Quantization of the Taub model with extrinsic time

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The paper addresses the quantization of cosmological models, with application to the Taub model. By reducing the model with extrinsic time, a formalism is developed in order to define a conserved Schrödinger inner product in the space of solutions of the Wheeler-DeWitt equation. A quantum version of classical canonical transformations is introduced for connecting the solutions of the Wheeler-DeWitt equation with the wave functions of the reduced system. Once this correspondence is established, boundary conditions on the space of solutions of the Wheeler-DeWitt equation are obtained to directly select the physical subspace.

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I. INTRODUCTION

General relativity is an example of a parametrized system, i.e., a system whose action is invariant under changes of the integrating parameter τ ''reparametrization,'' this invariance being a consequence of the covariance of the theory. This means that in general relativity there is no privileged time variable. On the contrary, in the ordinary formulation of quantum mechanics there is a time parameter besides the true degrees of freedom, and the inner product remains conserved in the time evolution of the system. This difference between general relativity and quantum mechanics, known as the problem of time [1–3], is one of the main obstacles for finding a quantum theory of gravity.

The evolution of a dynamical system is characterized by the way in which its dynamical variables evolve as a function of time. In this formulation, time is a relevant physical parameter, clearly distinct from the dynamical variables. There is nevertheless an alternative formulation of dynamics (parametrized systems) in which time is mixed with the dynamical variables [4]. A parametrized system can be obtained from an action $S(q^{\mu},p_{\mu})$ which is not invariant under reparametrizations by raising the time to the rank of a dynamical variable. Let us start with an action of the form

$$S[q^{\mu},p^{\mu}]=\int_{t_1}^{t_2} p_{\mu}dq^{\mu}-h(q^{\mu},p_{\mu},t)dt \qquad \mu=1,\ldots,n. \tag{1}$$

By identifying $q^0\equiv t,\ p_0\equiv -h$ one can rewrite the integrand as $p_i dq^i=p_i(dq^i/d\tau)d\tau,\ i=0,\ldots,n$. In this way the extended set of variables are left as functions of some physically irrelevant parameter τ. The set $\{q^i,p_i\}$ can be independently varied provided that the definition of p_t is incorporated to the action as a constraint $H=p_0+h(q^{\mu},p_{\mu},t)=0$, so yielding the following action:

$$S[q^i(\tau),p_i(\tau),N(\tau)]=\int_{\tau_1}^{\tau_2}\left(p_i\frac{dq^i}{d\tau}-NH(q,p)\right)d\tau, \tag{2}$$

where N is the Lagrange multiplier whose variation assures that the constraint does hold. This action is invariant under reparametrizations $\tau\rightarrow\tau+\varepsilon(\tau)$. The time variable t satisfies the Poisson brackets

$$\{t,H\}=1. \tag{3}$$

Once the system has been parametrized, it can be reduced using any time variable provided that it satisfies Eq. (3). These kinds of time variables are called global times [5]. In order to generalize this restriction let us suppose that we know a globally well defined time variable $\tilde{t}=\tilde{t}(q^i,p_i)$ which satisfies

$$\{\tilde{t},H\}|_{H=0}=f(q,p)>0. \tag{4}$$

The important fact is that f has a definite sign on the constraint surface (it could also be negative). In this case the variable $\tilde{t}$ is a global time associated with the Hamiltonian $\tilde{H}\equiv f^{-1}(q,p)H$.

The constraint $\tilde{H}=0$ could also be expressed in a set of variables in which the Hamiltonian H does not have the form $H=p_0+h$. In fact we can perform a canonical transformation

$$\{q^i,p_i\}=\{q^o=t,\ \ p_0=-h,q^{\mu},p_{\mu}\}\rightarrow\{Q^i,P_i\}$$

where now the time is hidden among the rest of the variables. In other words, a constraint of the form $H=p_0+h$ can be disguised by scaling it or by performing canonical transformations.

One of the main properties of the Hamiltonian formulation of general relativity is that the Hamiltonian is con-

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strained to be zero, making manifest that general relativity is a parametrized system. In cases like this, in which the theory is an already parametrized system, the invariance under reparametrizations means that there is no privileged time variable. To reduce the system means to select among the dynamical variables a proper global time, i.e., a variable which monotonically increases along any dynamical trajectory, to work as a physical clock. In this way we can express the evolution of the canonical variables as a function of this physical clock. The first step to reduce the system is thus to perform a canonical transformation in order to find a set of variables $\{q^i,p_i\}$ where the variables $q_0=t$ is a global time. The Hamilton equations are

$$\frac{dt}{d\tau}=Nf,$$

$$\frac{dq_\mu}{d\tau}=Nf\frac{\partial h}{\partial p^\mu},$$

$$\frac{dp^\mu}{d\tau}=-Nf\frac{\partial h}{\partial q_\mu}.$$

The dynamics of the system is thus undetermined unless one fixes a gauge, i.e., unless one chooses a physical clock. Choosing the gauge $\tau=t$ means choosing

$$N(\tau)=\frac{1}{f[q_\mu(\tau),p^\mu(\tau)]}.$$

One of the main approximations for quantizing general relativity begins by reformulating it under a Hamiltonian formulation [Arnowitt-Deser-Misner (ADM formalism [6]]. Within the framework of this canonical formalism or geometrodynamics it is supposed that the Lorentzian spacetime manifolds M are diffeomorphic to $R\times S$ where S represents a collection of spacelike hypersurfaces Σ parametrized by a real time parameter t (foliation). The Riemannian metric g_{ij} of one of these hypersurfaces Σ play the role of the configuration variable. The analogous of the configuration space R^n is the space of all the Riemannian metrics g_{ij} called superspace. The conjugated momenta π^{ij} are directly related with the way in which the hypersurface Σ is embedded in the manifold M, i.e., with the extrinsic curvature of the hypersurface Σ. The covariance of the theory under general coordinate transformations is reflected, within this formalism, in the presence of four constraints per each point of space-time. The so-called Hamiltonian constraint assures the invariance of the theory under a changing of the foliation, while the momenta constraints assure the invariance under a change of the spatial coordinates used to represent the spatial geometry of each hypersurface. The states of the corresponding quantum theory $\Psi[g_{ij}]$ are functionals of the spatial metric g_{ij} which satisfy the quantum version of the classical constraints in accordance with the Dirac method. The quantum version of the momenta constraints implies that the wave function depends on the geometry ${}^3g$ of the hypersurface but not on the particular metric tensor g_{ij} used to represent it. The quantum version of the Hamiltonian constraint is the so-called Wheeler-DeWitt equation.

Many of the tentatives for quantizing general relativity began addressing the analogy between the Wheeler-DeWitt equation and the Klein-Gordon equation. In fact both systems have Hamiltonian constraints which are hyperbolic in the momenta. The constraint associated with the motion of a particle in a pseudo-Riemannian geometry has the form

$$H_{particle}=g^{ij}(q^k)p_ip_j-m^2=0. \tag{5}$$

The space of solutions of the Klein-Gordon equation can be turned into a Hilbert space with a positive definite inner product only if the background is stationary. In this case the Hilbert space of the physical states will be the subspace of positive norm, this being equivalent to consider just one of the sheets of the hyperbolic constraint surface. Choosing the coordinates in a way that $g^{\mu 0}=0$ ($\Rightarrow g^{00}=g_{00}^{-1}$) and calling $\gamma^{\mu\nu}\equiv -g_{00}g^{\mu\nu}$ we can write Eq. (5) in the form

$$\begin{aligned}H_{particle}&=g^{00}(p_0p_0-\gamma^{\mu\nu}p_\mu p_\nu-g_{00}m^2)\\&=g^{00}(p_0-\sqrt{\gamma^{\mu\nu}p_\mu p_\nu+g_{00}m^2})\\&\quad\times(p_0+\sqrt{\gamma^{\mu\nu}p_\mu p_\nu+g_{00}m^2}).\end{aligned} \tag{6}$$

It is thus necessary to find a temporal Killing vector of the supermetric which also should be a symmetry of the potential term (this property could be relaxed to a conformal Killing vector). In this case the proper time variable to reduce the system is the parameter of the Killing vector. Otherwise there would be pair creation. In order to build a good analogy with the relativistic particle, it is also necessary to have a positive definite potential term for playing the role of the mass term. If the potential is positive definite, the momentum p_0 does not go to zero on the constraint surface $H=0$. This means that the Poisson bracket $\{q_0,H\}=2g^{00}p_0$ has a definite sign on each sheet of the constraint surface. If we choose $p_0+\sqrt{\gamma^{\mu\nu}p_\mu p_\nu+g_{00}m^2}=0$, the momentum p_o, and so $\{q_0,H\}$, will be negative on this sheet (provided that $g^{00}>0$). The other factor has then a definite sign on this sheet playing the role of the function f defined in Eq. (4). In this case f will be negative, being this the reason for $\{q_0=t,H\}<0$. But t is still the variable which monotonically increases on any dynamical trajectory because $\{t,\tilde{H}\}=1$ where

$$\tilde{H}=\frac{H}{g^{00}(p_0-\sqrt{\gamma^{\mu\nu}p_\mu p_\nu+g_{00}m^2})}.$$

The quantum physical states can be obtained by solving a Schrödinger equation with a positive definite operator $\hat{h}$ associated with the Hamiltonian of the reduced system $\sqrt{\gamma^{\mu\nu}p_\mu p_\nu+g_{00}m^2}$. As it was said before, to fix the gauge $t=\tau$ implies to choose $N=1/f$. The relation between proper time and the time variable chosen to represent the hypersurfaces of simultaneity is $dT=f^{-1}dt$ where T is the proper time, this being a consequence of the way in which the space-time interval is expressed in the ADM formalism. In geometrodynamics there is a conformal Killing vector of the supermetric [7], but this vector is not as well a symmetry of the potential term. This potential term is the spatial curvature [6], which can be negative in some regions of the configuration space. Thus it is not possible to associate an operator $\hat{h}$ with the square root, as one effectively does for the relativist particle.

There are two main approaches for achieving this quantization program. One possibility is to quantize the system without reducing it. The resulting Wheeler-DeWitt equation is a hyperbolic equation while the Schrödinger equation associated with the reduced system is a parabolic one. The former has then more solutions than the latter, so being necessary to define boundary conditions in order to select the physical solutions. It is not clear in this approach how to define a conserved inner product without having reduced the system, i.e., without knowing which variable plays the role of time. Another proposal is to perform a canonical transformation in order to find a Hamiltonian of the form (5) with a positive definite potential term independent of the variable q_0. The formalism of the relativistic particle can then be applied using q_0 as a proper time variable. The system can thus be quantized by means of the corresponding Schrödinger equation associated with one of the sheets of the constraint surface. The Hilbert space of the quantum states can be endowed with the natural inner product associated with the Schrödinger equation. This approach has the problem that different choices of global time variables can lead to different quantum theories.

II. PROPOSED FORMALISM

In this work we will address the quantization of minisuperspace cosmological models. The quantization program will be as follows. We will start with a Hamiltonian constraint such that none of the variables is a global time. We will suppose that it is possible to perform a coordinate transformation so that a subsystem depending on just one pair of canonical variables $\{q_1,p_1\}$ is separated in the Hamiltonian constraint

$$H_q = h_{q_1}(q_1,p_1) - h_{q_\mu}(q_\mu,p_\mu) \qquad \mu=2,\ldots,N \tag{7}$$

where the Hamiltonian $h_{q_1}(q_1,p_1)$ has the form

$$h_{q_1}(q_1,p_1) = \frac{4}{[d\ln V/dq_1]^2}p_1^2 + V(q_1) \tag{8}$$

with $V(q_1)>0$ and $h_{q_\mu}>0$. In order to find a global time, one should look for another canonical transformation $\{q_i,p_i\}\rightarrow\{Q_i,P_i\}$ $i=1,\ldots,N$ such that

$$\begin{aligned}
&Q_1 = t = t(q_1,p_1),\\
&P_1 = p_t = [h_{q_1}(q_1,p_1)]^{1/2},\\
&Q_\mu = q_\mu,\\
&P_\mu = p_\mu,\\
&\mu = 2,\ldots,N.
\end{aligned} \tag{9}$$

In this way we could separate an extrinsic time t, i.e., a global time variable which is a function of both original canonical coordinates and momenta. The generator function for the canonical transformation and the corresponding momenta are

$$F_1(q_1,t) = -\sinh(t)[V(q_1)]^{1/2},$$

$$\begin{aligned}
p_t &= \frac{\partial F_1}{\partial t} = -\cosh t[V(q_1)]^{1/2},\\
p_{q_1} &= \frac{\partial F_1}{\partial q_1} = -\tfrac{1}{2}\sinh t[V(q_1)]^{-1/2}\frac{dV}{dq_1}\\
&= -\tfrac{1}{2}\sinh t[V(q_1)]^{1/2}\frac{d(\ln V)}{dq_1}
\end{aligned} \tag{10}$$

so that

$$\begin{aligned}
\frac{4}{[d\ln V/dq_1]^2}p_{q_1}^2 + V(q_1) &= V(q_1)\sinh^2 t + V(q_1)\\
&= V(q_1)\cosh^2 t = p_t^2.
\end{aligned}$$

The Hamiltonian in the new set of variables has the form

$$H_Q = p_t^2 - h_{q_\mu}(q_\mu,p_\mu). \tag{11}$$

Thus, when the Hamiltonian $h_{q_\mu}(q_\mu,p_\mu)$ is positive definite and independent of time t one gets a constraint such that the analogy with the relativistic particle does hold, and one can quantize the reduced model by means of the parabolic Schrödinger equation associated with one of the sheets of the constraint surface.

Once the system was reduced and quantized we want to find out what kind of boundary conditions should be imposed on the solutions of the Wheeler-DeWitt equation expressed in the original set of variables. As was pointed out, the hyperbolic Wheeler-DeWitt has twice the number of independent solutions than the parabolic Schrödinger equation. It is then necessary to impose proper boundary conditions for selecting the physical solutions. In order to do that we will follow the lines of work used in Ref. [8]. Knowing the classical canonical transformation for reducing the model, its analogue in the quantum level can be defined. In Ref. [9] the conditions for relating the wave functions corresponding to a pair of quantum-mechanical systems whose classical Hamiltonians are canonically equivalent are studied. If one has two arbitrary Hamiltonians related at the classical level by a canonical transformation corresponding to the generating function $F_1(q,Q)$, the main issue is to find out what kind of integral transforms can be defined in order to relate the wave functions corresponding to each quantum-mechanical system. Generalizing the Fourier transform, a relationship of the following kind is proposed

$$\Theta_E(q)=N(E)\int_{-\infty}^{+\infty}dQ\,e^{iF(q,Q)}\Phi_E(Q) \tag{12}$$

where $F(q,Q)$ is not in general the generating function $F_1(q,Q)$ for the classical canonical transformation. In Ref. [9] it is shown, however, that this function coincides in fact with the generating function for the classical canonical transformation when the Hamiltonian operators satisfy the condition

$$H_q\left(-i\frac{\partial}{\partial q},q\right)e^{iF(q,Q)}=H_Q\left(i\frac{\partial}{\partial Q},Q\right)e^{iF(q,Q)} \tag{13}$$

where some proper boundary conditions in the integration limits are also assumed. If the canonical transformation cannot be represented by means of a generating function of the first kind, analogous integral transforms and conditions can be defined using the corresponding generating function. The inverse of the integral transform (12) is

$$\Phi_E(Q)=N\int_{-\infty}^{+\infty}dq\left|\frac{\partial^2F(q,Q)}{\partial q\partial Q}\right|e^{-iF(q,Q)}\Theta_E(q). \tag{14}$$

The canonical transformation defined by Eq. (10) does satisfy the condition (13). The function $F(q,Q)$ coincides then with the generating function $F_1(q,Q)$. Once defined, this ‘‘canonical quantum transformation’’ the physical solutions of the Wheeler-DeWitt equation can be found by transforming the solutions of the Schrödinger equation. Finally the question of defining proper boundary conditions for the solutions of the Wheeler-DeWitt equation without knowing how to reduce the system will be addressed.

III. APPLICATION TO THE TAUB MODEL

A. Reduction

We will study the application of the formalism displayed in the previous section to the particular case known as Taub model. In Bianchi cosmological models [10] the minisuperspace is a three dimensional manifold parametrized by two parameters (β_+,β_-) measuring the spatial anisotropy and a parameter α measuring the volume of the Universe (Misner parametrization). The Hamiltonian constraint for minisuperspace models has the form

$$H=e^{3\alpha}\{-p_\alpha^2+p_+^2+p_-^2+e^{-4\alpha}[V(\beta_+,\beta_-)-1]\} \tag{15}$$

where (p_α,p_+,p_-) are the momenta canonically conjugate to (α,β_+,β_-) and the potential $V(\beta_+,\beta_-)$ depends upon the particular Bianchi model.

The Taub model is a particular case of the Bianchi type IX model for $\beta_-=0$, $p_-=0$. For this case the resulting Hamiltonian constraint (scaled with the factor $e^{-3\alpha}$) is

$$H=-p_\alpha^2+p_+^2+12\pi^2e^{-4\Omega}(e^{-8\beta_+}-4e^{-2\beta_+}). \tag{16}$$

If we define the variables u and v by

$$\alpha=v-2u,$$

$$\beta_+=u-2v \tag{17}$$

the resulting Hamiltonian is (scaled with a constant)

$$H=\tfrac{1}{6}(p_v^2+36\pi^2e^{12v})-\tfrac{1}{6}(p_u^2+144\pi^2e^{6u}). \tag{18}$$

So, this coordinate transformation separates a subsystem depending on just one pair of canonical variables, which will work as a clock for the other subsystem. The canonical transformation (10) provides a global time and its conjugated momentum:

$$t=\text{Arc sinh}\left(-\frac{p_v}{6\pi}e^{-6v}\right), \tag{19}$$

$$p_t^2=\tfrac{1}{36}(p_v^2+36\pi^2e^{12v}). \tag{20}$$

The generator of this transformation is

$$F_1(v,t)=-\pi e^{6v}\sinh t. \tag{21}$$

The Hamiltonian in the new variables results to be

$$H=6p_t^2-\tfrac{1}{6}(p_u^2+144\pi^2e^{6u}). \tag{22}$$

This expression can be factorized in order to obtain a Hamiltonian linear in p_t, so giving

$$H=\left(\sqrt{6}p_t+\frac{1}{\sqrt{6}}\sqrt{p_u^2+\pi^2e^{6u}}\right)\times\left(\sqrt{6}p_t-\frac{1}{\sqrt{6}}\sqrt{p_u^2+144\pi^2e^{6u}}\right). \tag{23}$$

The constraint $H=0$ is fulfilled if one of the factors is null on the constraint surface. The other factor has, on the constraint surface, a definite sign, so playing the role of the factor f defined before. The scaled Hamiltonian is

$$\tilde{H}=\frac{H}{\sqrt{6}f}=p_t+\tfrac{1}{6}\sqrt{p_u^2+144\pi^2e^{6u}}=p_t+h_u \tag{24}$$

with

$$f=\left(\sqrt{6}p_t-\frac{1}{\sqrt{6}}\sqrt{p_u^2+144\pi^2e^{6u}}\right) \tag{25}$$

being h_u the Hamiltonian of the reduced system.

B. Quantization

In order to quantize the reduced system we will make the substitution $p_t \to -i\partial/\partial t$, $p_u \to -i\partial/\partial u$ and impose the constraint $\hat{H}\Psi(t,u)=0$ yielding the following Schrödinger equation

$$i\frac{\partial\Psi(t,u)}{\partial t}=\hat{h}_u\left(u,-i\frac{\partial}{\partial u}\right)\Psi(t,u). \tag{26}$$

Inserting solutions of the form $\Psi(t,u)=\phi(u)e^{-i\sqrt{\varepsilon}t}$ we obtain a modified Bessel equation for the function $\phi(u)$. The solutions of this equation are the modified Bessel functions

$$\phi(u)=CK_{2i\sqrt{\varepsilon}}(4\pi e^{3u})+DI_{2i\sqrt{\varepsilon}}(4\pi e^{3u}). \tag{27}$$

The functions $I_{2i\sqrt{\varepsilon}}(4\pi e^{3u})$ should be discarded because they diverge when $u\to\infty$ (classically forbidden zone). The solutions corresponding to the quantization of the reduced system are therefore

$$\Psi(t,u)=Ce^{-i\sqrt{\varepsilon}t}K_{2i\sqrt{\varepsilon}}(4\pi e^{3u}). \tag{28}$$

On the other hand the Wheeler-DeWitt equation associated with the Hamiltonian (18) is

$$\frac{1}{6}\left[\left(-\frac{\partial^2}{\partial v^2}+36\pi^2e^{12v}\right)-\frac{1}{6}\left(-\frac{\partial^2}{\partial u^2}+144\pi^2e^{6u}\right)\right]\varphi(v,u)=0 \tag{29}$$

whose solutions are

$$\varphi(v,u)=[AK_{i\sqrt{\varepsilon}}(\pi e^{6v})+BI_{i\sqrt{\varepsilon}}(\pi e^{6v})]\times[CK_{2i\sqrt{\varepsilon}}(4\pi e^{3u})+DI_{2i\sqrt{\varepsilon}}(4\pi e^{3u})]. \tag{30}$$

The functions $I_{2i\sqrt{\varepsilon}}(4\pi e^{3u})$ should be discarded because of the same reason as before. Nevertheless it would not be correct to impose the same kind of boundary conditions used to discard these functions to the functions of the variable, v. v is not a dynamical variable, but rather the variable associated with the clock of the system. It is by no means obvious that the physical solutions associated with this variable should go to zero in the classical forbidden zone.

In order to select the physical solutions we will try to apply the ''quantum canonical transformations'' defined in Eq. (14) to the solutions of the Wheeler-DeWitt equation. The physical solutions will be those whose transformed functions are the solutions of the Schrödinger equation $e^{-i\sqrt{\varepsilon}t}$. We will begin by transforming the functions which go to zero in the classically forbidden zone, i.e., the functions $\Theta(v)=K_{i\sqrt{\varepsilon}}(\pi e^{6v})$. The transformed functions are

$$\begin{aligned}\Phi(t)&=N\int_{-\infty}^{+\infty}dv\,6\pi e^{6v}\cosh t\,e^{i\pi e^{6v}\sinh t}K_{i\sqrt{\varepsilon}}(\pi e^{6v})\\&=\frac{\pi N}{4\sinh(\pi\sqrt{\varepsilon}/2)\cosh(\pi\sqrt{\varepsilon}/2)}\times[e^{\pi\sqrt{\varepsilon}/2}e^{i\sqrt{\varepsilon}t}-e^{-\pi\sqrt{\varepsilon}/2}e^{-i\sqrt{\varepsilon}t}].\end{aligned} \tag{31}$$

In this way it is manifest that the transformation of the functions $\Theta(v)=K_{i\sqrt{\varepsilon}}(\pi e^{6v})$ do not give the solutions of the Schrödinger equation $e^{-i\sqrt{\varepsilon}t}$. On the contrary they correspond to a combination of positive and negative energy states

$$K_{i\sqrt{\varepsilon}}(\pi e^{6v})\leftrightarrow\frac{\pi N}{4\sinh(\pi\sqrt{\varepsilon}/2)\cosh(\pi\sqrt{\varepsilon}/2)}\times[e^{\pi\sqrt{\varepsilon}/2}e^{i\sqrt{\varepsilon}t}-e^{-\pi\sqrt{\varepsilon}/2}e^{-i\sqrt{\varepsilon}t}]. \tag{32}$$

By transforming the right side of Eq. (32) one should recover the original function $\Theta(v)=K_{i\sqrt{\varepsilon}}(\pi e^{6v})$, so one obtains the factor $N=1/\sqrt{\pi}$, which does not depend on the energy.

As the functions $\Theta(v)=K_{i\sqrt{\varepsilon}}(\pi e^{6v})$ are not definite energy states, we will apply the transformation (14) to the other subspace of solutions, i.e., to the functions $I_{\pm i\sqrt{\varepsilon}}(\pi e^{6v})$, which diverge in the classically forbidden zone. The resulting integral has the form

$$\Phi(t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}dv\,6\pi e^{6v}\cosh t\,e^{i\pi e^{6v}\sinh t}I_{\pm i\sqrt{\varepsilon}}(\pi e^{6v}). \tag{33}$$

This integral diverges unless one gives an imaginary part η to t. Replacing $t\to t+i\pi/2$ in Eq. (33) one can actually perform the integration. Replacing $t\to t-i\pi/2$ in the result of the integral one can go back to the original real time variable, obtaining the correspondence

$$I_{\pm i\sqrt{\varepsilon}}(\pi e^{6v})\leftrightarrow\Phi(t)=\frac{i}{\sqrt{\pi}}e^{\mp\sqrt{\varepsilon}\pi/2}e^{\mp i\sqrt{\varepsilon}t}. \tag{34}$$

The functions $I_{i\sqrt{\varepsilon}}(\pi e^{6v})$ and $I_{-i\sqrt{\varepsilon}}(\pi e^{6v})$ do represent then the positive and negative energy states respectively.[1] In this way we could establish which subspace of the whole space

[1]This result can be verified by testing the consistence of Eq. (33) with Eq. (31) and the expression [11]

$$K_v(z)=\frac{\pi}{2}\frac{I_{-v}(z)-I_v(z)}{\sin(v\pi)}. \tag{35}$$

In fact, transforming the right side of Eq. (34) using Eq. (33) one does obtain the right side of Eq. (31), verifying in this way the coherence of the found correspondences (31) and (33) between both representations.

of solutions of the Wheeler-DeWitt equation is the physical one. It is remarkable that the functions in the selected subspace do not decay in the classically forbidden zone. The solutions to the Wheeler-DeWitt equation with the proper boundary conditions imposed are thus

$$\varphi(v,u)=AI_{i\sqrt{\varepsilon}}(\pi e^{6v})K_{2i\sqrt{\varepsilon}}(4\pi e^{3u}). \tag{36}$$

C. Boundary conditions

It would be interesting if one could define certain boundary conditions which would not rely on the fact that one knows how to reduce the system, and which would select the same physical functions imposed by the quantization of the reduced system. Let us start by considering the following Hamiltonian

$$H=p_1^2+V(q_1)-h_{q_\mu}(q_\mu,p_\mu), \quad \mu=2,\ldots,N, \tag{37}$$

and suppose that $\varphi(q_1,q_\mu)=\Theta(q_1)\phi(q_\mu)$ is the solution of the Wheeler-De Witt equation associated with this Hamiltonian. Instead of performing a canonical transformation so that the new Hamiltonian is quadratic in the new momentum p_t, we will study the wave functions obtained by solving the Wheeler-DeWitt equation associated with the Hamiltonian (37) in the region L where $V(q_1)$ tends to zero. In this region the Hamiltonian is

$$H=p_1^2-h_{q_\mu}(q_\mu,p_\mu). \tag{38}$$

The solutions of the quantum-mechanical system corresponding to the sheet in which p_1 is negative (positive energy solutions) will be combinations of $\varphi(q_1,q_\mu)=\phi(q_\mu)e^{-i\sqrt{\varepsilon}q_1}$. One would expect that these solutions do coincide with the asymptotic expressions in the region L of the functions $\varphi(q_1,q_\mu)=\Theta(q_1)\phi(q_\mu)$, i.e., it would be necessary that

$$\Theta(q_1)\to e^{-i\sqrt{\varepsilon}q_1},$$

$$q_1\to L. \tag{39}$$

The definite energy solutions will thus be those functions $\Theta(q_1)$ which do behave in the asymptotic region like a plane wave, ingoing *or* outgoing. This criterium relies on the fact that in the asymptotic region, time is (modulus a sign) the variable q_1.

In order to test this criterium in the case of the Taub model let us see the behavior of the Bessel functions in the region where $v\to-\infty$ $[V(q_1)\to 0]$. The asymptotic expressions are

$$I_v(z)\sim\frac{(\frac{1}{2}z)^v}{\Gamma(v+1)}, \tag{40}$$

$$K_v(z)\sim\frac{\pi}{2\sin(v\pi)}\left[\frac{(\frac{1}{2}z)^{-v}}{\Gamma(-v+1)}-\frac{(\frac{1}{2}z)^v}{\Gamma(v+1)}\right]. \tag{41}$$

The asymptotic expression for $K_v(z)$ was obtained by using the formula (35). The asymptotic expressions for the Taub model are thus

$$I_{i\sqrt{\varepsilon}}(\pi e^{6v})\sim\frac{(36\pi^2)^{i\sqrt{\varepsilon}/2}}{(12)^{i\sqrt{\varepsilon}}\Gamma(i\sqrt{\varepsilon}+1)}e^{i\sqrt{\varepsilon}6v} \tag{42}$$

$$K_{i\sqrt{\varepsilon}}(\pi e^{6v})\sim\frac{\pi}{2\sin(v\pi)}\left[\frac{(36\pi^2)^{-i\sqrt{\varepsilon}/2}}{(12)^{-i\sqrt{\varepsilon}}\Gamma(-i\sqrt{\varepsilon}+1)}\times e^{-i\sqrt{\varepsilon}6v}-\frac{(36\pi^2)^{i\sqrt{\varepsilon}/2}}{(12)^{i\sqrt{\varepsilon}}\Gamma(i\sqrt{\varepsilon}+1)}e^{i\sqrt{\varepsilon}6v}\right]. \tag{43}$$

In this way we confirm that the functions $K_{i\sqrt{\varepsilon}}(\pi e^{6v})$ do correspond to a combination of positive and negative energy states. The functions $I_{\pm i\sqrt{\varepsilon}}(\pi e^{6v})$ do correspond to states of positive or negative energy respectively. The proposed criterium establishes boundary conditions with the definite meaning of selecting the positive energy states. The space of solutions of these positive energy states can be endowed with the positive definite Schrödinger inner product. It is remarkable that the proposed boundary conditions coincides with the criterium proposed by Wald [12,13].

IV. CONCLUSIONS

In this work we addressed the question of quantizing minisuperspace models by studying the particular case known as the Taub model. The main two problems which arise in the canonical approach are the boundary conditions to be imposed on the solution of the Wheeler-DeWitt equation and the inner product to be defined in the corresponding Hilbert space. In the Taub model it is possible to perform a coordinate transformation in order to separate in the Hamiltonian constraint a subsystem depending on just one pair of canonical variables. In this new set of variables the Hamiltonian has thus the form $H=h_{q_1}(q_1,p_1)-h_{q_\mu}(q_\mu,p_\mu)$ where the subsystem h_{q_1} will work as the clock of the model. Performing a canonical transformation it is possible to transform the subsystem h_{q_1} in a free system $h_t=p_t^2$. This kind of time variables are known as ''extrinsic time'' because they are associated not only with the coordinates but also with the momenta [1,14]. Extrinsic times are specially important in quantum gravity because it is not possible to reduce the system by identifying an intrinsic time, i.e., a global time variable in the configuration space, as it happens for the relativistic particle. The new Hamiltonian $H=p_t^2-h_{q_\mu}(q_\mu,p_\mu)$

can be factorized in two disconnected sheets. In order to satisfy the constraint $H=0$ it is necessary that one of these factors goes to zero on the constraint surface. The other one has a definite sign on the constraint surface, therefore it is possible to scale the Hamiltonian constraint in order to find a Hamiltonian linear in the new momentum p_t. This Hamiltonian can be quantized by means of an ordinary Schrödinger equation. The canonical transformation used to reduce the system satisfies the necessary conditions to define the integral transforms which relate the wave functions corresponding to the quantization of both Hamiltonian systems. Thus it is possible to transform the positive energy solutions of the parabolic Schrödinger equation in order to find out those solutions of the hyperbolic Wheeler-DeWitt equation which are the physical ones. Armed with the knowledge of the physical solutions, we tried to define a criterion to select those solutions without using the fact that the reduced system is known. In order to do that, it was studied in the asymptotic behavior of the solutions of the Wheeler-DeWitt equation in the zone where $V(q_1)\rightarrow 0$. We argue that in that area the time is (modulo a sign) the variable q_1. It is thus necessary that the functions $\Phi(q_1)$ behave as an outgoing *or* ingoing plane wave, i.e., as definite energy states. The wave functions satisfying this criterion do coincide with the physical functions selected by reducing the system.

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