Back reaction in light cone QED

T. N. Tomaras* and N. C. Tsamis†
Department of Physics, and Institute of Plasma Physics, University of Crete and FO.R.T.H., P.O. Box 2208, 710 03 Heraklion, Crete, Greece

R. P. Woodard‡
Department of Physics, University of Florida, Gainesville, Florida 32611

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We consider the back reaction of quantum electrodynamics upon an electric field $E(x_+)=-A'_-(x_+)$ which is parallel to x^3 and depends only on the light cone coordinate $x_+=(x^0+x^3)/\sqrt{2}$. Novel features are that the mode functions have simple expressions for arbitrary $A_-(x_+)$ and that one cannot ignore the usual light cone ambiguity at zero + momentum. Each mode of definite canonical momentum k_+ experiences pair creation at the instant when its kinetic momentum $p_+=k_+-eA_-(x_+)$ vanishes, at which point operators from the surface at $x_-=-\infty$ play a crucial role. Our formalism permits a more explicit and complete derivation of the rate of particle production than is usually given. We show that the system can be understood as the infinite boost limit of the analogous problem of an electric field which is homogeneous on surfaces of constant x^0.

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I. INTRODUCTION

Many interesting things happen when quantum field theory is formulated on a non-trivial gauge field or metric background. One of these is that the background can cause virtual particles to move so as to engender currents or stresses which act to change it. This is the phenomenon of back reaction.

Our own fascination with back reaction concerns a quantum gravitational process which occurs on an inflating background. Superluminal expansion rips apart virtual pairs of gravitons—or any other effectively massless particle which is not conformally invariant. Although the total energy of these pairs grows exponentially with the co-moving time, the corresponding growth of the 3-volume results in only a constant energy density. The interesting secular effect comes at the next order when one considers the gravitational potentials engendered by the pairs. As each pair recedes these potentials remain behind to add with those of newly created pairs, and the accumulated gravitational self-interaction grows. Because gravity is attractive, this self-interaction must act to slow inflation. Because gravity is a weak interaction at typical inflationary scales, inflation can proceed for a very long time before the slowing becomes significant. Because the process is infrared, it can be studied by naively quantizing general relativity, without regard to that theory's lack of perturbative renormalizability. And explicit perturbative computations confirm that the slowing effect eventually becomes non-perturbatively strong, both for pure gravity [1] and for certain scalar models [2].

The potential phenomenological implications of this mechanism are staggering. It at once provides a realistic model of inflation *and* an explanation for why the currently observed cosmological constant is so small. If one forbids unnaturally light scalars, the model has only a single free parameter—the dimensionless product of Newton's constant and the bare cosmological constant. It can therefore make unique and cosmologically testable predictions in a way that scalar-driven inflation, with its arbitrary potential, can never do. This was exploited recently to make predictions for the tensor-to-scalar ratio and for the tensor and scalar spectral indices of anisotropies in the cosmic microwave background [3].

There is nonetheless a widespread dissatisfaction with the model. For one thing, its most interesting predictions are not easy to infer because they come after the slowing effect has become strong and perturbation theory has broken down. Even in the perturbative regime there are well-motivated objections to the use of gauge fixed expectation values in the explicit computations which have been done [4]. On a more subjective level there is the feeling that nothing can be understood about quantum gravity without first resolving the ultraviolet problem and that the new physics behind this should also resolve the problem of the cosmological constant. Finally, conventional particle physicists lack intuition about the locally de Sitter background in which the process occurs. For all these reasons it is interesting to study the phenomenon of back reaction in a simpler and more conventional setting for which there is no doubt either about what happens qualitatively or how it can be computed analytically. One such setting is the response of quantum electrodynamics to a homogeneous electric field.

What happens initially when a prepared state is released in the presence of a homogeneous electric field is that electron-positron pairs emerge from the vacuum to form a current which diminishes the electric field. If the state is released on a surface of constant x^0 with no initial charge, then the electric field at later times depends only upon x^0. This process was considered long before the ultraviolet problem of quantum electrodynamics was resolved [5,6]. Schwinger invented what we now know as the in-out back-

*Email address: tomaras@physics.uoc.gr
†Email address: tsamis@physics.uoc.gr
‡Email address: woodard@phys.ufl.edu

ground field effective action to compute the rate of particle production per unit volume in the presence of a strictly constant electric field [7]. Since then, a variety of articles [8–14] and monographs [15,16] have treated the issue of what happens when the effect becomes strong.

We cannot hope to add much to the physical picture which has emerged through the efforts of so many fine scientists. Indeed, our motive for studying this system is that the physics of what happens is *not* in doubt. However, we do have a technical contribution to make by working out the closely related process in which a source-free state is released on a surface of constant $x_+ \equiv (x^0 + x^3)/\sqrt{2}$ in the presence of an electric field which is parallel to x^3. The resulting evolution yields a homogeneous electric field which depends upon x_+ rather than x^0. An interesting feature of Dirac theory in *any* such background is that the mode functions are simple. This fact was noted recently by Srinivasan and Padmanabhan [17,18] for the special case of a charged scalar in a constant electric field, although we do not agree with their WKB solution.

It should be pointed out that our background is not the plane wave treated by Wolkow [19] and Schwinger [7]. In that background the electric field is perpendicular to x^3, there is a perpendicular magnetic field of the same magnitude, and the two together obey the free Maxwell equations. In our background the electric field is *parallel* to x^3, there is no magnetic field, and the free Maxwell equations are only obeyed when the field is constant. What we have instead is an explicit form for the fermion mode functions for a class of backgrounds which is general enough to include the actual evolution of the electric field as it changes under the impact of a quantum electrodynamic back reaction. By taking the expectation value of the current operator in this general class of backgrounds we obtain the source term for the effective field equation obeyed by the actual electric field. This is precisely what we should like to do for quantum gravity in order to treat the problem of what happens when the slowing effect becomes non-perturbatively strong. Therefore many of the same issues of gauge fixing, the use of expectation values, renormalization and the breakdown of perturbation theory can be examined in a setting where the answer is not in doubt.

This paper contains seven sections of which this introduction is the first. In Sec. II our light cone coordinate and gauge conventions are stated and we work out the dynamics of a classical charged particle moving in our general background. In Sec. III we give a complete operator solution for free QED in the presence of this background, expressed in terms of the field operators on the surfaces of $x_+ = 0$ and $x_- = -\infty$. It turns out that pair creation is a discrete event on the light cone. Each mode passes from positive to negative frequency at a certain value of x_+ depending upon the mode. At this instant each mode experiences a drop in amplitude with the missing amplitude taken up by operators from the surface at $x_- = -\infty$. We use these results in Sec. IV to give an explicit, analytic derivation for the rate of particle production per unit volume for our general background. In Sec. V we compute the one loop expectation value of the current induced by such a background. As expected, the ultraviolet divergence resolves itself into a renormalization of local terms in Maxwell's equations. Here, as in gravity, pair production and back reaction are infrared effects which can be studied without understanding the ultraviolet provided one subtracts the divergences and uses the physical couplings in the effective field equations. A peculiar feature of our one loop result is that the back reaction becomes infinitely strong infinitely fast. This is explained in Sec. VI by noting that our light cone system is the singular, infinite boost limit of the traditional system in which the state is prepared on a surface of constant x^0 and the electric field depends upon x^0 rather than x_+. Similar correspondence limits have been recognized since the earliest work on light cone quantum field theory [20]. Our conclusions comprise Sec. VIII.

II. CLASSICAL ELECTRODYNAMICS ON THE LIGHT CONE

All the analysis of this paper is done with a flat, timelike metric. We define the light cone coordinates as follows:

$$x_\pm \equiv \frac{1}{\sqrt{2}}(x^0 \pm x^3). \tag{1}$$

The other (''transverse'') components of x^μ comprise the 2-vector $\tilde{x}$. The same conventions apply to the momentum vector p^μ, so one might write

$$x^\mu p_\mu = x^0 p^0 - x^3 p^3 - \tilde{x}\cdot\tilde{p} = x_+ p_- + x_- p_+ - \tilde{x}\cdot\tilde{p}. \tag{2}$$

Note, however, that Eq. (1) results in derivatives with respect to x_+ and x_- having their natural expression in terms of derivatives with lowered indices:

$$\partial_\pm = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_3). \tag{3}$$

Since we define $\tilde{\nabla}$ as the transverse components of ∂_μ, one can write

$$p^\mu \partial_\mu = p^0\partial_0 + p^3\partial_3 + \tilde{p}\cdot\tilde{\nabla} = p_+\partial_+ + p_-\partial_- + \tilde{p}\cdot\tilde{\nabla}. \tag{4}$$

We define the light cone components of the vector potential A_μ in analogy with those of the derivative operator ∂_μ:

$$A_\pm \equiv \frac{1}{\sqrt{2}}(A_0 \pm A_3). \tag{5}$$

Our gauge condition is $A_+ = 0$ and we restrict our attention to configurations for which A_- and $\tilde{A}$ vanish at $x_+ = 0$. This means that only A_- is ever nonzero, and it depends only upon x_+. The nonzero components of the field strength tensor are

$$F^{30} = -F^{03} = F_{03} = -F_{30} = -A'_-(x_+). \tag{6}$$

Since we want the electric field $\vec{E} = \hat{z}F^{30}$ to be initially directed along the positive z axis, it follows that $A'_-(0) < 0$.

When necessary, we will therefore assume that $A_-(x_+)$ is a *decreasing* function of x_+. Since the electron's charge is negative ($e<0$), our nominal assumption is that $eA_-(x_+)$ is an *increasing* function of x_+.

It is instructive to consider the dynamics of a point particle of mass m and charge $e<0$ which moves under the influence of $A_-(x_+)$. From the differential form of the Lorentz force law,

$$dp^\mu = eF^{\mu\nu}dx_\nu, \tag{7}$$

we infer the following relations for the light cone coordinates and momenta:

$$dp_+ = -eA'_-(x_+)dx_+, \tag{8}$$

$$dp_- = eA'_-(x_+)dx_-, \tag{9}$$

$$d\tilde{p} = 0. \tag{10}$$

Since $A'_-(x_+)dx_+ = dA_-$, the relation for p_+ implies that

$$k_+ \equiv p_+(x_+) + eA_-(x_+) \tag{11}$$

is a conserved quantity. Since $dx_- = (p_-/p_+)dx_+$, the relation for p_- implies that the product $p_-(x_+)\times p_+(x_+)$ is also conserved. This product cannot involve $A_-(x_+)$, because the latter depends upon x_+, so the correspondence limit in which A_- vanishes determines the mass shell relation

$$2p_+(x_+)p_-(x_+) = \tilde{p}\cdot\tilde{p} + m^2 \equiv \tilde{\omega}^2. \tag{12}$$

In the free quantum field theory which corresponds to the motion of such a point particle, the conserved quantity k_+ is the Fourier conjugate to the coordinate x_- of the field which creates charge $-e$ and annihilates charge e. We shall follow the convention of Kluger *et al.* [11] in distinguishing between the constant *canonical momentum* k_+ and the x_+ dependent *kinetic momentum* $p_+(x_+) = k_+ - eA_-(x_+)$. We will also see that

$$p_-(x_+) = \frac{\tilde{\omega}^2/2}{p_+(x_+)} = \frac{\tilde{\omega}^2/2}{k_+ - eA_-(x_+)} \tag{13}$$

is indeed the eigenvalue of the operator $i\partial_+$. A fact of crucial importance is that it changes sign when $p_+(x_+)$ passes through zero.

We conclude by following the trajectory of a point particle of mass m and charge $e<0$ as it moves under the influence of $A_-(x_+)$. Since $dx_- = (p_-/p_+)dx_+$ we can integrate to find

$$x_-(x_+) = x_-(0) + \int_0^{x_+} \frac{\frac{1}{2}\tilde{\omega}^2 du}{[k_+ - eA_-(u)]^2}. \tag{14}$$

Under our nominal assumption that $eA_-(u)$ is an increasing function, $k_+ - eA_-(u)$ must pass through zero at some value $u_{\rm crit}>0$, at least for modes whose initial momentum k_+ is small. The integral diverges if $k_+ - eA_-(u)$ goes to zero even as fast as $\sqrt{u_{\rm crit}-u}$ — and note that $eA_-(x_+)$ is growing *linearly* at $x_+=0$. What this divergence means physically is that the electron accelerates to the speed of light and leaves the manifold moving parallel to the x_- axis as shown in Fig. 1.

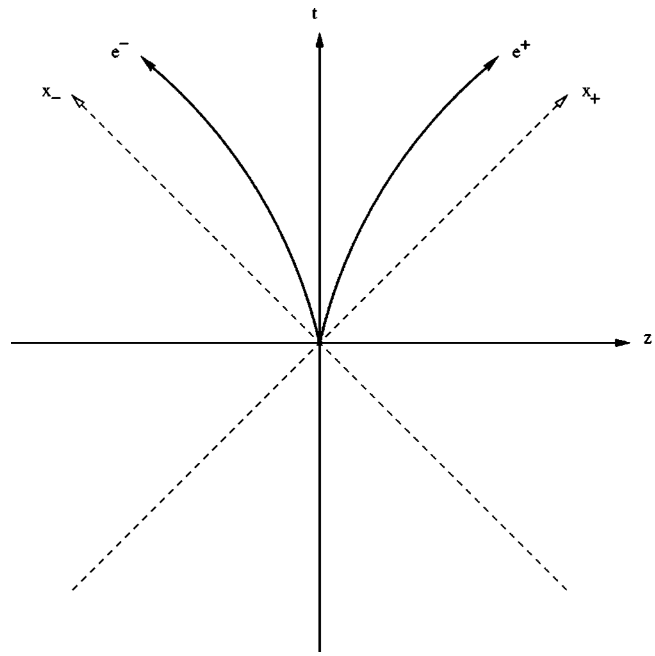

FIG. 1. The evolution of an e^+e^- pair created at $x_+ = x_- = 0$.

The result for positrons is obtained by simply changing e to $-e$. Note that although positrons also accelerate to the speed of light they move parallel to the x_+ axis and do not leave the manifold. We can therefore anticipate that, for $E(x_+)>0$, pair creation on the light cone manifests itself by the accumulation of a charge density of positrons whose electron partners have left the manifold. Since electrons exit the manifold by reaching the speed of light we can also anticipate that they induce an infinite current. These suspicions will be confirmed by the detailed calculations of Secs. IV and V. Why the light cone must show an infinite effect will be explained by the correspondence limit of Sec. VI.

III. QED ON THE LIGHT CONE

The light cone components of the gamma matrices are

$$\gamma_\pm \equiv \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^3). \tag{15}$$

Note that $(\gamma_\pm)^2 = 0$. We follow Kogut and Soper [20] in defining light cone spinor projection operators:

$$P_\pm \equiv \frac{1}{2}(I \pm \gamma^0\gamma^3) = \frac{1}{2}\gamma_\mp\gamma_\pm. \tag{16}$$

These act on the Dirac bispinor to give its "+" and "−" components:

$$\psi_\pm \equiv P_\pm \psi, \quad \psi_\pm^\dagger \equiv \psi^\dagger P_\pm . \tag{17}$$

It is convenient to Fourier transform on the transverse coordinates:

$$\tilde{\psi}_\pm(x_+,x_-,\tilde{k}) \equiv \int d^2\tilde{x} e^{-i\tilde{k}\cdot\tilde{x}} \psi_\pm(x_+,x_-,\tilde{x}). \tag{18}$$

Note that the transverse derivative operator $\tilde{\nabla}$ becomes $i\tilde{k}$ in the Fourier representation. Because transverse coordinates play so little role, we shall often omit $\tilde{k}$ from the argument list to simplify the notation.

With these conventions the Dirac equation becomes

$$(\gamma^\mu i\partial_\mu - \gamma^\mu e A_\mu - m)\tilde{\psi} = [\gamma_+ i\partial_+ + \gamma_-(i\partial_- - eA_-) - \tilde{\gamma}\cdot\tilde{k} - m]\tilde{\psi}, \tag{19}$$

where it should be noted that $e = -|e|$ is the charge of the electron. Multiplication alternately with γ_- and γ_+ gives

$$i\partial_+ \tilde{\psi}_+(x_+,x_-) = (m - \tilde{\gamma}\cdot\tilde{k})\frac{1}{2}\gamma_-\tilde{\psi}_-(x_+,x_-), \tag{20}$$

$$[i\partial_- - eA_-(x_+)]\tilde{\psi}_-(x_+,x_-) = (m - \tilde{\gamma}\cdot\tilde{k})\frac{1}{2}\gamma_+\tilde{\psi}_+(x_+,x_-). \tag{21}$$

One can integrate Eq. (20) from the initial value surface at $x_+ = 0$:

$$\tilde{\psi}_+(x_+,x_-) = \tilde{\psi}_+(0,x_-) - \int_0^{x_+} du (m - \tilde{\gamma}\cdot\tilde{k})\frac{i}{2}\gamma_-\tilde{\psi}_-(u,x_-). \tag{22}$$

A similar integration of Eq. (21) from the surface at $x_- = -L$ can be achieved by multiplying $e^{ieA_- x_-}$:

$$\tilde{\psi}_-(x_+,x_-) = e^{-ieA_-(x_+)(x_-+L)}\tilde{\psi}_-(x_+,-L) - e^{-ieA_-(x_+)x_-}\int_{-L}^{x_-} dv\, e^{ieA_-(x_+)v} \times (m - \tilde{\gamma}\cdot\tilde{k})\frac{i}{2}\gamma_+\tilde{\psi}_+(x_+,v). \tag{23}$$

Substituting this into the previous equation for $\tilde{\psi}_+$ and iterating gives the complete initial value solution for $\tilde{\psi}_+$ on the region $x_+ > 0$ and $x_- > -L$:

$$\tilde{\psi}_+(x_+,x_-) = \sum_{n=0}^\infty \left(-\frac{1}{2}\tilde{\omega}^2\right)^n \int_0^{x_+} du_1 e^{-ieA_-(u_1)x_-}\int_{-L}^{x_-} dv_1 e^{ieA_-(u_1)v_1}\cdots\int_0^{u_{n-1}} du_n e^{-ieA_-(u_n)v_{n-1}} \times \int_{-L}^{v_{n-1}} dv_n e^{ieA_-(u_n)v_n}\left\{\tilde{\psi}_+(0,v_n) - \int_0^{u_n} du\, e^{-iA_-(u)v_n}(m - \tilde{\gamma}\cdot\tilde{k})\frac{i}{2}\gamma_- e^{-ieA_-(u)L}\tilde{\psi}_-(u,-L)\right\}. \tag{24}$$

A similar expansion for $\tilde{\psi}_- = \tilde{\omega}^{-2}(m - \tilde{\gamma}\cdot\tilde{k})\gamma_+ i\partial_+\tilde{\psi}_+$ follows from Eq. (20).

Of course we are interested in the limit as L becomes infinite, in which case the series (24) can be summed. For $n > 0$ we first extend the integration over v_n to the full real line using the identity

$$\theta(v_{n-1} - v_n)e^{ieA_-(u_n)[v_n - v_{n-1}]} = \int_{-\infty}^\infty \frac{dk_+}{2\pi}\frac{ie^{i(k_+ + i\epsilon)[v_n - v_{n-1}]}}{k_+ - eA_-(u_n) + i\epsilon}. \tag{25}$$

Owing to the factor of $e^{-\epsilon v_n}$, the integration over v_n only makes sense provided the integration over k_+ is done first. To change the order of integration one must appropriately regulate the lower limit:

$$\int_{-\infty}^\infty dv_n F(v_n)\int_{-\infty}^\infty \frac{dk_+}{2\pi}G(k_+) = \lim_{\epsilon\to 0^+}\int_{-\infty}^\infty \frac{dk_+}{2\pi}G(k_+)\int_{-1/\epsilon}^\infty dv F(v). \tag{26}$$

The limit $\epsilon \to 0^+$ will be understood in all subsequent expressions, as per the usual convention (for a different ϵ) in quantum field theory.

The next step is to move the k_+ integration all the way to the left and perform the integrations over v_i successively, from $i = n-1$ to $i = 1$, using

$$\int_{-\infty}^{v_{i-1}} dv_i e^{-i[k_+ - eA_-(u_i) + i\epsilon]v_i} = \frac{ie^{-i[k_+ - eA_-(u_i) + i\epsilon]v_{i-1}}}{k_+ - eA_-(u_i) + i\epsilon}. \tag{27}$$

Since the integrand at this stage is the product over the same function of each u_i — $f(u_i)\equiv[k_+-eA_-(u_i)+i\epsilon]^{-1}$ — one can factor the u_i integrations:

$$\int_0^{x_+}du_1 f(u_1)\cdots\int_0^{u_{n-1}}du_n f(u_n)=\frac{1}{n!}\left[\int_0^{x_+}du_1 f(u_1)\right]^n, \tag{28}$$

$$\int_0^{x_+}du_1 f(u_1)\cdots\int_0^{u_n}du g(u)=\int_0^{x_+}du\frac{g(u)}{n!}\left[\int_u^{x_+}du_1 f(u_1)\right]^n. \tag{29}$$

The $n=0$ term can be included using the Fourier inversion theorem:

$$h(x_-)=\int_{-\infty}^{\infty}\frac{dk_+}{2\pi}e^{-i(k_++i\epsilon)x_-}\int_{-1/\epsilon}^{\infty}dv e^{i(k_++i\epsilon)v}h(v). \tag{30}$$

The resulting series gives an exponential. For the terms proportional to $\widetilde{\psi}_-$ we get

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left[-\frac{i}{2}\widetilde{\omega}^2\int_u^{x_+}\frac{du_1}{k_+-eA_-(u_1)+i\epsilon}\right]^n=\exp\left[-\frac{i}{2}\widetilde{\omega}^2\int_u^{x_+}\frac{du_1}{k_+-eA_-(u_1)+i\epsilon}\right]\equiv\mathcal{E}[A_-](u,x_+;k_+,\widetilde{k}). \tag{31}$$

The terms proportional to $\widetilde{\psi}_+$ give $\mathcal{E}[A_-](0,x_+;k_+,\widetilde{k})$.

It remains to perform the final integration over v. For the terms proportional to $\widetilde{\psi}_+$ this gives our ϵ-regulated Fourier transform

$$\Xi_0(k_+,\widetilde{k})\equiv\int_{-1/\epsilon}^{\infty}dv e^{i(k_++i\epsilon)v}\widetilde{\psi}_+(0,v,\widetilde{k}). \tag{32}$$

For the terms proportional to $\widetilde{\psi}_-$ the integral over v results in a delta sequence

$$\Delta(k_+-eA_-(u);\epsilon)\equiv\frac{ie^{-i[k_+-eA_-(u)+i\epsilon]/\epsilon}}{k_+-eA_-(u)+i\epsilon}, \tag{33}$$

whose distributional limit would be $2\pi\delta(k_+-eA_-)$ if it were multiplied by a test function. The final result is

$$\widetilde{\psi}_+(x_+,x_-,\widetilde{k})=\int_{-\infty}^{\infty}\frac{dk_+}{2\pi}e^{-i(k_++i\epsilon)x_-}\Big\{\mathcal{E}[A_-]\times(0,x_+;k_+,\widetilde{k})\Xi_0(k_+,\widetilde{k})-\int_0^{x_+}du\Delta(k_+-eA_-(u);\epsilon)\times\mathcal{E}[A_-](u,x_+;k_+,\widetilde{k})\Phi_\infty(u,\widetilde{k})\Big\}, \tag{34}$$

where we define

$$\Phi_\infty(u,\widetilde{k})\equiv\lim_{L\to\infty}(m-\widetilde{\gamma}\cdot\widetilde{k})\frac{i}{2}\gamma_-\widetilde{\psi}_-(u,-L,\widetilde{k})e^{-ieA_-(u)L}. \tag{35}$$

Because the factor of $\mathcal{E}[A_-](u,x_+;k_+,\widetilde{k})$ develops a singular phase as k_+ approaches $eA_-(u)$, the distributional limit of the delta sequence in the second term must be taken with care. We shall postpone this to the next section.

It is worth commenting on two exceptional properties of our solution (34). First, it is valid *for arbitrary* vector potential $A_-(x_+)$. If the state at $x_+=0$ is translation invariant in x_- and $\widetilde{x}$, then the back reaction will change the way A_- depends upon x_+ but it cannot induce other potentials or dependence upon other coordinates. Of course the photon propagator is not affected by the background, nor are the vertices. So we can evaluate the expectation value of the current operator — to as high an order in the loop expansion as is desired — for a class of vector potentials which certainly includes the actual solution. The only additional simplification one would obtain by making the electric field constant $[A_-(x_+)=-Ex_+]$ is that then the integral over u_1 in the mode functions (31) can be explicitly performed. We shall see, in Secs. IV and V, that this is not required in order to be able to compute either the rate of particle production or the expectation value of the current operator.

The second property is that our $i\epsilon$ prescription provides a precise definition for the ambiguity at zero + momentum which, for $m\neq 0$ and/or more than two spacetime dimensions, is traditionally left unresolved in light cone quantum field theory. (See, for example, footnote No. 12 in the work of Kogut and Soper [20].) One can usually avoid doing this because the analyticity of scattering amplitudes permits one to infer the zero momentum limit from the result for nonzero momentum. In our background the problem is aggravated by the fact that *every* mode with positive canonical momentum k_+ becomes singular when its kinetic momentum $p_+(x_+)=k_+-eA_-(x_+)$ passes through zero. At this instant the mode functions $\mathcal{E}[A_-]\times(0,x_+;k_+,\widetilde{k})$ oscillate with infinite rapidity and one requires the $i\epsilon$ prescription to precisely define what happens. Note too that we have *derived* it rather than simply making an *ad hoc* guess. As an essential part of the derivation we have found that $\psi_+(x_+,x_-,\widetilde{x})$ is determined not just by

$\psi_+(0,x_-,\tilde{x})$ but also by $\psi_-(x_+,-\infty,\tilde{x})$. When $A_-=0$ (and $m\neq 0$ and/or the number of spacetime dimensions is greater than 2) one can ignore the data from the surface at $x_-=-\infty$ because it remains segregated in the $k_+=0$ mode whose contribution to scattering processes is inferred by analytically continuing the result from $k_+\neq 0$. We shall see in the next section that these data cannot be ignored in our background and that they play an essential role in the process of particle production.

To complete our operator construction of free Dirac theory in the presence of $A_-(x_+)$ we must specify how the fundamental operators $\Xi_0(k_+,\tilde{k})$ and $\Phi_\infty(u,\tilde{k})$ act upon one another. Of course the operator algebra derives from canonical quantization. The Fourier transform (in $\tilde{x}$) of the Dirac Lagrangian is[1]

$$\mathcal{L}=\tilde{\psi}^\dagger\gamma^0(\gamma^\mu i\partial_\mu-\gamma^\mu eA_\mu-m)\tilde{\psi}, \tag{36}$$

$$=\sqrt{2}\tilde{\psi}_+^\dagger\left[i\partial_+\tilde{\psi}_+-(m-\tilde{\gamma}\cdot\tilde{k})\frac{1}{2}\gamma_-\tilde{\psi}_-\right] + \sqrt{2}\tilde{\psi}_-^\dagger\left[(i\partial_--eA_-)\tilde{\psi}_- -(m-\tilde{\gamma}\cdot\tilde{k})\frac{1}{2}\gamma_+\tilde{\psi}_+\right]. \tag{37}$$

The variable conjugate to $\tilde{\psi}_+$ under x_+ evolution is $i\sqrt{2}\tilde{\psi}_+^\dagger$, so we must have

$$\{\tilde{\psi}_+(x_+,x_-,\tilde{k}),\tilde{\psi}_+^\dagger(x_+,y_-,\tilde{q})\} =\frac{1}{\sqrt{2}}P_+\delta(x_--y_-)(2\pi)^2\delta^2(\tilde{k}-\tilde{q}). \tag{38}$$

Since the variable conjugate to $\tilde{\psi}_-$ under x_- evolution is $i\sqrt{2}\tilde{\psi}_-^\dagger$, we must similarly have

$$\{\tilde{\psi}_-(x_+,x_-,\tilde{k}),\tilde{\psi}_-^\dagger(y_+,x_-,\tilde{q})\} =\frac{1}{\sqrt{2}}P_-\delta(x_+-y_+)(2\pi)^2\delta^2(\tilde{k}-\tilde{q}). \tag{39}$$

Operators on an arbitrary surface of constant x_+ do *not* generally anti-commute with those on an arbitrary surface of constant x_-. However, by causality we know that the operators at $x_+=0$ do anti-commute with those at $x_-=-\infty$. So the only nonzero anti-commutators among the fundamental operators are

$$\{\Xi_0(k_+,\tilde{k}),\Xi_0^\dagger(q_+,\tilde{q})\}=\frac{1}{\sqrt{2}}P_+(2\pi)^3\delta(k_+-q_+)\times\delta^2(\tilde{k}-\tilde{q}), \tag{40}$$

$$\{\Phi_\infty(x_+,\tilde{k}),\Phi_\infty^\dagger(y_+,\tilde{q})\}=\frac{\tilde{\omega}^2}{2\sqrt{2}}P_+\delta(x_+-y_+)(2\pi)^2\times\delta^2(\tilde{k}-\tilde{q}). \tag{41}$$

IV. PARTICLE PRODUCTION ON THE LIGHT CONE

Equation (34) expresses the free field $\tilde{\psi}_+(x_+,x_-,\tilde{k})$ in terms of the fundamental operators $\Xi_0(k_+,\tilde{k})$ and $\Phi_\infty(u,\tilde{k})$. We have just seen in Eqs. (40),(41) how these fundamental operators act upon one another and upon their adjoints. Their particle interpretation in free field theory derives from the light cone ''Hamiltonian''—that is, from the generator of x_+ evolution. Since the Dirac Lagrangian vanishes as a consequence of the field equations, the Hamiltonian density is just the $p\dot{q}$ term:

$$\mathcal{H}(x_+,x_-,\tilde{x})=\sqrt{2}\psi_+^\dagger(x_+,x_-,\tilde{x})i\partial_+\psi_+(x_+,x_-,\tilde{x}). \tag{42}$$

The Hamiltonian is the integral of this over $\tilde{x}$ and our ϵ-truncated portion of the x_- axis. We can express it in terms of $\tilde{\psi}_+(x_+,x_-,\tilde{k})$ using Parseval's theorem

$$H(x_+)=\int_{-1/\epsilon}^{\infty}dx_-\int\frac{d^2\tilde{k}}{(2\pi)^2}\sqrt{2}\tilde{\psi}_+^\dagger(x_+,x_-,\tilde{k})i\partial_+\times\tilde{\psi}_+(x_+,x_-,\tilde{k}). \tag{43}$$

As might have been expected from this system's invariance under translations in x_- and $\tilde{x}$, the Hamiltonian becomes diagonal in momentum space. To see this we take the field's ϵ-regulated Fourier transform on x_-:

$$\Psi(x_+,k_+,\tilde{k})\equiv\int_{-1/\epsilon}^{\infty}dx_-e^{i(k_++i\epsilon)x_-}\tilde{\psi}_+(x_+,x_-,\tilde{k}) \tag{44}$$

$$=\mathcal{E}[A_-](0,x_+;k_+,\tilde{k})\Xi_0(k_+,\tilde{k}) -\int_0^{x_+}du\Delta(k_+-eA_-(u);\epsilon)\times\mathcal{E}[A_-](u,x_+;k_+,\tilde{k})\Phi_\infty(u,\tilde{k}). \tag{45}$$

In the limit of small ϵ the Hamiltonian becomes

$$H(x_+)=\int_{-\infty}^{\infty}\frac{dk_+}{2\pi}\int\frac{d^2\tilde{k}}{(2\pi)^2}\sqrt{2}\Psi^\dagger(x_+,k_+,\tilde{k})\times i\partial_+\Psi(x_+,k_+,\tilde{k}). \tag{46}$$

[1] Note that the quantity $\tilde{\psi}^\dagger$ is computed by Fourier transforming *first* and then taking the adjoint.

This last expression for $H(x_+)$ implies that the x_+-dependent ''energy'' carried by $\Psi(x_+,k_+,\widetilde{k})$ is its eigenvalue under $-i\partial_+$. From the first term of Eq. (45) we see that, if $\Psi(x_+,k_+,\widetilde{k})$ is an eigenfunction of $-i\partial_+$, its eigenvalue must be

$$-i\partial_+\ln\{\mathcal{E}[A_-](0,x_+;k_+,\widetilde{k})\}=\frac{-\widetilde{\omega}^2/2}{k_+-eA_-(x_+)+i\epsilon}. \tag{47}$$

When ϵ vanishes this is precisely minus the result (13) we found at the end of Sec. II for the p_- momentum of a classical charged particle moving in our vector potential. We therefore expect $\Psi(x_+,k_+,\widetilde{k})$ to annihilate electrons for $k_+>eA_-(x_+)$ and to create positrons for $k_+<eA_-(x_+)$.

It remains to show that $\Psi(x_+,k_+,\widetilde{k})$ is actually an eigenstate of $-i\partial_+$. Since the first term of Eq. (45) obviously has this property, our task reduces to taking the distributional limit of the delta sequence $\Delta(k_+-eA_-;\epsilon)$ in the second term. We shall do this under the assumption that k_+ is well separated from the singular points at $k_+=0$ and at $k_+=eA_-(x_+)$. Two pieces of notation we shall find useful are the inverse vector potential $X(k_+)$,

$$k_+=eA_-(X(k_+)), \tag{48}$$

and the dimensionless ratio of $\widetilde{\omega}^2$ to $(-2e$ times) the electric field:

$$\lambda(k_+,\widetilde{k})\equiv\frac{\widetilde{\omega}^2}{2eA_-'(X(k_+))}. \tag{49}$$

The first step in transforming the second term of Eq. (45) is to change variables from u to $z=[k_+-eA_-(u)]/\epsilon$,

$$-\int_l^U dz\,\frac{ie^{-i(z+i)}}{z+i}\mathcal{E}(X(k_+-\epsilon z),x_+;k_+,\widetilde{k})\times\frac{\Phi_\infty(X(k_+-\epsilon z),\widetilde{k})}{eA_-'(X(k_+-\epsilon z))}, \tag{50}$$

where the upper and lower limits are

$$U\equiv\frac{k_+}{\epsilon},\quad l\equiv\frac{k_+-eA_-(x_+)}{\epsilon}. \tag{51}$$

As ϵ approaches zero they go to positive and negative infinity, respectively, for k_+ in the range $0<k_+<eA_-(x_+)$. This is the only case in which one gets a nonzero result.

We can absorb the Jacobian in Eq. (50) by defining a new fundamental field

$$\Xi_\infty(k_+,\widetilde{k})\equiv\sqrt{\frac{2\pi}{\lambda(k_+,\widetilde{k})}}\frac{\Phi_\infty(X(k_+),\widetilde{k})}{eA_-'(X(k_+))}. \tag{52}$$

This brings us to the form

$$-\int_l^U\frac{dz}{2\pi}\frac{ie^{-i(z+i)}}{z+i}\mathcal{E}(X(k_+-\epsilon z),x_+;k_+,\widetilde{k})\times\sqrt{2\pi\lambda}\,\Xi_\infty(k_+-\epsilon z,\widetilde{k}). \tag{53}$$

Note from Eq. (41) that the anti-commutator of $\Xi_\infty(k_+,\widetilde{k})$ with its adjoint is the same as that of Ξ_0 with $\Xi_0^\dagger$:

$$\{\Xi_\infty(k_+,\widetilde{k}),\Xi_\infty^\dagger(q_+,\widetilde{q})\}=\frac{1}{\sqrt{2}}P_+(2\pi)^3\delta(k_+-q_+)\times\delta^2(\widetilde{k}-\widetilde{q}). \tag{54}$$

Now consider the mode function in expression (53):

$$\mathcal{E}(X(k_+-\epsilon z),x_+;k_+,\widetilde{k})=\exp\left[-\frac{i}{2}\widetilde{\omega}^2\int_{X(k_+-\epsilon z)}^{x_+}\frac{du_1}{k_+-eA_-(u_1)+i\epsilon}\right]. \tag{55}$$

For $z<0$ the lower limit of the integral is a little below the singular point where the real part of the denominator vanishes. For $z>0$ the lower limit is a little above this point. Straddling the singular point like this leads to great sensitivity with respect to z, even as ϵ goes to zero. To isolate this z dependence we factor the mode function

$$\mathcal{E}(X(k_+-\epsilon z),x_+;k_+,\widetilde{k})=\mathcal{E}(X(k_+-\epsilon z),X(k_+);k_+,\widetilde{k})\times\mathcal{E}(X(k_+),x_+;k_+,\widetilde{k}). \tag{56}$$

The second factor is independent of z and can be pulled outside the integral. We can also take ϵ to zero in $\lambda(k_+-\epsilon z,\widetilde{k})$ and in $\Xi_\infty(k_+-\epsilon z,\widetilde{k})$.

Taking the small ϵ limit of the first factor requires care. We first change variables in the exponent from u_1 to $y\equiv[k_+-eA_-(u_1)]/\epsilon$ and then expand the Jacobian for small ϵ:

$$-\frac{i}{2}\widetilde{\omega}^2\int_{X(k_+-\epsilon z)}^{X(k_+)}\frac{du_1}{k_+-eA_-(u_1)+i\epsilon}=-i\lambda(k_+,\widetilde{k})\int_0^z\frac{dy}{y+i}\times\frac{A_-'(X(k_+))}{A_-'(X(k_+)-\epsilon y)} \tag{57}$$

$$=-i\lambda(k_+,\widetilde{k})\ln(z+i)-\frac{\pi}{2}\lambda(k_+,\widetilde{k})+O(\epsilon). \tag{58}$$

Dropping the terms which vanish with ϵ and combining Eqs. (53), (56) and (58) gives the following result for the second term of Eq. (45):

$$-\theta(k_+)\theta(eA_-(x_+)-k_+)\times\mathcal{E}[A_-](X(k_+),x_+;k_+,\widetilde{k})\sqrt{2\pi\lambda}\,\gamma(\lambda)\Xi_\infty(k_+,\widetilde{k}), \tag{59}$$

where

$$\gamma(\lambda) \equiv e^{-(\pi/2)\lambda} \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{i e^{-i(z+i)}}{z+i} e^{-i\lambda \ln(z+i)}. \tag{60}$$

Substituting Eq. (59) into Eq. (45) results in the following for $\Psi(x_+, k_+, \tilde{k})$:

$$\Psi(x_+, k_+, \tilde{k}) \rightarrow \mathcal{E}[A_-](0, x_+; k_+, \tilde{k}) \Xi_0(k_+, \tilde{k}) - \theta(k_+)\theta(eA_- - k_+) \times \mathcal{E}[A_-](X, x_+; k_+, \tilde{k}) \sqrt{2\pi\lambda}\, \gamma(\lambda) \Xi_\infty(k_+, \tilde{k}). \tag{61}$$

We mention again that this is only valid for modes which are well separated from the singular points at $k_+ = 0$ and $k_+ = eA_-(x_+)$. If one wishes to study the behavior of modes which are arbitrarily near either point, there is no alternative to taking a new distributional limit for the delta sequence in Eq. (45).

Since $\mathcal{E}[A_-](X(k_+), x_+; k_+, \tilde{k})$ has the same $-i\partial_+$ eigenvalue (13) as the first mode function, $\Psi(x_+, k_+, \tilde{k})$ is indeed an eigenfunction of $-i\partial_+$. This means that it carries a definite energy:

$$[H(x_+), \Psi(x_+, k_+, \tilde{k})] = \frac{-\tilde{\omega}^2/2}{k_+ - eA_-(x_+)} \Psi(x_+, k_+, \tilde{k}). \tag{62}$$

That has implications for the fundamental operators from which it is constructed and for the state upon which they act. The latter is supposed to be ''empty'' at $x_+ = 0$. At that instant Eq. (45) implies

$$\Psi(0, k_+, \tilde{k}) = \Xi_0(k_+, \tilde{k}). \tag{63}$$

Since the potential vanishes at $x_+ = 0$, we can see from Eq. (62) that the modes with $k_+ > 0$ carry negative energy while those with $k_+ < 0$ carry positive energy. It follows that the state should obey

$$\Xi_0(k_+, \tilde{k}) |\Omega\rangle = 0 = \Xi_0^\dagger(-k_+, -\tilde{k}) |\Omega\rangle \quad \forall k_+ > 0. \tag{64}$$

The $\Xi_\infty(k_+, \tilde{k})$ operators [or, equivalently, the $\Phi_\infty(u, \tilde{k})$ operators] are not present at $x_+ = 0$. However, when they do appear — for $0 < k_+ < eA_-(x_+)$ — it is always with positive energy. It is therefore natural to regard them as creators and to define the state to be annihilated by their adjoints:

$$\Xi_\infty^\dagger(k_+, \tilde{k}) |\Omega\rangle = 0 \quad \forall k_+ > 0 \Leftrightarrow \Phi_\infty^\dagger(u, \tilde{k}) |\Omega\rangle = 0 \quad \forall u > 0. \tag{65}$$

What this seems to mean physically is that we allow no particles to enter the manifold from the surface at $x_- = -\infty$.

Now consider what happens as the system evolves in x_+. Under the assumption that $eA_-(x_+)$ is an increasing function of x_+, modes with $k_+ < 0$ begin as positron creation operators and remain that way, although their kinetic momenta increase according to the relation, $p_+(x_+) = -k_+ + eA_-(x_+)$. The associated mode functions begin as unity and retain unit magnitude in the limit that ϵ vanishes. For $k_+ > 0$ the picture is more complicated. These modes begin as electron annihilation operators, also with mode functions of unit magnitude. However, when $x_+ = X(k_+)$ the energy each mode carries passes from $-\infty$ to $+\infty$ and we must regard the mode as creating a positron. It is not possible to follow this process using Eq. (61) because that expression was derived under the assumption that the mode was not arbitrarily close to singularity. But we *can* use Eq. (61) a little before and a little after the singularity. Before the singularity $\Psi(x_+, k_+, \tilde{k})$ consists of only the term proportional to $\Xi_0(k_+, \tilde{k})$, and it has unit magnitude. After singularity the magnitude of this term has dropped to $e^{-\pi\lambda(k_+, \tilde{k})}$, and the term proportional to $\Xi_\infty(k_+, \tilde{k})$ has appeared. Let us pause at this point to evaluate the function $\gamma(\lambda)$ in order to show that the $\Xi_\infty(k_+, \tilde{k})$ term acquires the missing amplitude.

Evaluating $\gamma(\lambda)$ is complicated by the branch cut of the integrand. However, when $\lambda = -in$ the integrand is meromorphic and elementary methods give $\gamma(-in) = 1/n!$. By partial integration one can also derive the recursion relation $\gamma(\lambda) = (1 + i\lambda)\gamma(\lambda - i)$. These results together imply that we are dealing with an inverse gamma function

$$\gamma(\lambda) = \frac{1}{\Gamma(1 + i\lambda)}. \tag{66}$$

Its magnitude follows from a result of Lobachevskiy [24]

$$\frac{1}{\Gamma(1 + i\lambda)\Gamma(1 - i\lambda)} = \frac{e^{\pi\lambda} - e^{-\pi\lambda}}{2\pi\lambda}. \tag{67}$$

As previously noted, the magnitude of the first mode function $\mathcal{E}(0, x_+; k_+, \tilde{k})$ is $e^{-\pi\lambda}$ following the singularity. Because the integral in the exponent of the second mode function $\mathcal{E}(X(k_+), x_+; k_+, \tilde{k})$ begins precisely at the singularity, the magnitude of the second mode function is $e^{(-\pi/2)\lambda}$. Putting everything together gives the following result for the magnitude of the various terms multiplying $\Xi_\infty(k_+, \tilde{k})$:

$$\left\| \frac{\sqrt{2\pi\lambda}}{\Gamma(1 + i\lambda)} \mathcal{E}(X(k_+), x_+; k_+, \tilde{k}) \right\| = \sqrt{1 - e^{-2\pi\lambda}}. \tag{68}$$

Since $\Xi_0(k_+, \tilde{k})$ and $\Xi_\infty(k_+, \tilde{k})$ are independent and canonically normalized operators, this is precisely the correct factor for $\Psi(x_+, k_+, \tilde{k})$ to retain unit magnitude after singularity.

Heisenberg states cannot change but our interpretation of them can. Before the singularity $\Psi(x_+, k_+, \tilde{k})$ is proportional to $\Xi_0(k_+, \tilde{k})$, which annihilates $|\Omega\rangle$. Since $\Psi(x_+, k_+, \tilde{k})$ is an electron annihilation operator before the singularity, this means that both electron spin states with $p_+ = k_+ - eA_-(x_+)$ and $\tilde{p} = \tilde{k}$ are empty. After the singularity $\Psi(x_+, k_+, \tilde{k})$ must be a positron creation operator because it carries positive charge and energy. If $\Psi(x_+, k_+, \tilde{k})$ were still proportional to $\Xi_0(k_+, \tilde{k})$, it would annihilate $|\Omega\rangle$ and we should have to conclude that both positron spin states with $p_+ = -k_+ + eA_-(x_+)$ and $\tilde{p} = -\tilde{k}$ had been filled with unit probability. To see what actually happens pick the positron spin created by the ith spinor component of $\Psi(x_+, k_+, \tilde{k})$ and note that any state can be written as the sum of a state containing this particle and a state which does not contain it:

$$|\Omega\rangle = \sqrt{\mathrm{Prob}(k_+, \tilde{k})}|\mathrm{Full}\rangle + \sqrt{1 - \mathrm{Prob}(k_+, \tilde{k})}|\mathrm{Empty}\rangle. \tag{69}$$

Now act with $2^{1/4}\Psi_i(x_+, k_+, \tilde{k})$ and make sequential use of its expansion in terms of Ξ_0 and Ξ_∞ and the fact that it fills the one particle state with unit amplitude:

$$2^{1/4}\Psi_i(x_+, k_+, \tilde{k})|\Omega\rangle = \frac{2^{1/4}\sqrt{2\pi\lambda}}{\Gamma(1+i\lambda)}\mathcal{E}(X, x_+; k_+, \tilde{k})\Xi_{\infty i}|\Omega\rangle \tag{70}$$

$$= \sqrt{1 - \mathrm{Prob}(k_+, \tilde{k})}|\mathrm{Full}\rangle. \tag{71}$$

Use of the anti-commutation relations to compute the norm and comparison with Eq. (68) shows that the probability for the state to contain a positron of this spin is $\mathrm{Prob}(k_+, \tilde{k}) = e^{-2\pi\lambda(k_+, \tilde{k})}$.

Note that we do not see the electron of the electron-positron pair. This is because electrons and positrons are both created with $p_+ \sim 0^+$ on the light cone. As explained in Sec. II, the positrons accelerate in the $+z$ direction to $p_+ \to +\infty$, and eventually move parallel to the x_+ axis. But the electrons accelerate in the $-z$ direction to $p_+ = 0$ and therefore leave the manifold moving parallel to the x_- axis immediately after creation. We *will* see their contribution to the J_- current in Sec. V.

The picture we have just developed of particle production on the light cone is probably the most complete we shall ever have of this otherwise obscure phenomenon. To illustrate the power it confers we shall compute the rate per unit volume of particle production. For $x_+ > 0$ all modes with $0 < k_+ < eA_-(x_+)$ will have passed through singularity, so the probability for the entire state to still be in vacuum at this instant is

$$P_{\mathrm{vac}}(x_+) = \prod_{0<k_+<eA_-} \prod_{\tilde{k}} (1 - e^{-2\pi\lambda(k_+, \tilde{k})})^2, \tag{72}$$

$$= \exp\left[V_- \int_0^{eA_-(x_+)} \frac{dk_+}{2\pi}\tilde{V} \times \int \frac{d^2\tilde{k}}{(2\pi)^2} 2\ln(1 - e^{-2\pi\lambda(k_+, \tilde{k})}) \right], \tag{73}$$

$$= \exp\left[-V_-\tilde{V}\int_0^{eA_-(x_+)} dk_+ \times \frac{eA'_-(X(k_+))}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n\pi m^2/eA'_-(X)} \right], \tag{74}$$

where V_- and $\tilde{V}$ are the volumes of x_- and $\tilde{x}$ respectively. The rate of production per 4-volume is minus the logarithmic derivative of this probability:

$$-\frac{\partial \ln[P_{\mathrm{vac}}(x_+)]}{\partial x_+ \partial V_- \partial \tilde{V}} = \frac{eA'_-(x_+)^2}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n\pi m^2/eA'_-(x_+)}. \tag{75}$$

Note that we do not need to work asymptotically, like Schwinger [7]; nor do we require an *ad hoc* interpretation for the momentum integral, like Kluger *et al.* [14].

It is significant that our result (75) applies for any monotonically increasing function $eA_-(x_+)$. Although the restriction to increasing functions was made only for simplicity, and would be easy enough to remove, it has succeeded in concealing the essentially nonlocal character of particle creation. The system really preserves a memory of the extent to which each mode has been filled, and this must affect the subsequent rate of production. (In the literature this sort of effect is termed, ''non-Markovian'' [14,21,22].) Our formula shows no such effect for two reasons. First, particles of initial momentum k_+ are only created, on the light cone, at the instant when $eA_-(x_+) = k_+$. Second, under the assumption that $eA_-(x_+)$ is an increasing function of x_+, creation can occur at most once for any fixed spin and k_+. So there is never a previous Pauli blocking factor to overcome. Had we allowed $eA_-(x_+)$ to pass through the same value k_+ several times the probability of creation would depend upon what happened during previous passages.

Our formula (75) for the rate of particle production is also deceptively simple in that it is the same as Schwinger's with the instantaneous electric field $-A'_-(x_+)$ replacing the constant electric field he used. This is a special feature of electric fields which are homogeneous on surfaces of constant x_+. To see that it does not generalize even to electric fields which are homogeneous on surfaces of constant x^0 consider the recent work of Dunne and Hall [23]. Their formula (63) gives the imaginary part of the effective action, to first order in the derivative expansion, for the case of an electric field which is homogeneous on surfaces of constant x^0. There is no conflict between their result and ours; they merely explore the dependence upon different directions in the space of backgrounds.

V. BACK REACTION ON THE LIGHT CONE

The $\pm$ current operators are nominally $\sqrt{2}e\psi_\pm^\dagger\psi_\pm$. To enforce invariance under charge conjugation we take one-half of the commutator of the two field operators. To deal with the singularity of coincident operators we shall point split in the x_+

direction. Since $A_+=0$, this procedure is gauge invariant. Since point splitting *does* break Hermiticity, we shall take the real part

$$J_\pm(x_+,x_-,\tilde{x})\equiv\frac{e}{\sqrt{2}}\lim_{\Delta x_+\to 0}\mathrm{Re}\{\psi_\pm^\dagger(x_+,x_-,\tilde{x})\psi_\pm(x_++\Delta x_+,x_-,\tilde{x})-\mathrm{Tr}[\psi_\pm(x_++\Delta x_+,x_-,\tilde{x})\psi_\pm^\dagger(x_+,x_-,\tilde{x})]\}. \tag{76}$$

To compute the expectation value of J_+ it is sufficient to use the simplified expansion (61) derived in the last section:

$$\psi_+(x_++\Delta x_+,x_-,\tilde{x})\to\int_{-\infty}^{\infty}\frac{dk_+}{2\pi}e^{-ik_+x_-}\int\frac{d^2\tilde{k}}{(2\pi)^2}e^{i\tilde{k}\cdot\tilde{x}}\Big\{\mathcal{E}(0,\,x_++\Delta x_+;k_+,\tilde{k})\Xi_0(k_+,\tilde{k})-\theta(k_+)\times\theta(eA_-(x_++\Delta x_+)-k_+)\frac{\sqrt{2\pi\lambda}}{\Gamma(1+i\lambda)}\mathcal{E}(X(k_+),x_++\Delta x_+;k_+,\tilde{k})\Xi_\infty(k_+,\tilde{k})\Big\}, \tag{77}$$

$$\psi_+^\dagger(x_+,x_-,\tilde{x})\to\int_{-\infty}^{\infty}\frac{dk_+}{2\pi}e^{ik_+x_-}\int\frac{d^2\tilde{k}}{(2\pi)^2}e^{-i\tilde{k}\cdot\tilde{x}}\Big\{\mathcal{E}^*(0,\,x_+;k_+,\tilde{k})\Xi_0^\dagger(k_+,\tilde{k})-\theta(k_+)\times\theta(eA_-(x_+)-k_+)\frac{\sqrt{2\pi\lambda}}{\Gamma(1-i\lambda)}\mathcal{E}^*(X,x_+;k_+,\tilde{k})\Xi_\infty^\dagger(k_+,\tilde{k})\Big\}. \tag{78}$$

We note also two important identities concerning the mode function $\mathcal{E}$:

$$\mathcal{E}^*(0,\,x_+;k_+\tilde{k})\mathcal{E}(0,\,x_++\Delta x_+;k_+,\tilde{k})=e^{-2\pi\lambda\,\theta(k_+)\,\theta(eA_--k_+)}\mathcal{E}(x_+,x_++\Delta x_+;k_+,\tilde{k}) \tag{79}$$

$$=[\theta(-k_+)+\theta(k_+-eA_-)+\theta(k_+)\,\theta(eA_--k_+)e^{-2\pi\lambda}]\times\mathcal{E}(x_+,x_++\Delta x_+;k_+,\tilde{k}), \tag{80}$$

$$\mathcal{E}^*(X,x_+;k_+\tilde{k})\mathcal{E}(X,x_++\Delta x_+;k_+,\tilde{k})=e^{-\pi\lambda}\mathcal{E}(x_+,x_++\Delta x_+;k_+,\tilde{k}). \tag{81}$$

Combining these relations with the conditions (64),(65) which define the state and the anti-commutation relations (40) and (54) we obtain the following result for the expectation value of J_+:

$$\langle\Omega|J_+(x_+,x_-,\tilde{x})|\Omega\rangle=e\lim_{\Delta x_+\to 0^+}\int\frac{d^2\tilde{k}}{(2\pi)^2}\mathrm{Re}\Big\{\int_{-\infty}^{0}\frac{dk_+}{2\pi}+\int_0^{eA_-}\frac{dk_+}{2\pi}[1-e^{-2\pi\lambda(k_+,\tilde{k})}]-\int_0^{eA_-}\frac{dk_+}{2\pi}e^{-2\pi\lambda(k_+,\tilde{k})}-\int_{eA_-}^{\infty}\frac{dk_+}{2\pi}\Big\}\mathcal{E}(x_+,x_++\Delta x_+;k_+,\tilde{k}), \tag{82}$$

$$=-2e\int_0^{eA_-}\frac{dk_+}{2\pi}\int\frac{d^2\tilde{k}}{(2\pi)^2}e^{-2\pi\lambda(k_+,\tilde{k})}+e\lim_{\Delta x_+\to 0^+}\mathrm{Re}\Big\{\int_0^{\infty}\frac{dq}{2\pi}\int\frac{d^2\tilde{k}}{(2\pi)^2}\times[\mathcal{E}(x_+,\,x_++\Delta x_+;-q+eA_-,\tilde{k})-\mathcal{E}(x_+,x_++\Delta x_+;q+eA_-,\tilde{k})]\Big\}. \tag{83}$$

The final term in Eq. (83) vanishes. To see this first perform the integration over $\tilde{k}$,

$$\int\frac{d^2\tilde{k}}{(2\pi)^2}\mathcal{E}(x_+,x_++\Delta x_+;\pm q+eA_-,\tilde{k})=\frac{-i}{2\pi}\,\frac{e^{-(i/2)m^2I(\Delta x_+,\pm q)}}{I(\Delta x_+,\pm q)}, \tag{84}$$

where we define

$$I(\Delta x_+,\pm q)\equiv\int_{x_+}^{x_++\Delta x_+}\frac{du}{\pm q+eA_-(x_+)-eA_-(u)+i\epsilon}. \tag{85}$$

This brings the final term in Eq. (83) to the form

$$\frac{e}{4\pi^2}\lim_{\Delta x_+\to 0^+}\mathrm{Im}\left\{\int_0^\infty dq\left[\frac{e^{(-i/2)m^2 I(\Delta x_+,-q)}}{I(\Delta x_+,-q)}-\frac{e^{(-i/2)m^2 I(\Delta x_+,+q)}}{I(\Delta x_+,+q)}\right]\right\}. \tag{86}$$

The function $I(\Delta x_+,\pm q)$ can be expanded in powers of the splitting parameter Δx_+:

$$I(\Delta x_+,\pm q)=\frac{\Delta x_+}{\pm q+i\epsilon}\left\{1+\frac{1}{2}\left[\frac{eA'_-\Delta x_+}{\pm q+i\epsilon}\right]+\frac{1}{6}\left[\frac{eA''_-\Delta x_+^2}{\pm q+i\epsilon}\right]+\frac{1}{3}\left[\frac{eA'_-\Delta x_+}{\pm q+i\epsilon}\right]^2+O(\Delta x_+^3)\right\}. \tag{87}$$

Since $I(\Delta x_+,\pm q)$ goes to zero with Δx_+, and for large q, we can expand the exponentials of Eq. (86) inside the q integration. When this is done it is easy to see that every term vanishes either in taking the imaginary part or in taking Δx_+ to zero:

$$\lim_{\Delta x_+\to 0^+}\mathrm{Im}\left\{\frac{e^{(-i/2)m^2 I(\Delta x_+,-q)}}{I(\Delta x_+,-q)}-\frac{e^{(-i/2)m^2 I(\Delta x_+,+q)}}{I(\Delta x_+,+q)}\right\}=\lim_{\Delta x_+\to 0^+}\mathrm{Im}\left\{\frac{1}{I(\Delta x_+,-q)}-\frac{i}{2}m^2+\cdots-\frac{1}{I(\Delta x_+,+q)}+\cdots\right\} \tag{88}$$

$$=\lim_{\Delta x_+\to 0^+}\mathrm{Im}\left\{\frac{-q+i\epsilon}{\Delta x_+}-\frac{1}{2}eA'_-+\cdots-\frac{(q+i\epsilon)}{\Delta x_+}+\frac{1}{2}eA'_-+\cdots\right\} \tag{89}$$

$$=0. \tag{90}$$

J_+ gives the charge density on surfaces of constant x_+ and we have seen that its expectation value is

$$\langle\Omega|J_+(x_+,x_-,\tilde{x})|\Omega\rangle$$

$$=-2e\int_0^{eA_-}\frac{dk_+}{2\pi}\int\frac{d^2\tilde{k}}{(2\pi)^2}e^{-2\pi\lambda(k_+,\tilde{k})} \tag{91}$$

$$=-\frac{e}{\pi}\int_0^{x_+}du\left[\frac{eA'_-(u)}{2\pi}\right]^2\exp\left[-\frac{\pi m^2}{eA'_-(u)}\right]. \tag{92}$$

The first form (91) is actually the simplest to understand physically. It says that the charge density accumulates each of the two positron spin states with probability $e^{-2\pi\lambda}$ as the mode with canonical momenta k_+ and $\tilde{k}$ passes through singularity. As noted before, the electron partners in the pair creation event accelerate to the speed of light in the $-z$ direction and leave the manifold moving parallel to the x_- axis. It might seem that since the manifold becomes charged the vector potential must depend upon x_-, and we have therefore not solved the problem for a sufficiently general class of potentials to include the actual back reacted solution. However, we shall see that the response from J_- is actually infinite and infinitely fast — precisely *because* the electrons have exited by reaching the speed of light. This means that back-reaction drives the actual potential to zero infinitely fast, before J_+ can become nonzero.

Evaluating the expectation value of $J_-(x_+,x_-,\tilde{x})$ is complicated because the result must diverge as ϵ approaches zero. To see this note that since the expectation value of $\tilde{J}(x_+,x_-,\tilde{x})$ vanishes, current conservation and our result (91) for the expectation value of $J_+(x_+,x_-,\tilde{x})$ imply

$$\partial_-\langle\Omega|J_-(x_+,x_-,\tilde{x})|\Omega\rangle$$

$$=-\partial_+\langle\Omega|J_+(x_+,x_-,\tilde{x})|\Omega\rangle \tag{93}$$

$$=\frac{1}{\pi}e^2A'_-(x_+)\int\frac{d^2\tilde{k}}{(2\pi)^2}e^{-2\pi\lambda(eA_-,\tilde{k})}. \tag{94}$$

Integration from the lower limit of our ϵ-regulated range of x_- gives

$$\langle\Omega|J_-(x_+,x_-,\tilde{x})|\Omega\rangle=\langle\Omega|J_-(x_+,-\epsilon^{-1},\tilde{x})|\Omega\rangle+\left(x_-+\frac{1}{\epsilon}\right)\frac{1}{\pi}e^2A'_-(x_+)\int\frac{d^2\tilde{k}}{(2\pi)^2}\times e^{-2\pi\lambda(eA_-,\tilde{k})}. \tag{95}$$

The final term has a simple physical interpretation. J_- is a charge flux, so it must register the newly created electrons which rush off the manifold parallel to the x_- axis (because they are moving in the $-z$ direction at the speed of light). The rate at which this charge is created, per unit volume in x_- and $\tilde{x}$, is just $-\partial_+J_+$. An electron created at position $(x_+,x_-,\tilde{x})$ must pass through all points $(x_+,y_-,\tilde{x})$, for $y_->x_-$, on its way off the manifold. So the net electronic flux through any point x_- is the integral of $-\partial_+J_+(x_+,y_-,\tilde{x})$ over all points $y_-<x_-$. We have cut the lower limit off at $-1/\epsilon$, so the electronic contribution to the expectation value of J_- must diverge as ϵ goes to zero.

Although there is a good physical reason for it, the fact that the expectation value of J_- diverges like $1/\epsilon$ means that we must use special care in evaluating distributional limits which involve ϵ. For example, the field equations can be inverted to give $\tilde{\psi}_-$ in terms of $\tilde{\psi}_+$:

$$\tilde{\psi}_-(x_+,x_-,\tilde{k})=\left(\frac{m-\tilde{\gamma}\cdot\tilde{k}}{\tilde{\omega}^2}\right)\gamma_+ i\partial_+\tilde{\psi}_+(x_+,x_-,\tilde{k}). \tag{96}$$

However, we cannot simply substitute the x_+ derivative of expression (61) because the distributional limit in the second term of that formula was computed assuming that k_+ is separated from zero and $eA_-(x_+)$. When ∂_+ acts upon the second θ function in Eq. (61) it gives a δ function which invalidates that assumption by setting $k_+=eA_-(x_+)$. One can tell from the ultralocality of this term at $k_+=eA_-(x_+)$ that it is responsible for the electronic contribution computed above. Rather than forcing everything through from the cumbersome, initial expressions we shall just compute the expectation value of J_- without this term and then compensate by adding in the electron current found above.

Our computational shortcut amounts to making the following replacements:

$$\psi_-(x_++\Delta x_+,x_-,\tilde{x}) \rightarrow \int_{-\infty}^{\infty}\frac{dk_+}{2\pi}e^{-ik_+x_-}\int\frac{d^2\tilde{k}}{(2\pi)^2} \times e^{i\tilde{k}\cdot\tilde{x}}\frac{\frac{1}{2}(m-\tilde{\gamma}\cdot\tilde{k})\gamma_+\Psi(x_++\Delta x_+,k_+,\tilde{k})}{k_+-eA_-(x_++\Delta x_+)+i\epsilon}, \tag{97}$$

$$\psi_-^\dagger(x_+,x_-,\tilde{x}) \rightarrow \int_{-\infty}^{\infty}\frac{dk_+}{2\pi}e^{ik_+x_-}\int\frac{d^2\tilde{k}}{(2\pi)^2} \times e^{-i\tilde{k}\cdot\tilde{x}}\frac{\Psi^\dagger(x_+,k_+,\tilde{k})\,\frac{1}{2}\,\gamma_-(m+\tilde{\gamma}\cdot\tilde{k})}{k_+-eA_-(x_+)-i\epsilon}. \tag{98}$$

The integrand of this expression for ψ_- is just the same as for ψ_+ multiplied by a factor $\frac{1}{2}(m-\tilde{\gamma}\cdot\tilde{k})\gamma_+$ and divided by $p_++i\epsilon$ at the appropriate value of x_+. Hence the contribution to the expectation value of J_- is the same as to J_+ but with an additional factor of

$$\frac{\frac{1}{2}(m-\tilde{\gamma}\cdot\tilde{k})\gamma_+}{p_+(x_++\Delta x_+)+i\epsilon}\,\frac{\frac{1}{2}\gamma_-(m+\tilde{\gamma}\cdot\tilde{k})}{p_+(x_+)-i\epsilon} =\frac{\frac{1}{2}\tilde{\omega}^2P_-}{[p_+(x_++\Delta x_+)+i\epsilon][p_+(x_+)-i\epsilon]}. \tag{99}$$

The expectation value of J_-, without its electronic component, can therefore be obtained by simply including this factor in our previous expression (83) for the expectation value of J_+:

$$\langle\Omega|J_-(x_+,x_-,\tilde{x})|\Omega\rangle-(\text{electronic contribution})$$
$$=e\lim_{\Delta x_+\to0^+}\int\frac{d^2\tilde{k}}{(2\pi)^2}\text{Re}\Bigg\{\left[\int_{-\infty}^{0}\frac{dk_+}{2\pi}+\int_0^{eA_-}\frac{dk_+}{2\pi}(1-e^{-2\pi\lambda})-\int_0^{eA_-}\frac{dk_+}{2\pi}e^{-2\pi\lambda}-\int_{eA_-}^{\infty}\frac{dk_+}{2\pi}\right] \times\frac{\frac{1}{2}\tilde{\omega}^2\mathcal{E}(x_+,x_++\Delta x_+;k_+,\tilde{k})}{[k_+-eA_-(x_++\Delta x_+)+i\epsilon][k_+-eA_-(x_+)-i\epsilon]}\Bigg\} \tag{100}$$

$$=-2e\int_0^{eA_-}\frac{dk_+}{2\pi}\int\frac{d^2\tilde{k}}{(2\pi)^2}\frac{\frac{1}{2}\tilde{\omega}^2e^{-2\pi\lambda(k_+,\tilde{k})}}{[k_+-eA_-(x_+)]^2+\epsilon^2}+e\lim_{\Delta x_+\to0^+}\frac{\partial}{\partial\Delta x_+}\text{Re}\Bigg\{\int_0^{\infty}\frac{dq}{2\pi}\int\frac{d^2\tilde{k}}{(2\pi)^2} \times\left[-\frac{ie^{(-i/2)\tilde{\omega}^2I(\Delta x_+,-q)}}{q+i\epsilon}-\frac{ie^{(-i/2)\tilde{\omega}^2I(\Delta x_+,+q)}}{q-i\epsilon}\right]\Bigg\}, \tag{101}$$

where the function $I(\Delta x_+,q)$ is defined in Eq. (85).

The first term of Eq. (101) has a simple interpretation as the (ϵ-regulated) current due to the created positrons. Each of the two positron spin states is created with probability $e^{-\pi\lambda(k_+,\tilde{k})}$, and each contributes a factor of $-ep_-/p_+$ to the current density. It is simple to perform the integration over $\tilde{k}$ and to recast the remaining integration to one over x_+:

$$-2e\int_0^{eA_-}\frac{dk_+}{2\pi}\int\frac{d^2\tilde{k}}{(2\pi)^2}\frac{\frac{1}{2}\tilde{\omega}^2 e^{-2\pi\lambda(k_+,\tilde{k})}}{[k_+-eA_-(x_+)]^2+\epsilon^2}=-\frac{e}{8\pi^4}\int_0^{x_+}du[eA'_-(u)]^2\frac{[\pi m^2+eA'_-(u)]e^{-\pi m^2/eA'_-(u)}}{[eA_-(u)-eA_-(x_+)]^2+\epsilon^2}. \tag{102}$$

Although the positron current can diverge like $1/\epsilon$, it must vanish at $x_+=0$. This crucial fact distinguishes it from the electron current:

$$\left(x_-+\frac{1}{\epsilon}\right)\frac{1}{\pi}e^2A'_-(x_+)\int\frac{d^2\tilde{k}}{(2\pi)^2}e^{-2\pi\lambda(eA_-,\tilde{k})}=\left(x_-+\frac{1}{\epsilon}\right)\frac{e^3A'^2_-(x_+)}{4\pi^3}e^{-\pi m^2/eA'_-(x_+)}. \tag{103}$$

Even though the state is initially empty, there is no way to prevent particle production at $x_+=0$ because there are modes with k_+ arbitrarily close to zero. The electron current comes entirely from particles which are created moving with the speed of light at the same instant that the current is being measured, so it must be present even at $x_+=0$. This means that the negative electron current must initially dominate the positive positron current. Hence the back reaction acts in the physically sensible direction to reduce the initial electric field. Since the initial electron current is not only negative definite but infinite, as ϵ goes to zero, the back reaction becomes infinitely strong, infinitely fast.

The final term in Eq. (101) is the charge renormalization. One sees this because it contains the logarithmic ultraviolet divergence and because it is proportional to the right hand side of the relevant one of Maxwell's equations for this background:

$$-A''_-(x_+)=\langle\Omega|J_-(x_+,x_-,\tilde{x})|\Omega\rangle. \tag{104}$$

We evaluate it by the same strategy as for the analogous (vanishing) contribution to J_+. First perform the integration over $\tilde{k}$, then expand in powers of the function $I(\Delta x_+,\pm q)$, expand $I(\Delta x_+,\pm q)$ in powers of Δx_+ according to Eq. (87), and finally take the derivative, the real part and the limit inside the integration over q. The result is

$$\frac{e}{4\pi^2}\lim_{\Delta x_+\to 0^+}\frac{\partial}{\partial\Delta x_+}\mathrm{Re}\left\{\int_0^\infty dq\left[-\frac{e^{(-i/2)m^2I(\Delta x_+,-q)}}{(q+i\epsilon)I}-\frac{e^{(-i/2)m^2I(\Delta x_+,+q)}}{(q-i\epsilon)I}\right]\right\}$$

$$=\frac{e}{4\pi^2}\lim_{\Delta x_+\to 0^+}\frac{\partial}{\partial\Delta x_+}\mathrm{Re}\left\{\int_0^\infty dq\left[-\frac{1}{q+i\epsilon}\left(\frac{1}{I(\Delta x_+,-q)}-\frac{i}{2}m^2-\frac{1}{8}m^4I(\Delta x_+,-q)+\cdots\right)\right.\right.$$

$$\left.\left.-\frac{1}{q-i\epsilon}\left(\frac{1}{I(\Delta x_+,+q)}-\frac{i}{2}m^2-\frac{1}{8}m^4I(\Delta x_+,+q)+\cdots\right)\right]\right\} \tag{105}$$

$$=\frac{e}{4\pi^2}\lim_{\Delta x_+\to 0^+}\frac{\partial}{\partial\Delta x_+}\mathrm{Re}\left\{\int_0^\infty dq\left[\frac{1}{\Delta x_+}+\frac{\frac{1}{2}eA'_-}{q+i\epsilon}+\frac{\frac{1}{6}eA''_-\Delta x_+}{q+i\epsilon}-\frac{[\frac{1}{12}(eA'_-)^2+\frac{1}{8}m^4]\Delta x_+}{q^2+\epsilon^2}+\cdots-\frac{1}{\Delta x_+}\right.\right.$$

$$\left.\left.+\frac{\frac{1}{2}eA'_-}{q-i\epsilon}+\frac{\frac{1}{6}eA''_-\Delta x_+}{q-i\epsilon}+\frac{[\frac{1}{12}(eA'_-)^2+\frac{1}{8}m^4]\Delta x_+}{q^2+\epsilon^2}+\cdots\right]\right\} \tag{106}$$

$$=\frac{e}{12\pi^2}eA''_-(x_+)\int_0^\infty dq\,\frac{q}{q^2+\epsilon^2}. \tag{107}$$

It is interesting to contrast this ultraviolet divergence in the expectation value of J_- with the finite result (92) we found for J_+. Of course the expectation values of *both* current operators diverge for a general background; it is just that the divergent term happens to vanish for J_+ in the restricted class of backgrounds we considered. The one loop divergence in the expectation value of $J^\mu(x)$ must reside on $\partial_\nu F^{\nu\mu}(x)$. In the class of backgrounds we considered this happens to be $\partial_+^2A_-(x_+)$ for J_-. For J_+ it is $-\partial_+\partial_-A_-(x_+)=0$, which is why Eq. (92) is ultraviolet finite.

To get the one loop correction to Maxwell's equation (104) we must combine the constituents of the expectation value of J_-: expressions (102), (103) and (107). Since Eq. (107) is a charge renormalization, its proper place is on the left hand side of the equation. The result is

$$-\left[1+\frac{e^2}{12\pi^2}\int_0^\infty dq\,\frac{q}{q^2+\epsilon^2}\right]A''_-(x_+) = \left(x_-+\frac{1}{\epsilon}\right)\frac{e^3A'^2_-(x_+)}{4\pi^3}e^{-\pi m^2/eA'_-(x_+)} - \frac{e}{8\pi}\int_0^{x_+}du \times\left[\frac{eA'_-(u)}{\pi}\right]^2\frac{\left[m^2+\frac{eA'_-(u)}{\pi}\right]e^{-\pi m^2/eA'_-(u)}}{[eA_-(u)-eA_-(x_+)]^2+\epsilon^2}. \tag{108}$$

Now recall from standard QED that the renormalized charge e_R and field $A_R(x_+)$ are related to the unrenormalized ones by square roots of the field strength Z:

$$e_R\equiv\sqrt{Z}e, \quad A_R(x_+)\equiv\frac{1}{\sqrt{Z}}A_-(x_+). \tag{109}$$

Note particularly that $eA_-(x_+)=e_RA_R(x_+)$. Multiplying Eq. (108) by $\sqrt{Z}$ we obtain

$$-\left[Z+\frac{e_R^2}{12\pi^2}\int_0^\infty dq\,\frac{q}{q^2+\epsilon^2}\right]A''_R(x_+) = \left(x_-+\frac{1}{\epsilon}\right)\frac{e^3A'^2_-(x_+)}{4\pi^3}e^{-\pi m^2/eA'_-(x_+)} - \frac{e_R}{8\pi}\int_0^{x_+}du\left[\frac{e_RA'_R(u)}{\pi}\right]^2 \times\frac{\left[m^2+\frac{e_RA'_R(u)}{\pi}\right]e^{-\pi m^2/e_RA'_R(u)}}{[e_RA_R(u)-e_RA_R(x_+)]^2+\epsilon^2}. \tag{110}$$

If we recognize the one loop field strength renormalization as

$$Z=1-\frac{e_R^2}{12\pi^2}\int_0^\infty dq\,\frac{q}{q^2+\epsilon^2} \tag{111}$$

(up to finite renormalizations), then the equation assumes its standard form

$$-A''_R(x_+)=\left(x_-+\frac{1}{\epsilon}\right)\frac{e_R^3A'^2_R(x_+)}{4\pi^3}e^{-\pi m^2/e_RA'_R(x_+)} - \frac{e_R}{8\pi}\int_0^{x_+}du\left[\frac{e_RA'_R(u)}{\pi}\right]^2 \times\frac{\left[m^2+\frac{e_RA'_R(u)}{\pi}\right]e^{-\pi m^2/e_RA'_R(u)}}{[e_RA_R(u)-e_RA_R(x_+)]^2+\epsilon^2}. \tag{112}$$

For small ϵ (which we must take to zero anyway) the instantaneous electron current dominates the positron current and the equation becomes *local*:

$$-A''_R(x_+)\approx\frac{e_R^3A'^2_R(x_+)}{4\pi^3\epsilon}e^{-\pi m^2/e_RA'_R(x_+)}. \tag{113}$$

When compared with the sorts of equations one finds for the traditional problem of evolving from a surface of constant x^0 (for example, see Sec. III of [11]) expression (113) is almost unbelievably simple. We can simplify it further by rescaling both the evolution variable

$$\tau\equiv\left(\frac{e_Rm}{2\pi}\right)^2\frac{x_+}{\epsilon} \tag{114}$$

and the electric field,

$$F(\tau)\equiv\frac{e_RA'_R(x_+)}{\pi m^2}. \tag{115}$$

The result is a first order, ordinary differential equation

$$\frac{d}{d\tau}e^{F^{-1}}=1. \tag{116}$$

The solution is straightforward:

$$F(\tau)=\frac{1}{\ln(e^{1/F_0}+\tau)}. \tag{117}$$

Since τ approaches infinity for any fixed, positive value of x_+ our solution means that the back reaction forces the electric field to zero before any fixed, positive value of x_+. This is as far as the equations can be used because they were derived under the assumption that $e_RA_R(x_+)$ is an increasing function of x_+. Note that our solution also implies the vanishing of the vector potential before any fixed, positive value of x_+. So the expectation value of J_+ is really zero at the physical solution, and there is no need to consider backgrounds which depend upon x_-.

VI. INFINITE BOOST CORRESPONDENCE LIMIT

The results of the past section have a single unsatisfying feature: the factors of $1/\epsilon$ in the expectation value of J_-

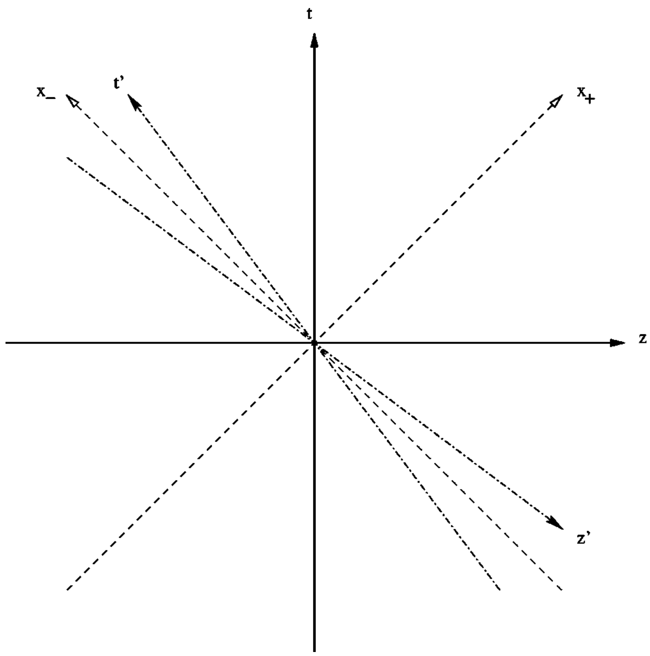

FIG. 2. The various coordinate systems.

mean that the back reaction on the light cone becomes infinitely strong, infinitely fast. This seems to be in dramatic distinction with what happens for the traditional problem in which the state is released on a surface of constant x^0. There the induced current grows smoothly from $x^0=0$, and it remains finite for finite x^0. The purpose of this section is to show that our result is not distinct from the traditional one. Rather the problem we have worked out can be viewed as the infinite boost limit of the traditional problem, in the same way that light cone quantum field theory can always be viewed as the infinite momentum frame [20].

To fix the notation let us consider two inertial frames. The one in which we have been working will be denoted the unprimed frame. The primed frame moves with speed β along the minus z axis, so the Lorentz transformation between the two systems is

$$t' = \gamma(t+\beta z), \tag{118}$$

$$z' = \gamma(z+\beta t), \tag{119}$$

where $\gamma \equiv 1/\sqrt{1-\beta^2}$. Note that the time coordinate of the primed frame has the following expression in terms of the light cone coordinates in the unprimed frame:

$$t' = \sqrt{\frac{1+\beta}{1-\beta}}\frac{x_+}{\sqrt{2}} + \sqrt{\frac{1-\beta}{1+\beta}}\frac{x_-}{\sqrt{2}}. \tag{120}$$

The relation between the two frames is shown in Fig. 2.

We wish to compare evolution in x_+ in the unprimed frame with evolution in primed frame in the limit that β approaches one. We assume that the vector potential and the current density of the primed frame depend only upon t' and have the form

$$A_0'(t')=0, \quad A_3'(t')=A(t'), \tag{121}$$

$$J'^0(t')=0, \quad J'^3(t')=J(t'). \tag{122}$$

Transforming the vector potential covariantly gives

$$A_0=\gamma(A_0'+\beta A_3')=\beta\gamma A(t'), \tag{123}$$

$$A_3=\gamma(A_3'+\beta A_0')=\gamma A(t'). \tag{124}$$

The current density transforms as a contravariant vector to give

$$J^0=\gamma(J'^0-\beta J'^3)=-\beta\gamma J(t'), \tag{125}$$

$$J^3=\gamma(J'^3-\beta J'^0)=\gamma J(t'). \tag{126}$$

The light cone components $A_\pm=(A_0\pm A_3)/\sqrt{2}$ of the unprimed frame vector potential are

$$A_+(x_+,x_-)=\frac{1}{\sqrt{2}}\sqrt{\frac{1+\beta}{1-\beta}}A\left(\sqrt{\frac{1+\beta}{1-\beta}}\frac{x_+}{\sqrt{2}}+\sqrt{\frac{1-\beta}{1+\beta}}\frac{x_-}{\sqrt{2}}\right), \tag{127}$$

$$A_-(x_+,x_-)=\frac{-1}{\sqrt{2}}\sqrt{\frac{1-\beta}{1+\beta}}A\left(\sqrt{\frac{1+\beta}{1-\beta}}\frac{x_+}{\sqrt{2}}+\sqrt{\frac{1-\beta}{1+\beta}}\frac{x_-}{\sqrt{2}}\right). \tag{128}$$

We can enforce our $A_+=0$ gauge condition with the tranformation

$$\hat{A}_\pm(x_+,x_-)=A_\pm(x_+,x_-)-\partial_\pm\int_0^{t'}ds'A(s'), \tag{129}$$

which gives

$$\hat{A}_-(x_+,x_-)=-\sqrt{2}\sqrt{\frac{1-\beta}{1+\beta}}A\left(\sqrt{\frac{1+\beta}{1-\beta}}\frac{x_+}{\sqrt{2}}+\sqrt{\frac{1-\beta}{1+\beta}}\frac{x_-}{\sqrt{2}}\right). \tag{130}$$

The (gauge invariant) electric field is

$$E(x_+,x_-)=-\partial_+\hat{A}_-(x_+,x_-)=A'\left(\sqrt{\frac{1+\beta}{1-\beta}}\frac{x_+}{\sqrt{2}}+\sqrt{\frac{1-\beta}{1+\beta}}\frac{x_-}{\sqrt{2}}\right). \tag{131}$$

The light cone components $J_\pm=(J^0\pm J^3)/\sqrt{2}$ of the unprimed frame current vector are

$$J_{+}(x_{+},x_{-})=\frac{1}{\sqrt{2}}\sqrt{\frac{1-\beta}{1+\beta}}J\left(\sqrt{\frac{1+\beta}{1-\beta}}\frac{x_{+}}{\sqrt{2}}+\sqrt{\frac{1-\beta}{1+\beta}}\frac{x_{-}}{\sqrt{2}}\right), \tag{132}$$

$$J_{-}(x_{+},x_{-})=\frac{-1}{\sqrt{2}}\sqrt{\frac{1+\beta}{1-\beta}}J\left(\sqrt{\frac{1+\beta}{1-\beta}}\frac{x_{+}}{\sqrt{2}}+\sqrt{\frac{1-\beta}{1+\beta}}\frac{x_{-}}{\sqrt{2}}\right). \tag{133}$$

The key relations are Eqs. (130)–(133). Let us consider them as β approaches 1; first under the assumption that the back reaction is turned off. In this case the electric field is constant, so the vector potential and the current density in the primed frame both grow linearly in t':

$$A(t')=E_0 t', \quad J(t')=J_0 t'. \tag{134}$$

From relations (130) and (131) we see that the light cone vector potential is also linear, and the electric field is also constant:

$$\hat{A}_{-}(x_{+},x_{-})\rightarrow -E_0 x_{+}, \quad E(x_{+},x_{-})\rightarrow E_0. \tag{135}$$

Relation (132) reveals a linearly growing light cone charge density,

$$J_{+}(x_{+},x_{-})\rightarrow \frac{1}{2}J_0 x_{+}, \tag{136}$$

just as our field theoretic computation produces for the case of a constant electric field. The really interesting relation is Eq. (133) which gives an *infinite* (and x_{-} dependent) result for J_{-}:

$$J_{-}(x_{+},x_{-})\rightarrow -\frac{1}{2}J_0\left(\frac{1+\beta}{1-\beta}x_{+}+x_{-}\right). \tag{137}$$

The physics of these results is quite simple. First note that any $e^{+}e^{-}$ pair which is created with finite speed in the primed frame must be moving at the speed of light in the $-z$ direction after the infinite boost needed to reach the unprimed frame. Recall that an on-shell particle has

$$p^3=\frac{1}{\sqrt{2}}\left(p_{+}-\frac{\widetilde{\omega}^2}{2p_{+}}\right), \tag{138}$$

so $p^3\rightarrow -\infty$ corresponds to $p_{+}=0^{+}$. This is why we only see particle production on the light cone at $p_{+}=0$. Since electrons must accelerate in the $-z$ direction, they immediately leave the manifold, moving parallel to the x_{-} axis. Positrons accelerate in the $+z$ direction, so they stay on the manifold and, at late values of x_{+}, move parallel to the x_{+} axis. This is why the light cone charge density J_{+} grows. The reason J_{-} tends to be *infinite* is that both particles of each pair are created moving at the speed of light, so they contribute an infinite p_{-}/p_{+}. Of course they tend to cancel by virtue of their opposite charges. The reason J_{-} is infinitely *negative* is that the electrons speed up while the positrons slow down. Finally, we can anticipate from the form of Eq. (120) that any nontrivial time dependence in the primed frame must give rise to infinitely rapid evolution in x_{+} on the unprimed frame.

Now consider the situation in the primed frame with the back reaction turned on. What one sees at one loop is an approximately oscillatory electric field and current [11]. Let us assume, for simplicity, that the behavior is exactly oscillatory and consistent with the Maxwell equation $-A''(t')=J(t')$:

$$A(t')=\frac{E_0}{\omega}\sin(\omega t'), \quad J(t')=\omega E_0\sin(\omega t'). \tag{139}$$

From relation (130) one sees that the vector potential oscillates infinitely fast with infinitely small amplitude:

$$\hat{A}_{-}(x_{+},x_{-})\rightarrow -\sqrt{1-\beta}\,\frac{E_0}{\omega}\sin\left(\frac{\omega x_{+}}{\sqrt{1-\beta}}\right). \tag{140}$$

The electric field oscillates with the same amplitude as in the primed frame but with infinite frequency:

$$E(x_{+},x_{-})\rightarrow E_0\cos\left(\frac{\omega x_{+}}{\sqrt{1-\beta}}\right). \tag{141}$$

From relation (132) we see that J_{+} goes to zero:

$$J_{+}(x_{+},x_{-})\rightarrow \frac{1}{2}\sqrt{1-\beta}\,\omega E_0\sin\left(\frac{\omega x_{+}}{\sqrt{1-\beta}}\right), \tag{142}$$

which means we do not need to consider vector potentials that depend upon x_{-} in addition to x_{+}. Of course the source of the infinitely rapid oscillations is the J_{-} current which has infinite amplitude in addition to infinite frequency:

$$J_{-}(x_{+},x_{-})\rightarrow \frac{-\omega E_0}{\sqrt{1-\beta}}\sin\left(\frac{\omega x_{+}}{\sqrt{1-\beta}}\right). \tag{143}$$

This all looks very much like what we found in the previous section.

VII. DISCUSSION

We have constructed a complete operator solution (34) for free QED in the presence of an electric field that depends arbitrarily upon the light cone coordinate x_{+}. This class of backgrounds is general enough to include the actual evolution of the electric field as it changes due to the back reaction from the current of electron-positron pairs which it induces. One determines the actual electric field (to some order in the loop expansion) by computing the expectation value of J_{-} (to this order), setting this equal to $-A_{-}''(x_{+})$, and solving the resulting equation. We did this to one loop order in Sec. VI and there is no essential obstacle to including higher loop effects. The vertices of QED do not even depend upon the background, nor does the photon propagator. And with our operator solution we have the essential elements of the electron propagator.

It might be useful to recapitulate the rather subtle way the equations of motion can be satisfied within our class of backgrounds. We started with the mode functions in a generic $A_-(x_+)$ gauge field. One consequence was expression (91) which states that the expectation value of J_+ grows with $eA_-(x_+)$. But then the Maxwell equation $\partial_-\partial_+A_-=J_+$ implies that A_- must depend on x_-, contradicting our initial ansatz. The resolution of this apparent contradiction derives from Eq. (112) for the renormalized expectation value of J_-. In the limit that ϵ goes to zero the leading contribution to this source is negative infinite and independent of x_-. Hence so too is ∂_+E. In other words, having a finite, positive electric field causes the x_+ derivative of the electric field to become infinitely negative, which of course drives the electric field to zero. At this point one of the assumptions of our formalism breaks down, but it is easy to see, on physical grounds, that the electric field must fall below zero and that the resulting negative electric field engenders a *positive* infinite J_- current. This would lift it back up through zero, whereupon the (not necessarily periodic) cycle would start again. Since the induced currents are infinite, the response time is zero. So the picture is of an electric field undergoing oscillations of finite amplitude with infinite frequency. Since our vector potential vanishes at $x_+=0$, we can recover it from the electric field by integration:

$$A_-(x_+)=-\int_0^{x_+}dy_+E(y_+). \tag{144}$$

But this integral must vanish for an electric field undergoing oscillations of finite amplitude with infinite frequency. Therefore our result (91) gives *zero* for the expectation value of J_+, and there is no need for the solution to depend upon x_-.

One of the novel features of our solution is that the phenomenon of pair creation is a discrete event on the light cone. Evolution is diagonal in the Fourier basis of k_+ and $\widetilde{k}$; however, it is the minimally coupled, *kinetic* momentum $p_+=k_+-eA_-(x_+)$ which determines whether a particular Fourier component creates or annihilates particles at any given value of x_+. When p_+ passes from negative to positive that particular Fourier component experiences pair creation with probability $e^{-2\pi\lambda(k_+,\widetilde{k})}$, where λ is given by Eq. (49). We exploited this at the end of Sec. IV to give a simple and explicit derivation of the particle production rate per unit volume, in real time and without resorting to *ad hoc* interpretations for formally meaningless expressions.

Why pair creation is so simple on the light cone was explained in Sec. VI. Quantum field theory on surface of constant x_+ can be viewed as the infinite boost limit of the conventional problem formulated on surfaces of constant t' [20]. Pair production is not localized in time when the electric field is homogeneous on surfaces of constant t'. Each of the various momentum modes has a nonzero probability of appearing in *any* time interval. However, when subject to an infinite boost one sees that the newly created particles must appear, to the light cone observer, to be moving with $p^3\rightarrow-\infty$. This corresponds to $p_+\rightarrow 0^+$, which is why particles are created on the light cone only when their kinetic momentum $p_+=k_+-eA_-(x_+)$ passes through zero.

Before a particular Fourier component undergoes pair production, the field at x_+ is a mode function of modulus unity times the same Fourier component of the field at $x_+=0$. After pair production the modulus of the mode function drops by a factor of $e^{-\pi\lambda(k_+,\widetilde{k})}$. The missing amplitude is acquired by new operators which come in from $x_-=-\infty$. This may be one of the more interesting features of our solution for light cone experts. It has long been known that specifying the fields on a surface of constant x_+ cannot completely determine their future evolution. This is obvious for massless fields in two spacetime dimensions. However, the problem has always been hidden at $k_+=0$ when either $m\neq 0$ or $D>2$. It never needed to be resolved if one only desired scattering amplitudes; these can be computed away from $k_+=0$ and then analytically continued. In our analysis the problem could not be avoided because more and more modes are pulled through zero kinetic momentum $p_+=k_+-eA_-(x_+)$ as long as the electric field remains positive.

What the light cone is not at all good for is studying plasma oscillations. In Sec. V we could follow the electric field only as far as its first zero owing to our assumption that $eA_-(x_+)$ is an increasing function. Had this assumption been relaxed we could, in prinicple have followed many oscillations, but they would still have come with infinite frequency in the limit that ϵ vanishes. We saw in Sec. VI that this is the right result for the light cone, but it still leaves us without quantitative control over the frequency of oscillation in the primed frame where the electric field is homogeneous on surfaces of constant time. Of course we *can* get the amplitude of the electric field, which is unchanged by the infinite boost limit, and we can count the number of oscillations. We should also be able to see dissipation by going to higher order. In $3+1$ dimensions one expects energy to flow out of the electron-positron plasma through the emission of photons. This should begin at two loop order in the Schwinger-Keldysh result for the expectation value of the current operator.

Our original motivation for studying this problem was to see what it can teach us about techniques for treating the related problem of quantum gravitational back reaction on inflation. It is worth summarizing what we have learned in that context. First, there does not seem to be any generic problem with using expectation values to study the back reaction. The results we obtained by doing this in Sec. VI have a transparently correct physical interpretation. We should caution, however, that the current operator is a gauge invariant, unlike the metric.

The second point of relevance is that the back reaction is an infrared effect. The important physics is associated with the finite range of modes whose kinetic momentum has passed through zero. We saw in Sec. VI that the ultraviolet divergent contribution to the expectation value of J_- comes from different terms and has a different dependence upon the fields. Had we merely subtracted these terms and replaced the bare charge and field everywhere with the renormalized ones we would have gotten the correct result. This *had* to work from the context of effective field theory, but it is comforting to see it actually do so.

Finally, there is at least the possibility that one can follow the system into the regime where the back reaction is a strong effect. This can happen if the one-particle-irreducible (1PI) diagrams past some finite order in the loop expansion make no large contribution to the effect. Then one will get the right result by simply solving the effective field equations obtained by evaluating the expectation value of the current operator to that finite order. Note especially that one does not have to simply do this and *hope* that it works. Once the solution from the truncated effective field equations is obtained one can always check to see whether the higher loop diagrams are in fact negligibly small in this background. So the way is open to making a potentially self-consistent calculation.

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