should be investigated in more detail, but it is not the focus of the present study. What we want to emphasize here is that two important questions regarding Eq. (4.22) have not yet been answered. First of all, it has not been shown that the problem is well defined. As we have already stated, one can show that the system of equations is overdetermined for barotropic stars. This means that Eq. (4.22) can only be relevant for nonbarotropic stars. But in order to show that the equation is, indeed, relevant, we must demonstrate that all other perturbation variables are uniquely specified given a solution for $h_{0,l}$ from Eq. (4.22). Given the relative complexity of the perturbed Einstein equation, this is not a trivial task. Second, we need to investigate whether Eq. (4.22) allows distinct mode solutions in addition to its continuous spectrum. After all, the true relativistic analogue to a Newtonian r mode ought to be a distinct mode with a well-defined eigenfunction.

We address the first issue by considering the perturbation equations that arise in the Eulerian formalism; cf. [48]. As far as the axial perturbation variables are concerned, the set of equations (4.19), (4.21), and (4.22) makes sense: We have three equations governing $h_{0,l}$, $h_{1,l}$, and U_l for all l. What is not so clear is whether the remaining $O(\Omega)$ perturbation equations yield unique K_l, $H_{0,l}$, $H_{2,l}$, δp_l, and $\delta\epsilon_l$ once the axial variables are specified. (For nonbarotropic stars, the variables W_l, V_l, and $H_{1,l}$ are assumed to be of order Ω^2, so they will not enter into our calculation.) In order to show that this is the case, we must show that the remaining equations can be reduced to five independent ones. In this effort we are immediately helped by the fact that, given the assumed ordering (4.18), (i) the two equations $\delta G_{t\varphi} = 8\pi\delta T_{t\varphi}$ and $\delta G_{t\theta} = 8\pi\delta T_{t\theta}$ both imply Eq. (3.22) at order Ω and (ii) $\delta G_{tr} - 8\pi\delta T_{tr} \sim \Omega^2$ and so is automatically satisfied at lower orders. This leaves us with six equations: The equation of state for the perturbations and, for example, the five remaining Einstein equations. In other words, the problem would seem to be overdetermined. However, for nonbarotropic stars the equation of state (2.13) that relates δp to $\delta\rho$ is of order Ω^2; that is, it fixes the $O(\Omega^2)$ quantity W_l. Thus we have five equations for our five unknown variables and the problem is well defined. In other words, if a discrete mode which is limited to a purely axial perturbation as $\Omega \to 0$ exists, it should follow from Eq. (4.22). For completeness, the perturbation equations for nonbarotropic stars (complete to order Ω) that follow from Eq. (4.18) are listed in Appendix B.

Let us now suppose that a distinct r-mode solution exists in the nonbarotropic case. One would intuitively expect this to be the case since there will then be an internal stratification in the star associated with the composition gradient. In the Newtonian case, this stratification leads to a single r mode for each combination of l and m at order Ω (these modes then become nondegenerate at order Ω^2 [26,52,53]), and it also leads to the presence of nontrivial polar g modes.

From the above discussion we know that a relativistic r mode of a nonbarotropic star should follow from Eq. (4.22). We begin our search for such solutions by deriving a constraint on the possible mode frequencies. We do this by first scaling out both Ω and m from the problem by expressing the frequency σ in terms of a real parameter α, such that

$$\sigma = -m\Omega\left[1 - \frac{2\alpha}{l(l+1)}\right]. \tag{4.23}$$

Then Eq. (4.22) can be written

$$(\alpha - \tilde{\omega})\left\{ e^{\nu-\lambda}\frac{d}{dr}\left[e^{-\nu-\lambda}\frac{dh_{0,l}}{dr}\right] - \left[\frac{l(l+1)}{r^2} - \frac{4M}{r^3} + 8\pi(p+\epsilon)\right]h_{0,l}\right\} + 16\pi(p+\epsilon)\alpha h_{0,l} = 0, \tag{4.24}$$

where we have used $\tilde{\omega} = \bar{\omega}/\Omega$. From this equation we see that the eigenvalues α and the corresponding eigenfunctions $h_{0,l}$ are not explicitly dependent on either Ω or m. The latter means that, if we find an acceptable mode solution to Eq. (4.24), it will be relevant for all $m \neq 0$ for each given multipole l. This would be in accordance with the nonbarotropic Newtonian case where one finds a single r mode for each combination of l and m at order Ω [26,52,53].

As we will now establish, nontrivial solutions to Eq. (4.24) may exist provided that $\alpha - \tilde{\omega}$ vanishes at at least one point in the interval $r\in[0,\infty]$. To show this we first assume that $\alpha-\tilde{\omega}$ does not have a zero in $r\in[0,\infty]$. Then we can define a new well-behaved function f through $h_{0,l}=r^2(\alpha-\tilde{\omega})f$. By introducing this definition in Eq. (4.24), multiplying by $r^2 f$, and integrating over $r\in[0,\infty]$, one can show that (as long as f vanishes both as $r\to 0$ and $r\to\infty$ as is required by the regularity conditions)

$$-\int_0^\infty(\alpha-\tilde{\omega})^2 r^4 e^{-\lambda-\nu}|f'|dr$$

$$=\int_0^\infty(\alpha-\tilde{\omega})^2[l(l+1)-2]r^2 e^{\lambda-\nu}|f|^2 dr. \quad (4.25)$$

Here both integrands are positive definite, and it follows that we can have no nontrivial solutions for f.

In other words, a nontrivial solution for $h_{0,l}$ can only exist if $\alpha-\tilde{\omega}=0$ at some point in the spacetime. That is, the eigenvalue α must lie somewhere in the range

$$\tilde{\omega}_c\leqslant\alpha\leqslant 1. \quad (4.26)$$

As already noticed by Kojima [30], this agrees well with the Newtonian result. As the star becomes less relativistic, $\tilde{\omega}_c\to 1$ and our integral relation then predicts a nontrivial solution only for $\alpha=1$, i.e., the Newtonian r-mode eigenvalue. We will attempt to find discrete r-mode solutions, with frequencies in the permissible range, in Sec. V B.

B. Barotropic case

As indicated above, the conservation of vorticity gives rise to a mixing of axial and polar modes at zeroth order in Ω. This suggests that the modes of barotropic stars will generically be of a hybrid nature, and as a consequence, the equations determining the modes are more complicated than those for r modes of nonbarotropic stars.

The relevant perturbation equations for the barotropic case follow from the spatial components of Eq. (4.16), which for barotropic stars becomes, simply,

$$\Delta\omega_{\alpha\beta}=0. \quad (4.27)$$

We begin by expressing this relation, i.e.,

$$0=\Delta\omega_{\alpha\beta}=\nabla_\alpha\left[\Delta\left(\frac{\epsilon+p}{n}u_\beta\right)\right]-\nabla_\beta\left[\Delta\left(\frac{\epsilon+p}{n}u_\alpha\right)\right], \quad (4.28)$$

in terms of ξ^α and $h_{\alpha\beta}$.

Making use of Eq. (2.11), we have

$$\Delta\left(\frac{\epsilon+p}{n}u_\alpha\right)=\frac{\epsilon+p}{n}\left[\Delta u_\alpha-\frac{1}{2}q^{\gamma\beta}\Delta g_{\gamma\beta}\left(\frac{\Gamma_1 p}{\epsilon+p}\right)u_\alpha\right], \quad (4.29)$$

where

$$\Delta u_\alpha\equiv\Delta(g_{\alpha\beta}u^\beta)=\Delta g_{\alpha\beta}u^\beta+g_{\alpha\beta}\Delta u^\beta. \quad (4.30)$$

The ordering (4.18) implies that $u^\alpha u^\beta h_{\alpha\beta}$ and $g^{\alpha\beta}h_{\alpha\beta}$ vanish to zeroth order in Ω, since the only zeroth-order metric components are h_{tr}, $h_{t\theta}$, and $h_{t\varphi}$. Therefore,

$$\tfrac{1}{2}u^\alpha u^\beta\Delta g_{\alpha\beta}=u^\alpha u^\beta\nabla_\alpha\xi_\beta, \quad (4.31)$$

$$\tfrac{1}{2}q^{\alpha\beta}\Delta g_{\alpha\beta}=q^{\alpha\beta}\nabla_\alpha\xi_\beta, \quad (4.32)$$

$$\Delta u^\alpha=u^\alpha u^\beta u^\gamma\nabla_\beta\xi_\gamma, \quad (4.33)$$

$$\Delta u_\alpha=h_{\alpha\beta}u^\beta+u^\beta\nabla_\beta\xi_\alpha+u^\beta\nabla_\alpha\xi_\beta$$
$$+u_\alpha u^\beta u^\gamma\nabla_\beta\xi_\gamma. \quad (4.34)$$

From Eqs. (2.11) and (3.6) and the relation

$$u^\alpha u^\beta\nabla_\alpha\xi_\beta=-\xi^\beta u^\alpha\nabla_\alpha u_\beta+u^\alpha\nabla_\alpha(u^\beta\xi_\beta)$$
$$=-\xi^\beta\nabla_\beta\nu+O(\Omega), \quad (4.35)$$

we obtain

$$\frac{1}{2}q^{\alpha\beta}\Delta g_{\alpha\beta}=\left(\frac{\epsilon+p}{\Gamma_1 p}\right)\nu' e^{-2\lambda}\xi_r, \quad (4.36)$$

$$u^\alpha u^\beta\nabla_\alpha\xi_\beta=-\nu' e^{-2\lambda}\xi_r, \quad (4.37)$$

to zeroth order in Ω. We will also use the explicit form of u_φ determined from Eq. (4.2),

$$u_\varphi=e^{-\nu}\overline{\omega}r^2\sin^2\theta, \quad (4.38)$$

and the components of Δu_α to zeroth order in Ω:

$$\Delta u_r=e^{-\nu}\left[h_{tr}+i\kappa\Omega\xi_r+\Omega r^2\partial_r\left(\frac{1}{r^2}\xi_\varphi\right)+\frac{e^{2\nu}}{r^2}\partial_r(r^2\omega e^{-2\nu})\xi_\varphi\right], \quad (4.39)$$

$$\Delta u_\theta=e^{-\nu}[h_{t\theta}+i\kappa\Omega\xi_\theta+\Omega\partial_\theta\xi_\varphi-2\overline{\omega}\cot\theta\xi_\varphi], \quad (4.40)$$

$$\Delta u_\varphi=e^{-\nu}[h_{t\varphi}+i\kappa\Omega\xi_\varphi+\Omega\partial_\varphi\xi_\varphi$$
$$+2\overline{\omega}\sin\theta\cos\theta\,\xi_\theta$$
$$+e^\nu\partial_r(r^2\overline{\omega}e^{-\nu})\sin^2\theta\, e^{-2\lambda}\xi_r]. \quad (4.41)$$

For completeness, we explicitly write down the components of $i\kappa\Omega\vec{\xi}$ to zeroth order in Ω [cf. Eq. (4.9)]:

$$i\kappa\Omega\xi^t=O(\Omega),$$

$$i\kappa\Omega\xi^\theta=\sum_l\frac{1}{r^2\sin\theta}[V_l\sin\theta\,\partial_\theta Y_l^m+mU_l Y_l^m]e^{i\sigma t},$$

$$i\kappa\Omega\xi^r=\sum_l \frac{1}{r}W_l Y_l^m e^{i\sigma t},$$

$$i\kappa\Omega\xi^\varphi=\sum_l \frac{i}{r^2\sin^2\theta}[mV_l Y_l^m+U_l\sin\theta\,\partial_\theta Y_l^m]e^{i\sigma t},$$

$$i\kappa\Omega\xi_t=0,$$

$$i\kappa\Omega\xi_\theta=\sum_l \frac{1}{\sin\theta}[V_l\sin\theta\,\partial_\theta Y_l^m+mU_l Y_l^m]e^{i\sigma t},$$

$$i\kappa\Omega\xi_r=\sum_l \frac{e^{2\lambda}}{r}W_l Y_l^m e^{i\sigma t},$$

$$i\kappa\Omega\xi_\varphi=\sum_i i[mV_l Y_l^m+U_l\sin\theta\,\partial_\theta Y_l^m]e^{i\sigma t}. \tag{4.42}$$

By making use of Eqs. (4.29)–(4.41) and expressions (4.42) and (4.12) for the components of $i\kappa\Omega\vec{\xi}$ and $h_{\alpha\beta}$, we may now write the spatial components of $\Delta\omega_{\alpha\beta}$. We will use Eq. (3.20) to eliminate $H_{1,l}$ (for all l) from the resulting expressions and drop the ''0'' subscript on the metric functions $h_{0,l}$, writing $h_{0,l}\equiv h_l$:

$$\begin{aligned}\Delta\omega_{\theta\varphi}&=\left(\frac{\epsilon+p}{n}\right)\left\{\partial_\theta\Delta u_\varphi-\partial_\varphi\Delta u_\theta-\partial_\theta\left[\frac{1}{2}q^{\alpha\beta}\Delta g_{\alpha\beta}\left(\frac{\Gamma_1 p}{\epsilon+p}\right)u_\varphi\right]\right\}\\&=\left(\frac{\epsilon+p}{n}\right)\frac{e^{-\nu}\sin\theta}{ik\Omega}\sum_l\Big\{[l(l+1)\kappa\Omega(h_l+U_l)-2m\bar{\omega}U_l]Y_l^m-2\bar{\omega}V_l\\&\quad\times[\sin\theta\,\partial_\theta Y_l^m+l(l+1)\cos\theta\,Y_l^m]+\frac{e^{2\nu}}{r}\partial_r(r^2\bar{\omega}e^{-2\nu})W_l[\sin\theta\,\partial_\theta Y_l^m+2\cos\theta\,Y_l^m]\Big\}e^{i\sigma t},\end{aligned}\tag{4.43}$$

$$\begin{aligned}\Delta\omega_{r\theta}&=\left(\frac{\epsilon+p}{n}\right)e^{\nu}[\partial_r(e^{-\nu}\Delta u_\theta)-\partial_\theta(e^{-\nu}\Delta u_r)]\\&=\left(\frac{\epsilon+p}{n}\right)\frac{e^{\nu}}{\kappa\,\Omega\sin\theta}\sum_l\Big\{m\kappa\Omega\partial_r[e^{-2\nu}(h_l+U_l)]Y_l^m-2\partial_r(\bar{\omega}e^{-2\nu}U_l)\cos\theta\sin\theta\,\partial_\theta Y_l^m\\&\quad+\frac{1}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})U_l[m^2+l(l+1)(\cos^2\theta-1)]Y_l^m-2m\partial_r(\bar{\omega}e^{-2\nu}V_l)\cos\theta\,Y_l^m\\&\quad+\frac{m}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})V_l\sin\theta\,\partial_\theta Y_l^m+\kappa\Omega\left[\partial_r(e^{-2\nu}V_l)+e^{-2\nu}\left(\frac{16\pi r(\epsilon+p)}{l(l+1)}-\frac{1}{r}\right)e^{2\lambda}W_l\right]\sin\theta\,\partial_\theta Y_l^m\Big\}e^{i\sigma t},\end{aligned}\tag{4.44}$$

$$\begin{aligned}\Delta\omega_{\varphi r}&=\left(\frac{\epsilon+p}{n}\right)e^{\nu}\left\{\partial_\varphi(e^{-\nu}\Delta u_r)-\partial_r(e^{-\nu}\Delta u_\varphi)+\partial_r\left[\frac{1}{2}q^{\alpha\beta}\Delta g_{\alpha\beta}\left(\frac{\Gamma_1 p}{\epsilon+p}\right)e^{-\nu}u_\varphi\right]\right\}\\&=\left(\frac{\epsilon+p}{n}\right)\frac{e^{\nu}}{i\kappa\Omega}\sum_l\Big\{m\kappa\Omega\left[\partial_r(e^{-2\nu}V_l)+e^{-2\nu}\left(\frac{16\pi r(\epsilon+p)}{l(l+1)}-\frac{1}{r}\right)e^{2\lambda}W_l\right]Y_l^m-2\partial_r(\bar{\omega}e^{-2\nu}V_l)\cos\theta\sin\theta\,\partial_\theta Y_l^m\\&\quad+\frac{m^2}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})V_l Y_l^m+\partial_r\left[\frac{1}{r}\partial_r(r^2\bar{\omega}e^{-2\nu})W_l\right](\cos^2\theta-1)Y_l^m-2m\partial_r(\bar{\omega}e^{-2\nu}U_l)\cos\theta\,Y_l^m\\&\quad+\kappa\Omega\partial_r[e^{-2\nu}(h_l+U_l)]\sin\theta\,\partial_\theta Y_l^m+\frac{m}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})U_l\sin\theta\,\partial_\theta Y_l^m\Big\}e^{i\sigma t}.\end{aligned}\tag{4.45}$$

These equations can be rewritten using the standard identities

$$\sin\theta\partial_\theta Y_l^m=lQ_{l+1}Y_{l+1}^m-(l+1)Q_l Y_{l-1}^m, \tag{4.46}$$

$$\cos\theta\,Y_l^m=Q_{l+1}Y_{l+1}^m+Q_l Y_{l-1}^m, \tag{4.47}$$

with Q_l defined as

$$Q_l\equiv\left[\frac{(l+m)(l-m)}{(2l-1)(2l+1)}\right]^{1/2}. \tag{4.48}$$

We then get, from $\Delta\omega_{\theta\varphi}=0$,

$$0=\sum_l \left\{[l(l+1)\kappa\Omega(h_l+U_l)-2m\bar{\omega}U_l]Y_l^m+\left[\frac{e^{2\nu}}{r}\partial_r(r^2\bar{\omega}e^{-2\nu})W_l-2l\bar{\omega}V_l\right](l+2)Q_{l+1}Y_{l+1}^m\right.$$
$$\left.-\left[\frac{e^{2\nu}}{r}\partial_r(r^2\bar{\omega}e^{-2\nu})W_l+2(l+1)\bar{\omega}V_l\right](l-1)Q_lY_{l-1}^m\right\}. \tag{4.49}$$

From $\Delta\omega_{r\theta}=0$ we have

$$0=\sum_l \left\{\left[-2\partial_r(\bar{\omega}e^{-2\nu}U_l)+\frac{(l+1)}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})U_l\right]lQ_{l+1}Q_{l+2}Y_{l+2}^m\right.$$
$$+\left[l\kappa\Omega\partial_r(e^{-2\nu}V_l)-2m\partial_r(\bar{\omega}e^{-2\nu}V_l)+\frac{lm}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})V_l+l\kappa\Omega e^{-2\nu}\left(\frac{16\pi r(\epsilon+p)}{l(l+1)}-\frac{1}{r}\right)e^{2\lambda}W_l\right]Q_{l+1}Y_{l+1}^m$$
$$+\left[m\kappa\Omega\partial_r[e^{-2\nu}(h_l+U_l)]+2\partial_r(\bar{\omega}e^{-2\nu}U_l)[(l+1)Q_l^2-lQ_{l+1}^2]+\frac{1}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})U_l[m^2+l(l+1)(Q_{l+1}^2+Q_l^2-1)]\right]Y_l^m$$
$$-\left[(l+1)\kappa\Omega\partial_r(e^{-2\nu}V_l)+2m\partial_r(\bar{\omega}e^{-2\nu}V_l)+\frac{m(l+1)}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})V_l+(l+1)\kappa\Omega e^{-2\nu}\left(\frac{16\pi r(\epsilon+p)}{l(l+1)}-\frac{1}{r}\right)e^{2\lambda}W_l\right]Q_lY_{l-1}^m$$
$$\left.+\left[2\partial_r(\bar{\omega}e^{-2\nu}U_l)+\frac{l}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})U_l\right](l+1)Q_{l-1}Q_lY_{l-2}^m\right\}. \tag{4.50}$$

From $\Delta\omega_{\varphi r}=0$ we have

$$0=\sum_l \left\{\left[\partial_r\left(\frac{1}{r}\partial_r(r^2\bar{\omega}e^{-2\nu})W_l\right)-2l\partial_r(\bar{\omega}e^{-2\nu}V_l)\right]Q_{l+2}Q_{l+1}Y_{l+2}^m\right.$$
$$+\left[l\kappa\Omega\partial_r[e^{-2\nu}(h_l+U_l)]-2m\partial_r(\bar{\omega}e^{-2\nu}U_l)+\frac{ml}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})U_l\right]Q_{l+1}Y_{l+1}^m$$
$$+\left[m\kappa\Omega\partial_r(e^{-2\nu}V_l)+2\partial_r(\bar{\omega}e^{-2\nu}V_l)[(l+1)Q_l^2-lQ_{l+1}^2]+\frac{m^2}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})V_l+\partial_r\left(\frac{1}{r}\partial_r(r^2\bar{\omega}e^{-2\nu})W_l\right)(Q_{l+1}^2+Q_l^2-1)\right.$$
$$\left.+m\kappa\Omega e^{-2\nu}\left(\frac{16\pi r(\epsilon+p)}{l(l+1)}-\frac{1}{r}\right)e^{2\lambda}W_l\right]Y_l^m-\left[(l+1)\kappa\Omega\partial_r[e^{-2\nu}(h_l+U_l)]+2m\partial_r(\bar{\omega}e^{-2\nu}U_l)+\frac{m(l+1)}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})U_l\right]$$
$$\left.\times Q_lY_{l-1}^m+\left[\partial_r\left(\frac{1}{r}\partial_r(r^2\bar{\omega}e^{-2\nu})W_l\right)+2(l+1)\partial_r(\bar{\omega}e^{-2\nu}V_l)\right]Q_{l-1}Q_lY_{l-2}^m\right\}. \tag{4.51}$$

Finally, let us rewrite these equations using the orthogonality relation for spherical harmonics:

$$\int Y_{l'}^{m'}Y_l^{m}{}^*d\Omega=\delta_{ll'}\delta_{mm'}, \tag{4.52}$$

where $d\Omega$ is the usual solid-angle element on the unit two-sphere.

From $\Delta\omega_{\theta\varphi}=0$ we have, for all allowed l,

$$0=[l(l+1)\kappa\Omega(h_l+U_l)-2m\bar{\omega}U_l]+(l+1)Q_l\left[\frac{e^{2\nu}}{r}\partial_r(r^2\bar{\omega}e^{-2\nu})W_{l-1}-2(l-1)\bar{\omega}V_{l-1}\right]$$
$$-lQ_{l+1}\left[\frac{e^{2\nu}}{r}\partial_r(r^2\bar{\omega}e^{-2\nu})W_{l+1}+2(l+2)\bar{\omega}V_{l+1}\right]. \tag{4.53}$$

Similarly, $\Delta\omega_{r\theta}=0$ leads to

$$
\begin{aligned}
0=&(l-2)Q_{l-1}Q_l\left[-2\partial_r(\overline{\omega}e^{-2\nu}U_{l-2})+\frac{(l-1)}{r^2}\partial_r(r^2\overline{\omega}e^{-2\nu})U_{l-2}\right]\\
&+Q_l\Bigg[(l-1)\kappa\Omega\partial_r(e^{-2\nu}V_{l-1})-2m\partial_r(\overline{\omega}e^{-2\nu}V_{l-1})\\
&+\frac{m(l-1)}{r^2}\partial_r(r^2\overline{\omega}e^{-2\nu})V_{l-1}+(l-1)\kappa\Omega e^{-2\nu}\left(\frac{16\pi r(\epsilon+p)}{(l-1)l}-\frac{1}{r}\right)e^{2\lambda}W_{l-1}\Bigg]\\
&+\Bigg[m\kappa\Omega\partial_r[e^{-2\nu}(h_l+U_l)]+2\partial_r(\overline{\omega}e^{-2\nu}U_l)((l+1)Q_1^2-lQ_{l+1}^2)\\
&+\frac{1}{r^2}\partial_r(r^2\overline{\omega}e^{-2\nu})U_l[m^2+l(l+1)(Q_{l+1}^2+Q_l^2-1)]\Bigg]\\
&-Q_{l+1}\Bigg[(l+2)\kappa\Omega\partial_r(e^{-2\nu}V_{l+1})+2m\partial_r(\overline{\omega}e^{-2\nu}V_{l+1})\\
&+\frac{m(l+2)}{r^2}\partial_r(r^2\overline{\omega}e^{-2\nu})V_{l+1}+(l+2)\kappa\Omega e^{-2\nu}\\
&\times\left(\frac{16\pi r(\epsilon+p)}{(l+1)(l+2)}-\frac{1}{r}\right)e^{2\lambda}W_{l+1}\Bigg]+(l+3)Q_{l+1}Q_{l+2}\left[2\partial_r(\overline{\omega}e^{-2\nu}U_{l+2})+\frac{(l+2)}{r^2}\partial_r(r^2\overline{\omega}e^{-2\nu})U_{l+2}\right],
\end{aligned}
\tag{4.54}
$$

and from $\Delta\omega_{\varphi r}=0$, we get

$$
\begin{aligned}
0=&Q_{l-1}Q_l\left[\partial_r\left(\frac{1}{r}\partial_r(r^2\overline{\omega}e^{-2\nu})W_{l-2}\right)-2(l-2)\partial_r(\overline{\omega}e^{-2\nu}V_{l-2})\right]\\
&+Q_l\left[(l-1)\kappa\Omega\partial_r[e^{-2\nu}(h_{l-1}+U_{l-1})]-2m\partial_r(\overline{\omega}e^{-2\nu}U_{l-1})+\frac{m(l-1)}{r^2}\partial_r(r^2\overline{\omega}e^{-2\nu})U_{l-1}\right]\\
&+\Bigg[m\kappa\Omega\partial_r(e^{-2\nu}V_l)+2\partial_r(\overline{\omega}e^{-2\nu}V_l)((l+1)Q_l^2-lQ_{l+1}^2)+\frac{m^2}{r^2}\partial_r(r^2\overline{\omega}e^{-2\nu})V_l+\partial_r\left(\frac{1}{r}\partial_r(r^2\overline{\omega}e^{-2\nu})W_l\right)(Q_{l+1}^2+Q_l^2-1)\\
&+m\kappa\Omega e^{-2\nu}\left(\frac{16\pi r(\epsilon+p)}{l(l+1)}-\frac{1}{r}\right)e^{2\lambda}W_l\Bigg]-Q_{l+1}\Bigg[(l+2)\kappa\Omega\partial_r[e^{-2\nu}(h_{l+1}+U_{l+1})]+2m\partial_r(\overline{\omega}e^{-2\nu}U_{l+1})\\
&+\frac{m(l+2)}{r^2}\partial_r(r^2\overline{\omega}e^{-2\nu})U_{l+1}\Bigg]+Q_{l+1}Q_{l+2}\left[\partial_r\left(\frac{1}{r}\partial_r(r^2\overline{\omega}e^{-2\nu})W_{l+2}\right)+2(l+3)\partial_r(\overline{\omega}e^{-2\nu}V_{l+2})\right].
\end{aligned}
\tag{4.55}
$$

It is instructive to consider the Newtonian limit [27]

$$\omega(r),\nu(r),\lambda(r),h_l(r)\rightarrow 0 \tag{5.56}$$

of these perturbation equations. We have already seen that Eq. (3.23) is the relativistic generalization of the Newtonian mass conservation equation (LF,13) [or Eq. (LF,42)] and that the other $O(1)$ perturbation equations (3.20) and (3.22) simply vanish in the Newtonian limit. Similarly, one can readily observe that the conservation of vorticity equations have as their Newtonian limit the corresponding equations from Lockitch and Friedman [27]:

Eq. (4.53)→Eq. (LF,38),

Eq. (4.54)→Eq. (LF,40),

Eq. (4.55)→Eq. (LF,39),

(and similarly for the other forms of these equations).

This correspondence leads us to expect the same structure for the relativistic modes as was found in the barotropic Newtonian case: we expect to find a discrete set of axial- and polar-led hybrid modes with opposite behavior under parity [27]. Further, we expect a one-to-one correspondence between these relativistic hybrid modes and the Newtonian modes, which the relativistic hybrids should approach in the Newtonian limit.

In deriving the components of the perturbed vorticity tensor in Newtonian gravity, (LF,38)–(LF,40), no assumptions about the ordering of the perturbation variables $(\delta\rho,\delta v^a)$ in powers of the angular velocity Ω were required. Thus the theorem concerning the character of the Newtonian modes (cf. Appendix A [27]) applies to any discrete normal mode of a uniformly rotating barotropic star with arbitrary angular velocity.

We conjecture that the perturbations of relativistic stars obey the same principle: If $(\xi^\alpha, h_{\alpha\beta})$ with $\xi^\alpha \neq 0$ is a discrete normal mode of a uniformly rotating stellar model obeying a one-parameter equation of state, then the decomposition of the mode into spherical harmonics Y_l^m has $l=m$ as the lowest contributing value of l, when $m\neq 0$, and has 0 or 1 as the lowest contributing value of l, when $m=0$.

However, in deriving the curl of the perturbed Euler equation for relativistic models, we have imposed assumptions that restrict its generality. We have derived Eqs. (4.53)–(4.55) in a form that requires a slowly rotating equilibrium model, assumes the ordering (4.18), and neglects terms of order Ω^2 and higher. Under these more restrictive assumptions, the following theorem holds.

Theorem 1. Let $(g_{\alpha\beta}(\Omega), T_{\alpha\beta}(\Omega))$ be a family of stationary, axisymmetric spacetimes describing a sequence of stellar models in uniform rotation with angular velocity Ω and obeying a one-parameter equation of state, where $(g_{\alpha\beta}(0), T_{\alpha\beta}(0))$ is a static spherically symmetric spacetime describing the nonrotating model. Let $(\xi^\alpha(\Omega), h_{\alpha\beta}(\Omega))$ with $\xi^\alpha \neq 0$ be a family of discrete normal modes of these spacetimes obeying the same one-parameter equation of state, where $(\xi^\alpha(0), h_{\alpha\beta}(0))$ is a stationary nonradial perturbation of the static spherical model. Let $(\xi^\alpha(\Omega_0), h_{\alpha\beta}(\Omega_0))$ be a member of this family with $\Omega_0 \ll \Omega_K$, the angular velocity of a particle in orbit at the star's equator. Then the decomposition of $(\xi^\alpha(\Omega_0), h_{\alpha\beta}(\Omega_0))$ into spherical harmonics Y_l^m [i.e., into (l,m) representations of the rotation group about its center of mass] has $l=m$ as the lowest contributing value of l, when $m\neq 0$, and $l=1$ as the lowest contributing value of l, when $m=0$.

We designate a nonaxisymmetric[9] mode with parity $(-1)^{m+1}$ an ''axial-led hybrid'' if ξ^α and $h_{\alpha\beta}$ receive contributions only from

$$\text{axial terms with } l=m,\ m+2,\ m+4,\ldots,$$

$$\text{polar terms with } l=m+1,\ m+3,\ m+5,\ldots\quad .$$

Similarly, we designate a nonaxisymmetric[10] mode with parity $(-1)^m$ a ''polar-led hybrid'' if ξ^α and $h_{\alpha\beta}$ receive contributions only from

$$\text{polar terms with } l=m,\ m+2,\ m+4,\ldots,$$

$$\text{axial terms with } l=m+1,\ m+3,\ m+5,\ldots\quad .$$

We prove the theorem separately for each parity class in Appendix C.

In essence, the theorem shows that if a mode of a slowly rotating barotropic star has a stationary nonradial perturbation as its spherical limit, then it is generically a hybrid mode with mixed axial and polar angular behavior. An immediate consequence of the theorem is that the r modes of barotropic stars (if they exist at all) must have $l=m$ (or $l=1$ if $m=0$), and it is well known that barotropic Newtonian stars retain a vestigial set of purely axial modes—the ''classical r modes''—whose angular behavior is a purely axial harmonic, having $l=m$. Let us address the question of whether or not such pure r-mode solutions also exist in barotropic relativistic stars.

We apply the perturbation equations for barotropic stars to the case of an axial mode belonging to a pure spherical harmonic of index l. In other words, let us assume that $h_l(r)$ and $U_l(r)$ (for some particular value of l) are the only nonvanishing coefficients in the spherical harmonic expansions (4.9) and (4.12) of the Lagrangian displacement ξ^α and the perturbed metric $h_{\alpha\beta}$, respectively.

The set of equations to be satisfied are the zeroth-order (spherical) equations (3.20), (3.22), and (3.23) and the order-Ω conservation of circulation equations (4.53)–(4.55), as well as suitable boundary conditions at infinity and at the surface of the star (Sec. IV C). Recall that as a result of Eq. (4.17), the conservation of circulation equations are linearly dependent and we need only satisfy two of them, say, Eqs. (4.53) and (4.54).

With $h_l(r)$ and $U_l(r)$ the only nonvanishing perturbation variables, Eqs. (3.20) and (3.23) vanish identically, while Eq. (3.22) remains unchanged. Equation (4.53) becomes

$$0=[l(l+1)\kappa\Omega(h_l+U_l)-2m\bar{\omega}U_l], \tag{4.57}$$

and Eq. (4.54) with $l\to l+2$, $l\to l$, and $l\to l-2$ gives the equations

$$0=lQ_{l+1}Q_{l+2}\left[-2\partial_r(\bar{\omega}e^{-2\nu}U_l)+\frac{(l+1)}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})U_l\right], \tag{4.58}$$

$$\begin{aligned}0=&\,m\kappa\Omega\partial_r[e^{-2\nu}(h_l+U_l)]\\&+2\partial_r(\bar{\omega}e^{-2\nu}U_l)[(l+1)Q_l^2-lQ_{l+1}^2]\\&+\frac{1}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})U_l[m^2+l(l+1)(Q_{l+1}^2+Q_l^2-1)],\end{aligned} \tag{4.59}$$

$$0=(l+1)Q_{l-1}Q_1\left[2\partial_r(\bar{\omega}e^{-2\nu}U_l)+\frac{l}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})U_l\right], \tag{4.60}$$

respectively.

[9]When $m=0$, there exists a set of modes with parity $+1$ that may be designated as ''axial-led hybrids,'' since ξ^α and $h_{\alpha\beta}$ receive contributions only from axial terms with $l=1,3,5,\ldots$ and polar terms with $l=2,4,6,\ldots$.

[10]When $m=0$, there exists a set of modes with parity -1 that may be designated as ''polar-led hybrids,'' since ξ^α and $h_{\alpha\beta}$ receive contributions only from polar terms with $l=1,3,5,\ldots$ and axial terms with $l=2,4,6,\ldots$. The family of modes for which ξ^α and $h_{\alpha\beta}$ receive contributions only from polar terms with $l=0,2,4,\ldots$ and axial terms with $l=1,3,5,\ldots$ would have parity $+1$ and could be designated ''polar-led hybrids.'' However, these modes require a more general theorem to establish their character.

Given the requirement that $l=m$ when $m>0$ (and $l=1$ when $m=0$), one readily finds that these equations can be satisfied only when $l=1$. *Thus no purely axial modes with $l=m\geqslant 2$ exist in barotropic relativistic stars* [33]. The dipole ($l=1$) solutions turn out to be stationary ($\sigma=0$) and have the natural physical interpretation of a uniform rotation of the star.[11]

In Newtonian barotropic stars there remained a large set of purely axial modes with $l=m$, the $l=m=2$ mode being the one expected to dominate the gravitational-wave-driven instability of sufficiently hot and rapidly rotating neutron stars [3,4]. In barotropic relativistic stars, however, we see that all such pure r modes with $l=m\geqslant 2$ are forbidden by the perturbation equations and instead must be replaced by axial-led hybrids. We explicitly construct these important hybrid modes to first post-Newtonian order in Sec. V C.

C. Boundary conditions

Having understood the general nature of the relativistic perturbation problem and derived the relevant perturbation equations for both barotropic and nonbarotropic stars, we want to determine mode solutions. Before we can do this, we need to discuss the boundary conditions that should be imposed.

For nonbarotropic stars, the zeroth-order variables are governed by the single equation (4.22), while for barotropic stars we have the set of perturbation equations (3.20), (3.22), (3.23), and (4.53)–(4.55). A physically reasonable solution $(\xi^\alpha, h_{\alpha\beta})$ to these equations must be regular everywhere in the spacetime. Of course, the fluid variables $W_l(r)$, $V_l(r)$, and $U_l(r)$ (for all l) have support only inside the star, $r\in[0,R]$. The metric functions $H_{1,l}(r)$ will also have support only inside the star (for all l), since they are directly proportional to $W_l(r)$ by Eq. (3.20). The metric functions $h_l(r)$, on the other hand, satisfy the nontrivial differential equation (3.22) in the exterior spacetime and will, therefore, have support on the whole domain $r\in[0,\infty]$. Let us now consider the boundary and matching conditions that our solutions must satisfy.

At the surface of the star, $r=R$, the perturbed pressure Δp must vanish. (This is how one defines the surface of the perturbed star.) The Lagrangian change in pressure is given by Eq. (2.11):

$$\Delta p = -\tfrac{1}{2}\Gamma_1 p q^{\alpha\beta}\Delta g_{\alpha\beta}. \tag{4.61}$$

[11]When $m=0$ and $l=1$, the solution corresponds to a small change in the angular velocity of the star about its original rotational axis. When $l=m=1$, the solution represents uniform rotation of the star about an axis perpendicular to its original rotational axis. These solutions are derived in detail by Lockitch [33] and are a generalization of the axial dipole modes studied in nonrotating relativistic stars by Campolattaro and Thorne [43].

Making use of Eq. (4.36) and the equilibrium equations (3.4) and (3.6), we find that, at $r=R$,

$$0=\Delta p = \frac{-\epsilon M_0}{R^2(R-2M_0)}\sum_l W_l(R) Y_l^m e^{i\sigma t}. \tag{4.62}$$

where $M_0=M(R)$ is the gravitational mass of the equilibrium star and satisfies $2M_0<R$.

For the equations of state we consider,[12] the energy density $\epsilon(r)$ either goes to a constant or vanishes at the surface of the star in the manner (this would be the behavior for a polytrope):

$$\epsilon(r)\sim\left(1-\frac{r}{R}\right)^k \tag{4.63}$$

(for some constant k). In both cases, it is required that

$$W_l(R)=0 \quad (\text{all } l). \tag{4.64}$$

If $\epsilon(R)\neq 0$, then Eq. (4.62) requires this directly. On the other hand, if ϵ vanishes as in Eq. (4.63), then $V_l(r)$ will diverge at the surface by Eq. (3.23) if Eq. (4.64) is not satisfied. By Eq. (3.20), this also implies that $H_{1,l}(r)$ vanishes at the surface of the star. This boundary condition is (obviously) relevant only in the barotropic case.

In the exterior vacuum spacetime, $r>R$, we have only to satisfy the single equation (3.22) for all l, which becomes

$$h_l''+\left[\frac{(2-l^2-l)}{r^2}e^{2\lambda}-\frac{2}{r^2}\right]h_l=0 \tag{4.65}$$

or

$$\left(1-\frac{2M_0}{r}\right)h_l''-\left[\frac{l(l+1)}{r^2}-\frac{4M_0}{r^3}\right]h_l=0, \tag{4.66}$$

where we have used $e^{-2\lambda}=(1-2M_0/r)$ for $r>R$.

Since this exterior equation does not couple $h_l(r)$ for different values of l, we can find its solution explicitly. The solution that is regular at spatial infinity can be written

$$h_l(r)=\sum_{s=0}^{\infty}\hat{h}_{l,s}\left(\frac{R}{r}\right)^{l+s}. \tag{4.67}$$

If we substitute this series expansion into Eq. (4.66), we find the following recursion relation for the expansion coefficients:

[12]This restriction can be dropped if the boundary condition $\Delta p(r=R)=0$ is replaced by $\Delta h(r=R)=0$, with the comoving enthalpy $h\equiv\int_0^p dp'/[\epsilon(p')+p']$.

$$\hat{h}_{l,s}=\left(\frac{2M_0}{R}\right)\frac{(l+s-2)(l+s+1)}{s(2l+s+1)}\hat{h}_{l,s-1}, \tag{4.68}$$

with $\hat{h}_{l,0}$ an arbitrary normalization constant. We therefore have the full solution to zeroth order in Ω of the perturbation equations in the exterior spacetime.

This exterior solution must be matched at the surface of the star to the interior solution for $h_l(r)$. One requires that the solutions be continuous at the surface,

$$\lim_{\varepsilon\to 0}[h_l(R-\varepsilon)-h_l(R+\varepsilon)]=0, \tag{4.69}$$

for all l, and that the Wronskian of the interior and exterior solutions vanish at $r=R$, i.e., that

$$\lim_{\varepsilon\to 0}[h_l(R-\varepsilon)h_l'(R+\varepsilon)-h_l'(R-\varepsilon)h_l(R+\varepsilon)]=0, \tag{4.70}$$

for all l.

Thus, in solving the perturbation equations to zeroth order in Ω, we need only work in the interior of the star (as in the Newtonian case). In the interior of a nonbarotropic star, the perturbation $(\xi^\alpha,h_{\alpha\beta})$ must only satisfy Eq. (4.22) together with the matching conditions (4.69) and (4.70). In the barotropic case we have the full set of coupled equations (3.20), (3.22), (3.23), and (4.53)–(4.55) for all l, subject to the boundary and matching conditions (4.64), (4.69), and (4.70).

Finally, we note that since we are working in linearized perturbation theory, there is a scale invariance to the equations. If $(\xi^\alpha,h_{\alpha\beta})$ is a solution to the perturbation equations, then $(K\xi^\alpha,Kh_{\alpha\beta})$ is also a solution for constant K. We will find it convenient to impose the following normalization condition in addition to the boundary and matching conditions just discussed:

$$U_m(r=R)=1 \quad \text{for axial hybrids and } r \text{ modes,}$$

$$U_{m+1}(r=R)=1 \quad \text{for polar hybrids.} \tag{4.71}$$

V. RELATIVISTIC CORRECTIONS TO THE *r* MODES OF UNIFORM-DENSITY STARS

In a future paper, we will consider the general problem of numerically solving for the r modes and hybrid modes of fully relativistic stars. Preliminary results have already been presented by Lockitch [33]. For now, we will focus on the post-Newtonian corrections to the Newtonian r modes. The equilibrium structure of a slowly rotating star with uniform density is particularly simple [47] and lends itself readily to such a post-Newtonian analysis. The results we obtain in this way provide important insights into the relativistic corrections to the familiar Newtonian r modes.

A. Post-Newtonian uniform-density model

For a spherically symmetric star with constant density,

$$\epsilon(r)=\frac{3M_0}{4\pi R^3}, \tag{5.1}$$

the equilibrium equations (3.3)–(3.7) have the well-known exact solution inside the star ($r\leqslant R$):

$$p(r)=\epsilon\left\{\frac{\left(1-\frac{2M_0}{R}\right)^{1/2}-\left[1-\frac{2M_0}{R}\left(\frac{r}{R}\right)^2\right]^{1/2}}{3\left[1-\frac{2M_0}{R}\left(\frac{r}{R}\right)^2\right]^{1/2}-\left(1-\frac{2M_0}{R}\right)^{1/2}}\right\}, \tag{5.2}$$

$$M(r)=M_0\left(\frac{r}{R}\right)^3, \tag{5.3}$$

$$e^{2\nu(r)}=\left\{\frac{3}{2}\left[1-\frac{2M_0}{R}\left(\frac{r}{R}\right)^2\right]^{1/2}-\frac{1}{2}\left(1-\frac{2M_0}{R}\right)^{1/2}\right\}^2, \tag{5.4}$$

$$e^{-2\lambda(r)}=1-\frac{2M_0}{R}\left(\frac{r}{R}\right)^2, \tag{5.5}$$

where M_0 is the gravitational mass of the star and R is its radius.

To find the equilibrium solution corresponding to a slowly rotating star, we must also solve Hartle's [46] equation (4.3):

$$0=r^2\overline{\omega}''+[4-r(\nu'+\lambda')]r\overline{\omega}'-4r(\nu'+\lambda')\overline{\omega} \tag{5.6}$$

(see also [47]), where we may use the spherical solution to write

$$r(\nu'+\lambda')=4\pi r^2(\epsilon+p)e^{2\lambda}=\frac{3\left(\frac{2M_0}{R}\right)\left(\frac{r}{R}\right)^2\left(1-\frac{2M_0}{R}\right)^{1/2}}{\left[1-\frac{2M_0}{R}\left(\frac{r}{R}\right)^2\right]\left\{3\left[1-\frac{2M_0}{R}\left(\frac{r}{R}\right)^2\right]^{1/2}-\left(1-\frac{2M_0}{R}\right)^{1/2}\right\}}. \tag{5.7}$$

To simplify the problem, we expand our equilibrium solution in powers of $(2M_0/R)$ and work only to linear order.[13] We will need the expressions

$$r(\nu'+\lambda')=\frac{3}{2}\left(\frac{r}{R}\right)^2\left(\frac{2M_0}{R}\right)+O\left(\frac{2M_0}{R}\right)^2 \tag{5.8}$$

and

$$e^{-2\nu}=1+\left[\frac{3}{2}-\frac{1}{2}\left(\frac{r}{R}\right)^2\right]\left(\frac{2M_0}{R}\right)+O\left(\frac{2M_0}{R}\right)^2. \tag{5.9}$$

Since we are also working to linear order in the star's angular velocity, we may set $\Omega=1$ without loss of generality. We write

$$\overline{\omega}=\sum_{i=0}^{\infty}\omega_i\left(\frac{r}{R}\right)^{2i} \tag{5.10}$$

and solve Eq. (5.6) subject to the boundary condition (4.6) at the surface of the star:

$$1=\Omega=\left[\overline{\omega}+\frac{1}{3}R\overline{\omega}'\right]_{r=R}. \tag{5.11}$$

To order $(2M_0/R)$ the solution is

$$\overline{\omega}(r)=1-\left(1-\frac{3r^2}{5R^2}\right)\left(\frac{2M_0}{R}\right)+O\left(\frac{2M_0}{R}\right)^2. \tag{5.12}$$

B. Nonbarotropic stars

In order to find the find the relativistic analogue to the familiar Newtonian r modes of nonbarotropic stars, we insert the above expressions in Eq. (4.22). We also assume that the mode frequency can be approximated as

$$\kappa=\frac{2m}{l(l+1)}\left[1+\kappa_1\left(\frac{2M_0}{R}\right)+O\left(\frac{2M_0}{R}\right)^2\right] \tag{5.13}$$

and that the eigenfunction takes the form

$$h_l\approx h_l^{(0)}(r)\left(\frac{2M_0}{R}\right)+O\left(\frac{2M_0}{R}\right)^2. \tag{5.14}$$

Solutions of this form would then lead to $U_l\sim O(1)$ via Eq. (4.21).

Under these assumptions, Eq. (4.22) is trivially satisfied to leading order. At order $(2M_0/R)^2$ we find an equation

$$\left[\kappa_1+1-\frac{3r^2}{5R^2}\right]\left\{h_l^{(0)''}-\frac{l(l+1)}{r^2}h_l^{(0)}\right\}+\frac{6}{R^2}h_l^{(0)}=0. \tag{5.15}$$

[13]This expansion will give us the first post-Newtonian (1PN) corrections to the Newtonian r modes.

Before proceeding, it is useful to compare our definition of the post-Newtonian eigenvalue κ_1 to the eigenvalue α we used in Sec. IV A. We then immediately see that

$$\alpha=1+\kappa_1\left(\frac{2M_0}{R}\right) \tag{5.16}$$

and deduce that the established range for possible eigenfrequencies translates into

$$-1<\kappa_1<0. \tag{5.17}$$

Within this range there are two possibilities. If $\kappa_1\leqslant-2/5$, we will have a singular eigenvalue problem, while for $-2/5<\kappa_1<0$, the problem is nonsingular. To determine the r modes to first order in $2M_0/R$ for the uniform-density model, one need only consider the simpler nonsingular situation, because the eigenvalues of the relativistic r modes turn out to be in the nonsingular range. The continuous part of the spectrum [30,31], as noted earlier, may be an artifact of an approximation in which the frequency is real.

We can rewrite Eq. (5.15) as

$$\frac{d^2h_l^{(0)}}{dr^2}-\left[\frac{l(l+1)}{r^2}+\frac{30}{3r^2-5R^2(\kappa_1+1)}\right]h_l^{(0)}=0. \tag{5.18}$$

As long as $-2/5<\kappa_1<0$, this equation can readily be integrated, and the solutions are well behaved at all values of r. We have integrated Eq. (5.18) using a fourth-order Runge-Kutta scheme, initiated from the appropriate regular power series solution close to the center of the star. That is, we use

$$h_l^{(0)}\approx Dr^{l+1}\left\{1-\frac{6r^2}{R^2(\kappa_1+1)[(l+2)(l+3)-l(l+1)]}\right\} \tag{5.19}$$

at an initial point close to $r=0$ and then integrate Eq. (5.18) to the surface $r=R$. At the surface we demand that $h_l^{(0)}$ and its derivative can be smoothly matched to the exterior solution according to Eq. (4.70). For each value of l, we then find a single acceptable solution, corresponding to a distinct eigenvalue κ_1. These eigenvalues, for $l=2-10$, are listed in Table I. It should be recalled that the tabulated eigenvalues correspond to mode frequencies (in the inertial frame) given by

$$\sigma\approx-m\Omega+\frac{2m\Omega}{l(l+1)}\left[1+\kappa_1\left(\frac{2M_0}{R}\right)\right]+O\left(\frac{2M_0}{R}\right)^2. \tag{5.20}$$

A typical eigenfunction, corresponding to $l=2$, is shown in Fig. 1.

We thus find a single post-Newtonian r-mode solution for each allowed combination of l and m. This is very much in accordance with the Newtonian r-mode results for nonbarotropic stars at order Ω (the degeneracy of these modes is not broken until at order Ω^2). The main difference in the relativistic case is that the post-Newtonian corrections (of order $2M_0/R$) break the degeneracy at order Ω and make it possible for us to determine the eigenfunctions.

TABLE I. Relativistic r-mode and hybrid-mode frequencies of uniform-density stars. We list the numerically determined values of the post-Newtonian r-mode frequency correction $\kappa_{1_{r\text{-mode}}}$ for the nonbarotropic star and compare the corresponding eigenvalues α_{pN} to ones obtained for fully relativistic uniform-density stars, α, for a star of compactness $2M_0/R=0.4$. In this case the value of the frame dragging at the surface of the star leads to $\overline{\omega}(R)/\Omega = 0.84424$ and we can see that the eigenvalues approach this value as l increases. It is also interesting to compare our post-Newtonian eigenvalues to the result we deduce for the hybrid $l=m$ modes of barotropic stars, $\kappa_{1_{\text{hybrid}}} = -4(m-1)(2m+11)/5(2m+1)(2m+5)$. The two results typically do not differ by more than a few percent. This is important since the two modes (for $l=m$) correspond to the relativistic analogue (for nonbarotropic and barotropic stars, respectively) of the same Newtonian r mode.

l	$\kappa_{1_{\text{hybrid}}}$	$\kappa_{1_{r\text{-mode}}}$	α_{pN}	α
2	−0.2667	−0.2629	0.8949	0.9086
3	−0.3532	−0.3428	0.8629	0.8699
4	−0.3897	−0.3734	0.8506	0.8561
5	−0.4073	−0.3868	0.8453	0.8502
6	−0.4163	−0.3931	0.8428	0.8474
7	−0.4211	−0.3962	0.8415	0.8460
8	−0.4235	−0.3979	0.8408	0.8453
9	−0.4247	−0.3988	0.8405	0.8448
10	−0.4251	−0.3993	0.8403	0.8446

Given these results, we expect similar r-mode solutions to exist also in the fully relativistic case. It is, in fact, easy to demonstrate this and we have extended our calculation for uniform density star to include all terms in Eq. (4.22). We then find that the mode eigenvalue is always such that $\alpha - \tilde{\omega} = \alpha - \overline{\omega}/\Omega \neq 0$ in the interior of the star (recall the discussion in Sec. IV A). The solutions to the problem are thus regular. The associated eigenvalues, for $l=2-10$, and a star with compactness $M/R=0.2$ are given and compared to the post-Newtonian results in Table I.

C. Barotropic stars

Having established that discrete r-mode solutions exist for nonbarotropic relativistic stars, we now turn to the barotropic case. As we have shown, we will then not have purely axial solutions (for $l \geqslant 2$). Instead, we need to calculate hybrid modes by solving Eqs. (3.20), (3.22), (3.23), and (4.53)–(4.55) subject to the boundary, matching, and normalization conditions (4.64), (4.69), (4.70), and (4.71). As in the nonbarotropic case, we seek the post-Newtonian corrections to the well-known Newtonian r modes. For barotropic stars such modes exist only for $l=m$ with frequency and radial dependence given by

$$\kappa = \frac{2}{(m+1)}, \qquad (5.21)$$

$$U_m = \left(\frac{r}{R}\right)^{m+1}. \qquad (5.22)$$

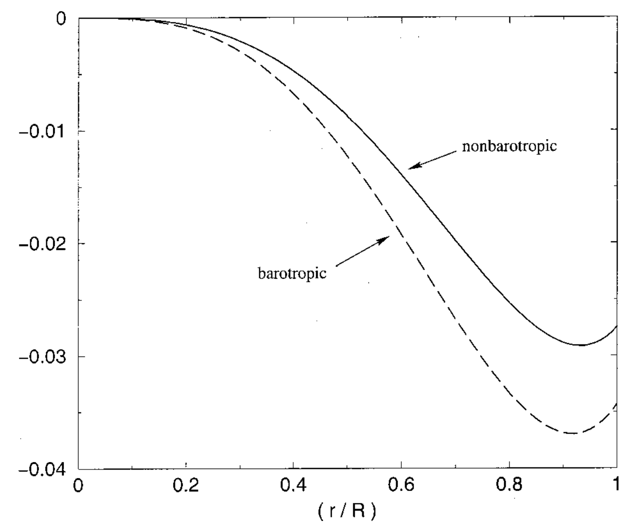

FIG. 1. Numerically determined post-Newtonian r-mode eigenfunction $h_l(r)$ for $l=2$ in the interior of a uniform-density star (solid line). The result is compared to the corresponding eigenfunction for the particular hybrid mode that is the relativistic counterpart of the Newtonian $l=m=2r$ mode in a barotropic star (shown as a dashed curve). Of course, in the barotropic case several other functions (such as W_{m+1} and V_{m+1}) are also nonzero (see Fig. 2). The functions are normalized in accordance with Eq. (4.71).

Therefore, let us make the following ansatz for our perturbed solution inside the star. We assume that the coefficients of the spherical harmonic expansions (4.9) and (4.12) of the Lagrangian displacement, ξ^α and the perturbed metric $h_{\alpha\beta}$, respectively, have the form

$$\kappa = \frac{2}{(m+1)}\left[1 + \kappa_1\left(\frac{2M_0}{R}\right) + O\left(\frac{2M_0}{R}\right)^2\right], \qquad (5.23)$$

$$U_m(r) = \left(\frac{r}{R}\right)^{m+1}\left[1 + u_{m,0}\left(1 - \frac{r^2}{R^2}\right)\left(\frac{2M_0}{R}\right) + O\left(\frac{2M_0}{R}\right)^2\right], \qquad (5.24)$$

$$h_m(r) = \left(\frac{r}{R}\right)^{m+1}\left[h_{m,0} + h_{m,1}\left(\frac{r}{R}\right)^2\right]\left(\frac{2M_0}{R}\right) + O\left(\frac{2M_0}{R}\right)^2, \qquad (5.25)$$

$$W_{m+1}(r) = w_{m,0}\left(\frac{r}{R}\right)^{m+1}\left(1 - \frac{r^2}{R^2}\right)\left(\frac{2M_0}{R}\right) + O\left(\frac{2M_0}{R}\right)^2, \qquad (5.26)$$

$$V_{m+1}(r) = \left(\frac{r}{R}\right)^{m+1}\left[v_{m,0} + v_{m,1}\left(\frac{r}{R}\right)^2\right]\left(\frac{2M_0}{R}\right) + O\left(\frac{2M_0}{R}\right)^2, \qquad (5.27)$$

$$U_{m+2}(r)=u_{m+2,0}\left(\frac{r}{R}\right)^{m+3}\left(\frac{2M_0}{R}\right)+O\left(\frac{2M_0}{R}\right)^2, \tag{5.28}$$

where κ_1, $u_{m,0}$, $h_{m,0}$, $h_{m,1}$, $w_{m+1,0}$, $v_{m+1,0}$, $v_{m+1,1}$, and $u_{m+2,0}$ are (as yet) unknown constants. We have chosen the form of $U_m(r)$ so as to automatically satisfy the normalization condition (4.71) and we have chosen the form of $W_{m+1}(r)$ so as to automatically satisfy the boundary condition (4.64). Note that we have assumed that h_l, $V_{l'}$, $W_{l'}$, and $U_{l''}$ are of order $(2M_0/R)^2$ or higher for all $l>m$, $l'>m+1$ and $l''>m+2$. We will justify this ansatz by showing self-consistently that such a solution satisfies the perturbation equations.

Observe that the exterior solution (4.67) for $h_m(r)$ already has a natural expansion in powers of $(2M_0/R)$ as a result of the recursion relation (4.68):

$$h_m(r)=\hat{h}_{m,0}\left(\frac{R}{r}\right)^{m}\left(\frac{2M_0}{R}\right)+O\left(\frac{2M_0}{R}\right)^2. \tag{5.29}$$

The normalization constant $\hat{h}_{m,0}$ is determined by the matching condition (4.69),

$$\hat{h}_{m,0}=h_{m,0}+h_{m,1}, \tag{5.30}$$

while Eq. (4.70) imposes the following condition on the interior solution:

$$0=\hat{h}_{m,0}\{-m(h_{m,0}+h_{m,1})-[(m+1)h_{m,0}+(m+3)h_{m,1}]\} \tag{5.31}$$

or

$$0=(2m+1)h_{m,0}+(2m+3)h_{m,1}. \tag{5.32}$$

We turn now to the barotropic perturbation equations (3.20), (3.22), (3.23), and (4.53)–(4.55). Recall that these latter three equations are not linearly independent, being related by Eq. (4.17). Also, because Eq. (3.20) merely expresses $H_{1,l}(r)$ in terms of $W_l(r)$, we may eliminate $H_{1,l}(r)$ from our system and ignore Eq. (3.20). Thus a complete set of perturbation equations is provided by Eqs. (3.22), (3.23), (4.53), and (4.54) for all allowed values of l.

We expand these equations to first post-Newtonian order using Eqs. (5.8), (5.9), and (5.12) to replace the equilibrium quantities and using our ansatz, Eqs. (5.23)–(5.28), to replace the various perturbation variables. The result is an algebraic system of seven independent equations, which together with our matching condition (5.32) allows us to uniquely find our eight unknown constants κ_1, $u_{m,0}$, $h_{m,0}$, $h_{m,1}$, $w_{m+1,0}$, $v_{m+1,0}$, $v_{m+1,1}$, and $u_{m+2,0}$. These equations are derived in detail by Lockitch [33]. Here we will simply present the resulting solution

$$\kappa=\frac{2}{(m+1)}\left[1-\frac{4(m-1)(2m+11)}{5(2m+1)(2m+5)}\times\left(\frac{2M_0}{R}\right)+O\left(\frac{2M_0}{R}\right)^2\right], \tag{5.33}$$

$$U_m(r)=\left(\frac{r}{R}\right)^{m+1}\left[1+u_{m,0}\left(1-\frac{r^2}{R^2}\right)\left(\frac{2M_0}{R}\right)+O\left(\frac{2M_0}{R}\right)^2\right], \tag{5.34}$$

$$h_m(r)=\left(\frac{r}{R}\right)^{m+1}\left[-\frac{3}{(2m+1)}+\frac{3}{(2m+3)}\times\left(\frac{r}{R}\right)^2\right]\left(\frac{2M_0}{R}\right)+O\left(\frac{2M_0}{R}\right)^2, \tag{5.35}$$

$$W_{m+1}(r)=(m+1)(m+2)K\left(\frac{r}{R}\right)^{m+1}\times\left(1-\frac{r^2}{R^2}\right)\left(\frac{2M_0}{R}\right)+O\left(\frac{2M_0}{R}\right)^2, \tag{5.36}$$

$$V_{m+1}(r)=K\left(\frac{r}{R}\right)^{m+1}\left[(m+2)-(m+4)\left(\frac{r}{R}\right)^2\right]\times\left(\frac{2M_0}{R}\right)+O\left(\frac{2M_0}{R}\right)^2, \tag{5.37}$$

$$U_{m+2}(r)=-KQ_{m+2}\frac{(m+1)^2(m+3)}{(2m+3)}\times\left(\frac{r}{R}\right)^{m+3}\left(\frac{2M_0}{R}\right)+O\left(\frac{2M_0}{R}\right)^2, \tag{5.38}$$

where we have defined the constant

$$K\equiv\frac{6(m-1)Q_{m+1}}{5(m+2)(2m+5)} \tag{5.39}$$

and where

$$u_{m,0}=-\frac{KQ_{m+1}}{24m(m+2)(2m+3)}\times\{48(m+1)^4(m+3)^2+(2m+3)^2(2m+5)\times[m(m+2)^2-48]\}. \tag{5.40}$$

Since our solution satisfies the full perturbation equations to order $(2M_0/R)$, our ansatz was self-consistent. Thus we have explicitly found the first post-Newtonian corrections to the $l=m$ Newtonian r modes of barotropic uniform-density stars.

The solution reveals the expected mixing of axial and polar terms in the spherical harmonic expansion of ξ^α. All of the barotropic Newtonian r modes with $m\geqslant 2$ pick up both axial and polar corrections[14] of order $(2M_0/R)$, becoming axial-led hybrid modes of the relativistic star. The $l=m=2$

[14] When $m=1$, the constant K vanishes and we recover the axial dipole solution mentioned in Sec. IV B.

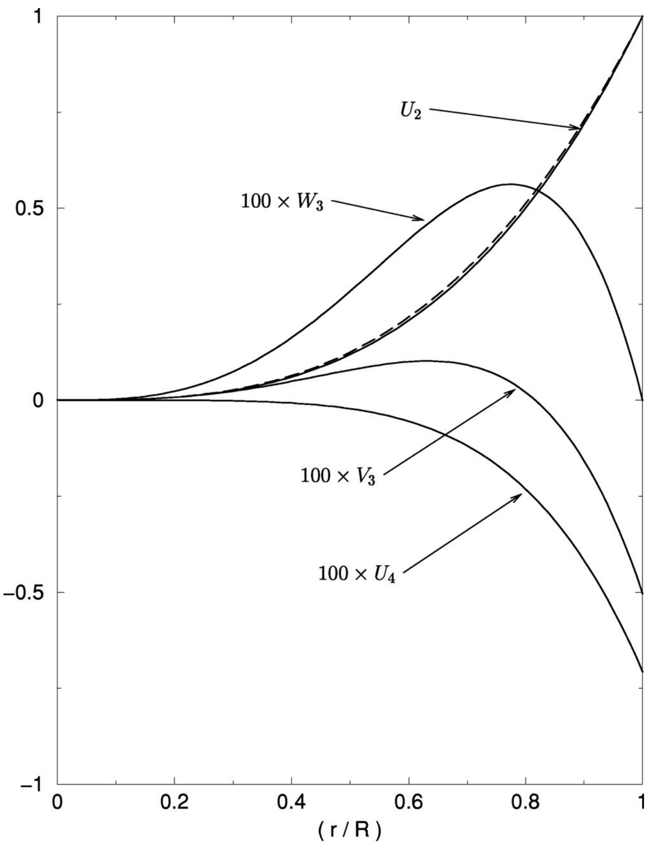

FIG. 2. The $(r/R)^3$ radial dependence of the Newtonian $l=m=2$ r mode is shown (dashed curve) together with the post-Newtonian corrections to this mode for a uniform-density star of compactness $2M/R=0.2$, i.e., the coefficients $U_l(r)$, $W_l(r)$, and $V_l(r)$ with $l\leqslant 4$ of the spherical harmonic expansion (4.9) (solid curves). The vertical scale is set by the normalization of $U_2(r)$ to unity at the surface of the star, and the other coefficients have been scaled by a factor of 100. Thus, while the relativistic corrections to the equilibrium structure of the star are of the order of 20%, the relativistic corrections to the r mode are only of the order of 1%.

hybrid mode is shown in Figs. 1 and 2, and compared to the corresponding r mode in a nonbarotropic star (see Sec. V B).

In addition, we see from Eq. (5.33) that the Newtonian r-mode frequency also picks up a small relativistic correction. The frequency decreases, just as it does in the nonbarotropic case (see Table I), and it is natural that general relativity will have such an effect. One reason is that gravitational redshift will tend to decrease the fluid oscillation frequencies measured by a distant inertial observer. Also, because these modes are rotationally restored, they will be affected by the dragging of inertial frames induced by the star's rotation. The Coriolis force is ''determined not by the angular velocity Ω of the fluid relative to a distant observer but by its angular velocity relative to the local inertial frame, $\overline{\omega}(r)$'' (Hartle and Thorne [54]). Thus, the Coriolis force decreases—and the modes oscillate less rapidly—as the dragging of inertial frames becomes more pronounced.

Finally, we note that the metric perturbation (whose radial behavior is determined by the function h_m) is of the same order as the post-Newtonian corrections to the fluid perturbation. Thus there is no justification for the Cowling approximation in constructing the hybrid-mode solutions. In Newtonian theory, the Cowling approximation corresponds to neglecting the variation in the gravitational potential. The original motivation for this [21] is that some pulsation modes (in particular the g modes) are mainly located in the less dense regions close to the surface of the star and do not involve large mass motion. Hence they will lead to variations in the gravitational potential that are small compared to the associated fluid velocities. The obvious generalization of this approach to general relativity would be to discard all metric perturbations [55]. However, as was pointed out by Finn [56], this approximation is not natural for relativistic g modes. The main reason is that, even though these modes involve small density perturbations, they could involve large fluid velocities. Hence Finn argues that one should keep those metric perturbations that can be associated with ''momentum transport'' in calculations of g modes. As is easy to see, similar arguments can be used for the modes we consider in the present paper. This would suggest that one should not discard the metric perturbations h_1, h_0, and H_1 in the relativistic Cowling approximation for r modes and hybrid modes. Interestingly, should we adopt this point of view, we retain the main perturbation equations we have used in the present paper. Hence this ''approximation'' would be consistent with our results. Furthermore, this would explain why the attempts to find relativistic r modes within the Cowling approximation (by neglecting all metric perturbations) have failed [32]. Of course, this discussion has little relevance for the present study. But it could be of crucial importance for attempts to find r modes in numerical simulations (by studying fluid motion in relativistic simulations with a ''frozen'' metric) that are currently under way [57,58].

VI. DISCUSSION

In this paper we have taken the first steps towards an understanding of both r modes and rotational hybrid modes of rotating relativistic stars. We have derived the perturbation equations that govern these modes to linear order in the rotation frequency Ω (at which the star is still spherical). For nonbarotropic stars we have focused on modes that have a purely axial limit as $\Omega\rightarrow 0$. These would be a natural relativistic generalization of the Newtonian r modes. For barotropic stars (and multipoles $l\geqslant 2$), we have shown that no such modes exist in the relativistic case, even though Newtonian stars retain a vestigial set corresponding to $l=m$. Instead, all modes of barotropic stars must have a hybrid nature. Having derived the relevant perturbation equations, we calculate relativistic corrections at the first post-Newtonian level (order $2M_0/R$) to the Newtonian r modes of both nonbarotropic and barotropic stars.

It is worth pointing out that, even though our results for barotropic and nonbarotropic stars are quite different, the particular modes that we have focused on (the analogues of the vestigial $l=m$ r modes that remain for barotropic Newtonian stars) are not too dissimilar. As is clear from the results given in Table I, the mode frequencies in the two cases we have considered do not differ by more than a few percent. Furthermore, we can see from Fig. 1 that the axial eigenfunctions $h_{0,l}$ are similar. There are, of course, still considerable differences between the two cases. In the nonbarotropic case we predict that purely axial modes exist for all combinations of l and $m\neq 0$, while in the barotropic case all modes are hybrids. Still, the fact that our results for the two cases seem consistent is encouraging. We anticipate that further work will eventually unveil a behavior quite similar to that of the Newtonian problem, for which the detailed barotropic limit has been investigated by Yoshida and Lee [59].

This paper represents progress in several important directions, but a considerable amount of work remains before we can claim to have a complete understanding of the nature of the rotational modes in relativity. For example, we have not yet discussed how the inferred changes in both mode frequency and eigenfunction will affect the strength of the gravitational-wave-driven instability. To do this we need to estimate the rate at which these modes radiate gravitational waves and also assess the strength of various dissipation mechanisms (like viscosity) that tend to damp an unstable mode. This is obviously an important issue, and we plan to address it once our ongoing work on fully relativistic hybrid modes of barotropic stars is completed. At that point it will also be appropriate to obtain and discuss results for different realistic equations of state.

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APPENDIX A: EQUATIONS DESCRIBING STATIONARY PERTURBATIONS OF SPHERICAL STARS

We have derived the various equations governing stationary perturbations of a spherical star using the Maple tensor package by substituting expressions (3.11)–(3.16) into Eqs. (3.8) and (3.10) [making liberal use of the equilibrium equations (3.4) through (3.7) to simplify the expressions]. The resulting equations are listed in the three distinct cases $l\geqslant 2$, $l=1$, and $l=0$ below.

1. Case $l\geqslant 2$

The nonvanishing components of the perturbed Einstein equation for $l\geqslant 2$ are as follows. We will use Eq. (A4) below to replace H_2 by H_0. From $\delta G^t_t=8\pi\delta T^t_t$ we have (using primes to denote derivatives with respect to r)

$$\begin{aligned}0=&e^{-2\lambda}r^2K''+e^{-2\lambda}(3-r\lambda')rK'\\&-[\tfrac{1}{2}\,l(l+1)-1]K-e^{-2\lambda}rH_0'\\&-[\tfrac{1}{2}\,l(l+1)+1-8\pi r^2\epsilon]H_0+8\pi r^2\delta\epsilon.\end{aligned}\tag{A1}$$

From $\delta G^r_r=8\pi\delta T^r_r$ we similarly have

$$\begin{aligned}0=&e^{-2\lambda}(1+r\nu')rK'-[\tfrac{1}{2}\,l(l+1)-1]K-e^{-2\lambda}rH_0'\\&+[\tfrac{1}{2}\,l(l+1)-1-8\pi r^2p]H_0-8\pi r^2\delta p.\end{aligned}\tag{A2}$$

From $\delta G^\theta_\theta+\delta G^\varphi_\varphi=8\pi(\delta T^\theta_\theta+\delta T^\varphi_\varphi)$ we have

$$\begin{aligned}0=&e^{-2\lambda}r^2K''+e^{-2\lambda}[r(\nu'-\lambda')+2]rK'\\&-16\pi r^2\delta p-e^{-2\lambda}r^2H_0''-e^{-2\lambda}(3r\nu'-r\lambda'+2)rH_0'\\&-16\pi r^2pH_0.\end{aligned}\tag{A3}$$

From $\delta G^\theta_\theta-\delta G^\varphi_\varphi=8\pi(\delta T^\theta_\theta-\delta T^\varphi_\varphi)$ we have

$$H_2=H_0.\tag{A4}$$

From $\delta G^\theta_r=8\pi\delta T^\theta_r$ we have

$$K'=e^{-2\nu}[e^{2\nu}H_0]'.\tag{A5}$$

From $\delta G^r_t=8\pi\delta T^r_t$ we have

$$0=H_1+\frac{16\pi(\epsilon+p)}{l(l+1)}e^{2\lambda}rW.\tag{A6}$$

From $\delta G^\theta_t=8\pi\delta T^\theta_t$ we have

$$0=e^{-(\nu-\lambda)}[e^{(\nu-\lambda)}H_1]'+16\pi(\epsilon+p)e^{2\lambda}V.\tag{A7}$$

From $\delta G^\varphi_t=8\pi\delta T^\varphi_t$ we have

$$\begin{aligned}&h_0''-(\nu'+\lambda')h_0'+\left[\frac{(2-l^2-l)}{r^2}e^{2\lambda}-\frac{2}{r}(\nu'+\lambda')-\frac{2}{r^2}\right]h_0\\&=\frac{4}{r}(\nu'+\lambda')U.\end{aligned}\tag{A8}$$

From $\delta G^\varphi_r=8\pi\delta T^\varphi_r$ we have

$$(l-1)(l+2)h_1=0.\tag{A9}$$

Finally, from $\delta G^\varphi_\theta=8\pi\delta T^\varphi_\theta$ we have

$$e^{-(\nu-\lambda)}[e^{(\nu-\lambda)}h_1]'=0.\tag{A10}$$

2. Case $l=1$

The $l=1$ case differs from $l\geqslant 2$ in two respects [43]. First, $H_2(r)\neq H_0(r)$, because the equation $\delta G^\theta_\theta-\delta G^\varphi_\varphi=8\pi(\delta T^\theta_\theta-\delta T^\varphi_\varphi)$ vanishes identically. Second, we may exploit the aforementioned gauge freedom for this case to eliminate the metric functions $K(r)$ and $h_1(r)$. [We note that Eq. (A9) implies $h_1(r)=0$ for $l\geqslant 2$ anyway.] With these two differences taken into account, the nonvanishing components of the perturbed Einstein equation for $l=1$ are as follows. From $\delta G^t_t=8\pi\delta T^t_t$ we have

$$0=e^{-2\lambda}rH_2'+(2-8\pi r^2\epsilon)H_2-8\pi r^2\delta\epsilon. \quad \text{(A11)}$$

From $\delta G^r_r=8\pi\delta T^r_r$ we have

$$0=e^{-2\lambda}rH_0'-H_0+(1+8\pi r^2 p)H_2+8\pi r^2\delta p. \quad \text{(A12)}$$

From $\delta G^\theta_\theta+\delta G^\varphi_\varphi=8\pi(\delta T^\theta_\theta+\delta T^\varphi_\varphi)$ we have

$$0=e^{-2\lambda}r^2H_0''+e^{-2\lambda}(2r\nu'-r\lambda'+1)rH_0'-H_0+e^{-2\lambda}(1+r\nu')rH_2'+(1+16\pi r^2p)H_2+16\pi r^2\delta p. \quad \text{(A13)}$$

From $\delta G^\theta_r=8\pi\delta T^\theta_r$ we

$$0=rH_0'+(r\nu'-1)H_0+(r\nu'+1)H_2. \quad \text{(A14)}$$

From $\delta G^r_t=8\pi\delta T^r_t$ we again have

$$0=H_1+8\pi(\epsilon+p)e^{2\lambda}rW. \quad \text{(A15)}$$

From $\delta G^\theta_t=8\pi\delta T^\theta_t$ we again have

$$0=e^{-(\nu-\lambda)}[e^{(\nu-\lambda)}H_1]'+16\pi(\epsilon+p)e^{2\lambda}V. \quad \text{(A16)}$$

Finally, from $\delta G^\varphi_t=8\pi\delta T^\varphi_t$ we have

$$h_0''-(\nu'+\lambda')h_0'-\left[\frac{2}{r}(\nu'+\lambda')+\frac{2}{r^2}\right]h_0=\frac{4}{r}(\nu'+\lambda')U. \quad \text{(A17)}$$

3. Case $l=0$

The $l=0$ case differs yet again from the previous two, being the case of stationary, spherically symmetric perturbations of a static, spherical equilibrium. To maximize the similarity to the preceding two cases, we will use the same form for the perturbed metric except that we may now exploit the gauge freedom for this case to eliminate the functions $K(r)$, $H_1(r)$, and $h_1(r)$. The nonvanishing components of the perturbed Einstein equation for $l=0$ are as follows.

From $\delta G^t_t=8\pi\delta T^t_t$ we have

$$0=e^{-2\lambda}rH_2'+(1-8\pi r^2\epsilon)H_2-8\pi r^2\delta\epsilon. \quad \text{(A18)}$$

From $\delta G^r_r=8\pi\delta T^r_r$ we have

$$0=e^{-2\lambda}rH_0'+(1+8\pi r^2p)H_2+8\pi r^2\delta p. \quad \text{(A19)}$$

From $\delta G^\theta_\theta+\delta G^\varphi_\varphi=8\pi(\delta T^\theta_\theta+\delta T^\varphi_\varphi)$ we have

$$0=e^{-2\lambda}r^2H_0''+e^{-2\lambda}(2r\nu'-r\lambda'+1)rH_0'+e^{-2\lambda}(1+r\nu')rH_2'+16\pi r^2pH_2+16\pi r^2\delta p. \quad \text{(A20)}$$

Finally, from $\delta G^r_t=8\pi\delta T^r_t$ we have

$$0=16\pi(\epsilon+p)W. \quad \text{(A21)}$$

APPENDIX B: PERTURBATION EQUATIONS FOR SLOWLY ROTATING NONBAROTROPIC STARS

The assumption of a purely axial perturbation in the limit $\Omega\to 0$ leads to the following equations [cf. Eq. (4.18)]: The three axial quantities follow from

$$\left[\sigma+m\Omega-\frac{2m\bar{\omega}}{l(l+1)}\right]\left\{e^{\nu-\lambda}\frac{d}{dr}\left[e^{-\nu-\lambda}\frac{dh_{0,l}}{dr}\right]-\left[\frac{l(l+1)}{r^2}-\frac{4M}{r^3}+8\pi(p+\epsilon)\right]h_{0,l}\right\}+16\pi(p+\epsilon)(\sigma+m\Omega)h_{0,l}=0, \quad \text{(B1)}$$

$$\left[\sigma+m\Omega-\frac{2m\bar{\omega}}{l(l+1)}\right]U_l+(\sigma+m\Omega)h_{0,l}=0 \quad \text{(B2)}$$

and

$$l(l+1)\left\{i(\sigma+m\omega)e^{-2\nu}\left[h_{0,l}'-2\frac{h_{0,l}}{r}\right]-\frac{(l-1)(l+2)h_{1,l}}{r^2}\right\}-2im\omega'e^{-2\nu}h_{0,l}=0 \quad \text{(B3)}$$

or, alternatively, Eq. (4.20).

The solutions to these three equations then serve as sources for three of the remaining (at order Ω) Einstein equations that determine the polar parity metric perturbations:

$$(l-1)l(l+1)(l+2)e^{2\nu}(H_{2,l}-H_{0,l})Y_l^m-\{r^2e^{-2\lambda}\omega'h_{0,l}'+[l(l+1)\omega-2re^{-2\lambda}\omega'-16\pi r^2(p+\epsilon)\bar{\omega}]h_{0,l}-16\pi i(p+\epsilon)r^2\bar{\omega}U_l\}\{2(l-1)(l+2)\times\sin\theta\,\partial_\theta Y_l^m+4l(l+1)\cos\theta\, Y_l^m\}=0, \quad \text{(B4)}$$

$$l(l+1)e^{2\nu}[r(K_l'-H_{0,l}')+(1-r\nu')H_{0,l}-(1+r\nu')H_{2,l}]Y_l^m-l(l+1)\{2r\omega h_{0,l}'+[r\omega'-2\omega(1+r\nu')]h_{0,l}\}\sin\theta\,\partial_\theta Y_l^m+2r\omega'h_{0,l}(\sin\theta\,\partial_\theta Y_l^m-l(l+1)\cos\theta\, Y_l^m)=0, \quad \text{(B5)}$$

and

$$l(l+1)e^{2\nu}\left[2\nu' e^{-2\lambda}\left(K_l' - \frac{H_{2,l}}{r^2}\right) - \frac{(l-1)(l+2)}{r^2}(k_l - H_{0,l}) + \left(\frac{6M}{r^3} - 8\pi\epsilon\right)H_{0,l}\right]Y_l^m - l(l+1)$$
$$\times\left[\omega' e^{-2\lambda}h_{0,l}' + \left(16\pi(p\bar{\omega} + \epsilon\Omega) - \omega\frac{(l-1)(l+2)r + 6M}{r^3}\right)h_{0,l}\right]\sin\theta\,\partial_\theta Y_l^m - \left(\frac{4}{r}\omega' e^{-2\lambda}h_{0,l} - 32\pi i(p+\epsilon)\bar{\omega}U_l\right)$$
$$\times(\sin\theta\,\partial_\theta Y_l^m - l(l+1)\cos\theta\, Y_l^m) + \frac{4l^2(l+1)^2}{r^2}\omega h_{0,l}\cos\theta\, Y_l^m = 0. \tag{B6}$$

These equations determine the polar metric perturbations K_l, $H_{0,l}$, and $H_{2,l}$ once $h_{0,l}$ and U_l are known for all l. Finally, the last two Einstein equations lead to the following equations for δp_l and $\delta\epsilon_l$ (recall that for nonbarotropic stars the equation of state determines the radial component W_l of δu^α, a quantity of order Ω^2):

$$\left[H_{0,l}'' - 2\nu' K_l' + \left(\frac{2}{r} + 2\nu' - \lambda'\right)H_{0,l}' + \nu' H_{2,l}' - \frac{l(l+1)}{r^2}e^{2\lambda}H_{0,l} + 8\pi(3p+\epsilon)e^{2\lambda}H_{2,l} + 8\pi e^{2\lambda}(3\delta p_l + \delta\epsilon_l)\right]e^{2\nu}Y_l^m$$
$$+\left[\left(\frac{4\omega}{r} + 2\omega' - 2\omega\nu'\right)h_{0,l}' + \left(4\omega(\nu')^2 - 2\omega'\nu' - \frac{8\omega}{r}\nu'\right)h_{0,l} + 32\pi i(p+\epsilon)\Omega e^{2\lambda}U_l\right]$$
$$\times\sin\theta\,\partial_\theta Y_l^m - \frac{4l(l+1)}{r^2}\omega e^{2\lambda}h_{0,l}\cos\theta\, Y_l^m = 0, \tag{B7}$$

$$\left[K_l'' + \left(\frac{2}{r} - 4\pi(p+\epsilon)re^{2\lambda}\right)K_l' + \frac{1}{r}(H_{0,l}' - H_{2,l}') - \frac{l(l+1)}{2r^2}e^{2\lambda}(H_{0,l} + H_{2,l}) + 8\pi(p+\epsilon)e^{2\lambda}H_{2,l} + 8\pi e^{2\lambda}(\delta p_l + \delta\epsilon_l)\right]Y_l^m$$
$$+\left[\frac{2}{r}\omega e^{-2\nu}h_{0,l}' + \left(\frac{2}{r}(\omega' - 2\omega\nu') + 16\pi e^{2\lambda}(p+\epsilon)\bar{\omega} - \frac{l(l+1)}{r^2}\omega e^{2\lambda}\right)e^{-2\nu}h_{0,l} + 16\pi i(p+\epsilon)\bar{\omega}e^{2\lambda-2\nu}U_l\right]$$
$$\times\sin\theta\,\partial_\theta Y_l^m - \frac{2l(l+1)}{r^2}\omega e^{2\lambda-2\nu}h_{0,l}\cos\theta\, Y_l^m = 0. \tag{B8}$$

APPENDIX C: PROOF OF THEOREM 1

1. Axial-led hybrids with $m>0$

Let l be the smallest value of l' for which $U_{l'} \neq 0$ in the spherical harmonic expansion (4.9) of the displacement vector ξ^α or for which $h_{l'} \equiv h_{0,l'} \neq 0$ in the spherical harmonic expansion (4.12) of the metric perturbation $h_{\alpha\beta}$. The axial parity of $(\xi^\alpha, h_{\alpha\beta})$, $(-1)^{l+1}$, and the vanishing of Y_l^m for $l<m$ implies $l \geq m$. That the mode is axial-led means $W_{l'} = 0$, $V_{l'} = 0$, and $H_{1,l'} = 0$ for $l' \leq l$. We show by contradiction that $l = m$.

Suppose $l \geq m+1$. From Eq. (4.53), $\int \Delta\omega_{\theta\varphi} Y_l^{*m} d\Omega = 0$, we have

$$0 = l(l+1)\kappa\Omega(h_l + U_l) - 2m\bar{\omega}U_l - lQ_{l+1}\left[\frac{e^{2\nu}}{r}\partial_r(r^2\bar{\omega}e^{-2\nu})W_{l+1} + 2(l+2)\bar{\omega}V_{l+1}\right], \tag{C1}$$

and from Eq. (4.55) with $l \to l-1$, $\int \Delta\omega_{\varphi r} Y_{l-1}^{*m} d\Omega = 0$, we have

$$0 = -\left\{(l+1)\kappa\Omega\,\partial_r[e^{-2\nu}(h_l + U_l)] + 2m\,\partial_r(\bar{\omega}e^{-2\nu}U_l) + \frac{m(l+1)}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})U_l\right\} + Q_{l+1}\left[\partial_r\left[\frac{1}{r}\partial_r(r^2\bar{\omega}e^{-2\nu})W_{l+1}\right] + 2(l+2)\partial_r(\bar{\omega}e^{-2\nu}V_{l+1})\right]. \tag{C2}$$

Together these give

$$0 = 2\partial_r(\bar{\omega}e^{-2\nu}U_l) + \frac{l}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})U_l \tag{C3}$$

$$= 2(r^2\bar{\omega}e^{-2\nu})^{-l/2}\partial_r[r^l(\bar{\omega}e^{-2\nu})^{(l+2)/2}U_l] \tag{C4}$$

or

$$U_l = K(\bar{\omega}e^{-2\nu})^{-(l+2)/2}r^{-l} \tag{C5}$$

(for some constant K), which is singular as $r \to 0$.

2. Axial-led hybrids with $m=0$

Let $m=0$ and let l be the smallest value of l' for which $U_{l'}\neq 0$ in the spherical harmonic expansion (4.9) of the displacement vector ξ^α or for which $h_{l'}\equiv h_{0,l'}\neq 0$ in the spherical harmonic expansion (4.12) of the metric perturbation $h_{\alpha\beta}$. Since $\nabla_a Y_0^0=0$, the mode vanishes unless $l\geqslant 1$. That the mode is axial-led means $W_{l'}=0$, $V_{l'}=0$, and $H_{1,l'}=0$ for $l'\leqslant l$. We show by contradiction that $l=1$.

Suppose $l\geqslant 2$. Then Eq. (4.54) with $l\to l-2$, $\int\Delta\omega_{r\theta}Y_{l-2}^{*0}d\Omega=0$, becomes

$$0=2\partial_r(\bar{\omega}e^{-2\nu}U_l)+\frac{l}{r^2}\partial_r(r^2\bar{\omega}e^{-2\nu})U_l \tag{C6}$$

$$=2(r^2\bar{\omega}e^{-2\nu})^{-l/2}\partial_r[r^l(\bar{\omega}e^{-2\nu})^{(l+2)/2}U_l] \tag{C7}$$

or

$$U_l=K(\bar{\omega}e^{-2\nu})^{-(l+2)/2}r^{-l} \tag{C8}$$

(for some constant K), which is singular as $r\to 0$.

3. Polar-led hybrids with $m\geqslant 0$

Let l be the smallest value of l' for which $W_{l'}\neq 0$ or $V_{l'}\neq 0$ in the spherical harmonic expansion (4.9) of the displacement vector ξ^α or for which $H_{1,l'}\neq 0$ in the spherical harmonic expansion (4.12) of the metric perturbation $h_{\alpha\beta}$. The polar parity of $(\xi^\alpha,h_{\alpha\beta}),(-1)^l$, and the vanishing of Y_l^m for $l<m$ implies $l\geqslant m$. That the mode is polar-led means $U_{l'}=0$ and $h_{l'}=0$ for $l'\leqslant l$. We show by contradiction that $l=m$ when $m>0$ and that $l=1$ when $m=0$.

Suppose $l\geqslant m+1$. From Eq. (4.53) with $l\to l-1$, $\int\Delta\omega_{\theta\varphi}Y_{l-1}^{*m}d\Omega=0$, we have

$$0=(l-1)Q_l\left[\frac{e^{2\nu}}{r}\partial_r(r^2\bar{\omega}e^{-2\nu})W_l+2(l+1)\bar{\omega}V_l\right]. \tag{C9}$$

Substituting for V_l using Eq. (3.23), we find

$$0=\frac{l}{r}\partial_r(r^2\bar{\omega}e^{-2\nu})W_l+2\bar{\omega}e^{-2\nu}\frac{e^{-(\nu+\lambda)}}{(\epsilon+p)}\times\partial_r[(\epsilon+p)e^{(\nu+\lambda)}rW_l], \tag{C10}$$

$$=2(r^2\bar{\omega}e^{-2\nu})^{-(l-2)/2}\frac{e^{-(\nu+\lambda)}}{r^2(\epsilon+p)}\partial_r[(r^2\bar{\omega}e^{-2\nu})^{l/2}\times(\epsilon+p)e^{(\nu+\lambda)}rW_l]. \tag{C11}$$

with the solution

$$W_l=K(\bar{\omega}e^{-2\nu})^{-l/2}\frac{e^{-(\nu+\lambda)}}{(\epsilon+p)}r^{-(l+1)} \tag{C12}$$

(for some constant K), which is singular as $r\to 0$.

When $m=0$ this argument fails to establish that l cannot be equal to 1, because Eq. (C9) is trivially satisfied for $l=1$ as a result of the overall $l-1$ factor. Instead, the argument proves that l cannot be greater than 1 in this case and therefore that $l=1$.

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