# Duality and noncommutative gauge theory

Ori J. Ganor*
*Department of Physics, Jadwin Hall, Princeton, New Jersey 08544*

Govindan Rajesh† and Savdeep Sethi‡
*School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540*

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We study the generalization of $S$ duality to noncommutative gauge theories. For rank-1 theories, we obtain the leading terms of the dual theory by Legendre transforming the Lagrangian of the noncommutative theory expressed in terms of a commutative gauge field. The dual description is weakly coupled when the original theory is strongly coupled if we appropriately scale the noncommutativity parameter. However, the dual theory appears to be noncommutative in space-time when the original theory is noncommutative in space. This suggests that locality in time for noncommutative theories is an artifact of perturbation theory.

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## I. INTRODUCTION

Noncommutative gauge theory [1] provides an interesting class of examples in which to explore the effects of spatial nonlocality. While it is easy to define the classical noncommutative gauge theory, it is much harder to determine whether the quantum theory exists. Since noncommutative gauge theories arise in particular string theory backgrounds, we know that these theories can be embedded consistently in string theory. The decoupling argument of Seiberg and Witten [2] suggests that some of these theories might exist as quantum theories independent of string theory.

We are primarily interested in four-dimensional gauge theories. Our goal is to understand how $S$ duality [3,4] generalizes to noncommutative gauge theory. The generalization is not a straightforward consequence of $S$ duality in type-IIB string theory. To see this, let us begin by briefly recalling how $S$ duality of $N=4$ Yang-Mills theory arises from string theory. In the limit $\alpha' \to 0$, the theory on coincident D3-branes is $N=4$ Yang-Mills theory. For simplicity, we set the Ramond-Ramond (RR) scalar $C^{(0)}$ to zero. The gauge theory coupling constant $g^2$ is then related to the closed string coupling constant $g_s = e^{\phi}$:

$$\frac{g^2}{4\pi} = g_s. \tag{1.1}$$

The conjectured SL$(2,\mathbb{Z})$ symmetry of string theory then descends to an SL$(2,\mathbb{Z})$ symmetry of the field theory.

To obtain noncommutative Yang-Mills theory, we consider a system of coincident D3-branes with Neveu-Schwarz–Neveu-Schwarz (NS-NS) $B$ field nonzero along the brane. In the decoupling limit [2], the theory on the brane has a coupling constant related to the open string coupling constant $G_s$ rather than the closed string coupling:

$$g^2 = 2\pi G_s. \tag{1.2}$$

In the decoupling limit, the closed string coupling constant goes to zero, while $G_s$ remains finite and dependent on the $B$ field. In this case, $S$ duality of the closed string theory does not descend to a symmetry of the field theory.

For a U(1) gauge theory, $S$ duality can be demonstrated directly with a purely field theoretic argument. We start with the Minkowski space action[1]

$$S = -\int \frac{1}{4g^2} F \wedge * F, \tag{1.3}$$

where $F = dA$ is the field strength. We want to perform a Legendre transformation with respect to $F$. To implement the Bianchi identity

$$dF = 0,$$

we introduce a dual gauge field $A_D$:

$$S = -\int \left( \frac{1}{4g^2} F \wedge * F + \frac{1}{2} A_D \wedge dF \right). \tag{1.4}$$

We can now treat $F$ as an independent variable and perform the path integral over $F$. This amounts to solving the field equations for $F$, which gives the relation

$$dA_D = \frac{1}{g^2} * F \tag{1.5}$$

and the resulting dual action

$$S = -\int \frac{g^2}{4} F_D \wedge * F_D. \tag{1.6}$$

The aim of this discussion is to generalize this purely field theoretic argument to the noncommutative rank-1 theory.

*Email address: origa@viper.princeton.edu
†Email address: rajesh@sns.ias.edu
‡Email address: sethi@sns.ias.edu

[1] We use $*$ to denote the Hodge dual of a form rather than the star product.

 

Unlike ordinary Abelian gauge theory, the coupling constant cannot be scaled away even for the rank-1 noncommutative theory.

In the following section, we explicitly show that the noncommutative action expressed in terms of a commutative gauge field contains only powers of $F$ to order $\theta^2$. In particular, the gauge field does not appear explicitly. It is not hard to argue that this must be true to all orders in $\theta$. This implies that we can obtain a dual description by Legendre transforming with respect to $F$. The resulting dual theory is classical since we neglect loops. However, to order $\theta$, we will see that no loops appear and the quantum and semiclassical dual descriptions agree. To order $\theta^2$, loops appear and the bosonic theory needs to be regulated. At this point, the computation should be performed in the full $N=4$ theory.

Fortunately, our primary observations are already visible at order $\theta$. We find that, under the duality transformation,

$$\theta \to \tilde{\theta} = g^2(*\theta). \tag{1.7}$$

That this transformation does not square to 1 is not so surprising since $(S)^2$ is not the identity operation, but charge conjugation. We will also find that $\tilde{\theta}$ must be held fixed if the dual theory is to have a perturbative expansion in $1/g$. Even more interesting is the observation that if $\theta$ is purely spatial, then $\tilde{\theta}$ involves a space direction and a time direction. The theory becomes noncommutative in space-time. Although we will not obtain the complete quantum dual description, it seems clear that this feature, visible at leading order in $\theta$, persists to higher orders. Space-time noncommutative theories are highly unusual; see [5] for a recent discussion. Our result suggests that we cannot avoid studying these theories if we are to understand theories which perturbatively have only spatial noncommutativity.

## II. DUALITY TRANSFORMATION

### A. Rewriting the noncommutative Lagrangian

The noncommutative theory is defined by the action

$$S = -\frac{1}{4g^2}\int \hat{F}\wedge *\hat{F}. \tag{2.1}$$

The change of variables given in [2] allows us to express $\hat{F}$ in terms of a commutative gauge field $A$. We assume that $\theta$ is purely spatial. The relation takes the form

$$\hat{F} = F + T_\theta(A) + T_{\theta^2}(A) + \cdots. \tag{2.2}$$

The terms of order $\theta$ are given by

$$T_\theta(A) = -F\theta F - A_k\theta^{kl}\partial_l F. \tag{2.3}$$

We follow the notation of [6] where $F\theta F = F_{ik}\theta^{kl}F_{lj}$. The expression for $T_{\theta^2}(A)$ is found in [6]:

$$T_{\theta^2}(A) = F\theta F g v F + \frac{1}{2}A_k\theta^{kl}(\partial_l A_m + F_{lm})\theta^{mn}\partial_n F + \theta^{kl}A_k\partial_l(F\theta F) + \frac{1}{2}\theta^{kl}\theta^{mn}A_kA_m\partial_l\partial_n F. \tag{2.4}$$

The expression for $\hat{F}$ explicitly contains $A$. However, we can manipulate the action (2.1) so that it takes the following form:

$$S = -\frac{1}{4g^2}\int [F\wedge *F + L_\theta(F) + L_{\theta^2}(F) + \cdots]. \tag{2.5}$$

The terms of order $\theta$ take the form

$$L_\theta(F) = 2\,\mathrm{tr}(\theta F^3) - \frac{1}{2}\mathrm{tr}(\theta F)\mathrm{tr}(F^2), \tag{2.6}$$

where we define $\mathrm{tr}(AB) = A_{ij}B^{ji}$. Since our theory is rank 1, there should be no confusion with traces over group indices. It is not too hard to find an expression for $L_{\theta^2}(F)$ which takes the form

$$L_{\theta^2}(F) = -2\,\mathrm{tr}(\theta F\theta F^3) + \mathrm{tr}(\theta F^2\theta F^2) + \mathrm{tr}(\theta F)\mathrm{tr}(\theta F^3) - \frac{1}{8}\mathrm{tr}(\theta F)^2\mathrm{tr}(F^2) + \frac{1}{4}\mathrm{tr}(\theta F\theta F)\mathrm{tr}(F^2). \tag{2.7}$$

While we have explicitly demonstrated that it is possible to express Eq. (2.1) in terms of $F$ to order $\theta^2$, it must be the case to all orders in $\theta$. The only gauge-invariant operator that can be constructed from $A$ is $F$. While $\hat{F}$ can depend on $A$ explicitly, the action must be gauge invariant under the commutative gauge invariance. This requires that the action be expressible in terms of $F$ alone.

### B. Duality at $O(\theta)$

Since the action can be expressed in terms of $F$, we can implement a duality transformation in essentially the way described in the Introduction. To perform the Legendre transform, we shift the action as before:

$$S \to S + \int \frac{1}{2}A_D\wedge dF. \tag{2.8}$$

The equation of motion for $F$ gives

$$g^2F_D = *F + \frac{1}{2}\frac{\delta L_\theta}{\delta F}(F) + O(\theta^2). \tag{2.9}$$

To lowest order in $\theta$, we can solve for $F$ in terms of $F_D$:

$$*F = g^2F_D - \frac{1}{2}\left.\frac{\delta L_\theta}{\delta F}\right|_{F=-*g^2F_D} + O(\theta^2). \tag{2.10}$$

At order $\theta$, loops play no role in the duality transformation, so the quantum and semiclassical dual descriptions are equivalent. Plugging Eq. (2.10) into the action (2.5) gives

$$S=-\frac{g^2}{4}\int\left(F_D\wedge *F_D+2\ \mathrm{tr}(\tilde{\theta}F_D^3)-\frac{1}{2}\mathrm{tr}(\tilde{\theta}F_D)\mathrm{tr}(F_D^2)\right)+O(\tilde{\theta}^2). \tag{2.11}$$

Note that we use $\tilde{\theta}=g^2(*\theta)$ as the new noncommutativity parameter. The factor of $g^2$ in $\tilde{\theta}$ is natural because of the following scaling argument: we can schematically expand $\hat{F}^2$,

$$\hat{F}^2\sim F^2\left(1+\sum_{n,l}\ \theta^{n+l}(\partial)^{2l}F^n\right), \tag{2.12}$$

on strictly dimensional grounds. This implies that, iteratively, we can express $F$ in schematic form:

$$F\sim -g^2*F_D\left(1+\sum_{n,l}\ \theta^{n+1}(\partial)^{2l}(g^2*F_D)^n\right). \tag{2.13}$$

In terms of $\tilde{\theta}$, we see that

$$F\sim -g^2*F_D\left(1+\sum_{n,l}\ \tilde{\theta}^{n+l}(\partial)^{2l}\left(\frac{1}{g^2}\right)^l(*F_D)^n\right). \tag{2.14}$$

The action now takes the form of a derivative expansion with higher derivatives of $F_D$ suppressed by powers of $g^{-1}$.

There are a number of observations at this point. Substituting even the lowest order expression

$$F=-g^2*F_D+O(\theta) \tag{2.15}$$

into Eq. (2.5) results in an infinite number of terms involving higher powers of $\tilde{\theta}$. While terms beyond $O(\tilde{\theta})$ will receive additional corrections from the $O(\theta)$ corrections to Eq. (2.15), it seems quite clear—barring miraculous cancellations—that there is no upper bound on the power of $\tilde{\theta}$ that appears in the dual action. This suggests that it will be difficult to quantize the theory nonperturbatively in any conventional way. We also note that the dual action to leading order in $\tilde{\theta}$, expressed in dual noncommutative variables, takes the form

$$S=-\frac{g^2}{4}\int\ \hat{F}_D\wedge\hat{F}_D+O(\tilde{\theta}^2). \tag{2.16}$$

As is natural, we define $\hat{F}_D$ with respect to a star product involving $\tilde{\theta}$. However, it is quite possible that the corrections to Eq. (2.16) of $O(\tilde{\theta}^2)$ are nonvanishing. It is not clear that the resulting dual action would then have a purely quadratic form.

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