Flow equations for quark-gluon interactions in light-front QCD

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The flow-equation method of continuous unitary transformations is used to eliminate the minimal quark-gluon interaction in the light-front quantized QCD Hamiltonian. The coupled differential equations in the two lowest Fock sectors correspond to the renormalization of the light-front gluon mass and the generation of an effective quark-antiquark (as well as gluon-gluon) interaction. The original gauge field coupling is completely eliminated, even in the presence of degenerate states connected by this interaction. Further, a more singular $1/q^4$ behavior for the quark and gluon effective interactions at small gluon momenta is obtained, due to the asymptotic behavior of the effective gluon mass for small cutoff. We discuss the consequences of this asymptotic behavior and possible confinement implications.

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I. INTRODUCTION

The perturbative aspects of quantum chromodynamics (QCD) were understood many years ago with the convincing documentation that QCD is asymptotically free. However, the calculational techniques for nonperturbative QCD are still under development. Thus, quantitative analysis of low energy and momentum transfer phenomena remains difficult even though the qualitative features are reasonably described due to chiral symmetry and the phenomenological success of the constituent quark picture. Nevertheless, it is widely believed that pure Yang-Mills theory, without dynamical quarks, exhibits confinement represented by a linear potential between static color sources. Confinement may be attributed to mass generation from transmutation of dimensions in QCD. Adding dynamical quarks also breaks chiral symmetry spontaneously. An ultimate aim of nonperturbative QCD studies is to understand both confinement and chiral symmetry breaking and how the constituent quark picture arises from fundamental QCD.

In the past few years there were several studies addressing the issue of confinement and mass gap generation in the framework of the Schrödinger picture [1,2]. Using path integral techniques, Ref. [1] utilized a vacuum wave functional ansatz suggested by Kogan and Kovner, and integrated over all possible gauge configurations. Since calculations are quite formidable, this study only solves a field theory problem in $1+1$ dimensions [2] and restricts $(3+1)$-dimensional [1] analyses to ground states. For earlier investigations, see Ref. [3].

Alternatively, nonperturbative studies have used a Hamiltonian framework with the QCD Hamiltonian quantized in a specific gauge. In particular the Coulomb gauge has recently been investigated [4,5]. In this paper we also use a Hamiltonian approach but utilize the light-front gauge, $A^+=0$ [6]. There are arguments that the light-front gauge may be the most suitable framework to study nonperturbative QCD [6]. This conjecture is also supported by the success of phenomenological constituent light-front quark models [7]. To provide further insight concerning this issue, we have applied the flow equations to the light-front QCD Hamiltonian to generate dynamical gluons and quarks as well as their effective interactions.

Previous light-front studies of confinement and bound states have been conducted using the methods of similarity renormalization [8] and transverse lattice [9]. Significantly, light-front QCD in $3+1$ rigorously contains a confining interaction in the form of the instantaneous four-fermion interaction, $1/q^{+2}$, which is the complete confining interaction in $1+1$ QCD for the light-front spatial dimension, x^-. Wilson suggested that a starting point for analyzing full QCD (with confinement) in light-front coordinates is the light-front infrared divergences. When an infrared (IR) cutoff is introduced, appropriate counterterms are necessary to restore the attending physics below the cutoff. Based on light-front power counting, these counterterms can involve color charge densities and exhibit an unknown nonlocal behavior in the transverse direction which represents a possible source for transverse confinement. In the similarity renormalization approach, it has been claimed [8] that IR divergences from the instantaneous gluon exchange potential are not completely canceled, leading to a remnant potential that increases logarithmically with either increasing separations $|x^-|$ or $|x_\perp|$. However, the issue of a nonzero gluon mass and local gauge invariance is not yet completely understood and the task of restoring rotational symmetry is still difficult to achieve.

In this work, we also introduce an IR longitudinal cutoff and generate a light-front counterterm which sets a scale for a dynamical mass gap and string tension. While our calculations for the flow equation are only to order g^2, our results challenge the conventional notion that weak-coupling Hamiltonians derived from QCD have only Coulomb-like potentials, and definitely no confining interactions. Introducing a longitudinal IR cutoff in light-front dynamics makes it impossible to create particles from a bare vacuum by a translationally invariant Hamiltonian and thus the number of constituents in a given eigenstate is fixed. However, light-front counterterms for the longitudinal IR cutoff dependence can generate a nonzero amplitude of particle creation and are therefore a possible source for features associated with a nontrivial vacuum structure in equal-time dynamics, including confinement and spontaneous symmetry breaking. Note that the small light-front longitudinal momentum fraction x corresponds to the high light-front energy. Therefore, in order to remove small x divergences and maintain the cutoff independence, one should use a renormalization group, which is appropriate to high energies.

In this work we address the above issue using the flow equation method [10]. More specifically, we adopt Susskind's idea of the ''long arm'' of the vacuum [11] which addresses chiral symmetry breaking in the light-front formulation. In the parton model, one pictures a high momentum hadron as a collection of constituents each also having large momentum, such that under a boost, both the hadron and the partons change their momenta by the same factor. Therefore, one can formulate an effective field theory on the axis of the light-front longitudinal momentum fraction x (or on the axis of rapidity which is logarithm of x). The partons, both valence and sea, have positive x and, according to Feynman and Bjorken, are distributed along the x axis with the density of sea partons increasing for small x according to dx/x. Here, the vacuum is at $x=0$. The fundamental property of light-front Hamiltonians is that under a rescaling of the light-front momentum, $x\rightarrow\alpha x$, the light-front Hamiltonian scales $H\rightarrow H/\alpha$. This may be interpreted as a dilatation symmetry along the x axis. This symmetry only holds classically and is broken on the quantum level by an anomaly. Now, the long arm of the vacuum occurs because the coupling, i.e., the interaction between neighboring partons, gets stronger and stronger as one approaches $x=0$ so rapidly that the system is able to hold itself together despite the fact that there is an infinite number of steps between $x=0$ and finite x [11]. To illustrate the long arm effect of the vacuum, we introduce a cutoff for small x. A natural cutoff is provided by $\delta x=\varepsilon^{+}x$, the minimal spacing between constituents, which plays the role of UV regulator. As long as the density of partons on the rapidity axis is not infinite, ε^{+} or δx is finite and one obtains finite matrix elements. The cutoff δx breaks dilatation symmetry in the x axis and generates an energy scale or mass gap governing the strength of the effective interaction between quarks, in our case a string tension in the quark-antiquark potential. Hence the long arm of the ''light-front vacuum'' enables the formation of a $q\bar{q}$ bound state through the breaking of an internal symmetry, analogous to the creation of Cooper pairs in a superconducter. Because the results must be independent of the cutoff, a renormalization group equation is required, which in this work is provided by the Hamiltonian flow equation method [10].

Incorporating effects from small x into an effective light-front Hamiltonian is equivalent to integrating over the high light-front-energy modes in the asymptotically free domain. In terms of the renormalization group, regulating small x introduces a mass gap, which together with asymptotic freedom leads to a renormalization group invariant scale and dimensional transmutation along the x axis. This transmutation in turn produces a linear effective interaction more singular than the Coulomb potential.

At the technical level our thrust is to use the flow equations to renormalize the gluon energy and eliminate the coupling between the gluon and quark Fock sectors. We focus upon zero gluon momentum and obtain a gap equation for the renormalized light-front gluon mass $\mu(\lambda)$. This equation can be solved by imposing a renormalization condition, $\mu_{ren}(\lambda=\lambda_0)=\mu_0$, where μ_0 is the ''physical'' value. The renormalized effective gluon mass, $\mu_{ren}(\lambda)$, is obtained by introducing a mass counterterm. As a result, the asymptotic behavior of $\mu_{ren}(\lambda)$ is obtained for small cutoff λ which approaches the renormalization point λ_0 from above ($\lambda\geqslant\lambda_0$). The properties of $\mu_{ren}(\lambda)$ ensure that the quark-gluon coupling is eliminated even in the degenerate case of zero gluon momenta. A similar idea was used for the spin-boson model by Kehrein, Mielke and Neu [12], who argued that the coupling to a bosonic bath is always eliminated by renormalizing the tunneling frequency. Also, in complete analogy to our problem, Lenz and Wegner [13] showed for interacting electrons in BCS theory that the elimination of electron-phonon coupling for all states is a direct consequence of renormalizing the phonon frequency. Finally, the flow of the gluon mass with λ produces for small momentum q a potential of form $1/q^4$ which is more singular than the Coulomb $1/q^2$.

In the next section (Sec. II), we consider flow equations as a renormalization group transformation for Hamiltonians and formulate them for one- and two-body sectors explicitly. In Sec. III the flow equations are applied to the light-front QCD Hamiltonian to generate a gluon gap equation and an effective $q\bar{q}$ (as well as gg) interaction. This section also addresses solving these equations. The concluding discussion follows in Sec. IV.

II. FLOW EQUATIONS FOR GAUGE THEORIES

A. Flow equations as renormalization group transformations

The basic element of the renormalization group transformation is a unitary transformation that renders the Hamiltonian matrix band diagonal; i.e., matrix elements with energy differences $|E_i-E_j|$ exceeding a cutoff λ are eliminated [14]. This procedure converges well when there is a hierarchy of scales in the problem. The goal is to decouple the high- and low-energy scales of the band-diagonal effective Hamiltonian, which is renormalized order by order in perturbation theory. Using Nth order perturbation theory to connect high and low energy states of an effective renormalized Hamiltonian with ultraviolet regulating cutoff Λ (which is the size of the full Hamiltonian matrix in the energy space, eventually taken to infinity) requires $N=2(\Lambda-\lambda)/\lambda$. Assuming a coupling constant $g<1$, $g^{2(\Lambda-\lambda)/\lambda}\ll 1$ for $\Lambda\rightarrow\infty$. One has therefore isolated the low energy scale effective Hamiltonian which can then be diagonalized nonperturbatively for the few lowest eigenstates. This should be simpler than solving the full Hamiltonian matrix exactly. The unitary transformation which connects Hamiltonian matrices with different widths, $H(\lambda_2,\Lambda)=U(\lambda_1,\lambda_2;\Lambda)H(\lambda_1;\Lambda)U^{\dagger}(\lambda_1,\lambda_2;\Lambda)$, is the renormalization group transformation formulated by Glazek and Wilson [14] called the similarity renormalization.

In terms of an effective Hamiltonian, one removes large energy differences by squeezing the width of the band, $\lambda_2<\lambda_1\ll\Lambda$. This is done sequentially from high to low energy to construct a low-energy effective Hamiltonian. Since the light-front energy has an inverse relationship to the light-front momentum, $E_{LF}\sim 1/P^{+}$, this scaling to lower λ is equivalent to moving from small x (high energy) to large x (low energy) for an effective light-front theory formulated on the x axis. The elimination of large energy differences is governed by a dimensionless form factor, called the similarity function, $f(|E_i-E_j|/\lambda)$, which is of order unity for small arguments (no elimination), and approaches zero for large arguments (complete elimination of the off-diagonal matrix elements). Note that squeezing the bandwidth to zero is equivalent to exactly diagonalizing the Hamiltonian matrix.

The flow equation approach entails an infinitesimal unitary transformation written in differential form

$$\frac{dH(l)}{dl}=[\eta(l),H(l)]$$

$$\eta(l)=[H_d(l),H_r(l)], \tag{2.1}$$

where l is the flow parameter related to the cutoff scale λ by $l=1/\lambda^2$. The choice of the transformation generator η has been suggested by Wegner [10] as the commutator of the diagonal, H_d, and off-diagonal (rest), H_r, Hamiltonian components. As shown by Wegner, for $l\to\infty$ the off-diagonal part of Hamiltonian is eliminated, producing an effective diagonal Hamiltonian.

One of the purposes for renormalization is to remove the UV divergent high-energy contributions. Since they arise from the particle-number-changing terms of the Hamiltonian (at least in gauge theories), only these terms should be eliminated. Wegner suggested assigning the off-diagonal part to the particle-number-changing term $H_r=H_{n\to m}$ and the diagonal part to the particle-number-conserving term $H_d=H_{n\to n}$. Thus, instead of diagonalization, one can implement flow equations to block-diagonalize the Hamiltonian in particle number space. Also, block diagonalization in particle number space precludes convergence problems associated with exact diagonalization in energy space [10]. Most noteworthy is the practical aspect of block-diagonalization when applied to field theory. By block-diagonalizing the Hamiltonian in particle number space the high and low Fock sectors are uncoupled, enabling separate eigenstate problems with an effective Hamiltonian in each particle number sector. For most calculations, solving an effective sector-uncoupled Hamiltonian should be simpler than solving the original Hamiltonian [15]. Note, however, that the flow equations eliminate the particle number changing matrix elements, not in one step, but rather continuously for different energy differences, sequentially from high to low energy (matrix elements between degenerate states, $E_i=E_j$, can also be eliminated by flow equations as discussed in the next subsection). Here, the link between the similarity and flow equation schemes and renormalization clearly emerges. The distinguishing feature for flow equations, however, is the separation of the particle-number-conserving and particle-number-changing terms, with only the particle-number-changing terms contributing in the renormalization of an effective Hamiltonian.

In this work, we run flow equations for the two lowest Fock sectors ($|q\bar{q}\rangle$ and $|g\rangle$) in a gauge theory and obtain equations for the single gluon energy and $q\bar{q}$ effective interaction.

B. Flow equations in the two lowest Fock sectors

We consider a canonical Hamiltonian operator for a gauge field theory, where the gauge field is minimally coupled to the matter field (for example, minimally coupled QED or simple Abelian QCD). In terms of bare quark and gluon fields the eigenfunctions of this Hamiltonian contain infinitely many Fock states. For the two lowest Fock states (neglecting the rest), the Hamiltonian matrix in particle number space is

$$H=\begin{pmatrix} PHP & PHQ \\ QHP & QHQ \end{pmatrix}, \tag{2.2}$$

where P and Q are projection operators for one- and two-body Fock states. For Abelian, minimally coupled QCD, P projects on a one-gluon state and Q on a quark-antiquark pair, $P|\psi\rangle=|g\rangle$ and $Q|\psi\rangle=|q\bar{q}\rangle$. One-quark states are omitted because there are no dynamical quarks in this analysis (we prefer to disentangle complexities from chiral symmetry breaking). Matrix elements of the operator PHQ describe quark-gluon minimal coupling, the PHP term is a gluon effective energy and QHQ describes the $q\bar{q}$ effective interaction. Flow equations for particle number changing matrix elements, h_{pq}, are given by

$$\frac{dh_{pq}(l)}{dl}=-[E_p(l)-E_q(l)]\eta_{pq}(l)$$

$$\eta_{pq}(l)=-\frac{h_{pq}(l)}{E_p(l)-E_q(l)}\frac{d}{dl}[\ln f(z_{pq}(l))]$$

$$z_{pq}(l)=l[E_p(l)-E_q(l)]^2, \tag{2.3}$$

where p and q span all (free single particle) energy states in the P and Q subspaces, respectively; $E_p(l)$ and $E_q(l)$ are diagonal matrix elements of block Hamiltonians PHP and QHQ. In the first equation the off-diagonal matrix elements $h_{pp'}$ and $h_{qq'}$ are neglected to leading order in coupling $O(h_{pq})$ (this uncouples the flow equations for H_d and H_r terms). With the generator η_{pq} defined by Eq. (2.3), elimination of the coupling between the P and Q sectors,

$$h_{pq}(l)=h_{pq}(0)f(z_{pq}(l)), \tag{2.4}$$

is governed by the similarity function $f(l[E_p(l)-E_q(l)]^2)$, which vanishes for matrix elements with energy differences exceeding the cutoff λ, $|E_p(l)-E_q(l)|\gg 1/\sqrt{l}=\lambda$. Also, $h_{pq}(0)$ is the initial value, and $E_p(l)$ and $E_q(l)$ flow too [see Eq. (2.6)]. Eliminating the coupling h_{pq} generates effective interactions in the particle number conserving P and Q sectors. The corresponding flow equation reads

$$\frac{dh_{pp'}(l)}{dl}=\sum_q[\eta_{pq}(l)h_{qp'}(l)-h_{pq}(l)\eta_{qp'}(l)], \tag{2.5}$$

with a similar equation for $h_{qq'}$. Using the generator defined in Eq. (2.3) these equations reduce to

$$\frac{dE_p(l)}{dl} = -\sum_q \frac{1}{E_p(l)-E_q(l)} \frac{d}{dl}[h_{pq}(l)h_{qp}(l)]$$

$$\frac{dh_{qq'}(l)}{dl} = -\sum_p \left(\frac{dh_{qp}(l)}{dl} \frac{1}{E_q(l)-E_p(l)} h_{pq'}(l) + h_{qp}(l) \frac{1}{E_{q'}(l)-E_p(l)} \frac{dh_{pq'}(l)}{dl} \right), \qquad (2.6)$$

with the coupling h_{pq} given by Eq. (2.4). Note that for the first equation in Eqs. (2.6) only the diagonal matrix elements in the P space enter. In our application these equations represent the gap equation for an effective gluon energy and an equation for an effective $q\bar{q}$ interaction, both investigated in the next section.

Generally, the set of Hamiltonian flow equations is not finite and the equations are coupled by kernels, which also flow with l, which are only known after solving the flow equations. Obviously, practical computations require truncating to the few lowest sectors, assuming that higher sectors are not very important at low energies. In the light-front framework, this approximation is valid because pair creation from the light-front vacuum is forbidden and generally the higher Fock components (with large particle number) carry large light-front energies. Neglecting the high Fock components reduces the problem to an effective low-energy theory. In this way, we close the set of equations and reduce the number of unknown couplings, leaving only the canonical operator couplings. Note that the truncation in number of particles participating in intermediate states is not equivalent to perturbation theory in coupling constant, but rather is closer to the Tamm-Dancoff approach. The flow equations are still coupled, and should be solved self-consistently including the flow dependences of the couplings.

In this work on minimally coupled gauge theory, the set of flow equations was truncated at the two-body sector, including at most three-particles in intermediate states. Hence the renormalization of the QCD strong coupling constant is not considered, but the single particle energy, Eq. (2.6) for $E_p(l)$, is renormalized. In particular, as shown in the next section, the renormalization of a gluon energy is important for constructing an effective $q\bar{q}$ interaction over all energy ranges.

Flow equations not only eliminate the quark-gluon coupling, which renders the Hamiltonian block diagonal, they also permit renormalization. Renormalizing the gluon effective energy yielding $E_p(l)$ leads to two important consequences. First, the quark-gluon coupling is eliminated for all energies, even for degenerate states. Second, eliminating the quark-gluon coupling with $E_p(l)$ for degenerate states, the effective $q\bar{q}$ interaction is obtained of the form $1/q^4$, which is more singular than the perturbative result, $1/q^2$. This potential provides confinement and bound state of quarks.

We conclude this section with a synopsis of previous flow equation applications in other fields. The situation with degenerate states was first treated by flow equations in solid state physics. Lenz and Wegner showed for systems of interacting electrons [13] that the original electron-phonon coupling can be completely eliminated, even when the states connected by this interaction are degenerate. Lenz and Wegner found [13] that the induced electron-electron interaction differs from Fröhlich's, whose unitary transformation is based on second order perturbation theory. Moreover, this interaction is attractive for any momentum, binding electrons in Cooper pairs. Also, Kehrein, Mielke and Neu [12] have shown for the spin-boson problem that flow equations allow a complete elimination of the coupling to the bosonic bath even for real processes. Finally, Kehrein and Mielke obtained similar modifications due to an l-dependent generator by eliminating the hybridization term in the Anderson model [16]. The authors showed that their approach generates a spin-spin interaction which differs from the one obtained by the famous Schrieffer-Wolff transformation. Their induced interaction has the right high-energy cutoff, as compared to Schrieffer-Wolff's result. Thus, within flow-equation approach it is possible to obtain new results which can not be obtained by perturbation theory.

In the next section, we use the above formulation to solve the flow equations for the effective gluon energy and the quark-antiquark interaction [see Eq. (2.6)] using the light-front quantized QCD Hamiltonian.

III. FLOW EQUATIONS IN LIGHT-FRONT QCD

A. Gluon gap equation

In the light-front formulation, the flow equation for a single particle energy $p^- = [p_\perp^2 + \mu^2(\lambda)]/p^+$ is actually written for the mass $\mu^2(\lambda)$ since the term $p_\perp^2/p^+$ is independent of flow. The set of the coupled light-front equations for the cutoff dependent quark and gluon masses was first derived by Glazek [17]. We uncouple this set of equations by assuming that the quark mass does not flow with the cutoff, $m(\lambda) = m$, where m is the current quark mass. The light-front gluon gap equation is

$$\frac{d\mu^2(\lambda)}{d\lambda} = -2T_f N_f \int_0^1 \frac{dx}{x(1-x)} \int_0^\infty \frac{d^2k_\perp}{16\pi^3} g_q^2(\lambda) \frac{1}{Q_2^2(\lambda)} \frac{df^2(Q_2^2(\lambda);\lambda)}{d\lambda} \left(\frac{k_\perp^2+m^2}{x(1-x)} - 2k_\perp^2 \right)$$
$$-2C_a \int_0^1 \frac{dx}{x(1-x)} \int_0^\infty \frac{d^2k_\perp}{16\pi^3} g_g^2(\lambda) \frac{1}{Q_1^2(\lambda)} \frac{df^2(Q_1^2(\lambda);\lambda)}{d\lambda} \left[k_\perp^2 \left(1 + \frac{1}{x^2} + \frac{1}{(1-x)^2} \right) \right], \qquad (3.1)$$

where

$$Q_1^2(\lambda)=\frac{k_\perp^2+\mu^2(\lambda)}{x(1-x)}-\mu^2(\lambda), \quad Q_2^2(\lambda)=\frac{k_\perp^2+m^2}{x(1-x)}-\mu^2(\lambda). \tag{3.2}$$

Here, the group factors for $SU(N_c)$ are $T_f\delta_{ab}=\mathrm{Tr}(T^aT^b)=\frac{1}{2}\delta_{ab}$ and the number of flavors is $N_f=6$. The adjoint Casimir is defined by $C_a\delta_{ab}=f^{acd}f^{bcd}=N_c\delta_{ab}$ with the number of colors $N_c=3$ (the subscripts in C_a and N_c should not be confused with the group indices a and c). In the integral kernel, the gluon couples to quark-antiquark pairs and gluon pairs with quark-gluon coupling, $g_q(\lambda)$, and three-gluon coupling, $g_g(\lambda)$, respectively. For non-zero λ these couplings are different functions of momenta. The light-front momentum flowing in the loops has components $(x,k_\perp)$. In our derivation, the connection between flow parameter and the cutoff, $l=1/\lambda^2$, is used.

In Eq. (3.1), the effective gluon mass is defined at transverse gluon momentum $p_\perp=0$, as in light-front perturbation theory for the gluon mass correction [18]. Following other gluon gap equation studies, we assume the effective gluon mass vanishes for large gluon momenta. Therefore, even though the effective gluon mass depends on momentum, only its λ cutoff dependence at zero momentum is given by Eq. (3.1), i.e., $\mu(\lambda,p(p_z,p_\perp)=0)$. However, the λ dependence of μ is the only relevant renormalization issue.

Generally, it is difficult to solve Eq. (3.1) because the running couplings, $g_q(\lambda)$ and $g_g(\lambda)$, are not known. Therefore, the coupling cutoff dependence is neglected. Also, the initial condition for Eq. (3.1) is not known. Accordingly, the following two renormalization conditions are imposed to determine the running gluon mass $\mu(\lambda)$. First, the effective Hamiltonian at scale λ has eigenstates with eigenvalues $p^-=(p_\perp^2+\mu_0^2)/p^+$ [17], satisfying

$$\frac{p_\perp^2+\mu_0^2}{p^+}\langle p'|p\rangle=\frac{p_\perp^2+\mu^2(\lambda)}{p^+}\langle p'|p\rangle - \int^\lambda d\lambda' \sum_q [\eta_{p'q}(\lambda')h_{qp}(\lambda') - h_{p'q}(\lambda')\eta_{qp}(\lambda')], \tag{3.3}$$

where μ_0 denotes the ''physical'' gluon mass, and $|p\rangle$ denotes a single effective gluon state [P subspace, Eq. (2.2)] with momentum $(p^+,p_\perp)$ and $\langle p'|p\rangle=16\pi^3p^+\delta^{(3)}(p'-p)$. The generator η_{pq}, given by Eq. (2.3), eliminates the quark-gluon (three-gluon) coupling h_{pq}. Second, the effective gluon mass, renormalized in second order perturbation theory, equals the ''physical'' mass

$$\mu_{ren}^2(\lambda=\lambda_0)=\mu_0^2, \tag{3.4}$$

at the renormalization point λ_0.

From the definition of flow equations, a resulting effective Hamiltonian is given in the limit of the flow parameter $l\to\infty$ or $\lambda=0$. In the parton picture described in the introduction, the boost may be regarded as a renormalization group operation and the renormalization group fixed point can be identified as the infinite momentum limit. The corresponding renormalization group fixed point in the flow equations is then $\lambda=0$. Therefore, in Eq. (3.4), the renormalization point is set to zero, $\lambda_0=0$, at the end of calculations. Also, the ''physical'' gluon mass, which is used as an arbitrary mass parameter, can be taken to zero to restore gauge invariance (see below).

Even with these simplifications, solving Eq. (3.1) is still quite involved. The solution can only be found numerically. In order to proceed analytically, a mass parameter μ_0 (''physical'' mass), which corresponds to the choice of the renormalization point at $p^2=\mu_0^2$, is substituted for $\mu(\lambda)$ in the integral kernel on the right-hand side of Eq. (3.1). The same procedure is used to calculate the perturbative mass correction in [19]. Therefore the energy differences in Eq. (3.2) are given by

$$\tilde{Q}_1^2=\frac{k_\perp^2+\mu_0^2}{x(1-x)}-\mu_0^2, \quad \tilde{Q}_2^2=\frac{k_\perp^2+m^2}{x(1-x)}-\mu_0^2. \tag{3.5}$$

Using Eq. (3.5), Eq. (3.1) is solved iteratively and $\mu^2(\lambda)=\mu_0^2$ is the first iteration.

Another problem in Eq. (3.1) is that the loop integrals have an UV divergence in the transverse directions $k_\perp$ and an IR divergence in the longitudinal direction x. The flow equations naturally regulate these divergences via the similarity function in each vertex [for example, the gluon loop in Eq. (3.1) is regulated by $Q_1^2(\lambda)\leqslant\lambda^2$]. This type of regularization is known as ''Jacobi cutoffs,'' because of the Jacobi momenta of a constituent (it is also called ''global'' regularization in [19]). The advantage of this regularization in the light-front approach is that it preserves both transverse and longitudinal boost invariance [18]. For a nonzero mass such as a current quark mass in the quark loop, this regularization restricts $k_\perp$ and the light-front x integrations. Thus, the similarity function $f(\tilde{Q}_2^2;\lambda)$, with $\tilde{Q}_2^2$ given by Eq. (3.5), ensures that $0\leqslant k_\perp\leqslant(\lambda^2+\mu_0^2)x(1-x)-m^2$ and $m^2/(\lambda^2+\mu_0^2)\leqslant x\leqslant 1-m^2/(\lambda^2+\mu_0^2)$. However, for zero gluon mass in the gluon loop, $k_\perp$ is restricted from above and x runs the entire range $0\leqslant x\leqslant 1$, contributing divergent terms at $x=0$ and $x=1$ [see the similarity function $f(\tilde{Q}_1^2;\lambda)$]. Even when the instantaneous diagrams are included, the gluon loop is still divergent in x [19]. For a massive case the cutoff λ has a lower bound from a mass in the theory, $\lambda\geqslant m\neq 0$, which limits the x integration. The reason why flow equations do not regulate the light-front divergences in a massless case is because the cutoff λ can evolve to $\lambda=0$ with the similarity function $f(z\to\infty)=1$, and x spans the entire 0 to 1 range, where the loop integral with massless intermediate states diverges at $x=0$ and $x=1$. To regulate these divergences in a massless case, Zhang and Harindranath suggested restricting the $k_\perp$ integration also from below by some scale u. This is equivalent in our approach to integrating the flow equation (3.1) for finite limits, from λ to u [19], which restricts intervals in the x axis to above 0 and below 1. It was also shown in [18] and [19] that for the scale u, even if $\mu_0=0$ is zero, the gluon mass correction does not vanish. Effectively, introducing the scale u mimics the situation of a non-zero mass, $m=u$, in intermediate states.

In terms of effective light-front theory formulated on the x axis, the Hamiltonian below the first light-front cutoff, $H_{0\leqslant x\leqslant\varepsilon}$, and the Hamiltonian above the second cutoff $x=1$, $H_{(1-\varepsilon)\leqslant x\leqslant 1}$, describe the strongly correlated high energy states. Thus, they can be replaced by the Hamiltonian vacuum expectation value, since strongly coupled configurations are frozen. This vacuum expectation value (VEV) replacement in gluodynamics is consistent with a composite field, ϕ, creating 0^+ glueballs having a finite VEV [2]. The Hamiltonian in the intermediate region, $H_{\varepsilon\leqslant x\leqslant(1-\varepsilon)}$, is then treated by flow equations.

Integrating the gluon flow equation (3.1) for finite limits $[u;\lambda]$ yields

$$\begin{aligned}\mu^2(\lambda)-\mu^2(u)=&-2g^2T_fN_f\int_0^1 dx\int_0^\infty\frac{d^2k_\perp}{16\pi^3}[f^2(\tilde{Q}_2^2;\lambda)-f^2(\tilde{Q}_2^2;u)]\\&\times\left[\frac{\mu_0^2(2x^2-2x+1)}{k_\perp^2+m^2-x(1-x)\mu_0^2}+\frac{2m^2}{k_\perp^2+m^2-x(1-x)\mu_0^2}+\left(-2+\frac{1}{x(1-x)}\right)\right]\\&-2g^2C_a\int_0^1 dx\int_0^\infty\frac{d^2k_\perp}{16\pi^3}[f^2(\tilde{Q}_1^2;\lambda)-f^2(\tilde{Q}_1^2;u)]\left(1+\frac{\mu_0^2x(1-x)}{k_\perp^2-x(1-x)\mu_0^2+\mu_0^2}-\frac{\mu_0^2}{k_\perp^2-x(1-x)\mu_0^2+\mu_0^2}\right)\\&\times\left(1+\frac{1}{x^2}+\frac{1}{(1-x)^2}\right),\end{aligned}\qquad(3.6)$$

where the renormalization point is $q^2=\mu_0^2$, and the energy differences are defined in Eq. (3.5). Though the quark loop does not diverge with x, the gluon flow equation is integrated the same way for the gluon and quark loops. Similarity functions restrict the transverse momenta to $k_{\perp min}\leqslant k_\perp\leqslant k_{\perp max}$, with

$$\begin{aligned}k_{\perp max}&=(\lambda^2+\mu_0^2)x(1-x)-\mu_0^2\\k_{\perp min}&=(u^2+\mu_0^2)x(1-x)-\mu_0^2,\end{aligned}\qquad(3.7)$$

and, for x,

$$\frac{\mu_0^2}{u^2+\mu_0^2}\leqslant x\leqslant 1-\frac{\mu_0^2}{u^2+\mu_0^2}.\qquad(3.8)$$

Analogous expressions hold for the quark loop.

Integration over momenta $(x,k_\perp)$ in Eq. (3.6) yields

$$\begin{aligned}\mu^2(\lambda)-\mu^2(u)=&-\frac{g^2T_fN_f}{4\pi^2}\left(\frac{1}{3}\mu_0^2\ln\frac{\lambda^2}{u^2}+m^2\ln\frac{\lambda^2}{u^2}+\frac{1}{3}(\lambda^2-u^2)\right)\\&-\frac{g^2C_a}{4\pi^2}\left[\mu_0^2\ln\frac{\lambda^2}{u^2}\left(-\frac{u^2}{\mu_0^2}+\ln\frac{u^2}{\mu_0^2}-\frac{5}{12}\right)+(\lambda^2-u^2)\left(\ln\frac{u^2}{\mu_0^2}-\frac{11}{12}\right)\right].\end{aligned}\qquad(3.9)$$

Assuming the renormalization conditions of Eqs. (3.3) and (3.4), the effective gluon mass at scale λ is

$$\mu^2(\lambda)=\mu_0^2+\delta\mu_{PT}^2(\lambda)+\delta\mu^2(\lambda,\lambda_0).\qquad(3.10)$$

Here, Eq. (3.3) fixes the integration constant [when integrating the flow equation (3.1)] to μ_0^2. In Eq. (3.10) the perturbative term reproduces the result from light-front perturbation theory [19], i.e.,

$$\delta\mu_{PT}^2(\lambda)=-\frac{g^2}{4\pi^2}\lambda^2\left[C_a\left(\ln\frac{u^2}{\mu_0^2}-\frac{11}{12}\right)+T_fN_f\frac{1}{3}\right].\qquad(3.11)$$

Renormalizing the effective Hamiltonian to second oder $O(g^2)$, the perturbative mass correction is absorbed by the mass counterterm, $m_{CT}^2(\Lambda_{UV})=-\delta\mu_{PT}^2(\Lambda_{UV})$ with $\Lambda_{UV}\to\infty$, and the renormalized effective gluon mass is

$$\mu_{ren}^2(\lambda)=\mu^2(\lambda)+m_{CT}^2(\lambda),\qquad(3.12)$$

for $\lambda=\Lambda_{UV}\to\infty$. Though perturbative renormalization is applied at large cutoff scales, Λ_{UV}, we assume that the leading cutoff dependence in second order is absorbed by the mass counterterm for all λ, and the renormalized gluon mass is given by Eq. (3.12) for any λ. Explicitly from Eq. (3.9), the renormalized gluon mass reads

$$\mu_{ren}^2(\lambda)=\mu_0^2+\delta\mu^2(\lambda,\lambda_0)=\mu_0^2+\sigma(\mu_0,u)\ln\frac{\lambda^2}{\lambda_0^2}$$

$$\sigma(\mu_0,u) = -\frac{g^2}{4\pi^2}\mu_0^2\left[C_a\left(-\frac{u^2}{\mu_0^2}+\ln\frac{u^2}{\mu_0^2}-\frac{5}{12}\right)+T_fN_f\left(\frac{1}{3}+\frac{m^2}{\mu_0^2}\right)\right], \tag{3.13}$$

where the renormalization condition, Eq. (3.4), at scale λ_0 is satisfied. Recall, this solution, Eq. (3.13), describes an effective gluon mass at zero gluon momentum. Since the effective Hamiltonian generated by flow equations is defined at cutoff scale $\lambda\to 0$, the resulting gluon mass equals the ''physical'' mass, $\mu_{ren}^2(\lambda=\lambda_0=0)=\mu_0^2$. In particular, when the ''physical'' mass is set to zero, $\mu_0=0$, the effective QCD Hamiltonian has zero mass gauge fields, and our unitary transformation no longer violates gauge invariance. Then, only for color singlet gauge invariant matrix elements are IR singularities associated with soft collinear gluons canceled exactly, satisfying the Kinoshita-Lee-Nauenberg (KLN) theorem [23]. This was demonstrated in detail perturbatively for the e^+e^- annihilation amplitude [24]. In our situation this zero gluon mass, gauge invariant limit also produces IR divergent quark and gluon self-energies. This may be interpreted as color confinement because the infinite self-energies suppress the quark and gluon propagation amplitudes. However and, consistent with the above theorem, in the Bethe-Salpeter equation (and the simpler Tamm-Dancoff equation), the IR divergent contributions from the kinetic self-energy and potential interaction kernel parts cancel exactly for color singlet states. Our generated two-body $q\bar{q}$ and gg potentials satisfy these conditions, providing finite masses for singlet states of mesons and glueballs.

From Eq. (3.13), one has, in the limit $\mu_0\to 0$,

$$\sigma = \lim_{\mu_0\to 0}\sigma(\mu_0,u) = u^2\frac{g^2C_a}{4\pi^2}, \tag{3.14}$$

where, as shown below, σ plays the role of the string tension between quark and antiquark. In Eq. (3.13), u^2 determines the rate at which the effective gluon mass asymptotically approaches the ''physical'' value, $\sim u^2\ln(\lambda/\lambda_0)$.

We note here how one may expect a difference in the result between QCD and QED. QCD has a non-Abelian gauge group and strong interactions, which distinguishes it from QED. Both canonical light-front QCD and QED Hamiltonians have the instantaneous interaction, $1/x^2$, which has a singular behavior for small x. For QED a small x regularization scheme can be found where divergent contributions from this instantaneous term are canceled in the matrix elements [19]. This is not true for QCD [19]. Because of the non-Abelian triple-gluon vertex, it is necessary to introduce a regulating scale u for the divergences at $x\sim 0$ and $x\sim 1$ in the gluon self-energy (the quark loop is regulated in the same fashion in order to combine quark and gluon loops). In fact, the string tension in Eq. (3.14) is proportional to the eigenvalue of quadratic Casimir operator in the adjoint representation, reflecting this non-Abelian character. There is no such scale in the QED self-energy operators, which are all regulated by a current electron mass m. One can therefore argue that the scale u carries information about strongly interacting fields and is present only in QCD.

In asymptotic free theories, such as QCD, the regulating scale can be related to the gauge invariant scale using the Callan-Symanzik equation. In our case, the scale u can be expressed in terms of Λ_{QCD} by solving the third order flow equations for the strong coupling constant, $\alpha_s(\lambda)$. Such calculations have been recently performed for an asymptotic free toy model [20] and for the three-gluon vertex in QCD [21]. However, here we forgo this procedure and simply assume u is given by the hadron scale, $u\sim\Lambda_{hadron}$, which also establishes the string tension scale.

The asymptotic behavior of the effective gluon mass near the renormalization point is important for self-consistently solving the flow equations for effective interactions, Eq. (2.6), at vanishing gluon momenta. In the next subsection, this dependence [Eq. (3.13)] is used to find an effective quark potential generated by the flow equations.

B. Effective quark-antiquark interaction

Eliminating the quark-gluon coupling generates an effective interaction in the quark-antiquark sector [the second equation in Eqs. (2.6)]. The energy transfers (i.e. energy denominators) along the quark $p_1\to p_2$ and antiquark $p_2'\to p_1'$ lines are given by $D_1=p_1^- - p_2^- - (p_1-p_2)^-$ and $D_2=p_2'^- - p_1'^- - (p_2'-p_1')^-$, respectively. We denote the exchange momentum as $q=(q^+,q_\perp)=p_1-p_2$. These energy transfers can be related to the corresponding square of the four-momenta as $Q_1^2=(p_1-p_2)^2=-q^+D_1$ and $Q_2^2=(p_2'-p_1')^2=-q^+D_2$ [22]. In the light-front formulation, the scalar product is defined $p\cdot k=(p^+k^-+p^-k^+)/2-p_\perp\cdot k_\perp$ and the four-momentum transfers for quark momentum $(x,k_\perp)$ and antiquark momentum $(1-x,-k_\perp)$ are given by

$$Q_1^2(\lambda)=Q_1^2+\mu_{ren}^2(\lambda)$$

$$Q_2^2(\lambda)=Q_2^2+\mu_{ren}^2(\lambda), \tag{3.15}$$

where the asymptotic form for the renormalized effective gluon mass, $\mu_{ren}^2(\lambda)=\mu_0^2+\sigma(\mu_0,u)\ln(\lambda^2/\lambda_0^2)$ [Eq. (3.13)], enters. The four-momentum transfers are now dependent on the cutoff λ and for zero gluon mass are given [22] by

$$Q_1^2=\frac{(x'k_\perp - xk_\perp')^2+m^2(x-x')^2}{xx'}$$

$$Q_2^2=Q_1^2|_{x\to(1-x);\ x'\to(1-x')}. \tag{3.16}$$

As shown below, in the instant formulation, energy transfers with zero gluon mass, Eq. (3.16), describe a three-momentum exchange, $\vec{q}^{\,2}$ with $\vec{q}(q_z,q_\perp)$, while energy transfers with non-zero cutoffs, Eq. (3.15), correspond to an effective energy (with effective mass) exchange, $\vec{q}^{\,2}+\mu_{ren}^2(\lambda)$. Near the renormalization point the energy transfers given by Eq. (3.15) and Eq. (3.16) coincide asymptotically, i.e. for $\lambda\to\lambda_0\sim 0$ and $\mu_0\sim 0$, $Q_i^2(\lambda)\to Q_i^2$, $i=1,2$. Using Eq. (3.15), the gluon four-momentum can be written as

$$q_\mu = p_{1\mu} - p_{2\mu} + \frac{\eta_\mu}{2q^+}Q_1^2(\lambda) = p'_{2\mu} - p'_{1\mu} + \frac{\eta_\mu}{2q^+}Q_2^2(\lambda), \tag{3.17}$$

where the light-front unit vector η_μ is $\eta^\mu=(2^-,0^+,0_\perp)$, with $\eta\cdot k = k^+$. In subsequent calculations, we use the average and difference of four-momenta transfers

$$Q^2(\lambda) = [Q_1^2(\lambda)+Q_2^2(\lambda)]/2 = Q^2 + \mu_{ren}^2(\lambda)$$
$$\delta Q^2(\lambda) = [Q_1^2(\lambda)-Q_2^2(\lambda)]/2 = \delta Q^2, \tag{3.18}$$

where $Q^2=(Q_1^2+Q_2^2)/2$ and $\delta Q^2=(Q_1^2-Q_2^2)/2$.

Solving the flow equations self-consistently, the energy transfers for non-zero λ define effective interactions between quarks as specified by Eq. (2.6). Thus, the similarity form factors in each vertex $f(Q_1^2(\lambda);\lambda)$ and $f(Q_2^2(\lambda);\lambda)$ provide an effective interaction (see below). However, for large λ only high energies are eliminated and the momenta transfer cutoff dependence, $Q_i^2(\lambda)$ is minimal. Reducing λ, energy transfers are eliminated continuously from high to low energy. Only for small λ and corresponding momentum transfers is the asymptotic behavior of $Q_i^2(\lambda)$ essential. This is important in solving for the effective interaction, since (see Sec. II B) this asymptotic behavior (the cutoff dependent gluon effective energy at small λ) eliminates the quark-gluon coupling even for vanishing gluon momenta, i.e., when $Q_1^2=Q_2^2=0$ ($x=x'$ and $k_\perp=k'_\perp$). If $Q^2=0$ and $\mu_0\to 0$, the similarity form factor $f=\exp[-Q^2(\lambda)/\lambda^2]$, with $Q^2(\lambda)$ given by Eq. (3.15), decays at small $\lambda\to 0$ (with $\lambda>\lambda_0$) as

$$f\sim\left(\frac{\lambda^2}{\lambda_0^2}\right)^{(-\sigma/\lambda^2)}. \tag{3.19}$$

Note this is a power suppression, not exponential, with σ specified by Eq. (3.14). Only for $\lambda=\lambda_0$ does $f=1$; however, for other values, $\lambda\neq\lambda_0$, f decays and the integral over λ in Eq. (2.6) is finite in the effective interaction.

In the light-front formulation, an effective quark-antiquark interaction is given by a matrix element of an effective light-front QCD Hamiltonian in the $q\bar{q}$ sector (Q space) and includes dynamical interactions generated by flow equations [the second equation in Eqs. (2.6)] as well as instantaneous terms present in the light-front gauge. These terms are embodied in the effective $q\bar{q}$ interaction

$$V_{q\bar{q}} = -4\pi\alpha_s C_f \langle\gamma^\mu\gamma^{\mu'}\rangle \lim_{(\mu_0,\lambda_0)\to 0} B_{\mu\mu'}, \tag{3.20}$$

where $C_f=T^aT^a=(N_c^2-1)/2N_c$ is the eigenvalue of Casimir operator in the fundamental representation, and the current-current term in the exchange channel is given by (quark helicity notation is suppressed)

$$\langle\gamma^\mu\gamma^{\mu'}\rangle = \frac{[\bar{u}(-k_\perp,(1-x))\gamma^\mu u(k_\perp,x)][\bar{v}(k'_\perp,x')\gamma^{\mu'}v(-k'_\perp,(1-x'))]}{\sqrt{xx'(1-x)(1-x')}}. \tag{3.21}$$

The interaction kernel, $B_{\mu\mu'}$, is analogous to the effective electron-positron interaction in the light-front QED [22], except for the cutoff dependence in the energy transfers

$$B_{\mu\mu'} = g_{\mu\mu'}(I_1+I_2) + \eta_\mu\eta_{\mu'}\frac{\delta Q^2}{q^{+2}}(I_1-I_2). \tag{3.22}$$

Here $g_{\mu\mu'}$ is the light-front metric tensor and η_μ is given previously [see Eq. (3.17)]. The cutoff dependent energy (four-momentum) transfer, Eq. (3.15), along the quark and antiquark lines is included in the integral

$$I_1 = \int_0^\infty d\lambda \frac{1}{Q_1^2(\lambda)}\frac{df(Q_1^2(\lambda);\lambda)}{d\lambda} f(Q_2^2(\lambda);\lambda). \tag{3.23}$$

I_2 is obtained by interchanging indices 1 and 2 in Eq. (3.23). As discussed previously, the sensitivity to the renormalization scheme and renormalization point is eliminated by setting both μ_0 and λ_0 to zero after integration.

For the following three explicit forms for the similarity function, the leading behavior of the integral factor, Eq. (3.23), is given by the following:

(1) Exponential:

$$f=\exp[-Q^2(\lambda)/\lambda^2],$$
$$I_1 = \frac{1}{Q_1^2(\lambda_0)+Q_2^2(\lambda_0)}\left(1+\frac{\sigma(\mu_0,u)}{Q_1^2(\lambda_0)}\right).$$

(2) Gaussian:

$$f=\exp[-Q^4(\lambda)/\lambda^4],$$
$$I_1 = \frac{Q_1^2(\lambda_0)}{Q_1^4(\lambda_0)+Q_2^4(\lambda_0)}\left(1+\frac{\sigma(\mu_0,u)}{Q_1^2(\lambda_0)}\right).$$

(3) Sharp:

$$f=\theta(\lambda^2-Q^2(\lambda)),$$

$$I_1=\frac{\theta(Q_1^2(\lambda_0)-Q_2^2(\lambda_0))}{Q_1^2(\lambda_0)}\left(1+\frac{\sigma(\mu_0,u)}{Q_1^2(\lambda_0)}\right), \tag{3.24}$$

where from Eq. (3.15) the four-momenta are given by $Q_i^2(\lambda_0)=Q_i^2+\mu_0^2$ with $i=1,2$. Several properties of the similarity function were used to approximate the integral in Eq. (3.23). The similarity function $f(z)$ decays for arguments $z\geqslant 1$; thus the integral saturates for values $0\leqslant\lambda^2\leqslant(Q_1^2(\lambda_0),Q_2^2(\lambda_0))$ at small energy transfer. Also, $\mu_0\sim\lambda_0\sim 0$. Note that Eq. (3.24) has a form consistent with the exchange of a gluon with an effective mass parameter μ_0 between quark and antiquark at distances $r\sim 1/\mu_0$. The resulting $q\bar{q}$ interaction is defined in the limit of vanishing mass parameter $\mu_0\to 0$ and at the zero renormalization cutoff point $\lambda_0\to 0$, Eq. (3.20). In this limit, the average four-momentum transfer reduces to $\lim_{(\mu_0,\lambda_0)\to 0}Q^2(\lambda_0)=Q^2$, Eq. (3.18), and the string tension is given by $\lim_{(\mu_0,\lambda_0)\to 0}\sigma(\mu_0,u)=\sigma$, Eq. (3.14). Using Eqs. (3.22) and (3.24), the resulting interaction kernel for the three similarity function choices is the following:

(1) Exponential:

$$\lim_{(\mu_0,\lambda_0)\to 0}B_{\mu\mu'}=g_{\mu\mu'}\left(\frac{1}{Q^2}+\frac{\sigma}{Q^4}\right)+\left(\frac{g_{\mu\mu'}}{Q^2}-\frac{\eta_\mu\eta_{\mu'}}{q^{+2}}\right)\frac{\sigma}{Q^2}\frac{\delta Q^4}{Q^4-\delta Q^4}.$$

(2) Gaussian:

$$\lim_{(\mu_0,\lambda_0)\to 0}B_{\mu\mu'}=g_{\mu\mu'}\left(\frac{1}{Q^2}+\frac{\sigma}{Q^4}\right)-\left[\frac{g_{\mu\mu'}}{Q^2}\left(1+\frac{\sigma}{Q^2}\right)-\frac{\eta_\mu\eta_{\mu'}}{q^{+2}}\right]\frac{\delta Q^4}{Q^4+\delta Q^4}.$$

(3) Sharp:

$$\lim_{(\mu_0,\lambda_0)\to 0}B_{\mu\mu'}=g_{\mu\mu'}\left(\frac{1}{Q^2}+\frac{\sigma}{Q^4}\right)-\left\{\frac{g_{\mu\mu'}}{Q^2}\left[1+\frac{\sigma}{Q^2}\left(1+\frac{Q^2}{Q^2+|\delta Q^2|}\right)\right]-\frac{\eta_\mu\eta_{\mu'}}{q^{+2}}\left(1+\frac{\sigma}{Q^2}\frac{Q^2}{Q^2+|\delta Q^2|}\right)\right\}\times\frac{|\delta Q^2|}{Q^2+|\delta Q^2|}, \tag{3.25}$$

where $Q^4=(Q^2)^2$ and $\delta Q^4=(\delta Q^2)^2$. The four-momenta in Eq. (3.25) can be represented in the ''mixed'' light-front and instant representations as

$$Q^2=\vec{q}^{\,2}-\frac{1}{4}(2x-1)(2x'-1)(M_1-M_2)^2$$

$$\delta Q^2=-\frac{1}{2}(x-x')(M_1^2-M_2^2), \tag{3.26}$$

where, together with the light-front momentum $(x,k_\perp)$, the instant momentum $\vec{k}=(k_z,k_\perp)$ enters with the connection

$$x=\frac{1}{2}\left(1+\frac{k_z}{\sqrt{\vec{k}^2+m^2}}\right), \tag{3.27}$$

and $\vec{k}^2=k_\perp^2+k_z^2$. The gluon three-momentum transfer is given by $\vec{q}=\vec{k}-\vec{k}'=(q_z,q_\perp)$ and the total energies of the initial and final $q\bar{q}$ states are given by

$$M_1^2=4(\vec{k}^2+m^2)$$

$$M_2^2=4(\vec{k}'^2+m^2). \tag{3.28}$$

Our results, Eq. (3.25), are not rotationally invariant due to truncation of Fock space. However, on the energy shell ($M_1^2=M_2^2$ and $\delta Q^2=0$), the second term in Eq. (3.25) is identically zero and the effective $q\bar{q}$ interaction is given by the first term

$$V_{q\bar{q}}=-\langle\gamma^\mu\gamma_\mu\rangle\left(C_f\alpha_s\frac{4\pi}{\vec{q}^2}+\sigma_f\frac{8\pi}{\vec{q}^4}\right). \tag{3.29}$$

It is significant to note that all three choices yield the same effective interaction. Here $\sigma_f=\sigma\alpha_sC_f/2$ with σ given by Eq. (3.14). Further, and interestingly, we also obtain a Cornell type interaction with Coulomb and linear confining potentials

$$V_{q\bar{q}}=\langle\gamma^\mu\gamma_\mu\rangle\left(-C_f\frac{\alpha_s}{r}+\sigma_f\cdot r\right). \tag{3.30}$$

If the external particles are all on the energy shell, our light-front formulation yields a rotationally invariant potential as expected. The confining term in Eq. (3.30) [or Eq. (3.29)] arises from the elimination of the quark-gluon coupling at small gluon momenta, which is governed by the asymptotic behavior of the effective gluon mass at small cutoff. It is also significant to note that the same quark effective interaction, Eqs. (3.29) and (3.30), emerges in the limit of static (infinite heavy $m\to\infty$) quarks.

The N_c behavior of the confining term in Eq. (3.30) with the string tension given by

$$\sigma_f=\frac{u^2}{8\pi^2}\frac{g^4C_aC_f}{4\pi}=\frac{u^2}{8\pi^2}\frac{g^4(N_c^2-1)}{8\pi} \tag{3.31}$$

can be compared with available $2+1$ lattice data [25] for the expectation value of the Wilson loop in the fundamental representation, $\langle W_f(C)\rangle=\exp(-\sigma_{2+1}S_C)$, where S_C is the area of the loop C. Though the data are for $2+1$, the same

N_c behavior is expected for $3+1$ and higher dimensions [25]. Monte Carlo calculations of the string tension in $2+1$ give the values $\sqrt{\sigma_{2+1}}/g^2=0.3354$, 0.553, 0.758 and 0.966 for the gauge groups $SU(2)$, $SU(3)$, $SU(4)$ and $SU(5)$ respectively [25] (note that the coupling constant in $2+1$ has dimension, $g^2\sim$energy). The corresponding values calculated from Eq. (3.31) are $2\pi\sqrt{2\sigma_f}/(ug^2)=0.345$, 0.564, 0.772 and 0.977. We see that there is excellent agreement (up to $\sim 3\%$) for this N_c behavior between (3.31) and the Monte Carlo results. It is further interesting to notice that our analytic expression for the string tension, Eq. (3.31), has the appropriate N_c dependence as expected from large N_c calculations.

Similarly the quark-quark potential can be obtained and is related to the quark-antiquark potential by

$$V_{qq}=-\frac{1}{2}V_{q\bar{q}}, \tag{3.32}$$

since only the commutator $[b^\dagger ba, b^\dagger ba^\dagger]$ contributes to V_{qq}, while $[b^\dagger ba, d^\dagger da^\dagger]$ and $[d^\dagger da, b^\dagger ba^\dagger]$ contribute to $V_{q\bar{q}}$. Assuming the relation Eq. (3.32), Basdevant and Boukraa [26] showed that the ground state of baryons can be calculated with good accuracy.

Note that, because we use perturbation theory, we do not claim to have completed the derivation of a confining potential in QCD. However, our $1/\vec{q}^4$ quark-antiquark potential might be a precursor to quark confinement. Perhaps equally as important, our results conflict with the notion that a weak-coupling expansion will never produce a confining potential.

Finally, there are corrections $O(\delta Q^2/Q^2)$ to the leading effective interaction, which depend on the direction $\vec{q}$ approaches zero (as was investigated in [22]). An important limiting case is the collinear limit

$$q^+\to 0. \tag{3.33}$$

This is special for light-front calculations and may cause divergences. From Eq. (3.26) in this limit, $\delta Q^2\sim q^+$, and for sufficiently smooth similarity functions $f(z)$, like exponential or Gaussian, the effective interaction Eq. (3.25) does not contain a collinear singularity, because $(\eta_\mu\eta_{\mu'}/q^{+2})\delta Q^4$ is finite. Thus the interaction only becomes singular if $\vec{q}$ approaches zero as $1/\vec{q}^2$ (''Coulomb'' singularity) or $1/\vec{q}^4$ (''confining'' singularity), namely the leading behavior given by Eq. (3.29). Note that for mass spectrums both singularities are controllable. The ''Coulomb'' singularity is integrable, because the integral $\int d^3q/q^2$ is finite for small q and the ''confining'' singularity can be regulated in the IR. However, this is not true for the sharp cutoff where the $\eta_\mu\eta_{\mu'}$ term diverges as $1/q^+$, because it appears in the combination $(\eta_\mu\eta_{\mu'}/q^{+2})|\delta Q^2|$. We do not attach physical significance to this singularity, which is a consequence of the arbitrary cutoff choice leading to a singular unitary transformation generator at small momenta, Eq. (2.3). In summary, the collinear singularity is completely absent for a smooth cutoff and only the rotationally invariant part of the effective interaction Eq. (3.29) remains in the limit $\vec{q}\to 0$, where the dominant contribution to the spectrum is obtained. Corrections from the $\eta_\mu\eta_{\mu'}$ term are finite and shift the entire spectrum by a constant since they are diagonal in spin space.

As numerically documented in the positronium QED effective interaction application [22], rotational symmetry holds with high accuracy even if the on-energy-shell condition for the external particles is removed. This holds for smooth cutoff functions and even for a sharp cutoff if the collinear singular part is subtracted. Based on our analysis here, we may anticipate similar results for the QCD effective interaction.

C. Effective gluon-gluon interaction

In full analogy with Sec. III B, eliminating the three-gluon coupling generates an effective interaction between two gluons. Again, in leading order, the asymptotic behavior of the effective mass of the exchanged gluon is included. Using the same notation as Sec. III B, an effective gluon-gluon interaction is again generated by the flow equations. It contains both dynamical and instantaneous gluon exchange and is given by

$$V_{gg}=-4\pi\alpha_s C_a\langle\Gamma^\mu\Gamma^{\mu'}\rangle\lim_{(\mu_0,\lambda_0)\to 0}B_{\mu\mu'}, \tag{3.34}$$

where now $C_a\delta_{ab}=f^{acd}f^{bcd}=N_c\delta_{ab}$ is the eigenvalue of Casimir operator in the adjoint representation. Here, the current-current term in the exchange channel is given by

$$\langle\Gamma^\mu\Gamma^{\mu'}\rangle=\frac{\Gamma^{\mu\nu\rho}(-q,p_1,-p_2)\Gamma^{\mu'\nu'\rho'}(q,p_1',-p_2')}{\sqrt{xx'(1-x)(1-x')}}\times\epsilon_\nu(p_1)\epsilon_\rho^*(p_2)\epsilon_{\nu'}(p_1')\epsilon_{\rho'}^*(p_2'), \tag{3.35}$$

where

$$\begin{aligned}\Gamma^{\mu\nu\rho}(-q,p_1,-p_2)&\Gamma^{\mu'\nu'\rho'}(q,p_1',-p_2')\\ &=[(p_1+p_2)^\mu g^{\rho\nu}+(-2p_1)^\rho g^{\mu\nu}+(-2p_2)^\nu g^{\mu\rho}]\\ &\quad\times[(p_1'+p_2')^{\mu'}g^{\rho'\nu'}+(-2p_1')^{\rho'}g^{\mu'\nu'}\\ &\quad+(-2p_2')^{\nu'}g^{\mu'\rho'}],\end{aligned} \tag{3.36}$$

and the light-front momenta are $p_1=(x,k_\perp)$, $p_1'=(1-x,-k_\perp)$ and $p_2=(x',k_\perp')$, $p_2'=(1-x',-k_\perp')$. The gluon polarizations are omitted for simplicity. The interaction kernel $B_{\mu\mu'}$ is given by Eq. (3.22), which is again defined for different cutoff functions in Eq. (3.25). The polarization vector property, $q\cdot\epsilon=\eta\cdot\epsilon=0$, and representation for the gluon momentum, Eq. (3.17), have been used to simplify the above expressions.

Both quark-antiquark and gluon-gluon interactions have the same kernel $B_{\mu\mu'}$, and differ only in prefactors. Following arguments detailed in Sec. III B, the leading gluon-gluon interaction is also independent of cutoff function form and is given by

$$V_{gg}=\langle\Gamma^{\mu}\Gamma_{\mu}\rangle\left(-C_a\frac{\alpha_s}{r}+\sigma_a\cdot r\right), \tag{3.37}$$

which is, again, a Coulomb plus linear potential between two color sources in the adjoint representation with $\sigma_a=\sigma\alpha_s C_a/2$. The only difference between the $q\bar{q}$, Eq. (3.30), and gg, Eq. (3.37), potentials is given by the ratio of the fundamental to adjoint Casimir operators,

$$V_{q\bar{q}}/V_{gg}=C_f/C_a=(N_c^2-1)/2N_c^2=4/9, \tag{3.38}$$

since both potentials have the same Coulomb plus confining behavior.

IV. CONCLUDING DISCUSSION

An effective QCD Hamiltonian in the light-front gauge has been obtained, solving the flow equations for the two lowest Fock sectors self-consistently. It has been shown that it is possible to eliminate the minimal quark-gluon interaction by using a continuous unitary transformation. In this elimination, the coupling functions of the Hamiltonian described by the flow equations are renormalized. In the two lowest Fock sectors this change of the couplings corresponds to the renormalization of the one-particle energies and to the generation of effective interactions between gluons and quarks, in particular the quark-antiquark interaction. In obtaining the flow differential equations, intermediate states with more than three particles were omitted. Fock number truncation in intermediate states is quite different from a perturbation coupling constant treatment and more similar to the Tamm-Dancoff approach.

Our approach has several advantages. First, the original gauge field coupling can be completely eliminated, even when the states connected by this interaction are degenerate. The continuous transformation is designed such that the transformed Hamiltonian does not contain any interactions between one (anti)quark and the creation or annihilation of one gluon. These unwanted interactions, connecting states with energy differences less than a cutoff scale $|E_p-E_q|\leqslant\lambda$, are present in the similarity renormalization approach because single particle energies are not renormalized. They can mix low and high Fock sectors, and are not amenable to a perturbative treatment. Further, ignoring these low-energy interactions may break gauge invariance and, in the light-front formulation, rotational symmetry as well. Second, an effective quark-antiquark interaction is rotationally symmetric when the external particles are on the energy shell. At small gluon momenta q, also the collinear singular terms $\sim 1/q^{+}$ and $\sim 1/q^{+2}$ cancel. Most interestingly, in addition to a Coulomb term $1/q^2$, which can be obtained in second order, there is also a more singular, confining term $1/q^4$. Our induced $q\bar{q}$ interaction also differs from the similarity renormalization result, where the collinear singular part of the remnant instantaneous interaction ($\sim 1/q^{+2}$) yields a logarithmic potential which is not rotationally symmetric. These differences stem from the coupling's flow parameter dependence.

Further, due to complete elimination of the quark-gluon coupling, the flow equation for an effective quark-antiquark interaction can be integrated for all cutoffs including $\lambda=0$. In the similarity renormalization approach one removes couplings perturbatively, reaching a minimum scale (cutoff), below which perturbation theory breaks down. The value of this cutoff depends on the problem considered, and might be ambiguous. For QCD this cutoff introduces a scale in the theory, which breaks gauge and rotational invariance [8]. In our approach, the regulator of small light-front x establishes a scale in the effective theory corresponding to the string tension in the effective quark-antiquark potential. Besides the nonzero scale, the resulting renormalized gluon mass vanishes asymptotically, maintaining gauge invariance. Also, the effective quark-antiquark interaction is rotationally symmetric when the external particles are on the energy shell. A shortcoming in this work is that the small light-front x cutoff scale u enters as an input parameter, fitted to the string tension from lattice calculations. This can be improved by relating the cutoff u to the renormalization group invariant scale Λ_{QCD} which necessitates higher Fock sector intermediate states. This would also permit confirmation of our regularization scheme independent results.

Our ultimate goal is to solve the coupled chain of flow equations in different sectors self-consistently. As shown in this work, even an approximate solution of the gluon gap equation combined with the flow equation for the effective $q\bar{q}$ interaction provides new information beyond standard perturbation theory. The next step is to address the quark sector, formulating the quark gap equation and obtaining the renormalized light-front quark mass and attending improved effective quark interaction.

Finally, reflecting on the above discussions, we note that the light-front flow equations can now avail themselves of self-consistent computer generated solutions. It would be quite interesting to confront hadronic data with large scale, but feasible applications of this approach.

In summary, the light-front formulation appears quite viable for examining the long range aspects of QCD. Implementing the flow equations within this framework has generated new insight regarding nonperturbative phenomena including confinement.

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