

Spectrum of the U(1) staggered Dirac operator in four dimensions

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We compare the low-lying spectrum of the staggered Dirac operator in the confining phase of compact U(1) gauge theory on the lattice to predictions of chiral random matrix theory. The small eigenvalues contribute to the chiral condensate similarly to the SU(2) and SU(3) gauge groups. Agreement with the chiral unitary ensemble is observed below the Thouless energy, which is extracted from the data and found to scale with the lattice size according to theoretical predictions.

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I. INTRODUCTION

In recent years the spectrum of the Dirac operator in QCD and related theories has been studied in great detail, with regard, in particular, to its relation to chiral random matrix theory (RMT) [1] and, more recently, partially quenched chiral perturbation theory [2]. Both the distribution of the small eigenvalues and the spectral correlations in the bulk of the spectrum are described by universal functions that can be computed analytically in these theories. Universal in this context means independent of dynamical details and only dependent on certain global symmetries (and their spontaneous breaking). The spectral correlations are only universal below a certain limiting energy, which is called the Thouless energy [3] because of analogous situations first studied for disordered mesoscopic systems. This picture has been verified numerically in great detail by lattice calculations for the gauge groups SU(2) and SU(3) in four and three dimensions, and for the Schwinger model in two dimensions, see Ref. [4] for a summary.

In this paper we study the staggered lattice Dirac operator in quenched U(1) gauge theory in four Euclidean dimensions. The bulk spectral correlations of this operator have been investigated earlier in Ref. [5]. Here, we concentrate on the low-lying eigenvalues. We compare their distribution to predictions of chiral RMT and estimate the Thouless energy. Our study is not done merely for the sake of completeness. We are also motivated by the fact that, because of the different topological structure of U(1), the physics governing the small Dirac eigenvalues may be different from the non-Abelian case.

The standard lattice action describing compact U(1) gauge theory in 4D is given by

$$S\{U_l\} = \beta \sum_p (1 - \cos \theta_p), \quad (1)$$

where $\beta = 1/g^2$, $U_l = U_{x,\mu} = \exp(i\theta_{x,\mu})$, and $\theta_p = \theta_{x,\mu} + \theta_{x+\hat{\mu},\nu} - \theta_{x+\hat{\nu},\mu} - \theta_{x,\nu}$ for $\nu \neq \mu$. For $\beta < \beta_c \approx 1.01$, the

theory is in the confinement phase, exhibiting a mass gap and monopole excitations [6]. For $\beta > \beta_c$, the theory is in the Coulomb phase with a massless photon [7]. There are many interesting questions concerning the order of the transition between the two phases and the possibility of a nontrivial continuum limit for $\beta \rightarrow \beta_c^-$ [8]. Only the confinement phase exhibits chiral symmetry breaking, which has been addressed in a number of recent numerical studies [9–11]. In the strong-coupling limit $\beta \rightarrow 0$, chiral symmetry breaking follows rigorously from infrared bounds [12] and has also been calculated explicitly [13]. The broken phase is characterized by a chiral condensate that is determined by the small eigenvalues of the Dirac operator according to the Banks-Casher relation [14].

In Sec. II, we compute the Dirac spectrum in both phases, and investigate in more detail the properties of the small Dirac eigenvalues in the confinement phase. Section III discusses the Thouless energy that limits the universal regime described by chiral RMT, and conclusions are drawn in Sec. IV.

II. SMALL DIRAC EIGENVALUES

The staggered Dirac operator is constructed from the gauge fields according to

$$D_{xy} = \frac{1}{2a} \sum_{\mu} [\eta_{\mu}(x) U_{\mu}(x) \delta_{y,x+\hat{\mu}} - \text{H.c.}], \quad (2)$$

where a is the lattice spacing, which we shall set to unity in the following, and the η_{μ} are the staggered phases. For the purpose of comparing the spectrum of D to RMT predictions, we note that D is in the symmetry class of the chiral unitary ensemble (chUE) of RMT because it has complex matrix elements and no antiunitary symmetries.

The nonzero eigenvalues of D come in pairs of $\pm i\lambda_n$ with real λ_n . For convenience, we refer to the λ_n as the eigenvalues. The spectral density of the Dirac operator is given by

$$\rho(\lambda) = \left\langle \sum_n \delta(\lambda - \lambda_n) \right\rangle, \quad (3)$$

where the average is over all gauge field configurations, weighted by $\exp(-S)$ in the quenched theory. If chiral symmetry is spontaneously broken, the vacuum is characterized by a nonzero order parameter—the chiral condensate $\langle \bar{\psi}\psi \rangle$. The Banks-Casher relation [14] states that

$$\Sigma \equiv |\langle \bar{\psi}\psi \rangle| = \lim_{\varepsilon \rightarrow 0} \lim_{V \rightarrow \infty} \pi \rho(\varepsilon) / V, \quad (4)$$

where V is the four-volume. The order of the limits in this equation is important. Note that the condensate is due to an accumulation of Dirac eigenvalues close to $\lambda = 0$. The Dirac operator can also have eigenvalues equal to zero, but this is not the case for the staggered Dirac operator at finite lattice spacing.

If the Dirac spectrum corresponds to one of the RMT universality classes and supports a nonzero value of Σ , then the distribution of the smallest Dirac eigenvalues is described by the microscopic spectral density [15]

$$\rho_s^{(\nu)}(z) = \lim_{V \rightarrow \infty} \frac{1}{V\Sigma} \rho^{(\nu)}\left(\frac{z}{V\Sigma}\right), \quad z = \lambda V\Sigma. \quad (5)$$

The quantity ρ_s is a universal function that depends only on the number of massless (or very light) flavors N_f and the topological charge ν , which is equal to the number of exact zero modes of D that are stable under small perturbations of the gauge field. The superscript (ν) in Eq. (5) means that the average, according to Eq. (3), is only over the configurations with topological charges equal to ν . In our case, we have $N_f = 0$ since we studied the quenched theory. Furthermore, we take $\nu = 0$ because we are using staggered fermions which do not have exact zero modes at finite lattice spacing [16]. This point has been discussed in Refs. [17,18], and the only situation where deviations from the result for $\nu = 0$ have been observed with staggered fermions is the Schwinger model in two dimensions at very weak coupling [19]. On the other hand, Neuberger's overlap Dirac operator [20] allows for exact topological zero modes on the lattice, and lattice simulations with this operator indeed find agreement with the RMT predictions for $\nu \neq 0$ [21].

The microscopic spectral density can be computed analytically. The prediction of the chUE of RMT for this quantity is, for $N_f = \nu = 0$, [22]

$$\rho_s(z) = \frac{z}{2} [J_0^2(z) + J_1^2(z)], \quad (6)$$

where J denotes the Bessel function. We also consider the distribution of the smallest eigenvalue of D , for which the RMT result for $N_f = \nu = 0$ reads [23]

$$P(\lambda_{\min}) = \frac{(V\Sigma)^2 \lambda_{\min}}{2} e^{-(V\Sigma \lambda_{\min}/2)^2}. \quad (7)$$

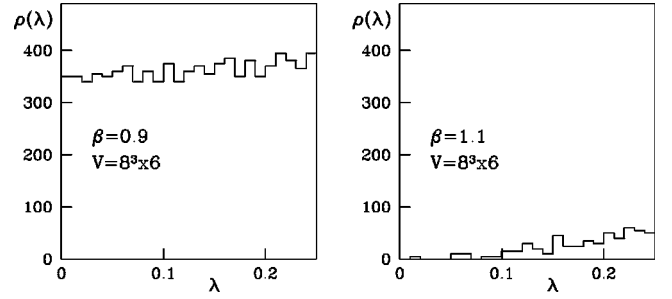


FIG. 1. Spectral density of the staggered Dirac operator on identical scales in the confinement (left) and Coulomb (right) phase, respectively. The average is over 20 configurations. A nonzero value of Σ is supported only in the confinement phase.

A comparison of lattice data with these predictions is only sensible if $\Sigma > 0$; i.e., if there is a sufficiently strong accumulation of small Dirac eigenvalues in the vicinity of $\lambda = 0$. In Fig. 1 we have plotted the spectral density of the staggered Dirac operator in this region, computed on an $8^3 \times 6$ lattice for $\beta = 0.9$ (confinement phase) and $\beta = 1.1$ (Coulomb phase), respectively. Clearly, a nonzero value of Σ is supported only in the confinement phase, and thus, the following analysis will be done only for this phase.

In non-Abelian gauge theories, the accumulation of the small Dirac eigenvalues is usually attributed to the presence of instantons. The argument is that the degeneracy of the exactly zero eigenvalues in the field of isolated instantons is lifted by interactions, leading to eigenvalue repulsion and a nonzero value of Σ . The topological structure of U(1) gauge theory in 4D is different, and evidence has been presented [9] which suggests that magnetic monopoles account for chiral symmetry breaking in the Abelian gauge theory. However, it is not quite clear whether the monopoles are really the driving mechanism, or if disorder alone would be sufficient, because it is difficult to disentangle disorder and monopole effects convincingly. Neither the rigorous arguments [12] nor the strong-coupling investigation [13] make use of any explicit mechanism.

Let us turn to the analysis of our data. We have computed the eigenvalues of the Dirac operator on lattices of size 4^4 , 6^4 , and $8^3 \times 6$ using $\beta = 0.9$ in the confinement phase. To compare the data to the RMT predictions, we determine the parameter Σ in Eq. (4) by extrapolating the spectral density many level spacings away from zero to $\lambda = 0$. This procedure is completely independent of RMT. As a check, we have also determined Σ via RMT. Using Eq. (7), the expectation value of λ_{\min} is given by

$$\langle \lambda_{\min} \rangle = \sqrt{\pi} / (V\Sigma), \quad (8)$$

which allows us to determine Σ from the numerical value of $\langle \lambda_{\min} \rangle$. Together with the numbers of configurations per parameter set, the values of Σ obtained from these two procedures are given in Table I. The two values of Σ are in excellent agreement, except for the smallest lattice size, where the agreement is not perfect but is still within error bars.

In Fig. 2, we have plotted the microscopic spectral density and the distribution of the smallest eigenvalue for all three

TABLE I. Summary of our simulations at $\beta=0.9$. The parameter Σ was obtained by two different procedures as described in the text.

V	Config.	Σ_{BC}	Σ_{RMT}
4^4	10,000	0.352(8)	0.345(2)
6^4	10,000	0.352(4)	0.353(2)
$8^3 \times 6$	3,745	0.353(3)	0.352(3)

lattice sizes, along with the predictions of the chUE of RMT. The lattice data for $P(\lambda_{\min})$ agrees perfectly with Eq. (7). The microscopic spectral density, Eq. (5), is also described well by RMT, Eq. (6), but the agreement breaks down for large values of z , with a critical value of z that increases with lattice size. This is essentially the Thouless energy, to which we now turn our attention.

III. THOULESS ENERGY

As mentioned earlier, the small Dirac eigenvalues are only described by universal functions for energies below a limiting scale—the Thouless energy. In QCD, this scale is

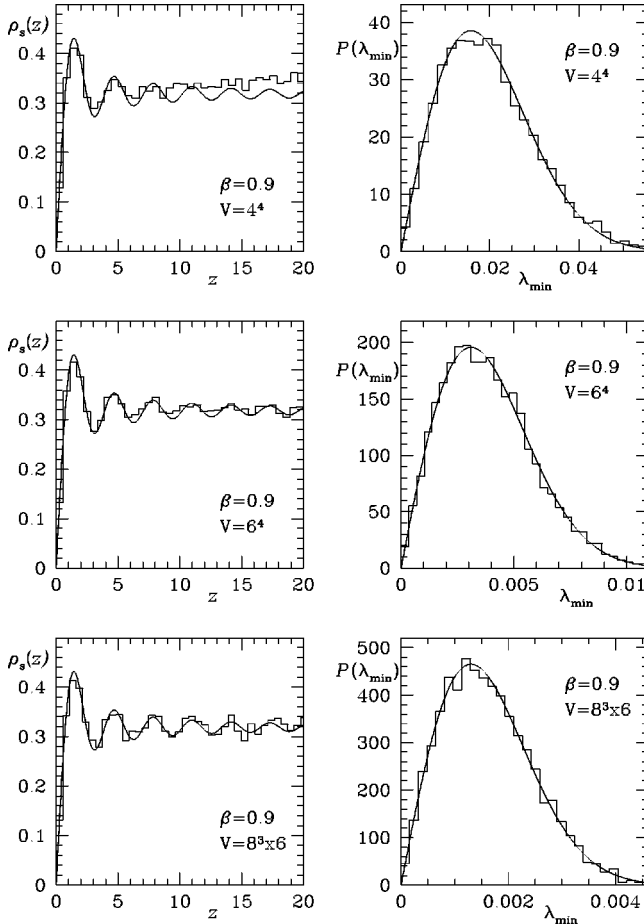


FIG. 2. Microscopic spectral density (left) and distribution of the smallest eigenvalue (right) of the Dirac operator for three different lattice sizes. The histograms represent lattice data, and the solid lines are the RMT predictions.

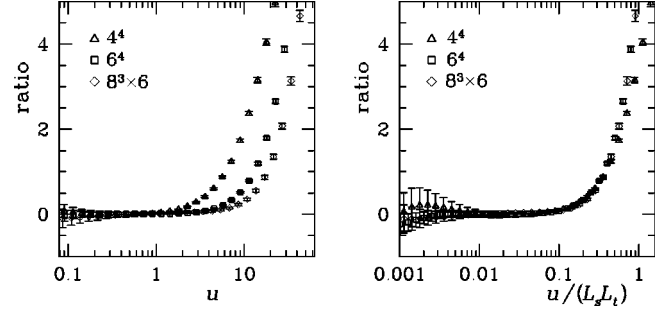


FIG. 3. The ratio of Eq. (12) plotted versus u (left) and $u/(L_s L_t)$ (right). The deviations of the ratio from zero, for very small values of u , are well-understood artifacts of the finite lattice size and finite statistics [27].

determined by the requirement that the inverse mass of the pion is equal to the largest linear size of the box with volume $V=L_s^3 \times L_t$, i.e., $1/m_\pi \approx \max(L_s, L_t)$ [24]. This can be translated to $E_c \sim f_\pi^2 / (\Sigma L_s^2)$ [25,26], where f_π is the pion decay constant and, in our case, we have $L_s \geq L_t$. A dimensionless estimate of the Thouless energy is obtained by expressing E_c in units of the mean level spacing at $\lambda=0$, given by $\Delta = 1/\rho(0) = \pi/(V\Sigma)$. This yields

$$u_c \equiv \frac{E_c}{\Delta} \sim \frac{1}{\pi} f_\pi^2 L_s L_t. \quad (9)$$

To test this prediction, we follow Ref. [27] and construct the disconnected scalar susceptibility, defined on the lattice by

$$\chi_{\text{latt}}^{\text{disc}}(m) = \frac{1}{N} \left\langle \sum_{k,l=1}^N \frac{1}{(i\lambda_k + m)(i\lambda_l + m)} \right\rangle - \frac{1}{N} \left\langle \sum_{k=1}^N \frac{1}{i\lambda_k + m} \right\rangle^2, \quad (10)$$

where m is a valence quark mass. The corresponding RMT result for the quenched chUE with $\nu=0$ reads [28]

$$\frac{\chi_{\text{RMT}}^{\text{disc}}(u)}{V\Sigma^2} = u^2 [I_0^2(u) - I_1^2(u)] [K_1^2(u) - K_0^2(u)], \quad (11)$$

where $u = mV\Sigma$ and I and K are modified Bessel and Hankel functions. The quantity $\chi_{\text{latt}}^{\text{disc}}$ should be described by Eq. (11) for $u < u_c$. The dimensionless Thouless energy can be extracted by inspecting the ratio [27]

$$\text{ratio} = (\chi_{\text{latt}}^{\text{disc}} - \chi_{\text{RMT}}^{\text{disc}}) / \chi_{\text{RMT}}^{\text{disc}}. \quad (12)$$

This quantity should be zero for $u < u_c$, and deviate from zero for $u > u_c$. The data for this ratio, computed at $\beta=0.9$ for our three lattice sizes, are shown in Fig. 3. Consider first the left plot. It is clear that u_c increases with increasing lattice size. To test the scaling predicted by Eq. (9), the same data are shown in the right plot, but now plotted versus

$u/(L_s L_t)$. The data for different lattice sizes now fall on the same curve, confirming the predicted scaling behavior of the Thouless energy.

IV. DISCUSSION

We have shown that in the confinement phase of compact U(1) gauge theory on the lattice, the distribution of the small Dirac eigenvalues is described by universal functions that can be computed in chiral RMT. The limiting energy, above which nonuniversal behavior emerges, scales with the lattice size as expected.

The origin of the small eigenvalues in U(1) gauge theory deserves further attention. The question may be less about a mechanism in the strong-coupling limit, where it appears that the disorder of the gauge fields could be sufficient. The important question is about a mechanism that could sustain chiral symmetry breaking for a large correlation length, eventually leading to a confined QED continuum theory for $\beta \rightarrow \beta_c^-$. There, U(1) monopoles could play a crucial role. Instantons appear to provide such a mechanism for SU(2) and SU(3) non-Abelian gauge theories.

In the Coulomb phase, chiral symmetry is restored, so the critical value for the chiral phase transition, β_c' , cannot be larger than β_c . However, we know of no strict argument that says confinement implies chiral symmetry breaking, so it is possible, at least in principle, that $\beta_c' < \beta_c$. (In supersymmetric theories, one can have confinement without chiral symmetry breaking [29].) Because the chiral condensate is directly related to the distribution of small eigenvalues, the chiral phase transition can be studied by observing the distribution of, say, the smallest positive eigenvalue for $\beta \rightarrow \beta_c^-$ [30]. This is another reason why it would be interesting to study this limit in future works.

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- [1] For a recent review and a list of references, see J. J. M. Verbaarschot and T. Wettig, hep-ph/0003017.
 - [2] J. C. Osborn, D. Toublan, and J. J. M. Verbaarschot, Nucl. Phys. **B540**, 317 (1999); P. H. Damgaard, J. C. Osborn, D. Toublan, and J. J. M. Verbaarschot, *ibid.* **B547**, 305 (1999).
 - [3] D. J. Thouless, Phys. Rep. **13**, 93 (1974).
 - [4] T. Wettig, hep-lat/9905020.
 - [5] B. A. Berg, H. Markum, and R. Pullirsch, Phys. Rev. D **59**, 097504 (1999).
 - [6] A. M. Polyakov, Phys. Lett. **59B**, 82 (1975); S. Mandelstamm, Phys. Rep. **23**, 245 (1976); G. 't Hooft, *High Energy Physics*, edited by A. Zichichi (Editrice Compositori, Bologna, 1976); T. Banks, R. J. Myerson, and J. Kogut, Nucl. Phys. **B129**, 493 (1977); T. A. DeGrand and T. Toussaint, Phys. Rev. D **22**, 2478 (1980).
 - [7] B. A. Berg and C. Panagiotakopoulos, Phys. Rev. Lett. **52**, 94 (1984).
 - [8] See, e.g., J. Jersák, C. B. Lang, and T. Neuhaus, Phys. Rev. Lett. **77**, 1933 (1996), and references therein.
 - [9] T. Bielefeld, S. Hands, J. D. Stack, and R. J. Wensley, Phys. Lett. B **416**, 150 (1998).
 - [10] J. Cox, W. Franzki, J. Jersák, C. B. Lang, and T. Neuhaus, Nucl. Phys. **B532**, 315 (1998).
 - [11] A. Hoferichter, V. K. Mitryushkin, M. Müller-Preussker, and H. Stüben, Phys. Rev. D **58**, 114505 (1998).
 - [12] M. Salmhofer and E. Seiler, Commun. Math. Phys. **139**, 395 (1991).
 - [13] H. Gausterer and C. B. Lang, Phys. Lett. B **263**, 476 (1991).
 - [14] T. Banks and A. Casher, Nucl. Phys. **B169**, 103 (1980).
 - [15] E. V. Shuryak and J. J. M. Verbaarschot, Nucl. Phys. **A560**, 306 (1992).
 - [16] J. Smit and J. C. Vink, Nucl. Phys. **B286**, 485 (1987).
 - [17] M. E. Berbenni-Bitsch, S. Meyer, A. Schäfer, J. J. M. Verbaarschot, and T. Wettig, Phys. Rev. Lett. **80**, 1146 (1998).
 - [18] P. H. Damgaard, U. M. Heller, R. Niclasen, and K. Rummukainen, Phys. Rev. D **61**, 014501 (2000).
 - [19] F. Farchioni, I. Hip, and C. B. Lang, Phys. Lett. B **471**, 58 (1999).
 - [20] H. Neuberger, Phys. Lett. B **417**, 141 (1998); R. Narayanan and H. Neuberger, Nucl. Phys. **B443**, 305 (1995).
 - [21] R. G. Edwards, U. M. Heller, J. Kiskis, and R. Narayanan, Phys. Rev. Lett. **82**, 4188 (1999).
 - [22] J. J. M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **70**, 3852 (1993).
 - [23] P. J. Forrester, Nucl. Phys. **B402**, 709 (1993).
 - [24] J. J. M. Verbaarschot, Phys. Lett. B **368**, 137 (1996).
 - [25] J. C. Osborn and J. J. M. Verbaarschot, Nucl. Phys. **B525**, 738 (1998); Phys. Rev. Lett. **81**, 268 (1998).
 - [26] R. A. Janik, M. A. Nowak, G. Papp, and I. Zahed, Phys. Rev. Lett. **81**, 264 (1998).
 - [27] M. E. Berbenni-Bitsch, M. Göckeler, T. Guhr, A. D. Jackson, J.-Z. Ma, S. Meyer, A. Schäfer, H. A. Weidenmüller, T. Wettig, and T. Wilke, Phys. Lett. B **438**, 14 (1998).
 - [28] M. Göckeler, H. Hehl, P. E. L. Rakow, A. Schäfer, and T. Wettig, Phys. Rev. D **59**, 094503 (1999).
 - [29] See section 4.3 of K. Intriligator and N. Seiberg, Nucl. Phys. B (Proc. Suppl.) **45B/C**, 1 (1996).
 - [30] F. Farchioni, P. de Forcrand, I. Hip, C. B. Lang, and K. Splittorff, Phys. Rev. D **62**, 014503 (2000); P. H. Damgaard, U. M. Heller, R. Niclasen, and K. Rummukainen, Nucl. Phys. **B583**, 347 (2000).