Unsharp degrees of freedom and the generating of symmetries

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While real degrees of freedom are usually described by operators which are self-adjoint, there are exceptions described by merely symmetric operators. It has been shown that such exceptional degrees of freedom generally display a form of ''unsharpness.'' Various studies in quantum gravity indicate that the widely expected unsharpness of space-time at very short distances can be described by such operators. It is also known, however, that unlike self-adjoint operators, merely symmetric operators do not generate unitary transformations, at least not straightforwardly. This raises the question of whether merely symmetric operators are able to play the important double role which self-adjoint operators often play, namely, both to represent a real degree of freedom and also to act as a symmetry generator. Here, we answer this question for a large class of symmetric non-self-adjoint operators X. We show that operators which coincide with such an X on the physical domain are even able to generate the entire unitary group of the Hilbert space.

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I. INTRODUCTION

As a rule, real degrees of freedom are described in quantum theory through operators which are self-adjoint. But there are exceptions to the rule. This is because the basic quantization requirement that a real degree of freedom of a classical theory become an operator Q whose expectation values $\langle Q\rangle$ are real does not strictly require that Q be self-adjoint. An operator whose expectation values are real is merely what is called a symmetric[1] operator.

A crucial property of merely symmetric operators is that they are not diagonalizable. As a consequence, a common feature of degrees of freedom which are described by merely symmetric operators is that they are ''unsharp'' or ''fuzzy,'' as opposed to self-adjoint operators, which always describe degrees of freedom which are absolutely ''sharp'': Namely, every real degree of freedom described by a self-adjoint operator Q is of course sharp in the sense that Q possesses a spectral resolution, or ''eigenbasis,'' and that for its eigenvectors $|q_n\rangle$ the uncertainty in Q vanishes,

$$\Delta Q(|q_n\rangle)=0, \tag{1}$$

with the usual definition, for normalized $|\psi\rangle$,

$$\Delta Q(|\psi\rangle):=\langle\psi|(Q-\langle\psi|Q|\psi\rangle)^2|\psi\rangle^{1/2}. \tag{2}$$

Of course, also when q is in the continuous spectrum, ΔQ can be made arbitrarily small.

On the other hand, when an operator Q is merely symmetric and therefore not diagonalizable, then this implies that there is an obstruction to precisely determining the physical quantity which this operator Q represents. As was first pointed out in [1] such degrees of freedom Q are therefore always ''unsharp'' in one of two ways. We will give a proper definition of the two types of unsharpness later. Roughly speaking, one type of unsharp degree of freedom has no (proper nor improper) eigenvectors, and the other type has ''eigenvectors'' which, however, are not orthogonal.

But do such unsharp real degrees of freedom actually occur? As will be explained in the next section, they do occur ubiquitously from simple quantum mechanical examples to even applied circumstances, such as in microscopy, in electronic communication, and, most interestingly, apparently also in theories of quantum gravity such as string theory. This need not be surprising, however, due to the generality of the argument: *Any* degree of freedom, in whichever theory, if described by a linear operator whose expectation values are real, can only be self-adjoint or merely symmetric, and it can therefore only be sharp or unsharp in the sense which we just discussed.

This applies, in particular, to any real space-time coordinate which is described by a linear operator X_i in any candidate quantum gravity theory. Every such coordinate X_i can only be either discrete or continuous, namely if X_i is self-adjoint with a discrete or a continuous spectrum respectively or the coordinate is unsharp in one of two ways, namely if X_i is merely symmetric.

In studies in quantum gravity such unsharp coordinates have indeed appeared, in particular in string theory and in noncommutative geometry; see e.g. [2–6]. The ultraviolet regularity of simple quantum field theories over specific choices of such unsharp coordinates was demonstrated in [7,8]. Recently, the ultraviolet properties of fields over one of the two types of unsharp coordinates was studied in full generality in [9]. It was found that fields over such coordinates are always continuous fields, but also that they are fields with ultraviolet regular properties much like fields over lattices. Namely, such fields are determined everywhere if known only on any one of a set of lattices with a certain (in general irregular) minimum spacing.

Interestingly, recent studies indicate that models of the structure of space-time at the Planck scale might conceivably be put to experimental tests in the foreseeable future:

For example, one of the successes of inflationary cosmology is that it predicts a spectrum of density perturbations

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[1]Recall that self-adjointness and symmetry coincide only in finite dimensional Hilbert spaces.

which matches the current experimental evidence. Crucially, however, inflationary models tend to inflate to the extent that initially sub-Planckian scales become cosmological scales and, as was recently demonstrated in [10,11], the inflationarily predicted perturbation spectrum is in general not robust under changes in the assumed structure of space-time at the Planck scale (apparently unlike Hawking radiation; see e.g. [24]). The experimental data on the perturbation spectrum, in particular from cosmic microwave background (CMB) observations, are set to further improve significantly.

In a different development it was also pointed out (see e.g. [12,13]) that the existence of large extra dimensions in which only gravity propagates could effectively bring the Planck scale down from 10^{-35} m to possibly just above the scale of 10^{-18} m which is currently accessible through accelerators (though this framework does not predict the actual value of the new Planck scale).

Here, our aim will not be to try to already match theory and experiment but rather to address a fundamental theoretical problem which necessarily arises with unsharp degrees of freedom that are described by merely symmetric operators.

Namely, *a priori*, it appears that merely symmetric operators fail to be able to play the important double role which self-adjoint operators often play, namely a role both as a real degree of freedom and a role as a generator of unitary transformations. For example, from a self-adjoint operator X we can obtain unitary operators such as $\exp[i\alpha_i(X)T^i]$ which might stand for local gauge transformations. It is known that merely symmetric operators cannot be exponentiated in this way.

Here, we will address this problem for one of the two types of unsharp degrees of freedom. We will find that any such symmetric operator X, defined on some physical Hilbert space domain D_{phys}, does generate unitaries, though in a subtle way: for each such X there exists a family of self-adjoint operators $\{X(u)\}$ which coincide with X on the physical domain D_{phys} and which generate unitaries in the usual way. We will find that those operators $X(u)$ are generating the entire unitary group of the Hilbert space.

Given the generality of the result that we aim at, we need to be as precise as possible about the assumptions and what follows from them. Therefore, after reviewing how unsharp degrees of freedom arise and after giving concrete and detailed examples of our claim, the main result will be formulated as a mathematical lemma, the proof of which, together with a corollary, we will give in the Appendices.

II. UNSHARP DEGREES OF FREEDOM

A. Examples

A simple example of an unsharp degree of freedom is given by the momentum operator $\mathbf{p}=-i\partial_x$ of a particle in a box, say in the one-dimensional interval $[-L,L]$. Because of the confining box potential, all physical wave functions $\psi(x)\in D_{phys}\subset H=L^2(-L,L)$ vanish at the boundary:

$$\psi(-L)=0=\psi(L). \tag{3}$$

The expectation values of $\mathbf{p}$ are real:

$$\langle\psi|\mathbf{p}|\psi\rangle\in\mathbb{R}, \quad \text{for all} \quad |\psi\rangle\in D_{phys}. \tag{4}$$

Thus, $\mathbf{p}$ is a symmetric operator. On the other hand, plane waves do not obey the boundary condition, Eq. (3). Thus, $\mathbf{p}$ does not possess (normalizable or nonnormalizable) eigenvectors—which implies that $\mathbf{p}$ is not self-adjoint but only symmetric. One can show that $\mathbf{p}$ is not even self-adjoint on any invariant subspace, which means that $\mathbf{p}$ is a so-called *simple symmetric* operator.

But even though the plane waves are not physical states, due to the boundary condition, one may wonder whether this really matters, because one can of course approximate plane waves by sequences of physical states which are plane waves within most of the interval $[-L,L]$ and which decay to zero towards the boundaries, such as to always obey the boundary condition, Eq. (3).

One may therefore be tempted to believe that $\mathbf{p}$ is still ''approximately'' self-adjoint and should therefore describe a sharp entity. This is, however, not the case. Namely, since Δx is bounded from above by the finite size of the box, we can expect from the uncertainty relation that the minimum uncertainty in momentum $\Delta\mathbf{p}_{min}$ is larger than zero. Indeed, $\mathbf{p}$ is unsharp in the sense that for all physical states $|\psi\rangle\in D_{phys}$ the momentum uncertainty is bounded from below by a fixed finite amount,

$$\Delta p(\psi)\geqslant\Delta p_{min}=\frac{\pi}{2L} \quad \text{for all normalized} \quad |\psi\rangle\in D_{phys}, \tag{5}$$

as is readily verified by a variational calculation; see e.g. [14]. Thus, perhaps surprisingly, while a sequence of physical wave functions can approximate a plane wave, the corresponding sequence of the states' uncertainty in momentum cannot converge to zero.

Intuitively, the reason is that the larger the part of the interval on which a physical wave function approximates a plane wave, the steeper the wave function must decay to zero towards the interval boundaries. But a steep decay necessarily yields a significant contribution to the derivative operator $\mathbf{p}$ and the calculation of $(\Delta\mathbf{p})^2=\langle\mathbf{p}^2\rangle-\langle\mathbf{p}\rangle^2$.

Technically, the phenomenon is related to the fact that $\mathbf{p}$ is an unbounded operator, as discussed e.g. in [15,16].

This type of unsharp operator actually occurs frequently. For example, as a little thought shows, the angular resolution of optical images is the resolution of the photon momentum in the direction parallel to the opening of the optical instrument. Thus, the aperture induced unsharpness of images, for example from a telescope, is technically the unsharpness of the momentum of a particle in a box.

Also, for example the uncertainty in the time resolution of band-limited electronic signals reduces to this case [9].

Most interestingly, there are also indications that unsharp degrees of freedom of this type occur with the description of space-time at the Planck scale, where, as is well known, various theoretical arguments have long indicated that space-time displays a fundamental ''foaminess'' (see [17]) or unsharpness.

In particular, several studies, see e.g. [2–5,18–23] suggest that the structure of space-time at the Planck scale, or the string scale, is characterized effectively by correction terms to the uncertainty relations and, in particular, by corrections of the type

$$\Delta X \Delta P \geq \frac{\hbar}{2}[1+k(\Delta P)^2+\cdots]. \tag{6}$$

As is easily verified, for any $k>0$, Eq. (6) implies the existence of a finite lower bound for ΔX, namely

$$\Delta X_{min}=\hbar\sqrt{k}. \tag{7}$$

Technically, it is clear that any operator X which obeys this type of uncertainty relation, in whichever theory, cannot possess eigenvectors since, by Eq. (1), the uncertainty ΔX would vanish for eigenvectors. Thus, any such X is merely symmetric.

Here, k is assumed to be a small positive constant which is related to the Planck scale or, in string theory, to the string scale. It has been shown that this type of cutoff could solve the trans-Planckian energy paradox of black hole radiation; see [24]. For general reviews of quantum gravity and string theory motivations of Eq. (6) see e.g. [4,5,25]. For a path integral approach to this type of modified uncertainty relations see [26].

A practical general characterization of this type of unsharp degree of freedom was first given in [1]: Namely, this type of unsharp degrees of freedom possesses a finite lower bound ΔQ_{min} on ΔQ which can be $\langle Q\rangle$ dependent; i.e., there is a function $\Delta Q_{min}(q)$ such that

$$\Delta Q_{|\psi\rangle} \geq \Delta Q_{min}(\langle\psi|Q|\psi\rangle) \tag{8}$$

for all $|\psi\rangle\in D_{phys}$. Intuitively, if such a Q were a position operator in nonrelativistic quantum mechanics, the interpretation would be that Q describes a space on which the position of a particle can be resolved only to a limited precision and that this limit can in general depend on the position where one tries to localize the particle along the Q coordinate.

Finally, let us also recall the precise definition of this type of unsharp degrees of freedom, as given in [1]. Namely, such Q are simple symmetric operators whose so-called deficiency indices $d_\pm := \dim((Q\pm i1)D_Q)^\perp$ (see e.g. [29]) are equal, $d_+=d_-$. For lack of a better name, the type of unsharpness described by these operators was called fuzzy A. The only other type of unsharp real degree of freedom—represented by simple symmetric operators whose deficiency indices $d_\pm$ are unequal—might be called fuzzy B. Concrete examples of fuzzy B type degrees of freedom arise for example in optics from diffraction at an edge.

B. Unsharpness from kinematics and dynamics

Let us briefly consider the general circumstances in which unsharp degrees of freedom can occur.

To this end, let us assume that S and T are two self-adjoint operators which do not commute, such as **x** and **p**, or such as the noncommuting coordinates which have recently been much discussed; see e.g. [27,28]. The corresponding uncertainty relation

$$\Delta S \Delta T \geq \frac{1}{2}|\langle[S,T]\rangle| \tag{9}$$

implies of course that, if say $[S,T]=i1$, and if we know for example that ΔT is smaller than some value, say $\Delta T \leq t_{max}$, then ΔS cannot be made arbitrarily small but possesses instead a finite lower bound: $\Delta S \geq 1/2t_{max}$. For example, the recently discussed UV/IR connection in noncommutative geometries is of this form. But how can this fuzzyness of S be compatible with the fact that self-adjoint operators can *always* be resolved to arbitrary precision?

The reason is that to require $\Delta T \leq t_0$ is to restrict the Hilbert space to only those vectors for which $\Delta T \leq t_0$ holds true. On this sub-domain, the restriction of the operator S is not self-adjoint—it is merely symmetric.

The example illustrates that the noncommutativity of operators induces an interplay between their domains which then affects whether these operators are self-adjoint or merely symmetric. While this seems obvious, let us however also point out that not only the kinematical commutation relations between operators, but in the same way also the dynamical operator equations of motion affect the domains of the operators involved. Thus, it appears that there is *a priori* no reason to exclude the possibility that, for example in a fundamental theory of quantum gravity, the sharpness or unsharpness of real degrees of freedom, such as those of space-time, can also vary *dynamically* by this mechanism.

III. EXAMPLES OF THE GENERATING OF UNITARIES

Our aim now is to clarify whether or not symmetric operators, while possessing the interesting ability to describe unsharp real degrees of freedom, are also able to play a role in the generating of symmetries.

We will address this problem for the class of simple symmetric operators which describe fuzzy-A type short-distance structures, i.e. which have equal deficiency indices, and we will find the following:

For each simple symmetric operator X with equal deficiency indices, acting on a physical domain D_{phys} which is dense in a Hilbert space H, there exists a family of self-adjoint operators $X(\alpha)$ which coincide with X on the physical domain. We will prove that these operators $X(\alpha)$, together, generate even the full unitary group of the Hilbert space. Let us first discuss this phenomenon for the simple example of the momentum operator of the particle in a box versus the case of the particle on the full real line.

For a particle residing on the entire real line the operators **x** and **p** are of course self-adjoint and can be exponentiated to yield unitaries: The operators **x** and **p** are represented irreducibly as the self-adjoint multiplication and differentiation operators $\mathbf{x}\cdot\psi(x)=x\psi(x)$ and $\mathbf{p}\cdot\psi(x)=-i\partial_x\psi(x)$ acting on a dense domain in the Hilbert space H of square integrable wave functions $\psi(x)$ over the real line and, as is well known, **x** and **p**, *together*, generate all unitary operators

U on the Hilbert space H, via the Weyl formula

$$U=\int\int\frac{dsdt}{2\pi\hbar}u(s,t)\exp[i(s\mathbf{x}+t\mathbf{p})/\hbar]. \quad (10)$$

The $u(s,t)$ are suitable complex-valued functions. In fact, all bounded operators $B\in B(H)$ can be generated in this way.

On the other hand, we can also represent **x** and **p** reducibly, for example, as the self-adjoint multiplication and differentiation operators

$$\mathbf{x}\cdot\psi_i(x)=x\psi_i(x) \quad \text{and} \quad \mathbf{p}\cdot\psi_i(x)=-i\partial_x\psi_i(x) \quad (11)$$

acting on a Hilbert space of wave functions $\psi_i(x)$ on the real line which then possess an additional ''isospinor'' index, running $i=1,\ldots,n$.

The scalar product of wave functions then contains an iso-sum:

$$\langle\psi|\phi\rangle=\sum_{i=1}^{n}\int_{-\infty}^{\infty}dx\,\psi_i^*(x)\phi_i(x). \quad (12)$$

Clearly, **x** and **p** are acting diagonally in the isospinor space. Therefore, **x** and **p** do not generate the $U(n)$ of the isorotations. Thus, in this case, the Weyl formula, Eq. (10), does not yield all bounded operators; nor does it yield all unitaries on the Hilbert space. Only if we supplemented the operators **x** and **p** by additional Hermitian $n\times n$ matrices, T_i, could we generate $U(n)$ on the isospinor space and therefore all of $B(H)$.

Let us now consider the case where the particle is confined to the interval $[-L,L]$. As we saw above, the momentum operator $\mathbf{p}=-i\partial_x$ is then no longer self-adjoint and it is instead simple symmetric of type fuzzy A. What difference does this make?

Our proposition in this case is that there exists a one-parameter family of self-adjoint operators $\mathbf{p}(\alpha)$, say $(0\leqslant\alpha<2\pi)$, such that

(i) each $\mathbf{p}(\alpha)$ coincides with **p** on the physical domain, i.e.

$$\mathbf{p}(\alpha)|\psi\rangle=\mathbf{p}|\psi\rangle \quad \text{for all} \quad |\psi\rangle\in D_{phys},$$

(ii) the $\mathbf{p}(\alpha)$, together, (weakly) generate the algebra $B(H)$ of bounded operators on the Hilbert space, which includes of course the full unitary group on H.

Indeed, we claim that, unlike in the Weyl formula, the operator **x** is now no longer needed to generate $B(H)$, because the operators $\mathbf{p}(\alpha)$ alone already generate $B(H)$—even though each $\mathbf{p}(\alpha)$ coincides with **p** on the physical domain $D_{\mathbf{p}}$, which is dense in the Hilbert space. We will also consider the case where the wave functions of the particle in the box carry an isospinor index: then, **p** is again simple symmetric of type fuzzy A and our lemma applies. We then claim that there exists a multi-parameter set of self-adjoint operators $\mathbf{p}(u)$, which again all coincide with **p** on physical states and which generate all of $B(H)$. This means that there is no need to introduce isospin rotation generators T_j by hand, since the $\mathbf{p}(u)$ are able to generate all: translations, phase rotations and isorotations.

A. Scalar case

We show this first for a scalar particle in a box, i.e. without an isospinor index.

As discussed above, the momentum operator **p**, acting as $\mathbf{p}\cdot\psi(x)=-i\partial_x\psi(x)$ on the physical wave functions $\psi\in D_{phys}$ over the interval, is a simple symmetric operator. Although all physical wave functions vanish at the boundary, they are a dense set in the Hilbert space of square integrables $\overline{D_{phys}}=H=L^2(-L,L)$.

Let us now construct a family of operators $\mathbf{p}(\alpha)$ which coincide with **p** on the physical domain D_{phys}, but whose domain is larger and who are self-adjoint on this larger domain. This is simple functional analysis: The operators $\mathbf{p}(\alpha)$ are obtained by extending the domain D_{phys} such as to include wave functions which are periodic up to a phase

$$\psi(-L)=e^{i\alpha}\psi(L). \quad (13)$$

For each arbitrary fixed phase $e^{i\alpha}$ the boundary terms of the partial integrations in the equation $\langle\psi_1|(\mathbf{p}(\alpha)|\psi_2\rangle)=(\langle\psi_1|\mathbf{p}(\alpha))|\psi_2\rangle$ vanish. This makes $\mathbf{p}(\alpha)$ self-adjoint with an eigenbasis of plane waves $\psi_n^{(\alpha)}(x)$ which obey the corresponding boundary condition:

$$\psi_n^{(\alpha)}(x)=e^{i\omega_n x} \quad \text{where} \quad \omega_n=\frac{2\pi n-\alpha}{2L}, \quad n\in\mathbb{Z}. \quad (14)$$

What are the implications for the generating of unitaries? Were the wave functions not restricted to the interval, **p** would be self-adjoint and **p** could be exponentiated to obtain a unitary operator, say $U(a):=\exp(ia\mathbf{p})$, for some $a\geqslant 0$, whose action is of course to translate wave functions by the amount a to the right:

$$U(a)\cdot\psi(x)=e^{a\partial_x}\psi(x)=\psi(x+a). \quad (15)$$

In the case where the particle is confined to the box, however, i.e. where the Hilbert space only consists of wave functions on the interval, the operator **p** is not self-adjoint and cannot be exponentiated: The formal expression $U(a)=\exp(ia\mathbf{p})$ is now not a unitary transformation, because it would translate beyond the interval boundaries, which is not defined in the Hilbert space.

Nevertheless, for the particle in a box, there exists, as we saw, a whole family of self-adjoint extensions $\mathbf{p}(\alpha)$ of **p**. Since each $\mathbf{p}(\alpha)$ is self-adjoint, each can be exponentiated and the resulting operator

$$U_\alpha(a):=\exp[ia\mathbf{p}(\alpha)] \quad (16)$$

is unitary. Are these unitaries translating wave functions beyond the interval boundaries? The action of $U_\alpha(a)$ on wave functions is indeed again to translate wave functions to the right (say for $a>0$), as in Eq. (15). Now, however, as a result of the boundary condition, Eq. (13), the part of the wave function which would be translated beyond the right interval boundary reappears into the interval from the left, with the same modulus, but phase shifted by the phase $e^{i\alpha}$.

Crucially now, the product

$$U_{\alpha'}(-a)U_{\alpha}(a) \tag{17}$$

is a unitary operator which does not translate wave functions. This is because the first factor translates by a and the second factor translates back by the same amount. Nevertheless, since the two factors translate with different phase shifts, the product is not the identity operator. Namely, $U_{\alpha'}(-a)U_{\alpha}(a)$ is the unitary operator whose action is to leave the modulus of wave functions unchanged, but to phase shift the wave functions on a part of the interval.

E.g., choosing some $a\in[0,2L]$, the action is

$$U_{\alpha'}(-a)U_{\alpha}(a)\cdot\psi(x) = \begin{cases} \psi(x) & \text{for } x\in[-L,L-a], \\ e^{i(\alpha-\alpha')}\psi(x) & \text{for } x\in[L-a,L]. \end{cases} \tag{18}$$

By suitable composition of operators $U_{\alpha}(a)$ for various a and α it is therefore possible to generate unitaries which yield *arbitrary* local phase rotations of wave functions.

For example, we can in this way compose an operator which phase rotates wave functions by $e^{i\alpha}$ in an interval $[L-a,L-a+b]$ (where $0<b<a<2L$), and leaves the wave functions invariant outside that interval:

$$U_{\alpha}(-(a-b))U_0(-b)U_{\alpha}(a)\cdot\psi(x) = \begin{cases} \psi(x) & \text{for } x\in[-L,L-a]\cup[L-a+b,L], \\ e^{i(\alpha)}\psi(x) & \text{for } x\in[L-a,L-a+b]. \end{cases} \tag{19}$$

Thus, remarkably, the set of self-adjoints which coincide with $\mathbf{p}$ on the physical domain is able to generate all translations *and* also all local phase rotations, while we recall that in the case where $\mathbf{p}$ is self-adjoint, the operator $\mathbf{x}$ is needed order to generate local phase rotations, namely through $e^{if(\mathbf{x})}\psi(x)=e^{if(x)}\psi(x)$.

B. Case with isospin

We consider again a particle constrained to the interval $[-L,L]$. The particle's wave function $\psi_i(x)$ shall now carry an isospinor index $i=1,\ldots,n$. The scalar product in the Hilbert space of square integrables on the interval then includes an iso-sum:

$$\langle\psi|\phi\rangle=\sum_{i=1}^{n}\int_{-L}^{L}dx\,\psi_i^*(x)\phi_i(x). \tag{20}$$

Because of the box potential, the physical wave functions, $|\psi\rangle\in D_{phys}$, again obey the boundary condition

$$\psi_i(-L)=0=\psi_i(L)\quad(i=1,\ldots,n). \tag{21}$$

The action of $\mathbf{p}$ is diagonal in iso-space: $\mathbf{p}\cdot\psi_i(x)=-i\partial_x\psi_i(x)$. Again, there are no plane waves in the physical domain and therefore the momentum operator on the physical domain is not self-adjoint. Instead, $\mathbf{p}$ is simple symmetric [with deficiency indices (n,n)]. Self-adjoint extensions $\mathbf{p}(u)$ are now obtained by enlarging the domain of $\mathbf{p}$ to include wave functions which obey the boundary condition

$$\psi_i(-L)=\sum_{i=1}^{n}u_{ij}\psi_j(L) \tag{22}$$

where u_{ij} is any unitary $n\times n$ matrix, generalizing the phase $e^{i\alpha}$ of the scalar case above. As is readily checked, the proof of self-adjointness of the $\mathbf{p}(u)$ requires again the cancellation of the boundary terms which arise through the partial integrations needed to show that $(\langle\psi|\mathbf{p}(u))|\phi\rangle=\langle\psi|(\mathbf{p}(u)|\phi\rangle)$, and this cancellation is achieved exactly by the boundary conditions of the form of Eq. (22).

As in the scalar case, while $\mathbf{p}$ does not directly yield unitaries, each of the self-adjoint $\mathbf{p}(u)$ which reduce to $\mathbf{p}$ on the physical domain does generate unitaries, e.g. by exponentiation (a real):

$$U_u(a)\coloneqq e^{iap(u)}. \tag{23}$$

The unitaries $U_u(a)$ again act on wave functions by translating them by the amount a, and, because of the self-adjoint extension's boundary conditions, any part of the wave function which hits a boundary reappears from the other side into the interval, now iso-rotated by the matrix u (or by u^{-1} if a is negative).

It is possible to proceed as in the scalar case, composing such unitaries to translate the wave functions back and forth, using different self-adjoint extensions. It is clear that in this way arbitrary local isorotations can be generated.

Thus, the set of self-adjoint operators which reduce to $\mathbf{p}$ on the physical domain indeed generates not only translations but also arbitrary local phase rotations and—if an isospinor index is present—then they even generate all local isorotations.

Let us remark that $\mathbf{p}$ has this property of course no matter how large L, is i.e. no matter how small $\Delta\mathbf{p}_{min}$ is, as long as they are finite. Changing context, this means that for example position coordinates X in a theory of quantum gravity with some unmeasurably small but finite fuzziness ΔX_{min} would be able to in this sense provide also the generators T_i of isorotations in the Hilbert space on which they act. This perhaps indicates the possibility of a mechanism by which internal degrees of freedom could arise from Planck scale structure.

IV. GENERAL CASE

Let us begin by clarifying the definitions: A symmetric operator X is called *simple symmetric* if X is not self-adjoint and if it possesses no invariant subspace such that the restriction of X to this subspace yields a self-adjoint operator. Our examples above are simple symmetric. Further, we recall that the *Cayley transformed* operator S of a symmetric operator X, defined as

$$S\coloneqq(X-i1)(X+i1)^{-1}, \tag{24}$$

is isometric. An isometric operator is called *simple isometric* if it cannot be reduced to an invariant subspace such that the reduced operator is unitary. It is known that a subspace reduces a symmetric operator X if and only if it reduces its Cayley transform; see, for example, [29]. Note, however, that not every isometric operator is the Cayley transform of a symmetric operator.

Now a precise statement of our claim is the following:

Lemma. Let X be a closed simple symmetric operator with equal deficiency indices, defined on a domain $\mathbf{D}_X$ which is dense in a complex Hilbert space $\mathbf{H}$. Then, the self-adjoint extensions $X(\alpha)$ of X generate a $*$-algebra $\mathcal{A}$ which is weakly dense in $\mathcal{B}(\mathbf{H})$. Thus, in particular, the self-adjoint extensions generate the full unitary group $\mathcal{U}(\mathbf{H})$ of the Hilbert space.

The full proof is given in Appendix A. To summarize, the proof begins by using the $X(\alpha)$ to generate a suitable set $\mathcal{M}$ of unitaries, which in turn generate an algebra $\mathcal{A}$. The main part of the proof is then to show that the commutant $\mathcal{A}'$ of the algebra $\mathcal{A}$ is $\mathcal{A}'=\mathbf{C}1$. This implies that its double commutant is $\mathcal{A}''=\mathcal{B}(\mathbf{H})$. The lemma then follows since, with von Neumann (see, e.g., [37]), the double commutant of any $*$-algebra is its weak closure.

V. CONCLUSIONS AND OUTLOOK

We discussed that real degrees of freedom which are described by linear operators can only be self-adjoint, in which case they are ''sharp'' entities—or they are merely symmetric, in which case they display one of two types of ''unsharpness.''

We then considered the problem that self-adjoint operators physically often play an important double role, namely both representing a real degree of freedom and also acting as a generator of unitary transformations—while merely symmetric operators are known not to generate unitaries in a straightforward way.

This led us to investigate whether and to what extent the operators which arise with the description of ''fuzzy'' degrees of freedom are also able to generate unitary transformations.

Studying the class of unsharp degrees of freedom of the type fuzzy A (i.e. those described by operators with equal deficiency indices) we found that these possess a remarkable property:

For any such symmetric operator X defined on some physical Hilbert space domain D_{phys} there is always a set of self-adjoint operators which coincide with X on D_{phys}, and which together generate all unitaries in the Hilbert space. We conclude that, in this way, at least the operators of type fuzzy A can play a role in all aspects of symmetries in the Hilbert space in which they act.

This result applies quite generally because we did not make any further technical or physical assumptions about these unsharp degrees of freedom or about the physical theory in which they occur.

We are covering, for example, the case of the matrix model for M theory (see e.g. [30]), which employs symmetric operators, X_i, to encode space-time information. In this case, the matrix elements of the X_i are interpreted in terms of coordinates of D0-branes. Initially, the X_i are finite dimensional, say $N\times N$ matrices. The quantization and the necessary limit $N\rightarrow\infty$ are highly nontrivial, but it is clear that the resulting operators will still be at least symmetric. The short-distance structure which they describe will therefore fall into one of the types which we discussed. If these operators are found to be of type fuzzy A, then our present results show how they relate to the unitary group of the Hilbert space on which they act.

Also studies in the context of quantum groups (see e.g. [31,32]) and in the wider field of noncommutative geometry have yielded new approaches to building models for space-time at the Planck scale (see e.g. [33–35]), some of which are related to string theory (see e.g. [27,28,36]). As far as these models of space-time use linear operators to describe real entities such as coordinates, which they of course usually do, we are covering these operators. It should be interesting to investigate our present results in those contexts.

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APPENDIX A: PROOF OF THE LEMMA

We begin by choosing a suitable set of unitaries which are generated by the self-adjoint extensions $X(\alpha)$ of X. To this end, consider the isometric Cayley transform S of X,

$$S:=(X-i1)(X+i1)^{-1}, \tag{A1}$$

with domain

$$\mathbf{D}_S=(X+i1)\mathbf{D}_X. \tag{A2}$$

We define the *local group* $\mathcal{T}$ as the set of all unitaries which map the deficiency space $\mathbf{D}_S^{\perp}=((X+i1)\mathbf{D}_X)^{\perp}$ onto itself and which act as the identity on $\mathbf{D}_S$, i.e.

$$\mathcal{T}:=\{T|T:\mathbf{D}_S\rightarrow\mathbf{D}_S, T:\mathbf{D}_S^{\perp}\rightarrow\mathbf{D}_S^{\perp}, T_{|\mathbf{D}_S}=1, TT^{\dagger}=T^{\dagger}T=1\}. \tag{A3}$$

It is clear that the local group, $\mathcal{T}$, is isomorphic to the unitary group $U(n)$, where n is the deficiency index $n:=\dim(\mathbf{D}_S^{\perp})$.

Since, by assumption, both deficiency indices are equal, i.e. both spaces

$$L_{\pm}=((X\pm i1)\mathbf{D}_X)^{\perp} \tag{A4}$$

are of equal dimension, there exist unitary extensions of S.

Let U be one of the unitary extensions of S:

$$U^{\dagger}U=UU^{\dagger}=1, \quad U:L_{+}\rightarrow L_{-}, \quad U_{|\mathbf{D}_S}=S. \tag{A5}$$

We consider now the coset

$$\mathcal{M}:=\{M|M=UT, T\in\mathcal{T}\} \tag{A6}$$

of unitary extensions of S.

Indeed, as is well known, each unitary extension of the Cayley transform S of a symmetric X, i.e. here each element of $\mathcal{M}$ is indeed generated, via the Cayley transform, by a self-adjoint extension $X(\alpha)$ of X.

We will now show that the *-algebra $\mathcal{A}$ generated by $\mathcal{M}$ is weakly dense in $\mathcal{B}(\mathbf{H})$. As mentioned, this follows from von Neumann's double commutant theorem if we can prove that only multiples of the identity operator commute with $\mathcal{M}$, i.e. with U and all elements of $\mathcal{T}$.

To this end, let us consider an operator V which obeys

$$\|V\|<\infty \quad \text{and} \quad [V,U]=0=[V,T], \quad \forall \ T\in\mathcal{T}. \tag{A7}$$

We need to show that V is a multiple of the identity operator.

Since the closure of X implies the closure of the deficiency space $\mathbf{D}_S^\perp$ and of $\mathbf{D}_S$, we can use $\mathbf{H}=\mathbf{D}_S\oplus\mathbf{D}_S^\perp$ to write V and the elements $T\in\mathcal{T}$ in block form:

$$T=\begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix}, \quad V=\begin{pmatrix}V_{\mathbf{D}_S\mathbf{D}_S} & V_{\mathbf{D}_S\mathbf{D}_S^\perp}\\ V_{\mathbf{D}_S^\perp\mathbf{D}_S} & V_{\mathbf{D}_S^\perp\mathbf{D}_S^\perp}\end{pmatrix}. \tag{A8}$$

Here, $t=T_{|\mathbf{D}_S^\perp}$, i.e. $t:\mathbf{D}_S^\perp\to\mathbf{D}_S^\perp$ and, e.g., $V_{\mathbf{D}_S\mathbf{D}_S^\perp}:\mathbf{D}_S^\perp\to\mathbf{D}_S$. In this notation, $[T,V]=0$ reads

$$\begin{pmatrix}0 & V_{\mathbf{D}_S\mathbf{D}_S^\perp}(1-t)\\ (t-1)V_{\mathbf{D}_S^\perp\mathbf{D}_S} & [t,V_{\mathbf{D}_S^\perp\mathbf{D}_S^\perp}]\end{pmatrix}=0. \tag{A9}$$

Equation (A9) holds for all $T\in\mathcal{T}$, and in particular it holds for unitaries $t:\mathbf{D}_S^\perp\to\mathbf{D}_S^\perp$ for which the value 1 is a regular point, e.g. $t=-1$. Thus, $V_{\mathbf{D}_S\mathbf{D}_S^\perp}=0$ and $V_{\mathbf{D}_S^\perp\mathbf{D}_S}=0$.

Further, $\mathcal{T}$ is the full unitary group on $\mathbf{D}_S^\perp$. It is therefore irreducibly represented on $\mathbf{D}_S^\perp$. Thus, $[t,V_{\mathbf{D}_S^\perp\mathbf{D}_S^\perp}]=0,\forall t$ implies with Schur (see, e.g. [37]) that V acts on $\mathbf{D}_S^\perp$ as a multiple of the identity, i.e. $V_{\mathbf{D}_S^\perp\mathbf{D}_S^\perp}=\lambda 1$ where $\lambda\in\mathbf{C}$. In block matrix form, V therefore reads

$$V=\begin{pmatrix}V_{\mathbf{D}_S\mathbf{D}_S} & 0\\ 0 & \lambda 1\end{pmatrix}. \tag{A10}$$

Consider now the kernel

$$\mathbf{K}:=\ker(V-\lambda 1). \tag{A11}$$

By construction, $\mathbf{D}_S^\perp\subset\mathbf{K}$ and $\mathbf{K}^\perp\subset\mathbf{D}_S$. As the kernel of a closed operator, $\mathbf{K}$ is closed. We wish to show that in fact $\mathbf{K}=\mathbf{H}$ and $\mathbf{K}^\perp=\emptyset$, which is to say that $V=\lambda 1$.

To this end, let us assume the opposite, namely that $\mathbf{K}^\perp\neq\emptyset$.

We can then use $\mathbf{H}=\mathbf{K}^\perp\oplus\mathbf{K}$ to write both V and U in a new block form:

$$V=\begin{pmatrix}V_{\mathbf{K}^\perp\mathbf{K}^\perp} & 0\\ V_{\mathbf{K}\mathbf{K}^\perp} & \lambda 1\end{pmatrix}, \quad U=\begin{pmatrix}U_{\mathbf{K}^\perp\mathbf{K}^\perp} & U_{\mathbf{K}^\perp\mathbf{K}}\\ U_{\mathbf{K}\mathbf{K}^\perp} & U_{\mathbf{K}\mathbf{K}}\end{pmatrix}. \tag{A12}$$

The relation $[V,U]=0$ now reads

$$\begin{pmatrix}\cdots & (V_{\mathbf{K}^\perp\mathbf{K}^\perp}-\lambda 1)U_{\mathbf{K}^\perp\mathbf{K}}\\ \cdots & V_{\mathbf{K}\mathbf{K}^\perp}U_{\mathbf{K}^\perp\mathbf{K}}\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}. \tag{A13}$$

On the other hand, $U_{\mathbf{K}^\perp\mathbf{K}}\mathbf{K}\subset\mathbf{K}^\perp$; i.e., the range of $U_{\mathbf{K}^\perp\mathbf{K}}$ is not in the kernel of the operator $(V-\lambda 1)$:

$$(V-\lambda 1)|w\rangle=\begin{pmatrix}(V_{\mathbf{K}^\perp\mathbf{K}^\perp}-\lambda)|w\rangle\\ V_{\mathbf{K}\mathbf{K}^\perp}|w\rangle\end{pmatrix}\neq 0, \quad \forall|\omega\rangle\neq 0,|\omega\rangle \in U_{\mathbf{K}^\perp\mathbf{K}}\mathbf{K}. \tag{A14}$$

Thus, the existence of any nonzero vector $|w\rangle\in\mathbf{K}^\perp$ in the range $U_{\mathbf{K}^\perp\mathbf{K}}\mathbf{K}$ would contradict Eq. (A13). Consequently, the range of $U_{\mathbf{K}^\perp\mathbf{K}}\mathbf{K}$ is empty, i.e. $U_{\mathbf{K}^\perp\mathbf{K}}=0$.

Therefore, $\mathbf{K}$ is an invariant subspace for U. Since also $[U^{-1},V]=0$, it follows analogously that $\mathbf{K}$ is an invariant subspace for U^{-1}. Thus, $\mathbf{K}$ and $\mathbf{K}^\perp$ both reduce U:

$$U=\begin{pmatrix}U_{\mathbf{K}^\perp\mathbf{K}^\perp} & 0\\ 0 & U_{\mathbf{K}\mathbf{K}}\end{pmatrix}. \tag{A15}$$

Since $U_{|\mathbf{D}_S}=S$ and $\mathbf{K}^\perp\subset\mathbf{D}_S$, we have $U_{\mathbf{K}^\perp\mathbf{K}^\perp}=S_{\mathbf{K}^\perp\mathbf{K}^\perp}$. This implies that $\mathbf{K}^\perp$ is an invariant subspace for S, on which S is unitary. However, the simplicity of X implies that also S is simple; i.e., S does not have any invariant subspace on which it would be unitary.

Thus, in fact, $\mathbf{K}^\perp=\emptyset$ and $\mathbf{K}=\mathbf{H}$. Consequently, $V=\lambda 1$, which had to be shown.

With von Neumann this implies that the weak closure of the *-algebra $\mathcal{A}$ generated by $1,U$ and the elements of $\mathcal{T}$ is the algebra $\mathcal{B}(\mathbf{H})$ of all bounded operators on the Hilbert space, and $\mathcal{B}(\mathbf{H})$ includes of course all unitaries. We recall that this means that for each bounded operator $B\in\mathcal{B}(\mathbf{H})$ there exist sequences of operators $B_n\in\mathcal{A}$ such that

$$\lim_{n\to\infty}\langle\psi|B-B_n|\phi\rangle=0 \quad \forall|\psi\rangle, \quad |\phi\rangle\in H.$$

Thus, for any simple symmetric X with equal deficiency indices the set of self-adjoint operators which coincide with X on its domain generate indeed (e.g. via generating the coset $\mathcal{M}$) the full unitary group of the Hilbert space.

APPENDIX B: A COROLLARY

As we mentioned before, in finite dimensional Hilbert spaces every symmetric operator; i.e., every operator whose expectation values are real, i.e. every matrix obeying $X_{ij}=X_{ji}^*$, is also self-adjoint. Therefore, in finite dimensional Hilbert spaces, there are no simple symmetric operators; i.e., our lemma cannot be applied.

Let us add, however, that the above proof yields as a corollary that any simple isometric operator with equal deficiency indices has the property that its unitary extensions, together, generate all unitaries and $\mathcal{B}(H)$. And indeed, there exist simple isometric operators also in finite dimensional Hilbert spaces.

As an illustration, let us consider the simple case of the two dimensional Hilbert space spanned by normalized vectors e_1,e_2. We define a linear operator, S, as the map which maps $S:e_1\to e_2$. Clearly, S is not unitary, because of its limited domain $D_S := \mathbf{C}e_1$ and range $\mathbf{C}e_2$. Also, S does not have any invariant proper subspace. S is norm preserving where it is defined. Thus, S is a simple isometric operator. The dimensions of its deficiency spaces, i.e. of the orthogonal complements of its domain and range, are both 1; i.e., they are equal. Thus, S is an operator to which the corollary of our lemma applies. The claim is that the unitary extensions of S generate all 2×2 matrices, including of course the unitaries.

To see this, we begin by choosing one unitary extension U of S, e.g.

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{B1}$$

The elements $T(\alpha)$ of the local group $\mathcal{T}$ of all unitaries which act as the identity on D_S and which act as a unitary on $D_S^{\perp}$ are of the form

$$T(\alpha) := \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \tag{B2}$$

where $e^{i\alpha}$ is any arbitrary phase. Thus, each unitary extension of S is of the form $UT(\alpha)$ for some α. Indeed, the algebra generated by $1,U$ and the unitary extensions $T(\alpha)$ are all of $M_2(\mathbf{C})$, as is clear because it contains for example the Pauli matrices:

$$\sigma_1 = U, \quad \sigma_2 = iUT(\pi), \quad \sigma_3 = T(\pi). \tag{B3}$$

We also observe that the inverse Cayley transform X of S does exist,

$$X = i(S+1)(S-1)^{-1} = -i\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \tag{B4}$$

but is not symmetric, which demonstrates, as we mentioned, that the inverse Cayley transform of a simple isometric operator is not necessarily simple symmetric.

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