Summing Sudakov logarithms in $B \to X_s \gamma$ in effective field theory

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We construct an effective field theory valid for processes in which highly energetic light-like particles interact with collinear and soft degrees of freedom, using the decay $B \to X_s \gamma$ near the end point of the photon spectrum, $x = 2E_\gamma/m_b \to 1$, as an example. Below the scale $\mu = m_b$ both soft and collinear degrees of freedom are included in the effective theory, while below the scale $\mu = m_b\sqrt{x-y}$, where $1-y$ is the light cone momentum fraction of the b quark in the B meson, we match onto a theory of bilocal operators. We show that at one loop large logarithms cancel in the matching conditions, and that we recover the well-known renormalization group equations that sum leading Sudakov logarithms.

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I. INTRODUCTION

Effective field theories (EFT's) provide a simple and elegant method for calculating processes with several relevant energy scales [1]. Part of the utility of EFT's is that they dramatically simplify the summation of powers of logarithms of ratios of mass scales, which would otherwise make perturbation theory poorly behaved. For example, in a theory with a very heavy particle of mass M, one-loop corrections will typically be enhanced by $\log(M/\lambda)$, where λ is a low scale in the problem. In the EFT in which the heavy particle has been removed from the theory, such logarithms are replaced by factors of $\log(\mu/\lambda)$ (where μ is the renormalization scale in dimensional regularization, or the cutoff in cutoff regularization), and the complete series of leading logarithms $\alpha_s^n \log^n(\mu/\lambda)$ is straightforward to sum via the renormalization group.

The situation is more complicated for processes with highly energetic light particles. In this case, there are both collinear and infrared divergences in the theory, which give rise to the familiar Sudakov double logarithms [2]. For example, the perturbative expansion of the Nth moment of the photon spectrum in inclusive $b \to X_s \gamma$ decay is of the form

$$\sum_n \sum_{m \leqslant 2n} C_{n,m} \alpha_s^n \log^m N. \tag{1}$$

Although the arguments of these logarithms are not obviously the ratio of two scales, they arise because the typical energy and invariant mass of light particles are widely separated, and they may be summed via well-known techniques based on factorization theorems [3] into the form

$$\exp\left[\sum_n (a_n \alpha_s^n \log^{n+1} N + b_n \alpha_s^n \log^n N) + \cdots\right]. \tag{2}$$

The terms $\alpha_s^n \log^{n+1} N$ are referred to as the leading logarithmic contribution, the terms $\alpha_s^n \log^n N$ are referred to as the next-to-leading logarithmic contribution, and the remaining terms are called subdominant.

Recently there has been some discussion in the literature of summing Sudakov logarithms using effective field theory techniques [4–6]. Such an approach could have several advantages over the conventional method; in particular, while factorization formulas are based on perturbation theory, EFT's, by construction, are valid beyond perturbation theory, and by including higher dimension operators it should be straightforward (if tedious) to go beyond the leading twist approximation. In the various versions of the EFT approach which have been suggested, the effective theory is the so-called "large energy effective theory" (LEET) [7], which describes light-like particles coupled to soft degrees of freedom. However, a difficulty with the approaches presented to date is that, as pointed out in Refs. [6], in the minimal subtraction (MS) scheme logarithms arising at one loop in LEET do not match logarithms arising at one loop in QCD for any choice of the matching scale μ; hence these logarithms may not be summed using the renormalization group equations (RGE's).

In this paper we consider this problem in the context of $B \to X_s \gamma$ decays.[1] We show that the problem of matching scales may be resolved by introducing a new intermediate effective theory containing both soft and collinear degrees of freedom, which is then matched onto LEET (effectively integrating out the collinear modes) at a lower scale. We show that the matching conditions onto both effective theories contain no large logarithms at one loop. We then calculate the RGE's in the two theories summing the leading logarithms and a certain subset of the next-to-leading logarithms. To this order the expression obtained for the resummed Sudakov logarithms is identical to that derived in Refs. [9], [10].

II. SUDAKOV LOGARITHMS IN $B \to X_s \gamma$ AND LEET

Inclusive decays of heavy quarks have been well understood for many years in the context of an operator product

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[1] In fact, the authors of [8] argued that the resummation of subleading Sudakov logarithms is not necessary for practical purposes for this decay. Nevertheless, it provides a simple example in which we may compare our results to those in the literature.

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expansion (OPE) in the inverse mass of the heavy quark [11]. At leading order in the $\Lambda_{\rm QCD}/m_b$ expansion the B meson decay rate is equal to the b quark decay rate, and nonperturbative effects are suppressed by at least two powers of $\Lambda_{\rm QCD}/m_b$. However, the OPE only converges for sufficiently inclusive observables. Unfortunately, experimental cuts on measurements of rare decays such as $B\to X_s\gamma$, $B\to X_s l^+l^-$, and $B\to X_u l\bar{\nu}$ are required, restricting the available phase space considerably. Since all of these decays are of phenomenological interest, either in the determination of $|V_{ub}|$ or detection of new physics, understanding inclusive decays in restricted regions of phase space is important.

If the phase space is restricted such that the final hadronic state is dominated by only a few resonances, the breakdown of the OPE simply reflects the fact that an inclusive treatment based on local duality is no longer appropriate. This is the case for the dilepton invariant mass spectrum in inclusive $B\to X_s l^+l^-$ and $B\to X_u l\bar{\nu}$ decays [12]. However, when the kinematic cut is in a region of phase space dominated by highly energetic, low invariant mass final states, the OPE breaks down even for quantities smeared over a parametrically larger region of phase space, where the decay is not resonance dominated. This situation arises in the end-point region of the electron energy spectrum and the low hadronic invariant mass region in semileptonic $B\to X_u l\bar{\nu}$ decay, as well as the end-point region of the photon spectrum in $B\to X_s\gamma$ decay [8,13].

Consider the dominant contribution to the decay $B\to X_s\gamma$, which arises from the magnetic penguin operator [14]

$$\hat{O}_7=\frac{e}{16\pi^2}m_b\bar{s}\sigma^{\mu\nu}\frac{1}{2}(1+\gamma_5)bF_{\mu\nu}, \tag{3}$$

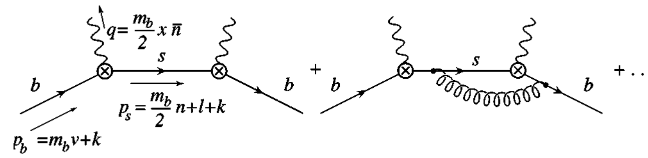

FIG. 1. The OPE for $B\to X_s\gamma$.

where the strange quark mass has been set to zero.[2] The OPE for this decay is illustrated in Fig. 1. We write the momenta of the b quark, photon, and light s quark jet as

$$p_b^\mu=m_bv^\mu+k^\mu,\quad q^\mu=\frac{m_b}{2}x\bar{n}^\mu,\quad p_s^\mu=\frac{m_b}{2}n^\mu+l^\mu+k^\mu, \tag{4}$$

where, in the rest frame of the B meson,

$$v^\mu=(1,\vec{0}),\quad n^\mu=(1,0,0,-1),\quad \bar{n}^\mu=(1,0,0,1). \tag{5}$$

Here k^μ is a residual momentum of order $\Lambda_{\rm QCD}$, and $l^\mu=m_b/2(1-x)\bar{n}^\mu$, where $x=2E_\gamma/m_b$. The invariant mass of the light s-quark jet

$$p_s^2\approx m_b n\cdot(l+k)=m_b^2(1-x+\hat{k}^+) \tag{6}$$

(where $\hat{k}^+=k^+/m_b$) is $O(m_b^2)$ except near the end point of the photon energy spectrum where $x\to 1$. Inclusive quantities are calculated via the OPE by taking the imaginary part of the graphs in Fig. 1 and expanding in powers of $k^\mu/\sqrt{p_s^2}$. As long as x is not too close to the end point, this is an expansion in powers in k^μ/m_b, which matches onto local operators. This leads to an expansion for the photon energy spectrum as a function of x in powers of α_s and $1/m_b$ [15]:

$$\frac{d\Gamma}{dx}=\Gamma_0\Bigg\{\left[1-\frac{\alpha_sC_F}{4\pi}\left(2\log\frac{\mu^2}{m_b^2}+5+\frac{4}{3}\pi^2\right)\right]\delta(1-x)+\frac{\alpha_sC_F}{4\pi}\left[7+x-2x^2-2(1+x)\log(1-x)-\left(4\frac{\log(1-x)}{1-x}+\frac{7}{1-x}\right)_+\right]$$
$$+\frac{1}{2m_b^2}\left[(\lambda_1-9\lambda_2)\delta(1-x)-(\lambda_1+3\lambda_2)\delta'(1-x)-\frac{\lambda_1}{3}\delta''(1-x)\right]\Bigg\}+O(\alpha_s^2,1/m_b^3), \tag{7}$$

where

$$\Gamma_0=\frac{G_F^2|V_{tb}V_{ts}^*|^2\alpha|C_7(\mu)|^2}{32\pi^4}m_b^5\left[\frac{m_b(\mu)}{m_b}\right]^2, \tag{8}$$

and the subscript "+" denotes the usual plus distribution,

$$\frac{1}{(1-x)_+}\equiv\lim_{\beta\to 0}\left\{\frac{1}{1-x}\theta(1-x-\beta)+\log(\beta)\delta(1-x-\beta)\right\}, \tag{9}$$

$$\left(\frac{\log(1-x)}{(1-x)}\right)_+\equiv\lim_{\beta\to 0}\left\{\frac{\log(1-x)}{1-x}\theta(1-x-\beta)+\frac{1}{2}\log^2(\beta)\delta(1-x-\beta)\right\}.$$

The parameters λ_1 and λ_2 are matrix elements of local dimension-five operators.

Near the end point of the photon spectrum, $x\to 1$, both the

[2]Throughout this work we will ignore the contribution of operators other than $\hat{O}_7$ to the decay.

perturbative and nonperturbative corrections are singular and the OPE breaks down. The severity of the breakdown is most easily seen by integrating the spectrum over a region $1-\Delta<x<1$. When $\Delta\lessdot\Lambda_{\rm QCD}/m_b$ the most singular terms in the $1/m_b$ expansion sum up into a nonperturbative shape function of characteristic width $\Lambda_{\rm QCD}/m_b$ [16]. The perturbative series is of the form

$$\frac{1}{\Gamma_0}\int_{1-\Delta}^{1}\frac{d\Gamma}{dx}=1+\frac{\alpha_s C_F}{4\pi}(-2\log^2\Delta-7\log\Delta+\cdots)+O(\alpha_s^2),\tag{10}$$

where the ellipses denote terms that are finite as $\Delta\rightarrow 0$. These Sudakov logarithms are large for $\Delta\ll 1$, and can spoil the convergence of perturbation theory. The full series has been shown to exponentiate [9,10] and the leading and next-to-leading logarithms must be resummed for $\Delta\lessdot\exp(-\sqrt{\pi/\alpha_s(m_b)})$, which is parametrically larger than $\Lambda_{\rm QCD}/m_b$ in the $m_b\rightarrow\infty$ limit [17].

In general, ''phase space'' logarithms are to be expected whenever a decay depends on several distinct scales. For example, in $b\rightarrow X_c e\bar{\nu}_e$ decay the rate calculated with the OPE performed at $\mu=m_b$ contains large logarithms of m_c/m_b. In [18] an EFT was used to run from m_b to m_c, summing phase space logarithms of the ratio m_c/m_b. Similarly, in $b\rightarrow X_s\gamma$ near the end point of the photon energy spectrum the invariant mass of the light quark jet scales as $m_b\sqrt{1-x}$, and is widely separated from the scale $\mu=m_b$ where the OPE is performed. In order to sum logarithms of Δ [or the more complicated plus distributions in the differential spectrum, Eq. (7)] we would expect to have to switch to a new effective theory at $\mu=m_b$, use the renormalization group to run down to a scale of order $m_b\sqrt{1-x}$, at which point the OPE is performed. (In fact, we will see that the situation is slightly more complicated than this.)

We are then left with the question of the appropriate theory below the scale m_b. The simplest possibility is to expand the theory in powers of k^μ/m_b and l^μ/m_b. The heavy quark is then treated in the heavy quark effective theory (HQET) [19], while the light quark propagator is treated in the large energy effective theory (LEET) proposed many years ago by Dugan and Grinstein [7]. Expanding the s quark propagator in powers of $1/m_b$, we find the LEET propagator

$$\frac{i\not{p}_s}{p_s^2}=\frac{\not{n}}{2}\,\frac{i}{n\cdot(l+k)}+O\left(\frac{l^\mu+k^\mu}{m_b}\right).\tag{11}$$

LEET is an effective theory of lightlike Wilson lines, much as heavy quark effective theory (HQET) is an effective theory of timelike Wilson lines [4]. The hope would then be to match QCD onto LEET and then use the renormalization group to sum the Sudakov logarithms. This is the approach taken in [6]. However, a simple attempt at matching shows that this does not sum the appropriate logarithms.

Consider the one-loop matching of the operator $\hat{O}_7$ from QCD to LEET. We regulate ultraviolet divergences with dimensional regularization ($d=4-2\epsilon$). We introduce a small invariant mass p_s^2 for the s quark which regulates all infrared

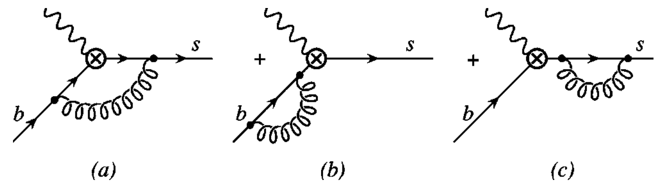

FIG. 2. One-loop corrections to the matrix element of $\hat{O}_7$ in QCD.

(IR) divergences except that in the heavy-quark wave function diagram, Fig. 2(b). This IR divergence is regulated using dimensional regularization. The vertex diagram, Fig. 2(a), yields

$$A_{\rm QCD}^{(a)}=-C_7(\mu)\bar{s}\Gamma^\mu b\,\frac{\alpha_s C_F}{4\pi}\left[\log^2\frac{p_s^2}{m_b^2}+2\log\frac{p_s^2}{m_b^2}+\cdots\right],\tag{12}$$

where

$$\Gamma^\mu=\frac{e}{8\pi^2}m_b\sigma^{\mu\nu}\frac{(1+\gamma_5)}{2}q_\nu.\tag{13}$$

$C_7(\mu)$ is the Wilson coefficient of $\hat{O}_7$ and the dots denote (here and in the rest of the paper) finite terms which are not logarithmically enhanced. Including a factor of $\sqrt{Z}$ for each external field

$$Z_b=1-\frac{\alpha_s C_F}{4\pi}\left[\frac{3}{\epsilon}+3\log\frac{\tilde{\mu}^2}{m_b^2}+\cdots\right],\tag{14}$$

$$Z_s=1-\frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon}-\log\frac{p_s^2}{m_b^2}+\log\frac{\tilde{\mu}^2}{m_b^2}+\cdots\right],\tag{15}$$

where

$$\tilde{\mu}^2\equiv 4\pi\mu^2 e^{-\gamma_E}\tag{16}$$

and adding the counterterm required to subtract off the UV divergence

$$Z_7=1+\frac{\alpha_s C_F}{4\pi}\,\frac{1}{\epsilon},\tag{17}$$

we find

$$A_{\rm QCD}=C_7(\mu)\bar{s}\Gamma^\mu b\left[1-\frac{\alpha_s C_F}{4\pi}\left(\log^2\frac{p_s^2}{m_b^2}+\frac{3}{2}\log\frac{p_s^2}{m_b^2}+\frac{1}{\epsilon}+2\log\frac{\tilde{\mu}^2}{m_b^2}+\cdots\right)\right].\tag{18}$$

The corresponding LEET diagram is shown in Fig. 3. Neither of the wave-function graphs gives a contribution, since the light quark wave function in Feynman gauge[3] is

[3]We will work in Feynman gauge throughout this paper.

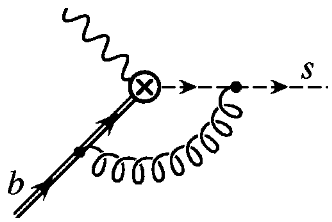

FIG. 3. One-loop correction to the $bs\gamma$ vertex in LEET.

proportional to $n^2=0$, and the heavy quark wave function vanishes in dimensional regularization. Thus the only contribution is from the vertex graph. Denoting the coefficient of the corresponding operator in LEET as $C^{(0)}[1+(\alpha_s C_F/4\pi)C^{(1)}+\cdots]$, we find

$$A_{\rm LEET}=C^{(0)}(\tilde\mu)\bar\xi_n\Gamma^\mu h\left\{1-\frac{\alpha_s C_F}{4\pi}\left[\left(4\pi\frac{\mu^2 m_b^2}{p_s^4}\right)^\epsilon\right.\right.$$
$$\left.\left.\times\frac{\Gamma(1+2\epsilon)\Gamma(1-2\epsilon)\Gamma(1+\epsilon)}{\epsilon^2}-C^{(1)}(\tilde\mu)\right]\right\}$$
$$=C^{(0)}(\tilde\mu)\bar\xi_n\Gamma^\mu h\left[1-\frac{\alpha_s C_F}{4\pi}\left(\frac{1}{\epsilon^2}-\frac{2}{\epsilon}\log\frac{p_s^2}{m_b\tilde\mu}\right.\right.$$
$$\left.\left.+2\log^2\frac{p_s^2}{m_b\tilde\mu}+\cdots-C^{(1)}(\tilde\mu)\right)\right],\qquad(19)$$

where p_s^2/m_b is the soft scale. So

$$C^{(0)}(\tilde\mu)=C_7(\tilde\mu),$$
$$C^{(1)}(\tilde\mu)=\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\left(2\log\frac{p_s^2}{m_b\tilde\mu}+1\right)+2\log^2\frac{p_s^2}{m_b\tilde\mu}$$
$$-\log^2\frac{p_s^2}{m_b^2}-\frac{3}{2}\log\frac{p_s^2}{m_b^2}-2\log\frac{\tilde\mu^2}{m_b^2}+\cdots.\qquad(20)$$

We immediately notice two problems.[4]

(1) There is no matching scale $\tilde\mu$ at which all the large single and double logarithms in $C^{(1)}$ vanish. Thus, there are logarithms in the rate which cannot be summed using the renormalization group in LEET.

(2) $C^{(1)}$ contains a divergence proportional to $(1/\epsilon)\log p_s^2$. Since p_s^2 is an infrared scale in the problem, it is not clear how to sensibly renormalize this term. In Ref. [6] this divergence was cancelled by a nonlocal counterterm in the inclusive rate; however, this term indicates that LEET cannot be used for exclusive processes [20]. Furthermore, the matching of the inclusive rate performed in [6] still leaves large logarithms in the coefficient of the operator.

The problem is that LEET only describes the coupling of light-like particles to soft gluons, but does not describe the splitting of an energetic particle into two almost collinear particles. Thus, by matching onto LEET, one is integrating out the collinear modes which also contribute to infrared physics. As we will show below, once collinear degrees of freedom are included, both of the above problems are resolved.

III. THE COLLINEAR-SOFT THEORY

A. Collinear and soft modes

It is convenient to work in light-cone coordinates $p^\mu=(p^+,p^-,p_\perp^i)$, where $p^+=n\cdot p$ and $p^-=\bar n\cdot p$, and to define a power-counting parameter $\lambda=\sqrt{1-x}$ that becomes small in the limit $x\to1$. The momentum of the light-quark jet then scales as

$$p_s^\mu\sim m_b(\lambda^2,1,\lambda).\qquad(21)$$

This scaling is unchanged by emission of either soft or collinear degrees of freedom, with momenta scaling as

$$p_{\rm soft}\sim m_b(\lambda^2,\lambda^2,\lambda^2),\quad p_{\rm collinear}\sim m_b(\lambda^2,1,\lambda).\qquad(22)$$

and so emission of both modes is kinematically allowed. It is the presence of infrared sensitive graphs with collinear loop momentum that makes this EFT more complicated than other, more familiar, EFT's, where infrared sensitivity comes purely from soft modes. This is similar to the situation in nonrelativistic QCD (NRQCD) [21], in which power counting is complicated by the fact that a given amplitude receives contributions from loop momenta which are small compared to the heavy quark mass, but which have parametrically different dependence on the heavy quark velocity v. In NRQCD, the relevant scales are known as soft, ultrasoft, and potential, and must be treated separately in order to obtain consistent power counting [22,23].

We follow a similar approach here, and introduce separate fields for both soft and collinear degrees of freedom.[5] Between the scales m_b and $m_b\lambda$ the effective theory contains separate fields for both collinear and soft modes, while at scales below $\sim m_b\lambda$ (the exact scale depends on the operator under consideration, as will be discussed in the next section), the collinear modes are integrated out of the theory and it is matched onto LEET. We will refer to this intermediate theory as the collinear-soft theory, and resist the urge to create another acronym.

There is an important difference between the approach taken here and the one taken in Refs. [24,25] where logarithms of v are summed in NRQCD and NRQED. In the latter case no intermediate theory is introduced; instead the running is performed in one step through the velocity RGE. In NRQED these two approaches differ at subleading order [25,26], and it may be that such one-step running is needed here at two loops.

[4]Note that $C_7(\mu)$ includes a factor of $\alpha_s C_F/(4\pi)\log(m_W/\mu)$, which converts one of the factors of $\log(\mu/m_b)$ in Eq. (12) to $\log(m_W/m_b)$. This is not important for our argument.

[5]At two loops an additional gluon field scaling as $(\lambda,\lambda,\lambda)$ might have to be included [27].

TABLE I. Power counting rules for fields in the collinear-soft theory in Feynman gauge, where $\lambda=\sqrt{1-x}$.

Factor	Scaling
Soft gluon A^s_μ	λ^2
Collinear gluon A^c_μ	λ
Heavy quark h	λ^3
Collinear quark ξ	λ
Collinear volume element, d^4x_c	λ^{-4}
Soft volume element d^4x_s	λ^{-8}

The power counting rules in the collinear-soft theory may be obtained by a field rescaling, analogous to that performed in [23]. The scaling of the fields is chosen such that the propagators are all $O(1)$, putting the λ dependence into the interaction terms. For example, in the kinetic term for a soft gluon,

$$\sim\int d^4x(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu)^2 \tag{23}$$

the typical length scale associated with soft excitations scales as $\lambda^{-2}\sim p^{-1}_{\text{soft}}$, so the factor of d^4x scales as λ^{-8}. Each derivative scales as $p_{\text{soft}}\sim\lambda^2$, so the soft gluon field must scale as λ^2 for the kinetic term to be $O(1)$.

Since the various collinear momentum components scale differently with λ, power counting for collinear gluons is gauge dependent (this is easily seen from the propagator, since in a covariant gauge the components of the $k^\mu k^\nu$ term scale differently). In this paper we are working in Feynman gauge, in which case the different components of collinear gluons have the same scaling. Performing a similar analysis for the other fields, we obtain the power-counting rules given in Table I.

Rather than write down the effective Lagrangian for the various fields, which is quite lengthy, we will instead just give the Feynman rules, which are obtained by expanding the QCD amplitudes in powers of λ. The spinors in the collinear-soft theory are related, at leading order in λ, to the QCD spinors via

$$h_v=P_+u,\quad \xi_n=P_nu,\quad \xi_{\bar n}=P_{\bar n}u, \tag{24}$$

where we have defined the projection operators

$$P_+=\frac{\not{v}+1}{2},\quad P_n=\frac{\not{n}\not{\bar n}}{4},\quad P_{\bar n}=\frac{\not{\bar n}\not{n}}{4}, \tag{25}$$

which project out the heavy quark spinor, a massless spinor in the n direction, and a massless spinor in the $\bar n$ direction, respectively. The propagators for the different fields are shown in Fig. 4.

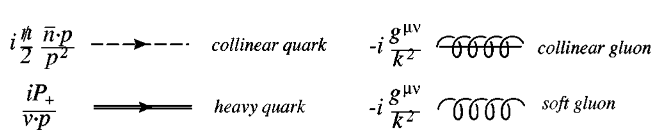

FIG. 4. Propagators in the collinear-soft effective theory.

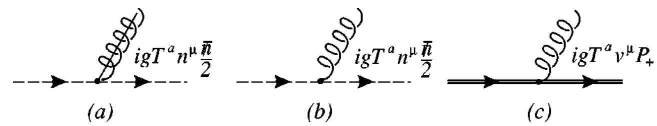

FIG. 5. Leading-order quark-gluon interactions in the collinear-soft effective theory: (a) collinear-collinear, (b) collinear-soft, and (c) heavy-soft. Applying the rules from Table I, the vertices scale as (a) λ^{-1}, (b) λ^0, and (c) λ^0.

The interactions leading in λ which we will need in this paper are shown along with their Feynman rules and scaling in Fig. 5. Note that the interaction of a soft particle with a collinear particle leaves the minus and perpendicular momenta of the collinear particle unchanged, since they are parametrically larger for the collinear particle. This is analogous to the multipole expansion which is performed in NRQCD [28]. As a result at one loop, soft-collinear interactions in this theory are equivalent to LEET, since collinear propagators in soft loops reduce to LEET propagators:

$$\frac{\not{n}}{2}\frac{\bar n\cdot(p-k)}{(p-k)^2}\sim\frac{\not{n}}{2}\frac{p^-}{(p-k)^+p^--(p^\perp)^2}=-\frac{\not{n}}{2}\frac{1}{n\cdot k}, \tag{26}$$

where p is a collinear momentum, k is a soft momentum, and $p^2=0$ from the equations of motion. Once again, this is analogous to NRQCD, where in ultrasoft loops the Feynman rules reduce to those for HQET. By the same token, in soft-collinear interactions, the appropriate volume element is the collinear volume element, scaling as λ^{-4}.

Because the leading purely collinear interaction, Fig. 5(a), scales as λ^{-1}, power counting for collinear loops is less simple than for soft loops. Terms which would scale as λ^{-2}, such as the purely collinear wave-function graph in Fig. 6, are proportional to $n^2=0$ and so vanish in the effective theory. However, the $1/\lambda$ coupling enhances terms which would naively be suppressed. In fact, although the λ counting looks complicated, graphs with *only* collinear lines are identical to the corresponding graphs in QCD. This is because in any graph in which all the lines have the same scaling (and there are no purely soft graphs, so this only refers to purely collinear graphs), expanding in powers of λ does not change the propagators. Since the locations of poles in the propagators are unaffected, it is irrelevant whether one calculates the full graph in QCD and then expands in powers of λ, or calculates each order in λ in the collinear-soft theory. Thus, for purely collinear graphs, such as the wave function graph in Fig. 6, we will not bother to write down the complete set of operators, but simply calculate the graph in QCD and expand.

There is one important subleading operator, shown in Fig. 7, which can be enhanced by the $1/\lambda$ piece of the purely

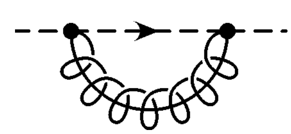

FIG. 6. Pure collinear wave function graph. The $O(\lambda^{-2})$ contribution vanishes.

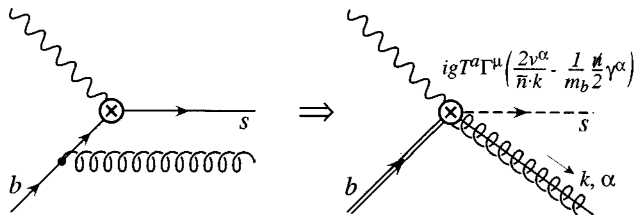

FIG. 7. Nonlocal vertex at $O(\lambda)$.

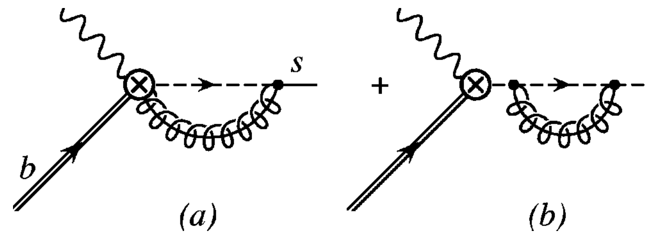

FIG. 8. The one-loop collinear gluon corrections to the vertex V^μ.

collinear coupling. By momentum conservation, there is no vertex coupling two heavy quarks and a collinear quark, since a heavy quark cannot emit a collinear gluon and stay on its mass shell. However, expanding the diagram in Fig. 7 in powers of λ gives the nonlocal $O(\lambda)$ interaction shown in the figure. (This is similar to the nonlocal operators found in [29].) Though it is formally subleading, in graphs such as Fig. 8(a) it gives an $O(1)$ effect.

B. Matching onto the collinear-soft theory

We now proceed to compute the matching conditions for the operator $\hat{O}_7$, and demonstrate that there are no large logarithms in the matching coefficients. At tree level, the matching is trivial. Defining the current in the effective theory by

$$V^\mu=\bar{\xi}_n\Gamma^\mu h_v\,, \tag{27}$$

where Γ^μ is given in Eq. (13), the Wilson coefficient C_V at the tree level is

$$C_V=1+O(\alpha_s). \tag{28}$$

To perform this matching at one loop, we repeat the one-loop matching calculation discussed in Sec. II, but now using the collinear-soft theory instead of LEET, hence including collinear modes. The calculation is simplest if we set the invariant mass of the s quark to zero; this introduces additional infrared divergences to the calculation which cancel in the matching conditions. The one-loop matrix element of $\hat{O}_7$ in full QCD can be calculated from the diagrams in Fig. 2, and we find the amplitude

$$A_{\rm QCD}=\bar{s}\Gamma^\mu b\left[1-\frac{\alpha_s C_F}{4\pi}\left(\frac{1}{\epsilon^2}+\frac{\log(\tilde{\mu}^2/m_b^2)}{\epsilon}+\frac{5}{2\epsilon}+\frac{1}{2}\log^2\frac{\tilde{\mu}^2}{m_b^2}+\frac{7}{2}\log\frac{\tilde{\mu}^2}{m_b^2}+\cdots\right)\right], \tag{29}$$

where all the $1/\epsilon$ divergences are infrared in origin. The one-loop correction in the collinear-soft theory can be calculated from the Feynman diagrams in Figs. 3 and 8. In pure dimensional regularization all graphs are zero, as there is no scale present in the loop integrals. Thus, we find the matching condition

$$C_V Z_V=1+\frac{\alpha_s C_F}{4\pi}\left(\frac{1}{\epsilon^2}+\frac{\log(\tilde{\mu}^2/m_b^2)}{\epsilon}+\frac{5}{2\epsilon}+\cdots\right), \tag{30}$$

where Z_V is the counterterm required to subtract the UV divergences in the collinear-soft theory.

This derivation of course assumes that the collinear-soft theory reproduces the infrared behavior of QCD. We can check this by instead introducing a small invariant mass for the s quark, as in Sec. II, and explicitly verifying that the dependence on the invariant mass in the collinear-soft theory is identical to that in full QCD given in Eq. (18). The soft gluon contribution in the collinear-soft theory is identical to the LEET result, given in Eq. (19):

$$A_s=-C_V\bar{\xi}_n\Gamma^\mu h\,\frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon^2}-\frac{2}{\epsilon}\log\frac{p_s^2}{m_b\tilde{\mu}}+2\log^2\frac{p_s^2}{m_b\tilde{\mu}}+\cdots\right]. \tag{31}$$

The collinear vertex diagram, Fig. 8(a), gives

$$\begin{aligned}A_c^{(v)}&=C_V\bar{\xi}_n\Gamma^\mu h\,\frac{\alpha_s C_F}{2\pi}\left(4\pi\frac{\mu^2}{p_s^2}\right)^\epsilon\frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}\,\frac{1}{\epsilon^2}\\&=-C_V\bar{\xi}_n\Gamma^\mu h\,\frac{\alpha_s C_F}{4\pi}\left[-\frac{2}{\epsilon^2}-\frac{2}{\epsilon}+\frac{2}{\epsilon}\log\frac{p_s^2}{\tilde{\mu}^2}-\log^2\frac{p_s^2}{\tilde{\mu}^2}+2\log\frac{p_s^2}{\tilde{\mu}^2}+\cdots\right].\end{aligned} \tag{32}$$

As previously discussed, the leading piece of the wave function graph Fig. 8(b) is $O(1/\lambda^2)$, but fortunately vanishes. Expanding to higher orders in λ, the graph gives the same result as in full QCD, Eq. (15). We therefore obtain for the contribution of the collinear gluons

$$A_c=-C_V\bar{\xi}_n\Gamma^\mu h\,\frac{\alpha_s C_F}{4\pi}\left[-\frac{2}{\epsilon^2}-\frac{3}{2\epsilon}+\frac{2}{\epsilon}\log\frac{p_s^2}{\tilde{\mu}^2}-\log^2\frac{p_s^2}{\tilde{\mu}^2}+\frac{3}{2}\log\frac{p_s^2}{\tilde{\mu}^2}+\cdots\right]. \tag{33}$$

Adding the soft and collinear contributions, as well as the counterterm given in Eq. (30), we obtain

$$A_{cs}=-C_V\bar{\xi}_n\Gamma^\mu h\,\frac{\alpha_s C_F}{4\pi}\left[\log^2\frac{p_s^2}{m_b^2}+\frac{3}{2}\log\frac{p_s^2}{m_b^2}+\frac{1}{\epsilon}-\frac{1}{2}\log^2\frac{\tilde{\mu}^2}{m_b^2}-\frac{3}{2}\log\frac{\tilde{\mu}^2}{m_b^2}+\cdots\right]. \tag{34}$$

Note that the troublesome divergence $\sim(1/\epsilon)\log p_s^2$ cancels once the two contributions (31) and (33) are added. Thus, both collinear and soft modes are required for the theory to be renormalized sensibly. Comparing to the full theory result (18), we see that the collinear-soft theory reproduces the IR physics of QCD, and that at the scale $\tilde{\mu}=m_b$ all nonanalytic terms vanish. This determines the matching scale to be m_b, confirming the result found by calculating in pure dimensional regularization (30).

C. Renormalization group equations

From the counterterm given in Eq. (30) it is simple to extract the anomalous dimension of the operator V^μ in the collinear-soft theory. From the definition

$$\gamma_V=Z_V^{-1}\left(\tilde{\mu}\frac{\partial}{\partial\tilde{\mu}}+\beta\frac{\partial}{\partial g}\right)Z_V, \tag{35}$$

we have

$$\tilde{\mu}\frac{\partial}{\partial\tilde{\mu}}Z_V=\frac{\alpha_s(\tilde{\mu})C_F}{2\pi\epsilon},$$

$$\beta\frac{\partial}{\partial g}Z_V=-\frac{\alpha_s(\tilde{\mu})C_F}{2\pi}\left(\frac{1}{\epsilon}+\log\frac{\tilde{\mu}^2}{m_b^2}+\frac{5}{2}\right), \tag{36}$$

where we have used $\beta=-g\epsilon+O(g^3)$. This give the anomalous dimension

$$\gamma_V=-\frac{\alpha_s(\tilde{\mu})C_F}{2\pi}\left(\log\frac{\tilde{\mu}^2}{m_b^2}+\frac{5}{2}\right). \tag{37}$$

Note that the divergent piece of the anomalous dimension cancels between the two terms [6]. The RGE for the coefficient of the operator V^μ is therefore

$$\tilde{\mu}\frac{d}{d\tilde{\mu}}C_V(\tilde{\mu})=\gamma_V(\tilde{\mu})C_V(\tilde{\mu}). \tag{38}$$

Solving this RGE we obtain

$$C_V(\tilde{\mu})=\left(\frac{\alpha_s(\tilde{\mu})}{\alpha_s}\right)^{(C_F/2\beta_0)(5-8\pi/\beta_0\alpha_s)}\left(\frac{\tilde{\mu}^2}{m_b^2}\right)^{-C_F/\beta_0}C_V(m_b), \tag{39}$$

where $\alpha_s\equiv\alpha_s(m_b)$, $\beta_0=11-2/3n_f$, and $C_V(m_b)=1+O(\alpha_s(m_b))$. Note that in deriving the anomalous dimension (37) we have assumed that the nonlocal vertex given in Fig. 7 has the same running as the QCD coupling. This assumption needs to be checked in subsequent work.

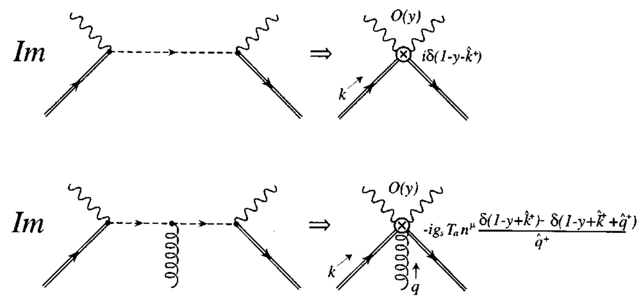

FIG. 9. Diagrammatic representation of the OPE, as well as the zero and one gluon Feynman rules for the resulting nonlocal operator $O(y)$.

IV. THE SOFT THEORY

A. Matching

The collinear-soft effective theory is valid down to $\tilde{\mu}\approx m_b\sqrt{1-x}$, the typical invariant mass of the light s-quark jet. At this scale we integrate out the collinear modes, and perform an OPE to calculate the inclusive b decay rate. Diagrammatically, this is illustrated in Fig. 9. This results in a nonlocal OPE in which the two currents are separated along a light-like direction. As in Eq. (4), we write the momentum of the eikonal line as

$$p_s^\mu=\frac{m_b}{2}n^\mu+k^\mu+\frac{m_b}{2}(1-y)\bar{n}^\mu, \tag{40}$$

where k^μ is the residual momentum of the heavy quark (note that we distinguish y from x, the rescaled photon momentum, since beyond tree level they will differ). The imaginary piece of the first graph is then proportional to $\delta(1-y+\hat{k}^+)$ (where, as usual, careted variables are divided by m_b), so the OPE is in terms of an infinite number of nonlocal operators, labeled by y:

$$O(y)=\overline{h_v}\,\delta(1-y+i\hat{D}^+)h_v\,. \tag{41}$$

Feynman rules for nonlocal operators of this type were obtained in [30], by writing them as the Fourier transform of operators in position space, and expanding out the path-ordered exponential in powers of the gauge field. Equivalently, the Feynman rules may be obtained by taking the imaginary piece of the time-ordered product in LEET with additional gluons; the single gluon Feynman rule is given in Fig. 9.

The matrix element of $O(y)$ between heavy quark states with residual momentum k is

$$\langle b(k)|\overline{h_v}\,\delta(1-y+i\hat{D}^+)h_v|b(k)\rangle=\delta(1-y+\hat{k}^+)+O(\alpha_s), \tag{42}$$

while its matrix element between hadrons is the well-known structure function [16]

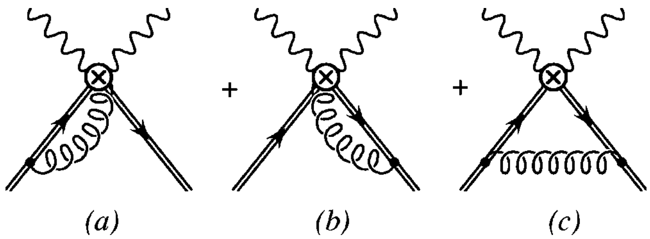

FIG. 10. Feynman diagrams contributing to the one-loop matrix element of $O(y)$.

$$f(y)=\frac{\langle B|\overline{h_v}\,\delta(1-y+i\hat{D}^+)h_v|B\rangle}{\langle B|\overline{h_v}h_v|B\rangle}. \tag{43}$$

Thus, LEET consists of a continuous set of operators labeled by y. Each operator has a coefficient that depends on the kinematic variable x, and the differential rate for $B\to X_s\gamma$ is given by the integral

$$\frac{d\Gamma}{dx}=\Gamma_0\int dy C(y,x;\mu)f(y;\mu), \tag{44}$$

where the $C(y,x;\mu)$'s are the coefficients of the OPE.

To match onto LEET at one loop we compare the differential decay rate in the parton model, $b\to X_s\gamma$, which in LEET is

$$\left.\frac{d\Gamma}{dx}\right|_{k^+}=\Gamma_0\int dy C(y,x;\mu)\langle b(k)|O(y;\mu)|b(k)\rangle. \tag{45}$$

We therefore need the one-loop matrix element of $O(y)$ between quark states. This may be calculated from the diagrams shown in Fig. 10.

Again all divergences are regulated in dimensional regularization. As an example, Fig. 10(a) gives

$$\langle b(k)|O^{(a)}(y)|b(k)\rangle=iC_Fg^2\left(\frac{\mu}{m_b}\right)^{4-d}\int\frac{d^{d-2}\hat{q}_\perp}{(2\pi)^{d-2}}\frac{d\hat{q}^-}{2\pi}\frac{d\hat{q}^+}{2\pi}\frac{\delta(\hat{k}^++1-y)-\delta(\hat{k}^++\hat{q}^++1-y)}{(\hat{q}^+\hat{q}^--\hat{q}_\perp^2+i\epsilon)(\hat{q}^++\hat{q}^-+i\epsilon)\hat{q}^+}. \tag{46}$$

The first term is proportional to

$$\begin{aligned}\int\frac{d^d\hat{q}}{(2\pi)^d}\frac{1}{(\hat{q}^2+i\epsilon)(\hat{q}\cdot v+i\epsilon)(\hat{q}\cdot n)}&=8\int\frac{d^d\hat{q}}{(2\pi)^d}\int_0^1dx\int_0^\infty d\lambda\frac{\lambda}{[\hat{q}^2+2\lambda\hat{q}\cdot(v(1-x)+xn)]^3}\\&=-\frac{4i}{(4\pi)^{d/2}}\Gamma(3-d/2)\int_0^1dx\int_0^\infty d\lambda\lambda^{d-5}[(1-x)^2+2(1-x)]^{d/2-3}.\end{aligned} \tag{47}$$

The λ integral vanishes in dimensional regularization, so this term vanishes. After performing the trivial $\hat{q}^+$ integration in the second term, we are left with

$$\begin{aligned}\langle b(k)|O^{(a)}(y)|b(k)\rangle&=i\frac{C_Fg^2}{2\pi}\left(\frac{\mu}{m_b}\right)^{4-d}\frac{1}{\hat{k}^++1-y}\int\frac{d^{d-2}\hat{q}_\perp}{(2\pi)^{d-2}}\frac{d\hat{q}^-}{2\pi}\frac{1}{\hat{q}^-(\hat{k}^++1-y)+\hat{q}_\perp^2-i\epsilon}\\&\quad\times\frac{1}{\hat{q}^--(\hat{k}^++1-y)+i\epsilon}\\&=\frac{C_Fg^2}{2\pi}\left(\frac{\mu}{m_b}\right)^{4-d}\frac{\theta(\hat{k}^++1-y)}{\hat{k}^++1-y}\int\frac{d^{d-2}\hat{q}_\perp}{(2\pi)^{d-2}}\frac{1}{\hat{q}_\perp^2+(\hat{k}^++1-y)^2}\\&=\frac{C_Fg^2}{8\pi^2}\left(4\pi\frac{\mu^2}{m_b^2}\right)^\epsilon\Gamma(\epsilon)\frac{\theta(\hat{k}^++1-y)}{(\hat{k}^++1-y)^{1+2\epsilon}}.\end{aligned} \tag{48}$$

Using the identity

$$\frac{\theta(y-x)}{(y-x)^{1+2\epsilon}}=-\frac{1}{2\epsilon}\delta(y-x)+\theta(y-x)\left[\frac{1}{(y-x)_+}-2\epsilon\left(\frac{\log(y-x)}{(y-x)}\right)_++O(\epsilon^2)\right], \tag{49}$$

we find

$$\langle b(k)|O^{(a)}(y)|b(k)\rangle=-\frac{\alpha_s C_F}{4\pi}\left\{\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\log\frac{\tilde{\mu}^2}{m_b^2}+\frac{1}{2}\log^2\frac{\tilde{\mu}^2}{m_b^2}+\cdots\right)\delta(1-y+\hat{k}^+)-\theta(1-y+\hat{k}^+)\right.$$
$$\left.\times\left[\left(\frac{2}{\epsilon}+2\log\frac{\tilde{\mu}^2}{m_b^2}\right)\frac{1}{(1-y+\hat{k}^+)_+}-4\left(\frac{\log(1-y+\hat{k}^+)}{1-y+\hat{k}^+}\right)_+\right]\right\}. \tag{50}$$

The diagram in Fig. 10(b) gives the same result as 10(a), while the diagram in Fig. 10(c) gives

$$\langle b(k)|O^{(c)}(y)|b(k)\rangle=-\frac{\alpha_s C_F}{4\pi}\left[\left(-\frac{2}{\epsilon}-2\log\frac{\tilde{\mu}^2}{m_b^2}\right)\delta(1-y+\hat{k}^+)+4\frac{\theta(1-y+\hat{k}^+)}{(1-y+\hat{k}^+)_+}\right]. \tag{51}$$

In dimensional regularization the wave function diagrams vanish. Since the decay rate is infrared finite, including the wave function graphs simply converts an infrared $1/\epsilon$ divergence to an ultraviolet divergence. Therefore, we may neglect the wave function counterterm, and combining all graphs we find the bare matrix element

$$\langle b(k)|O^{\mathrm{bare}}(y)|b(k)\rangle=\left[1-\frac{\alpha_s C_F}{4\pi}\left(\frac{2}{\epsilon^2}-\frac{2}{\epsilon}+\frac{2}{\epsilon}\log\frac{\tilde{\mu}^2}{m_b^2}-2\log\frac{\tilde{\mu}^2}{m_b^2}+\log^2\frac{\tilde{\mu}^2}{m_b^2}+\cdots\right)\right]\delta(1-y+\hat{k}^+)$$
$$-\frac{\alpha_s C_F}{4\pi}\theta(1-y+\hat{k}^+)\left[\left(-\frac{4}{\epsilon}-4\log\frac{\tilde{\mu}^2}{m_b^2}+4\right)\frac{1}{(1-y+\hat{k}^+)_+}+8\left(\frac{\log(1-y+\hat{k}^+)}{1-y+\hat{k}^+}\right)_+\right], \tag{52}$$

where all divergences are ultraviolet. The renormalized operator $O(y;\mu)$ is related to the bare operator by

$$O^{\mathrm{bare}}(y)=\int dy' Z(y',y;\tilde{\mu})O(y',\tilde{\mu}). \tag{53}$$

Renormalizing in MS (generalized in the obvious way to cancel the $1/\epsilon^2$ divergences), we find

$$Z(y',y;\tilde{\mu})=\left\{\left[1-\frac{\alpha_s(\tilde{\mu})C_F}{2\pi}\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\log\frac{\tilde{\mu}^2}{m_b^2}-\frac{1}{\epsilon}\right)\right]\delta(y'-y)+\frac{\alpha_s(\tilde{\mu})C_F}{\pi}\frac{1}{\epsilon}\frac{1}{(y'-y)_+}\theta(y'-y)\right\}. \tag{54}$$

Note that the counterterm consists of a diagonal piece which is proportional to $\delta(y'-y)$, and an off-diagonal piece proportional to $\theta(y'-y)$. This latter term mixes the operator $O(y)$ with all operators $O(y')$ with $y'>y$.

Inserting the one-loop matrix element of the renormalized operator into Eq. (45) we find the the differential decay rate in the parton model $b\to X_s\gamma$

$$\left.\frac{d\Gamma}{dx}\right|_{k^+}=\Gamma_0\int dy C(y,x;\tilde{\mu})\langle O(y;\tilde{\mu})\rangle=\Gamma_0\int dy C(y,x;\tilde{\mu})\left\{\left[1-\frac{\alpha_s C_F}{4\pi}\left(\log^2\frac{\tilde{\mu}^2}{m_b^2}-2\log\frac{\tilde{\mu}^2}{m_b^2}+\cdots\right)\right]\delta(1-y+\hat{k}^+)\right.$$
$$\left.-\frac{\alpha_s C_F}{4\pi}\theta(1-y+\hat{k}^+)\left[\left(4-4\log\frac{\tilde{\mu}^2}{m_b^2}\right)\frac{1}{(1-y+\hat{k}^+)_+}+8\left(\frac{\log(1-y+\hat{k}^+)}{1-y+\hat{k}^+}\right)_+\right]\right\}. \tag{55}$$

One might worry about the appearance in Eq. (55) of logarithmic terms that depend on m_b, since this scale has been integrated out and thus should not be present in the effective theory. These terms are due to our choice of factoring the heavy quark mass out of the soft scale $m_b(1-y+\hat{k}^+)$ by writing our expressions in terms of careted quantities. The logarithms of m_b cancel in the matching coefficient.

The Wilson coefficients $C(y,x;\mu)$ are determined by matching the collinear-soft theory onto LEET. In the collinear-soft theory, the Feynman diagrams for the forward scattering matrix element are shown in Fig. 11. As with LEET, all divergences are regulated in dimensional regularization. Expanding the expression for the forward scattering amplitude obtained from these graphs in powers of $(1-x+\hat{k}^+)$, we find for the differential decay rate

$$\left.\frac{d\Gamma}{dx}\right|_{k^+}=C_V^2(\tilde{\mu})\Gamma_0\Bigg\{\left[1+\frac{\alpha_s C_F}{4\pi}\left(\log^2\frac{\tilde{\mu}^2}{m_b^2}+5\log\frac{\tilde{\mu}^2}{m_b^2}+\cdots\right)\right]\delta(1-x+\hat{k}^+)-\frac{\alpha_s C_F}{4\pi}\theta(1-x+\hat{k}^+)$$
$$\times\left[4\left(\frac{\log(1-x+\hat{k}^+)}{1-x+\hat{k}^+}\right)_+ +7\left(\frac{1}{1-x+\hat{k}^+}\right)_+\right]\Bigg\}. \tag{56}$$

Comparing Eqs. (56) and (55) gives the short-distance coefficient $C(y,x;\mu)$. At the tree level, the matching is trivial, and we write

$$C(y,x;\tilde{\mu})=C_V^2(\tilde{\mu})\left[\delta(y-x)+\frac{\alpha_s C_F}{4\pi}C^{(1)}(y,x;\tilde{\mu})\right]+O(\alpha_s^2), \tag{57}$$

where μ is the matching scale. At one loop, we find

$$C^{(1)}(y,x;\tilde{\mu})=\left(2\log^2\frac{\tilde{\mu}^2}{m_b^2}+3\log\frac{\tilde{\mu}^2}{m_b^2}+\cdots\right)\delta(y-x)-\left(3+4\log\frac{\tilde{\mu}^2}{m_b^2}\right)\frac{\theta(y-x)}{(y-x)_+}+4\,\theta(y-x)\left(\frac{\log(y-x)}{y-x}\right)_+$$
$$=\left(2\log^2\frac{\tilde{\mu}^2}{m_b^2(y-x)}+3\log\frac{\tilde{\mu}^2}{m_b^2(y-x)}+\cdots\right)\delta(y-x)-4\,\frac{\theta(y-x)}{y-x}\log\frac{\tilde{\mu}^2}{m_b^2(y-x)}-3\,\frac{\theta(y-x)}{y-x}. \tag{58}$$

At the scale $\tilde{\mu}=m_b\sqrt{y-x}$ the logarithmic terms vanish, and we find

$$C^{(1)}(y,x;m_b\sqrt{y-x})=-3\,\frac{\theta(y-x)}{y-x}+\cdots. \tag{59}$$

The matching scale is therefore different for each operator $O(y)$.

B. Renormalization group

The differential decay rate in LEET given in Eq. (55) may be written as

$$\frac{d\Gamma}{dx}=\Gamma_0\int dy\,C(y,x;\tilde{\mu})\Bigg\{\left[1-\frac{\alpha_s C_F}{4\pi}\left(\log^2\frac{\tilde{\mu}^2}{m_b^2(1-y+\hat{k}^+)^2}-2\log\frac{\tilde{\mu}^2}{m_b^2(1-y+\hat{k}^+)^2}\right)\right]\delta(1-y+\hat{k}^+)$$
$$+\frac{\alpha_s C_F}{4\pi}\left(\frac{4}{1-y+\hat{k}^+}\log\frac{\tilde{\mu}^2}{m_b^2(1-y+\hat{k}^+)^2}-\frac{4}{1-y+\hat{k}^+}\right)\Bigg\}, \tag{60}$$

and so the large logarithms in the matrix element of $O(y;\tilde{\mu})$ vanish at the scale $\tilde{\mu}=m_b(1-y+\hat{k}^+)$. [This expression looks highly singular, but as can be seen from Eq. (55), the delta functions combine with the other terms to form plus functions.] Thus, in order to sum all logarithms of μ we must continue to run the operator $O(y)$ in LEET. From Eqs. (53) and (54) we obtain the renormalization group equation

$$\mu\frac{d}{d\mu}C(y,x;\tilde{\mu})=\int dy'\,\gamma(y,y';\tilde{\mu})C(y',x;\tilde{\mu}), \tag{61}$$

where $\gamma(y,y';\tilde{\mu})$ is the continuous anomalous dimension matrix

$$\gamma(y,y';\tilde{\mu})=\frac{\alpha_s(\tilde{\mu})C_F}{\pi}\left[\left(\log\frac{\tilde{\mu}^2}{m_b^2}-1\right)\delta(y'-y)-\frac{2}{(y'-y)_+}\theta(y'-y)\right]. \tag{62}$$

Solving Eq. (61) analytically, however, is nontrivial and beyond the scope of this work [8]. Instead, we may diagonalize the anomalous dimension matrix by taking high moments of the spectrum. This will allow us to compare our results to those of Refs. [9,10]. Note that in Refs. [9,10] both leading and next-to-leading logarithms were resummed. This requires the two-loop contribution to the $1/\epsilon^2$ counterterm, the full one-loop matching condition, and the two-loop running of α_s, none of which have been included here. As a result

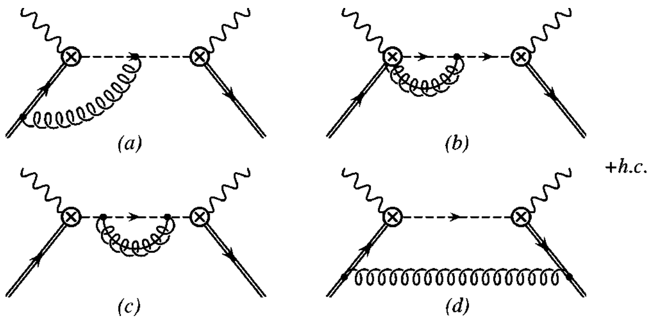

FIG. 11. Collinear-soft theory Feynman diagrams which contribute to the forward scattering amplitude through $O(\alpha_s)$.

our calculation only resums the leading logarithms and a class of the subleading logarithms. However, it is straightforward to extract from the literature a resummation of exactly the same set of logarithms.

To calculate the moments we set the residual momentum k to zero. (This residual momentum can easily be incorporated by boosting from the rest frame of the b quark, $p_b=m_b v$, to the frame $p=m_b v+k$.) Taking moments unconvolutes the expression for the differential decay rate in LEET, Eq. (45), and we obtain

$$\begin{aligned}\Gamma(N)&=\int_0^1 dx x^{N-1}\frac{d\Gamma}{dx}\\&=\Gamma_0\int_0^1 dx x^{N-1}\int_{-\infty}^{\infty}dy C(y-x;\mu)\langle O(y;\mu)\rangle\\&=\Gamma_0\int_0^1 dz z^{N-1}C'(1-z;\tilde{\mu})\int_0^1 dy y^{N-1}\langle O(y;\tilde{\mu})\rangle\\&\equiv\Gamma_0 C(N;\tilde{\mu})\langle O(N;\tilde{\mu})\rangle,\end{aligned}\tag{63}$$

where we have used

$$C(y-x)=\frac{1}{y}C'\left(1-\frac{x}{y}\right)\theta(y-x),\tag{64}$$

since $C(y-x)$ just contains delta functions and plus distributions. Thus, the operator $O(N;\mu)$ is just a linear combination of the set of operators $O(y;\mu)$. The matching from the collinear-soft theory onto LEET at tree level is trivial, and we find

$$C(N;\tilde{\mu})=C_V^2(\tilde{\mu})\left[1+\frac{\alpha_s C_F}{4\pi}C^{(1)}(N;\tilde{\mu})\right]+O(\alpha_s^2).\tag{65}$$

Determining $C^{(1)}(N;\mu_0)$ requires the one-loop expression of $\Gamma(N)$ in the collinear-soft theory and the one-loop matrix element of $O(N;\mu)$ between partonic states. The one-loop expression for the differential decay rate in the collinear-soft theory is given in Eq. (56). Setting k_+ to zero and taking moments we obtain

$$\begin{aligned}\Gamma(N)&=\int_0^1 x^{N-1}\frac{d\Gamma}{dx}\\&=\Gamma^0 C_V^2(\tilde{\mu})\left\{1-\frac{\alpha_s C_F}{4\pi}\left[2\log^2\frac{N}{n_0}-7\log\frac{N}{n_0}\right.\right.\\&\qquad\left.\left.-\log^2\frac{\tilde{\mu}^2}{m_b^2}-5\log\frac{\tilde{\mu}^2}{m_b^2}\right]\right\}+\cdots,\end{aligned}\tag{66}$$

where $n_0=e^{-\gamma_E}$. This needs to be compared to the one-loop matrix element of $\langle O(N;\mu)\rangle$, which can be obtained from Eq. (55):

$$\begin{aligned}\langle O(N;\tilde{\mu})\rangle&=\int_0^1 dy y^{N-1}\langle O(y;\tilde{\mu})\rangle\\&=1-\frac{\alpha_s C_F}{4\pi}\left[4\log^2\frac{\tilde{\mu}N}{m_b n_0}-4\log\frac{\tilde{\mu}N}{m_b n_0}\right]+\cdots.\end{aligned}\tag{67}$$

The one-loop matching coefficient is now easily determined using Eqs. (63), (66), and (67) and we find

$$C^{(1)}(N;\tilde{\mu})=\frac{\alpha_s C_F}{4\pi}\left[2\log^2\frac{\tilde{\mu}^2 N}{m_b^2 n_0}+3\log\frac{\tilde{\mu}^2 N}{m_b^2 n_0}\right]+\cdots.\tag{68}$$

At the matching scale $\tilde{\mu}=m_b\sqrt{n_0/N}$ all logarithms in this matching coefficient vanish. Furthermore, from Eq. (67) it is clear that the matrix element $\langle O(N;\tilde{\mu})\rangle$ contains no large logarithms of N at the scale $\tilde{\mu}=m_b n_0/N$. Thus we run in the collinear-soft theory from m_b to $m_b\sqrt{n_0/N}$, perform the OPE, and run $C(N;\tilde{\mu})$ from $m_b\sqrt{n_0/N}$ to $m_b n_0/N$.

The running of the coefficient C_V in the collinear-soft theory from the scale m_b to the scale $m_b\sqrt{n_0/N}$ is obtained by setting $\tilde{\mu}=m_b\sqrt{n_0/N}$ in Eq. (39). The running in LEET is determined by the RGE for $C(N;\tilde{\mu})$

$$\mu\frac{d}{d\mu}C(N;\tilde{\mu})=\gamma(N;\tilde{\mu})C(N;\tilde{\mu}),\tag{69}$$

where the anomalous dimension is given by

$$\begin{aligned}\gamma(N;\tilde{\mu})&=\int_0^1 dz z^{N-1}\gamma(z;\tilde{\mu})\\&=-\frac{\alpha_s(\tilde{\mu})C_F}{\pi}\left[1-2\log\left(\frac{\tilde{\mu}N}{m_b n_0}\right)\right].\end{aligned}\tag{70}$$

The solution to this equation is

$$C\left(N;\frac{m_b n_0}{N}\right)=C_V^2\left(m_b\sqrt{\frac{n_0}{N}}\right)\left(\frac{\alpha_s\left(m_b\frac{n_0}{N}\right)}{\alpha_s\left(m_b\sqrt{\frac{n_0}{N}}\right)}\right)^{\frac{2C_F}{\beta_0}\left(1+\frac{4\pi}{\beta_0\alpha_s}-2\log\frac{N}{n_0}\right)}\left(\frac{n_0}{N}\right)^{\frac{2C_F}{\beta_0}}. \tag{71}$$

This sums perturbative logarithms of N into the coefficient $C(N)$. We can then substitute this into Eq. (63) to obtain an expression for the resummed moments of the differential decay rate.

Using the result for $C_V(\mu)$ given in Eq. (39) and taking the matrix element of $O(N;\mu)$ between hadronic states, we find the resummed expression for large photon energy moments of the decay $B\to X_s\gamma$

$$\Gamma(N)=\Gamma_0 f(N;m_b n_0/N)\left(\frac{\alpha_s\left(m_b\sqrt{\frac{n_0}{N}}\right)}{\alpha_s}\right)^{\frac{C_F}{\beta_0}\left(\frac{5-8\pi}{\beta_0\alpha_s}\right)}\left(\frac{\alpha_s\left(m_b\frac{n_0}{N}\right)}{\alpha_s\left(m_b\sqrt{\frac{n_0}{N}}\right)}\right)^{\frac{2C_F}{\beta_0}\left(1+\frac{4\pi}{\beta_0\alpha_s}-2\log\frac{N}{n_0}\right)}. \tag{72}$$

Logarithms are explicitly summed in this expression and only long distance physics is contained in the function $f(N;m_b n_0/N)$.

We can easily compare our results to those in the literature. A resummed expression for $\Gamma(N)$ is given in Ref. [9]:

$$\Gamma(N)=\Gamma_0 f(N;m_b/N)\exp\left[-\int_{n_0/N}^{1}\frac{dy}{y}\left(2\int_{m_b y}^{m_b\sqrt{y}}\frac{d\mu}{\mu}\Gamma_c(\mu)+\Gamma(m_b y)+\gamma(m_b\sqrt{y})\right)\right], \tag{73}$$

where, at one loop,

$$\Gamma_c(\mu)=\frac{\alpha_s(\mu)C_F}{\pi},$$
$$\Gamma(\mu)=-\frac{\alpha_s(\mu)C_F}{\pi}, \tag{74}$$
$$\gamma(\mu)=-\frac{3\alpha_s(\mu)C_F}{4\pi}.$$

Note that the cusp anomalous dimension $\Gamma_c(\mu)$ is the contribution to the anomalous dimension from the $1/\epsilon^2$ counterterm. Using only the one-loop cusp anomalous dimension, tree level matching, and the one-loop running of α_s, Eq. (73) resums leading logarithms and the same class of next-to-leading logarithms we resum in our calculation. Performing the integrations in the exponent we reproduce Eq. (72). Thus the approach presented here, based on an effective field theory, is in agreement with the factorization formalism approach for summing perturbative logarithms.

V. CONCLUSIONS

In the specific case of $\bar{B}\to X_s\gamma$ we have shown how Sudakov logarithms can be summed within an effective field theory framework. First we construct an intermediate theory which includes both collinear and soft degrees of freedom. By performing a one-loop calculation we show that this collinear-soft theory can be matched onto QCD at the scale m_b without introducing logarithmic terms into the short-distance coefficient. In addition we determine the one-loop anomalous dimension and solve the RGE. Next we integrate out collinear modes at the scale $m_b\sqrt{y-x}$ by switching to LEET. We perform an OPE in powers of $(y-x)$ which leads to the appearance of a nonlocal operator where two vertices are separated along the light cone. The matrix element of this operator between B meson states is the structure function. We perform the OPE at one-loop in the collinear-soft theory and match onto the nonlocal operator in LEET. At the scale $m_b\sqrt{y-x}$ no logarithmic terms are introduced into the short-distance coefficient.

In order to compare to the factorization formalism results in the literature we repeat our analysis for large moments of the decay rate. In this case we find that the collinear-soft theory matches onto LEET at the scale $m_b\sqrt{n_0/N}$, and that there are no large logarithms in the matrix element of the bilocal operator at the scale $m_b n_0/N$. Using the renormalization group equations in the collinear-soft theory we sum logarithms of N between the scales m_b and $m_b\sqrt{n_0/N}$. We then switch to LEET and sum logarithms of N between the scales $m_b\sqrt{n_0/N}$ and $m_b n_0/N$. This sums all perturbative logarithms of N. We find that our result agrees with the results presented in the literature. This gives us confidence that we have constructed the correct effective field theory.

Though we have presented this work entirely in the context of $B\to X_s\gamma$ our approach is general. It should be straightforward to apply the collinear-soft theory and LEET to other processes in which Sudakov logarithms arise. Furthermore, this approach could also be applied to exclusive decays, in which case one does not perform the final OPE onto LEET, but remains in the collinear-soft theory. This could be applied to recent results on factorization in nonleptonic decays [31], as well as LEET-based relations between form factors in decays to highly energetic final states [32]. Since these latter results depend only on the spin symmetry of LEET, which is also present in the collinear-soft theory, they should remain valid in the present approach.

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