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Renormalization group flow with unstable particles

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The renormalization group flow of an integrable two-dimensional quantum field theory which contains unstable particles is investigated. The analysis is carried out for the Virasoro central charge and the conformal dimensions as a function of the renormalization group flow parameter. This allows us to identify the corresponding conformal field theories together with their operator content when the unstable particles vanish from the particle spectrum. The specific model considered is the $SU(3)_2$-homogeneous sine-Gordon model.

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The study of two-dimensional quantum field theories (2D-QFT) has turned out to be a fruitful venture for almost three decades. In particular, when exploiting integrability many nonperturbative methods have been developed over the years. In addition to the challenge to understand the underlying mathematical structures and the intriguing physical applications in two dimensions itself, e.g., to describe measurable quantities of carbon nanotubes [1], the ultimate goal is to extrapolate ones findings to higher dimensions. In particular, for the celebrated c-theorem of Zamolodchikov [2], which originally describes the renormalization group trajectory of a function which corresponds to the Virasoro central charge at the renormalization group fixed point, various counterparts have been developed in higher dimensions, e.g., [3].

Fairly recently a class of massive integrable quantum field theories, the homogeneous sine-Gordon models (HSG) [4], has been proposed, introducing the feature of possessing unstable particles inside its particle spectrum. Despite the fact that theories containing resonances have been treated before in the context of two-dimensional massive quantum field theories, e.g., [5], the HSG models are somewhat special since they constitute the first examples of theories which admit a well-defined Lagrangian description. In general the HSG models are associated with integrable perturbations of G-parafermions of level k [6], i.e., Weiss-Zumino-Novikov-Witten (WZNW) coset theories of the form $G_k/U(1)^l$ with l being the rank of a compact Lie group G. As free parameters the model contains l different mass scales and $l-1$ different scales for the resonance parameter σ, which enters the Breit-Wigner formula [7]. In general an unstable particle of type $\tilde{c}$ is described by complexifying the physical mass of a stable particle by adding a decay width $\Gamma_{\tilde{c}}$, such that it corresponds to a pole in the S-matrix as a function of the Mandelstam variable s at $s=M_R^2=(M_{\tilde{c}}-i\Gamma_{\tilde{c}}/2)^2$ (for a more detailed discussion see, e.g., [8]). As mentioned in [8], whenever $M_{\tilde{c}} \gg \Gamma_{\tilde{c}}$, the quantity $M_{\tilde{c}}$ admits a clear cut interpretation of the physical mass. However, since this assumption is only required for interpretational reasons we will not rely on it. As is usual in this context, transforming from s to the rapidity plane and describing the scattering of two stable particles of type a and b with masses m_a and m_b by an S matrix $S_{ab}(\theta)$ as function of the rapidity θ, the resonance pole is situated at $\theta_R=\sigma-i\bar{\sigma}$. Identifying the real and imaginary parts of the pole then yields

$$M_{\tilde{c}}^2-\frac{\Gamma_{\tilde{c}}^2}{4}=m_a^2+m_b^2+2m_a m_b\cosh\sigma\cos\bar{\sigma}, \tag{1}$$

$$M_{\tilde{c}}\Gamma_{\tilde{c}}=2m_a m_b\sinh|\sigma|\sin\bar{\sigma}. \tag{2}$$

Eliminating the decay width from Eqs. (1) and (2), we can express the mass of the unstable particles $M_{\tilde{c}}$ in the model as a function of the masses of the stable particles m_a, m_b, and the resonance parameter σ. Assuming σ to be large gives

$$M_{\tilde{c}}^2\sim\tfrac{1}{2}m_a m_b(1+\cos\bar{\sigma})e^{|\sigma|}. \tag{3}$$

One recognizes the occurrence of the variable $me^{|\sigma|/2}$, which was introduced originally in [9] in order to describe massless particles, i.e., one may safely perform the limit $m\to 0,\sigma\to\infty$, and one might therefore be tempted to describe flows related to Eq. (3) as massless flows. In [10] the relative mass scales between the unstable and stable particles and the stable particles themselves were investigated by computing the finite size scaling function from the thermodynamic Bethe ansatz (TBA). A consistent physical picture was obtained for the overall identification of the flow between different coset models. It remained, however, an open question of how to identify the operator content. In general this question is left unanswered in the context of the TBA. For theories with certain properties, it is sometimes possible to determine at least the dimension of the perturbing operators by investigating periodicities in the so-called Y systems [11]. Resorting to a different method, namely, by appealing to sum rules which are expressible in terms of correlation functions, the major part of the operator content was successfully identified for some of the HSG models [12]. The purpose of this paper is, on one hand, to confirm and refine the TBA results by the latter method, i.e., by investigating the renormalization group flow described by the Zamolodchikov c-function [2]. We will precisely study the onset of the mass scale of the unstable particles and investigate how a particular coset flows to another one. On the other hand, we also study the flow of the operator content of one conformal field theory to another one by exploiting the flow provided by the Δ sum rule of Delfino, Simonetti, and Cardy [13].

Denoting by r the radial distance and by $t = \ln r^2$ the renormalization group parameter, the functions $c(t)$ and $\Delta(t)$ were defined in [2] and [13], respectively, obeying the differential equations

$$\frac{dc(t)}{dt} = -\frac{3}{4} e^{2t} \langle \Theta(t)\Theta(0)\rangle, \tag{4}$$

$$\frac{d\Delta(t)}{dt} = \frac{1}{\langle \mathcal{O}(0)\rangle} e^{t} \langle \Theta(t)\mathcal{O}(0)\rangle. \tag{5}$$

The right-hand side of these equations involve the two-point correlation functions of the trace of the energy-momentum tensor Θ and an operator $\mathcal{O}$, which is a primary field in the sense of [14]. In general these equations are integrated from $t = -\infty$ to $t = \infty$, and one consequently compares the difference between the ultraviolet and the infrared fixed points. In order to exhibit the quantitative onset of the mass scale of the unstable particles we instead integrate these equations from some finite value t_0 to infinity. Restricting our attention to purely massive theories we use the fact that for those theories the infrared central charges are zero, such that

$$c(r_0) = \frac{3}{2}\int_{r_0}^{\infty} dr\, r^3 \langle \Theta(r)\Theta(0)\rangle. \tag{6}$$

Instead of the integral representation, Eq. (6), the c-function is equivalently expressible in terms of a sum of correlators also involving other components of the energy momentum tensor [2]. In deriving Eq. (4) these terms have been eliminated by means of the conservation law of the energy momentum tensor. We find Eq. (6) most convenient. The flow of $c(r_0)$ will surpass various steps: Starting with $r_0 = 0$ the theory will leave its ultraviolet fixed point and at a certain definite value, say, $r_0 = r_u$, the unstable particle will become massive such that $c(r_0 > r_u)$ can be associated to a different conformal field theory. It appears natural to identify the mass $M_{\tilde{c}}$ as the point at which $c(r_0)$ is half the difference between the two coset values of c. As a consequence of Eq. (3) we may relate the masses of the unstable particles at different values of the resonance parameter σ, σ' and expect $M_{\tilde{c}}(r_u, \sigma) = M_{\tilde{c}}(r'_u, \sigma')$. We will employ the latter equality evaluated in the form (3) not only as a consistency requirement, but also as a confirmation of the fact that the renormalization group flow is indeed achieved by $m \to r_0 m$. Increasing r_0 further, the energy scale of the stable particles will eventually be reached at, say, at $r_0 = r_a, r_b, \ldots, r_n$. Depending on the relative mass scales between the stable particles these points may coincide. Finally the flow will reach its infrared fixed point $c(r_0 = r_{ir}) = 0$.

Likewise we can integrate Eq. (5),

$$\Delta(r_0) = -\frac{1}{2\langle \mathcal{O}(0)\rangle}\int_{r_0}^{\infty} dr\, r \langle \Theta(r)\mathcal{O}(0)\rangle, \tag{7}$$

which allows us to keep track of the manner with which the operator contents of the various conformal field theories are mapped into each other. We used the idea that all conformal dimensions vanish in the infrared limit. Fortunately, we have $\langle \Theta(r)\mathcal{O}(0)\rangle \sim \langle \mathcal{O}(0)\rangle$ in many applications such that the vacuum expectation value $\langle \mathcal{O}(0)\rangle$ cancels often. One should note, however, that Eq. (7) is only applicable to those operators for which its two-point correlator with the trace of the energy momentum tensor is nonvanishing, such that one may not be in a position to investigate the flow of the entire operator content by means of Eq. (7).

In order to evaluate Eqs. (6) and (7) we have to compute the two-point correlation functions in some way. In 2D-QFT this is probably most efficiently achieved by expanding them in terms of n-particle form factors, i.e., the matrix elements of some local operator $\mathcal{O}(\vec{x})$ located at the origin between a multiparticle in-state and the vacuum denoted by

$$\langle 0|\mathcal{O}(0)|V_{\mu_1}(\theta_1)V_{\mu_2}(\theta_2)\ldots V_{\mu_n}(\theta_n)\rangle_{\text{in}} =: F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n).$$

Here the $V_\mu(\theta)$ are some vertex operators representing a particle of species μ. Abbreviating the sum of the on-shell energies as $E = \sum_{i=1}^{n} m_{\mu_i} \cosh\theta_i$, one may write

$$\langle \mathcal{O}(r)\mathcal{O}'(0)\rangle = \sum_{n=1}^{\infty}\sum_{\mu_1\ldots\mu_n}\int_{-\infty}^{\infty}\frac{d\theta_1\ldots d\theta_n}{n!(2\pi)^n} \times e^{-rE} F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n) \times (F_n^{\mathcal{O}'|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n))^*. \tag{8}$$

Using this expansion we replace the correlation functions in the expression of the c-function $c(r_0)$ and the scaled conformal dimension $\Delta(r_0)$ and perform the r integrations thereafter. Thus we obtain

$$c(r_0) = 3\sum_{n=1}^{\infty}\sum_{\mu_1\ldots\mu_n}\int_{-\infty}^{\infty}\frac{d\theta_1\ldots d\theta_n}{n!(2\pi)^n} \times e^{-r_0E}|F_n^{\Theta|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n)|^2 \times \frac{(6+6r_0E+3r_0^2E^2+r_0^3E^3)}{2E^4} \tag{9}$$

and

$$\Delta(r_0) = -\sum_{n=1}^{\infty}\sum_{\mu_1\ldots\mu_n}\int_{-\infty}^{\infty}\frac{d\theta_1\ldots d\theta_n}{n!(2\pi)^n}\frac{(1+r_0E)e^{-r_0E}}{2E^2} \times F_n^{\Theta|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n) \times (F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n))^*/\langle \mathcal{O}(0)\rangle. \tag{10}$$

We will now analyze Eqs. (9), (10), and (3) for the $SU(3)_2$-HSG model. This model only contains two self-conjugate solitons which we denote by $+$, $-$, and one unstable particle, which we call $\tilde{u}$. The corresponding scattering matrix was found [16] to be $S_{\pm\pm}=-1, S_{\pm\mp}(\theta)=\pm\tanh(\theta\pm\sigma-i\pi/2)/2$, which means the resonance pole is situated at $\theta_R=\mp\sigma-i\pi/2$. Stable bound states may not be formed. Note that for the corresponding value of $\bar{\sigma}=\pi/2$ and arbitrary σ, the condition $M_{\tilde{u}}\gg\Gamma_{\tilde{u}}$ is not satisfied. However, as indicated above this condition only helps to obtain a clearer identification of the mass parameter. For the HSG models this condition starts to hold when the level is large, which indicates that in these types of models this interpretation is, in fact, a semiclassical one.

A huge class of form factors corresponding to various operators related to this model were constructed in [15,12]. Labeling an operator by four quantum numbers μ,ν,τ,τ', the general n-particle solution reads

$$F_{2s+\tau,2t+\tau'}^{\mathcal{O}_{\tau,\tau'}^{\mu,\nu}|M^+M^-}(\theta_1,\ldots,\theta_n)=H_{2s+\tau,2t+\tau'}^{\mathcal{O}_{\tau,\tau'}^{\mu,\nu}|M^+M^-}\det\mathcal{A}_{2s+\tau,2t+\tau'}^{\mu,\nu}(\sigma_{2s+\tau}^{+})^{s-t+(\tau-1-\nu)/2}\times(\sigma_{2t+\tau'}^{-})^{(1+\tau-\tau'-\mu/2)-t}\prod_{i<j}\hat{F}^{\mu_i\mu_j}(\theta_{ij}). \quad (11)$$

We used here a particular ordering by starting with $2s+\tau$ particles of the type $\mu=+$ followed by $2s+\tau'$ particles of the type $\mu=-$, collected in the sets $M^{\pm}=\{\pm,\ldots,\pm\}$. Once these expressions are known, all other form factors related to it by permutations of the particles may be constructed trivially by exploiting Watson's equations [17] (see [15,12] for details concerning the HSG models). The functions $\hat{F}^{\mu_i\mu_j}$ for all combinations of the μ's are

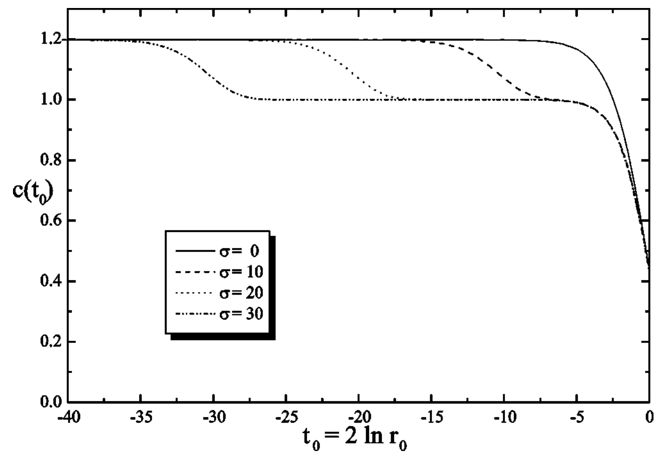

FIG. 1. Renormalization group flow for the Virasoro central charge $c(r_0)$ for various values of the resonance parameter σ.

$$\hat{F}^{\pm\pm}(\theta)=-i/2\tanh\frac{\theta}{2}\exp(\mp\theta/2), \quad (12)$$

$$\hat{F}^{\pm\mp}(\theta)=2^{1/4}e^{i\pi(1\mp1)/4\pm\theta/4-(G/\pi)-\int_0^\infty(dt/t)\sin^2[(i\pi-\theta\mp\sigma)t/2\pi]/\sinh t\cosh t/2}, \quad (13)$$

with $G=0.91597\ldots$ being the Catalan constant. The $(t+s)\times(t+s)$ matrix

$$(\mathcal{A}_{2s+\tau,2t+\tau'}^{\mu,\nu})_{ij}=\begin{cases}\sigma_{2(j-i)+\mu}^{+}, & 1\leq i\leq t\\ \hat{\sigma}_{2(j-i)+2t+\nu}^{-}, & t<i\leq s+t\end{cases} \quad (14)$$

has as its entries elementary symmetric polynomials (see, e.g., [18] for properties) depending on different sets of variables. We use the notation $\sigma^{\pm}$ when they depend on the variable $x=\exp\theta$ associated to the sets $M^{\pm}$ and $\hat{\sigma}$ to indicate that all variables are multiplied by a factor $ie^{-\sigma}$. The overall constant was computed to

$$H_{2s+\tau,2t+\tau'}^{\mathcal{O}_{\tau,\tau'}^{\mu,\nu}|M^+M^-}=i^{s(2\tau+\tau'+\nu+2)}2^{s(2s-2t-\tau'-1+2\tau)}\times e^{s\sigma(2t+\tau')/2}H_{\tau,2t+\tau'}^{\mathcal{O}_{\tau,\tau'}^{\mu,\nu}}, \quad (15)$$

where the value of $H_{\tau,2t+\tau'}^{\mathcal{O}_{\tau,\tau'}^{\mu,\nu}}$ is fixed by the lowest nonvanishing form factor. In particular we need

$$F_{2s,2t}^{\Theta}=\sigma_1(x_1,\ldots,x_n)\sigma_1(x_1^{-1},\ldots,x_n^{-1})F_{2s,2t}^{\mathcal{O}_{2,2}^{1,1}}. \quad (16)$$

Having assembled all the ingredients we can evaluate the expressions Eqs. (9) and (10). We carry out the integrals by means of a Monte Carlo computation. For $c(r_0)$ we take contributions up to the 4-particle form factor into account, and we display our results in Fig. 1.

Following the renormalization group flow from the ultraviolet to the infrared, Fig. 1 illustrates the flow from the $SU(3)_2/U(1)^2$- to the $SU(2)_2/U(1)\otimes SU(2)_2/U(1)$-coset when the unstable particle becomes massive. This qualitatively confirms the previous observation of the TBA analysis [10]. Here we also want to compare the value of the mass of the unstable particle at different points of the resonance parameter σ and t_0. Taking now the mass scales of the stable particles to be the same, i.e., $m_+=m_-=m$, we compute the mass of the unstable particle according to Eq. (3), i.e., $M_{\tilde{u}}(t_u,\sigma)\sim m/\sqrt{2}\exp((|\sigma|+t_u)/2)$. This means for different values of the resonance parameter we may still have the same value for the mass of the unstable particle when changing t_u; indeed, we find

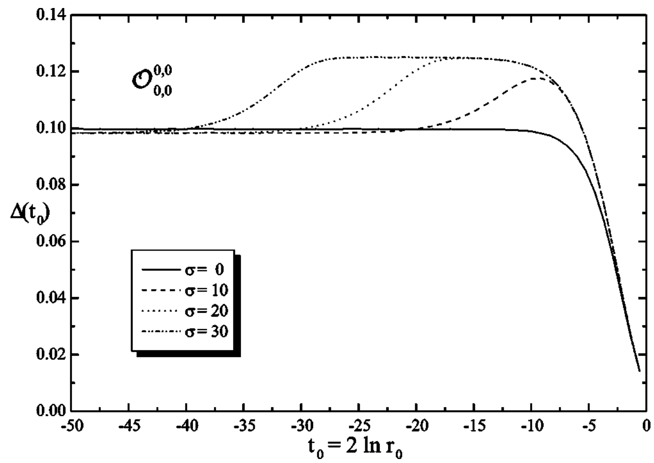

FIG. 2. Renormalization group flow for the conformal dimension $\Delta(r_0)$ of the operator $\mathcal{O}^{0,0}_{0,0}$ for various values of the resonance parameter σ.

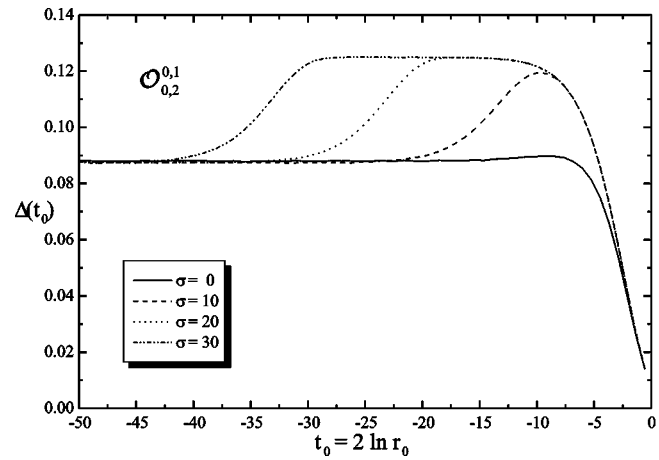

FIG. 3. Renormalization group flow for the conformal dimension $\Delta(r_0)$ of the operator $\mathcal{O}^{0,1}_{0,2}$ for various values of the resonance parameter σ.

$$M_{\tilde{u}}(-30.8,30) = M_{\tilde{u}}(-20.8,20) = M_{\tilde{u}}(-10.8,10). \qquad (17)$$

Since the flow between the two cosets is smooth and takes place over some range of t_0, we had to select one particular point t_u. As already indicated in general, it is convenient to identify $M_{\tilde{u}}$ as the point at which $c(t_0)$ is half the difference between the two coset values of c. It is clear from Fig. 1 that, since the overall shape of the curves between two values of c is identical for different values of σ, any other value in the interval would lead to the same results in comparative considerations. This also means that when evaluating Eq (17) the resulting value $0.47m$, which apparently violates the energetically necessary condition $M_{\tilde{u}} > m_a + m_b$, should not be taken too literally since the point t_u is only chosen because it is easy to fix. Equation (17) confirms our general assertions outlined above.

For the evaluation of the scaled conformal dimension, Eq. (10), we proceed similarly. For the solutions corresponding to the operators $\mathcal{O}^{0,0}_{0,0}$, $\mathcal{O}^{0,1}_{0,2}$, and $\mathcal{O}^{1,0}_{2,0}$, whose conformal dimension in the ultraviolet limit was identified [12] to be 1/10, we take up to six-particle form factors into account. For the former two operators our results are presented in Figs. 2 and 3.

We observe that the conformal dimension of the operator $\mathcal{O}^{0,0}_{0,0}$ flows to the value 1/8, which is twice the conformal dimension of the disorder operator μ in the Ising model. The factor two is expected from the mentioned coset structure, i.e., we find two copies of $SU(2)_2/U(1)$. The nature of the operator is also anticipated, since by construction $F_n^{\mathcal{O}^{0,0}_{0,0}|M^+M^-}$ of the $SU(3)_2$-HSG model coincides precisely with F_n^{μ} of the thermally perturbed Ising model when one of the sets $M^{\pm}$ is empty. It is also clear that we could alternatively obtain Eq. (17) from the analysis of $\Delta(r_0)$.

Despite the fact that the explicit expressions for the form factors of $\mathcal{O}^{0,1}_{0,2}$ and $\mathcal{O}^{1,0}_{2,0}$ differ, the values of $\Delta(r_0)$ are hardly distinguishable and we therefore omit the plots for the latter case. We also note the previously observed fact [12] that the higher particle contributions for the latter operators are more important than for $\mathcal{O}^{0,0}_{0,0}$, which explains the fact that the starting point at the ultraviolet fixed point is not quite 0.1. The operators also flow to the value 1/8, such that the degeneracy of the $SU(3)_2$-HSG model disappears surjectively when the unstable particles become massive.

In comparison with other methods it would be extremely desirable to elaborate on the precise relationship between $c(r_0)$ and the finite size scaling function of the thermodynamic Bethe ansatz. Also, the relation to the intriguing proposal in [19] of a renormalization group flow between Virasoro characters remains unclarified. The analog of $\Delta(r_0)$ still needs to be identified in the TBA as well as in the context of [19]. In addition one may pose the question whether there exists higher dimensional counterparts of the function $\Delta(r_0)$ in an analogy to the results obtained in [3] for $c(r_0)$. Concerning the specific status of the HSG models it remains a challenge to extend the results to other Lie groups [20].

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