Multicritical phenomena of Reissner-Nordström anti–de Sitter black holes

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We present a study of the thermodynamic critical behavior of a Reissner-Nordström anti–de Sitter (RNAdS) black hole in the vicinity of certain critical points in the set of black hole parameters (M,Q) at which the heat capacity at constant charge of the black hole becomes divergent, a characteristic which suggests that the second order phase transition may occur for a RNAdS black hole. Critical exponents of the relevant thermodynamical quantites are computed. Thermal fluctuation in the canonical ensemble near criticality is calculated and anomalous behavior is found for the mean square fluctuation of certain thermodynamical variables. Scaling symmetry for the free energy near a van der Waals–like critical point is also found from which scaling laws among the critical exponents are derived. The thermodynamic analogy of a RNAdS black hole with a van der Waals's liquid-gas system is then discussed and its possible relevance to our microscopic understanding of black hole physics is speculated.

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I. INTRODUCTION

For a Kerr Newman black hole whose charge and angular momentum are not both zero, it is known that at a certain critical temperature the heat capacity at constant charge and angular momentum of the black hole becomes divergent, a characteristic shared by thermodynamical systems known to undergo a second order phase transition [1], such as for example the 2D Ising model. However, unlike other physical systems which exhibit critical behavior, the heat capacity at constant charge and angular momentum of a Kerr Newman black hole is not of constant sign when its charge and angular momentum are not both zero. At the critical temperature, a Kerr Newman black hole changes from a thermally stable thermodynamical object with positive heat capacity to a thermally unstable one with negative heat capacity or vice versa. The statistical mechanics underlying this singular thermodynamical behavior of a Kerr Newman black hole still remains to be understood.

Motivated by the AdS/CFT duality conjecture in string theory [2], there has been recent interest in investigating the critical phenomena of black holes in a more general context of asymptotically anti–de Sitter black holes [3]. Preliminary study in the case of Reissner-Nordström AdS (RNAdS) black holes reveals a richer thermodynamic phase structure than that of its RN counterpart. In a way perhaps not entirely expected, a RNAdS black hole displays multi-critical phenomena which bear certain remarkable resemblances to that of a van der Waals liquid gas system. It is the aim of the present work to gain a better understanding of this multi-critical behavior of a four dimensional RNAdS black hole and the thermodynamic analogy between a RNAdS black hole and a van der Waals liquid gas system. To this end, we shall compute the critical exponents of the relevant thermodynamical quantites and then discuss the scaling symmetry of the free energy as well as the thermal fluctuation near criticality. Among the results we get, we shall see that though the critical behavior of a RNAdS black hole resembles that of a van der Waals liquid gas system qualitatively, these two systems are actually not in the same universality class in the sense that the critical exponents calculated are not exactly the same for these two systems.

The present work is organized as follows. Section II is a brief review of the background of the problem discussed in the present work. In Sec. III, we study the thermodynamic behavior of a RNAdS black hole in the vicinity of the critical points at which there is a change of thermodynamic stability for the black hole. Section IV is devoted to the study of the critical behavior of a RNAdS black hole when its charge reaches certain critical value and the black hole becomes thermodynamically stable. Section V is a discussion of the thermodynamic analogy between a RNAdS black hole and a van der Waals liquid gas system. The work is concluded in Sec. VI with some remarks and discussion.

II. PHASE STRUCTURE OF A RNAdS BLACK HOLE

The theme of the present Sec. is to give an overview of the singular behavior of the heat capacity at constant charge of a RNAdS black hole which forms the background of this work. For more details, see [3]. Throughout we shall adopt Planck units in which $G=\hbar=c=k=1$ where all symbols have their usual meanings.

Consider a RNAdS black hole whose spacetime metric is

$$ds^2=-V(r)dt^2+\frac{dr^2}{V(r)}+r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{2.1}$$

where

$$V(r)=1-\frac{2M}{r}+\frac{Q^2}{r^2}+\frac{r^2}{l^2}$$

and $\Lambda=-3/l^2$ is the cosmological constant. For later convenience, further define $\lambda=3/l^2=-\Lambda$.

The mass of the black hole is given by

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$$M=\frac{1}{2}\left(r_{+}+\frac{Q^{2}}{r_{+}}+\frac{r_{+}^{3}}{l^{2}}\right) \tag{2.2}$$

where r_+ is the radius of the spherical event horizon and

$$r_{+}>\left(\frac{-1+\sqrt{1+4\lambda Q^{2}}}{2\lambda}\right)^{1/2}$$

for a non-extreme black hole. Using the thermodynamical relation $T=(\partial M/\partial S)_Q$ and the Bekenstein-Hawking formula $S=\pi r_+^2$, it may be deduced that

$$T=\frac{1}{4\pi}\frac{\lambda r_{+}^{4}+r_{+}^{2}-Q^{2}}{r_{+}^{3}} \tag{2.3}$$

which may be regarded as the equation of state of the black hole.

Denote the heat capacity at constant charge by C_Q. We have

$$C_{Q}=\frac{2\pi r_{+}^{5}T}{\lambda r_{+}^{4}-r_{+}^{2}+3Q^{2}}. \tag{2.4}$$

As $2\pi r_+^5 T>0$ for a non-extreme black hole, it may be seen from Eq. (2.4) that C_Q will become singular for a certain set of black hole parameters (M,Q) at which

$$\lambda r_{+}^{4}-r_{+}^{2}+3Q^{2}=0. \tag{2.5}$$

Solving the quadratic equation (2.5), we then find that the critical points are given in terms of the radius of the event horizon as

$$r_{1}^{2}=\frac{1-\sqrt{1-12\lambda Q^{2}}}{2\lambda},$$

$$r_{2}^{2}=\frac{1+\sqrt{1-12\lambda Q^{2}}}{2\lambda}\quad \text{when}\quad 12Q^{2}\lambda<1, \tag{2.6}$$

$$r_{c}^{2}=\frac{1}{2\lambda}\quad \text{when}\quad 12Q^{2}\lambda=1. \tag{2.7}$$

For fixed Q so that $12Q^2\lambda<1$, when T is regarded as a function of r_+, it may further be inferred from Eqs. (2.3) and (2.6) that T attains local maximum and minimum respectively at the two distinct roots r_1 and r_2 (see Fig. 1). Since $C_Q<0$ when $r_1<r_+<r_2$ and $C_Q>0$ when $r_+<r_1$ and $r_+>r_2$, so across the critical points at r_1 and r_2, there is a change of thermodynamic stability of a black hole.

In the limit $\Lambda\to 0$, the critical point at r_1 corresponds to the critical point discovered independently by Davies and Hut [1] for a RN black hole while the critical point at r_2 disappears from the (T,r_+) phase plane because $\lim_{\Lambda\to 0} r_2^2<0$. When Q approaches zero, we recover from the critical point at r_2 the the Hawking-Page critical point [4] for a Schwarzchild AdS black hole and the critical point at r_1 disappears because $\lim_{Q\to 0} r_1^2<0$.

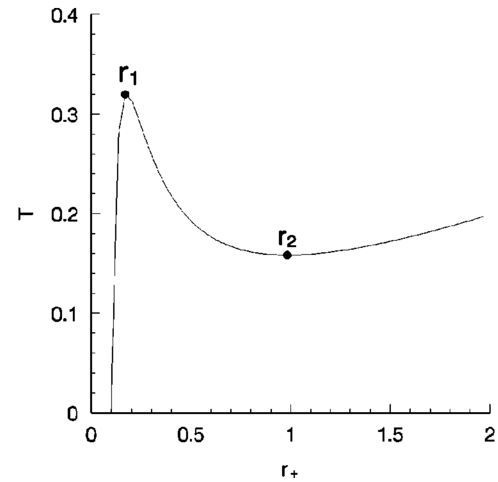

FIG. 1. The isocharge curve with $Q<Q_c$ in the (T,r_+) phase plane. The local maxima and minima at r_1 and r_2 respectively correspond to critical points of C_Q. The segment of curve between r_1 and r_2 corresponds to an unstable phase of a RNAdS black hole along which $C_Q<0$.

In the limit Q^2 approaches the critical value $1/12\lambda$, both r_1 and r_2 degenerate into r_c given in Eq. (2.7). The two critical points at r_1 and r_2 coalesce into a single critical point which is a horizontal point of inflection of the isocharge curve at which $12\lambda Q^2=1$ in the (T,r_+) phase diagram (see Fig. 2).

Along the critical isocharge curve, C_Q remains positive and the unstable phase of a black hole disappears. When Q^2 is greater than the critical value $1/12\lambda$, a RNAdS black hole exhibits no critical behavior.

III. TRANSITION BETWEEN STABLE AND UNSTABLE PHASES

In this section, we shall begin our study of the critical behavior of a RNAdS black hole when $12\lambda Q^2<1$ and a change of thermodynamic stability of the black hole takes

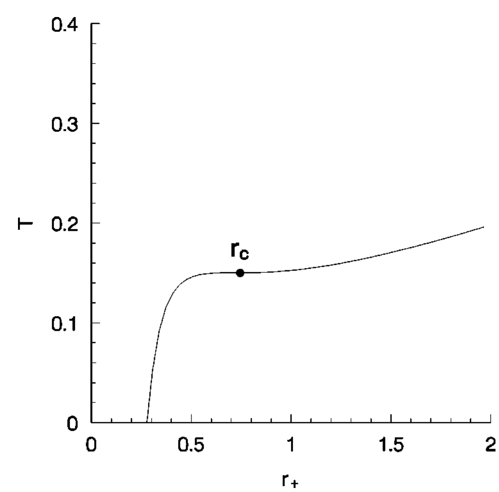

FIG. 2. The critical isocharge curve with $Q=Q_c=1/\sqrt{12\lambda}$. The point of inflection at r_c on the critical isocharge curve is a critical point of C_Q.

place across the critical points at r_1 and r_2. To this end, we shall first work out the critical equation of state of a RNAdS black hole valid in the vicinity of the critical points at r_1 and r_2.

A. Critical equation of state

Denote by T_1 and T_2 the critical temperature at r_1 and r_2 respectively given by Eq. (2.3). Near the critical points, let

$$T = T_i(1+\epsilon) \quad \text{and} \quad r_+ = r_i(1+\Delta), \quad i=1,2, \tag{3.1}$$

where $|\epsilon|,|\Delta| \ll 1$. In what follows, the index i always runs from 1 to 2. Like in the critical behavior of Kerr Newman black holes [5], although C_Q is singular at r_i, with Q fixed, T remains a smooth function of r_+ at r_i.

From Eqs. (2.3) and (2.6), we have

$$\left(\frac{\partial T}{\partial r_+}\right)_Q\bigg|_{r=r_i} = \frac{1}{4\pi}\frac{\lambda r_i^4 - r_i^2 + 3Q^2}{r_i^4} = 0$$

$$\left(\frac{\partial^2 T}{\partial r_+^2}\right)_Q\bigg|_{r=r_i} = \frac{1}{2\pi}\frac{r_i^2 - 6Q^2}{r_i^5}$$

with

$$\left(\frac{\partial^2 T}{\partial r_+^2}\right)_Q\bigg|_{r=r_1} < 0, \quad \left(\frac{\partial^2 T}{\partial r_+^2}\right)_Q\bigg|_{r=r_2} > 0.$$

The difference in the sign between $(\partial^2 T/\partial r_+^2)_Q$ at r_1 and r_2 arises from the fact that, as a function of r_+, T at r_1 is a local maximum while T at r_2 is a local minimum (see Fig. 1).

In a sufficiently small neighborhood of r_i, we may expand T in terms of r_+ to give

$$T_i(1+\epsilon) = T_i + \left(\frac{\partial^2 T}{\partial r_+^2}\right)_Q\bigg|_{r_i} r_i^2\Delta^2 + O(\Delta^3). \tag{3.2}$$

It then follows from Eq. (3.2) that

$$\epsilon = D_i\Delta^2 + O(\Delta^3) \tag{3.3}$$

where

$$D_i = \frac{r_i^2}{T_i}\left(\frac{\partial^2 T}{\partial r_+^2}\right)_Q\bigg|_{r=r_i} = \frac{1}{2\pi T_i}\frac{r_i^2 - 6Q^2}{r_i^3}. \tag{3.4}$$

Equation (3.3) may be regarded as the critical equation of state for a RNAdS black hole when the event horizon radius r_+ is sufficiently close to r_i. As far as the critical behavior of a RNAdS black hole is concerned, it is sufficient to consider the lowest order nontrivial term of Δ in Eq. (3.3) which dominates the thermodynamics near r_i.

B. Calculation of critical exponents

To shed further light on the critical behavior of the black hole near r_1 and r_2, we shall undertake the calculation of the critical exponents of certain thermodynamical quantites which become singular at the critical points. With the help of the critical equation of state given in Eq. (3.3), the calculations turn out to be quite straightforward.

By definition, we have

$$C_Q = T\left(\frac{\partial S}{\partial T}\right)_Q, \quad C_\Phi = T\left(\frac{\partial S}{\partial T}\right)_\Phi, \quad \kappa_T = \left(\frac{\partial \Phi}{\partial Q}\right)_T \tag{3.5}$$

which are the analogs of C_P, C_V and the isothermal compressibility respectively of a liquid-gas system. On further calculations using Eqs. (2.2) and (2.3), we have from Eqs. (3.5) that

$$C_Q = 2\pi r_+^2 \frac{\lambda r_+^4 + r_+^2 - Q^2}{\lambda r_+^4 - r_+^2 + 3Q^2} \tag{3.6}$$

$$C_\Phi = 2\pi r_+^2 \frac{\lambda r_+^4 + r_+^2 - Q^2}{\lambda r_+^4 - r_+^2 + Q^2} \tag{3.7}$$

$$\kappa_T = \frac{1}{r_+}\frac{\lambda r_+^4 - r_+^2 + Q^2}{\lambda r_+^4 - r_+^2 + 3Q^2}. \tag{3.8}$$

From Eqs. (3.7) and (2.6), we see that C_Φ displays no singular behavior at the critical points located at r_1 and r_2, to calculate the critical exponents of C_Q and κ_T which are manifestly singular at r_1 and r_2.

Substitute Eq. (3.1) into Eq. (3.6); we have for $|\epsilon|,|\Delta| \ll 1$, the asymptotic behavior of C_Q near the critical point r_i is given in terms of Δ as

$$C_Q = \pi\frac{\lambda r_i^4 + r_i^2 - Q^2}{2\lambda r_i^2 - 1}\frac{1}{\Delta} + O(\Delta). \tag{3.9}$$

Together with the critical equation of state given in Eq. (3.3), we may infer from Eq. (3.9) that near r_i, the critical behavior of C_Q is described by

$$\begin{aligned} C_Q &\approx \frac{A_i}{|\epsilon|^{1/2}} \quad \text{at the stable phase} \\ &\approx \frac{-A_i}{|\epsilon|^{1/2}} \quad \text{at the unstable phase} \end{aligned} \tag{3.10}$$

where

$$A_i = 2\pi\left|\frac{\lambda r_i^4 + r_i^2 - Q^2}{2\lambda r_i^2 - 1}\right||D_i|^{1/2} \tag{3.11}$$

and D_i are as that given in Eq. (3.4).

Using Eqs. (3.8), (3.2) and (3.3), it may also be deduced in a way similiar to the case of C_Q that, for $|\epsilon|,|\Delta|\ll 1$, the asymptotic behavior of κ_T near r_i may be given as

$$\kappa_T \approx \frac{-B_i}{|\epsilon|^{1/2}} \quad \text{at the stable phase}$$

$$\approx \frac{B_i}{|\epsilon|^{1/2}} \quad \text{at the unstable phase} \tag{3.12}$$

where

$$B_i = \left| \frac{2}{r_i^2} \frac{\lambda r_i^4 - r_i^2 + Q^2}{2\lambda r_i^2 - 1} \right| |D_i|^{1/2}. \tag{3.13}$$

So the critical exponents of C_Q and κ_T near the critical regime of both r_1 and r_2 may be read off from Eqs. (3.10) and (3.12) respectively as $-\frac{1}{2}$.

As a consistency check, we see that in the limit $\lambda \to 0$, we recover from Eq. (3.10) the critical exponent of C_Q in the case of RN black holes for the critical point at r_1 [5]. Further, in the limit $Q \to 0$, the critical point at r_2 becomes the Hawking-Page critical point for a Schwarzchild AdS black hole. Equations (3.10) and (3.12) then also give the critical exponets of C_Q and κ_T for a Schwarzchild AdS black hole which, as far as we know, have not been given before.

C. Definition of the order parameter

In the case of a RN black hole, a change of thermodynamic stability of the black hole across the critical point (which corresponds to the limiting case of $\lambda \to 0$ of that at r_1 in the RNAdS context) enables us to introduce an order parameter which measures the gradual phase change of a RNAdS black hole across the critical point [6]. The definition of the order parameter may be carried over to the RNAdS context without difficulty and described as follows.

Across r_i, C_Q changes sign. So in the thermally stable phase at which $C_Q > 0$, a quasistatic absorption of a particle without charge (for e.g. a Hawking particle of a neutral scalar field backscattered by the gravitational potential near the event horizon) will be followed by a corresponding increase of the mass of the black hole, while in the thermally unstable phase at which $C_Q < 0$, the corresponding absorption process will lead to a suppression of the black hole temperature.

Consider a RNAdS black hole at the temperature $T = T_i(1+\epsilon)$ with $|\epsilon| \ll 1$. Denote by p the probability that a RNAdS black hole will absorb quasistatically a particle of mass ΔM without charge followed by an increase in its temperature characterized by $\Delta\epsilon$. Further, assume $\Delta\epsilon$ is sufficiently small so that

$$\Delta S \approx T_i C_Q \Delta\epsilon \approx \frac{T_i A_i}{\sqrt{|\epsilon|}} \Delta\epsilon \quad \text{at the stable phase}$$

$$-\frac{T_i A_i}{\sqrt{|\epsilon|}} \Delta\epsilon \quad \text{at the unstable phase} \tag{3.14}$$

according to Eq. (3.10). Subject to the Boltzmann-Einstein hypothesis that the entropy S is a measure of the number of internal states of a RNAdS black hole that correspond to a particular macrostate characterized by definite M and Q, it follows from Eq. (3.14) that p may be given as

$$p \propto \exp\left(\frac{T_i A_i}{\sqrt{|\epsilon|}} \Delta\epsilon \right) \quad \text{at the stable phase}$$

$$= 0 \quad \text{at the unstable phase} \tag{3.15}$$

where the second equality follows from $C_Q < 0$ in the unstable phase at which an absorption process is always followed by a decrease in temperature of the black hole. Equation (3.15) then suggests the definition of the following order parameter:

$$\eta_i = \exp\left(-\frac{T_i A_i}{\sqrt{|\epsilon|}} \right) \quad \text{at the stable phase}$$

$$= 0 \quad \text{at the unstable phase.} \tag{3.16}$$

It may be checked from Eq. (3.16) that $\eta_i \neq 0$ in the stable phase and η_i vanishes in the unstable phase. Further, across the critical point at r_i, η_i remains continuous. So η_i satisfies the requirements common to the order parameters of other physical systems which exhibit criticality.

As far as the physical meaning is concerned, η_i is the inverse of the statistial weight factors which determine the probability of an absorption process which raises the temperature of a RNAdS black hole near the critical points r_i.

D. Thermodynamic critical fluctuation

We shall now go on to compute the thermal fluctuation of the mass and entropy near the critical points located at r_1 and r_2 when a RNAdS black hole is in a canonical ensemble. It will be shown that the mean square fluctuation of the mass and entropy of a RNAdS black hole behaves anomalously and becomes divergent as the critical points at r_1 and r_2 are approached.

Consider the thermal fluctuation for a RNAdS black hole immersed in a bath of radiation in thermal equilibrium with the black hole such that Q is kept constant. Subject to the appropiate asymptotic boundary conditions, the thermal equilibrium involving the black hole and the heat bath is stable [7].

Near r_i, the black hole is at a temperature $T = T_i(1+\epsilon)$ where $|\epsilon| \ll 1$ and T_i is the critical temperature at r_i. With Q fixed, subject to the equation of state given in Eq. (2.3), an exchange of energy of the black hole with the heat bath leads inevitably to a deviation of the temperature of the black hole from that of the heat bath. Denote this deviation of temperature by $\Delta\epsilon$. Suppose further that the black hole temperature is only weakly perturbed in the sense that

$$-c_1<\Delta\epsilon<c_2 \tag{3.17}$$

where c_1, c_2 are sufficiently small positive constants. Since T_1 is the local maximum temperature near r_1, c_2 is further bounded by $c_2<|\epsilon|$. When we come to consider the critical point r_2, as T_2 is the local minimum near r_2 this time, c_1 is bounded by $|c_1|<|\epsilon|$. However, these restrictions will not affect our calculations in what follows as long as c_1 or c_2 remains finite and not both zero as the limit $\epsilon\to 0$ is approached.

Denote by p the fluctuation probability of a RNAdS black hole immersed in a heat bath. For $|\epsilon|\ll 1$, p is given by [8]

$$p\propto \exp\left(-\frac{\Delta F}{T_i(1+\epsilon)}\right)\approx \exp\left(-\frac{\Delta F}{T_i}\right) \tag{3.18}$$

where ΔF is the change of free energy of the black hole induced by $\Delta\epsilon$. For $\Delta\epsilon$ sufficient small and $|\epsilon|\ll 1$,

$$\begin{aligned}\Delta F&=T_iS_i[1+O(|\epsilon|^{1/2})]\Delta\epsilon\\ &\approx T_iS_i\Delta\epsilon.\end{aligned} \tag{3.19}$$

As a result, we have from Eq. (3.18) and (3.19) that

$$p\propto \exp(S_i\Delta\epsilon). \tag{3.20}$$

Subject to Eq. (3.17), Eq. (3.20) may be further normalized to be

$$p=\frac{S_i}{\exp(S_ic_1)-\exp(-S_ic_2)}\exp(S_i\Delta\epsilon). \tag{3.21}$$

On the other hand, from Eq. (2.2), we find, in a sufficiently small neighborhood of r_i,

$$\Delta M=T_iC_Q\Delta\epsilon\approx \begin{cases}\dfrac{T_iA_i}{\sqrt{|\epsilon|}}\Delta\epsilon & \text{at the stable phase}\\[2ex] -\dfrac{T_iA_i}{\sqrt{|\epsilon|}}\Delta\epsilon & \text{at the unstable phase}\end{cases} \tag{3.22}$$

according to Eq. (3.10). From Eq. (3.22) and the normalized fluctuation probability given in Eq. (3.21), the mean square fluctuation of M may be worked out to be

$$\begin{aligned}\langle(\Delta M)^2\rangle&=\frac{S_i}{\exp(S_ic_1)-\exp(-S_ic_2)}\frac{T_i^2A_i^2}{|\epsilon|}\\ &\quad\times\int_{-c_1}^{c_2}x^2\exp(S_ix)dx.\end{aligned} \tag{3.23}$$

It may be inferred from Eq. (3.23) that, modulo a non-zero constant independent of ϵ, asymptotically near the critical point at r_i,

$$\langle(\Delta M)^2\rangle\sim\frac{1}{|\epsilon|}. \tag{3.24}$$

Moreover, from the definition $C_Q=(\partial M/\partial T)_Q=T(\partial S/\partial T)_Q$, it may further be deduced that, for $|\epsilon|\ll 1$,

$$\Delta M\approx T_i\Delta S. \tag{3.25}$$

From Eqs. (3.24) and (3.25), we see that in the limit $|\epsilon|\to 0$, both $\langle(\Delta M)^2\rangle$ and $\langle(\Delta S)^2\rangle$ become divergent.

The singular behavior of the mean square fluctuation of mass and entropy calculated above suggests that, in a canonical ensemble, the number of copies of black holes which deviate from the average thermodynamic behavior becomes very large when the critical points at r_1 or r_2 is approached. This indicates a breakdown of thermodynamic description of a RNAdS black hole near the critical points.

IV. THERMAL PHASE TRANSITION AND CRITICAL VALUE OF Q

As described in Sec. II, when the charge of a RNAdS black hole reaches the critical value $Q^2=1/12\pi\lambda$, the critical points at r_1 and r_2 studied in the preceding section degenerate into a single critical point located at r_c. The thermally unstable phase of a RNAdS black hole disappears and the black hole becomes thermally stable along the isocharge curve at which $Q^2=1/12\pi\lambda$ (see Fig. 2). The theme of this section is to study the critical thermodynamic behavior of a RNAdS black hole near r_c. To this end, we shall first review a thermodynamic analogy between a RNAdS black hole and a van der Waals liquid gas system first discovered in [3]. The analogy, though incomplete, will still serve as a very useful guide in the study of the critical behavior of a RNAdS black hole in the vicinity of r_c.

A. Thermodynamic analogy with a van der Waals liquid gas system

Given the electromagnetic potential at the event horizon $\Phi=Q/r_+$, the equation of state (2.3) may be rewritten as

$$T=\frac{1}{4\pi}\frac{\Phi^2-\Phi^4+Q^2}{Q\Phi}. \tag{4.1}$$

In terms of the thermodynamical variables (Q,Φ), we have

$$C_Q=2\pi\left(\frac{Q}{\Phi}\right)^2\frac{-\Phi^4+\Phi^2+\lambda Q^2}{3\Phi^4-\Phi^2+\lambda Q^2} \tag{4.2}$$

and

$$\left(\frac{\partial Q}{\partial\Phi}\right)_T=\frac{Q}{\Phi}\frac{3\Phi^4-\Phi^2+\lambda Q^2}{\Phi^4-\Phi^2+\lambda Q^2}. \tag{4.3}$$

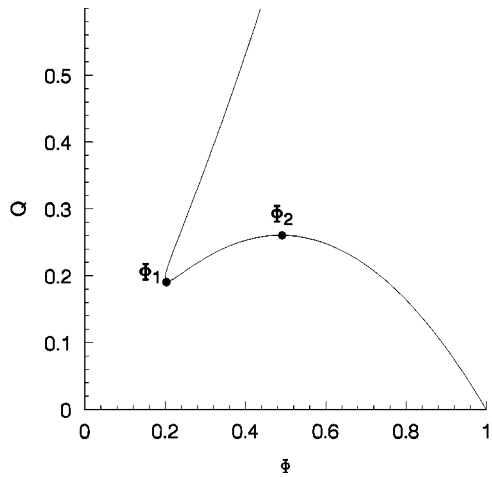

FIG. 3. The isotherm of a RNAdS black hole along which $T > T_c$. The local maxima and minima located respectively at Φ_1 and Φ_2 are critical points of C_Q. For $\Phi \in (\Phi_1, \Phi_2)$, the black hole is unstable with $(\partial Q/\partial \Phi)_T > 0$.

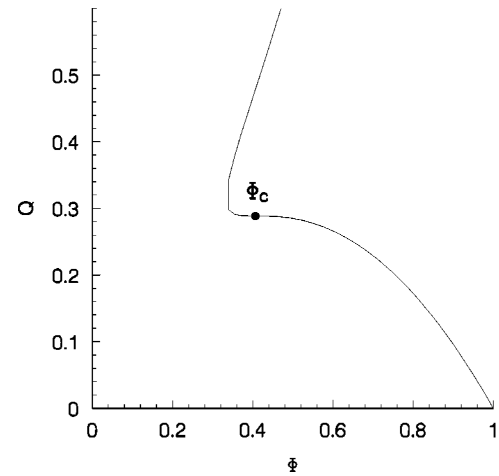

FIG. 4. The critical isotherm along which $T = T_c$. The point of inflection located at Φ_c is a critical point of C_Q, $C_Q > 0$ along the critical isotherm.

It then follows from Eq. (4.2) that along an isotherm, C_Q become divergent at

$$\Phi_1^2 = \frac{1 - \sqrt{1 - 12\lambda Q^2}}{2\lambda},$$

$$\Phi_2^2 = \frac{1 + \sqrt{1 - 12\lambda Q^2}}{2\lambda} \quad \text{when} \quad T > T_c, \tag{4.4}$$

$$\Phi_c^2 = \frac{1}{2\lambda} \quad \text{at} \quad T_c. \tag{4.5}$$

Further, it may be inferred from Eq. (4.3) that, like a subcritical isotherm of a van der Waals liquid gas system in the (P, V) phase plane (see e.g. [9]), an isotherm of a RNAdS black hole with $T > T_c$ also has a local maxima and minima located respectively at Φ_1 and Φ_2 given in Eq. (4.4). Along the segment of the isotherm between Φ_1 and Φ_2, a RNAdS black hole is in a thermally unstable phase with $(\partial Q/\partial \Phi)_T > 0$ (see Fig. 3).

In the limit when T_c is reached, the shape of the isotherm undergo noticable change (see Fig. 4) and the critical points located at Φ_1 and Φ_2 on a subcritical isotherm coalesce into a single critical point located at Φ_c [given in Eq. (4.5)] at the critical isotherm. The critical point at Φ_c coincides with that located at r_c on the critical isocharge curve with $Q = 1/\sqrt{12\lambda}$.

Like the case of the van der Waals liquid gas system, the critical point at the critical isotherm (along which $T = T_c$) of a RNAdS black hole is also a point of inflection of the critical isotherm and may be characterized by

$$\left(\frac{\partial Q}{\partial \Phi}\right)\bigg|_c = 0$$

$$\left(\frac{\partial^2 Q}{\partial \Phi^2}\right)\bigg|_c = 0$$

equation of state in Eq. (4.1)

where the subscript c denotes the corresponding quantity evaluated at the critical point at r_c from now on. In view of the above similiarities, if we formally identify the variables (Q, Φ) of a RNAdS black hole with (V, P) of a van der Waals liquid gas system, then we see that, at least at a qualitative level, the phase structure of a RNAdS black hole does bear certain remarkable resemblences to that of a van der Waals liquid gas system.

B. Choice of order parameter

In analogy to a van der Waals liquid gas system, an order parameter in the RNAdS context which measures the phase change across the critical point at r_c may also be defined in terms of the Maxwell equal-area law. To do so, in the (Q, Φ) phase plane, fix a subcritical isotherm and draw a horizontal line which interests the subcritical isotherm at points a, d, b (see Fig. 5) such that the area bounded by the horizontal line segment ad and the isotherm is equal to that bounded by the line segment db and the isotherm.

As in the case of a van der Waals liquid gas system, define

$$\eta = \Phi_b - \Phi_a \tag{4.6}$$

as the order parameter to describe the phase change of a RNAdS black hole near r_c.

C. Critical exponents

Near the critical point at the critical isotherm, the critical behavior of a van der Waals liquid gas system may be described in terms of

$$(1) \quad P - P_c \sim (V - V_c)^{\delta}$$

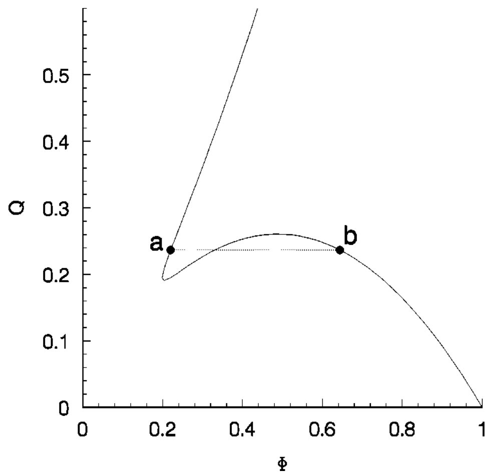

FIG. 5. A horizontal line is drawn which connects points a and b of the subcritical isotherm. The area bounded by the line segment ad and the isotherm is equal to that bounded by the line segment db and the isotherm.

$$(2) \quad \frac{V_g - V_l}{V_c} \sim (-\epsilon)^{\beta}$$

$$(3) \quad C_P \sim (-\epsilon)^{-\alpha'} \quad (T < T_c)$$

$$\sim \epsilon^{-\alpha} \quad (T > T_c)$$

$$(4) \quad \kappa_T \sim (-\epsilon)^{-\gamma'} \quad (T < T_c)$$

$$\sim \epsilon^{-\gamma} \quad (T > T_c).$$

Along the critical isotherm, we further have

$$(5) \quad C_P \sim (P - P_c)^{-\phi} \ \ (\text{for } \ \epsilon = 0)$$

$$(6) \quad \kappa_T^{-1} \sim (P - P_c)^{1-1/\delta} \quad (\text{for } \ \epsilon = 0)$$

$$(7) \quad S - S_c \sim \epsilon^{1-\alpha} \quad (\text{for } \ \Delta P = 0)$$

$$\sim (P - P_c)^{\psi} \quad (\text{for } \ \epsilon = 0).$$

With the formal correspondence $(\Phi,Q) \leftrightarrow (V,P)$ as described in the preceding subsection, analogous quantities may also be defined for a RNAdS black hole. The concrete values of the corresponding critical exponents in the case of a RNAdS black hole may also be worked out as follows.

1. Calculation of δ

Using the equation of state (4.1), we have

$$\left(\frac{\partial Q}{\partial \Phi}\right)_T\bigg|_c = 0$$

$$\left(\frac{\partial^2 Q}{\partial \Phi^2}\right)_T\bigg|_c = 0$$

$$\left(\frac{\partial^3 Q}{\partial \Phi^3}\right)_T\bigg|_c \neq 0.$$

So the critical point at r_c is a point of inflection of the critical isotherm. Standard Taylor expansion then gives in a sufficiently small neighborhood of r_c,

$$Q - Q_c = \left(\frac{\partial^3 Q}{\partial \Phi^3}\right)_T\bigg|_c (\Phi - \Phi_c)^3 + O((\Phi - \Phi_c)^4).$$

This means

$$\delta = 3 \tag{4.7}$$

which is identical to that of a van der Waals liquid gas system. From Eq. (4.7), it may also be inferred that along the critical isotherm,

$$\kappa_T^{-1} \sim (Q - Q_c)^{2/3}.$$

2. Calculation of β

Let $\Delta_1 = \Phi_a - \Phi_c$ and $\Delta_2 = \Phi_b - \Phi_c$. The order parameter defined in Eq. (4.6) may be rewritten as

$$\eta = \Delta_1 + \Delta_2. \tag{4.8}$$

Using Eq. (4.1) again, we have

$$\left(\frac{\partial T}{\partial \Phi}\right)_Q\bigg|_c = 0$$

$$\left(\frac{\partial^2 T}{\partial \Phi^2}\right)_Q\bigg|_c = 0$$

$$\left(\frac{\partial^3 T}{\partial \Phi^3}\right)_Q\bigg|_c \neq 0$$

and therefore

$$\epsilon = \left(\frac{\partial^3 T}{\partial \Phi^3}\right)_Q\bigg|_c (\Delta_1^3 + \Delta_2^3) + \text{higher order terms of } \ \Delta_1, \Delta_2. \tag{4.9}$$

Close enough to the critical point, we have $\Delta_1 \sim \Delta_2$. Together with Eqs. (4.8) and (4.9), we then find

$$\beta = \frac{1}{3}. \tag{4.10}$$

3. Calculation of α, α'

It may be deduced from Eq. (2.3) that

$$\left(\frac{\partial T}{\partial r_+}\right)_Q\bigg|_c = 0$$

$$\left(\frac{\partial^2 T}{\partial r_+^2}\right)_Q\Bigg|_c = 0$$

$$\left(\frac{\partial^3 T}{\partial r_+^3}\right)_Q\Bigg|_c \neq 0.$$

Like in the derivation of the critical equation of state (2.3) near the critical points at r_1 and r_2 in Sec. III A, we have, in a sufficiently small neighborhood of r_c,

$$\epsilon \approx A_c \Delta^3 \tag{4.11}$$

where

$$A_c = \frac{r_c^3}{T_c}\left(\frac{\partial^3 T}{\partial r_+^3}\right)_Q\Bigg|_c .$$

On the other hand, recall from Eq. (3.5) that

$$C_Q = 2\pi r_+^2 \, \frac{\lambda r_+^4 + r_+^2 - Q^2}{\lambda r_+^4 - r_+^2 + 3Q^2}.$$

Near the critical point at r_c, let

$$r_+ = r_c(1+\Delta) \quad \text{where} \quad |\Delta| \ll 1. \tag{4.12}$$

Substitute Eq. (4.12) into Eq. (3.5) and bear in mind $r_c^2 = 1/2\lambda$; we have

$$C_Q \approx \frac{2\pi}{3\lambda}\,\frac{1}{\Delta^2}. \tag{4.13}$$

Equations (4.11) and (4.13) together then imply

$$C_Q \sim |\epsilon|^{-2/3}$$

immediately below or above T_c and therefore

$$\alpha = \alpha' = \frac{2}{3}. \tag{4.14}$$

4. Calculation of γ, γ'

From Eq. (3.8), we have

$$\left(\frac{\partial \Phi}{\partial Q}\right)_T = \frac{1}{r_+}\,\frac{\lambda r_+^4 - r_+^2 + Q^2}{\lambda r_+^4 - r_+^2 + 3Q^2}.$$

Sufficiently close to r_c, using Eq. (4.12) and $r_c^2 = 1/2\lambda$, we have

$$\left(\frac{\partial \Phi}{\partial Q}\right)_T \approx -\frac{1}{6\sqrt{2\lambda}}\,\frac{1}{\Delta^2}. \tag{4.15}$$

From Eqs. (4.11) and (4.15), we may then infer that, near r_c,

$$\left(\frac{\partial \Phi}{\partial Q}\right)_T \sim \epsilon^{-2/3}.$$

This means

$$\gamma = \gamma' = \frac{2}{3}. \tag{4.16}$$

5. Calculation of ϕ

From Eq. (2.3), we get

$$\left(\frac{\partial Q}{\partial r_+}\right)_T\Bigg|_c = 0$$

$$\left(\frac{\partial^2 Q}{\partial r_+^2}\right)_T\Bigg|_c = 0$$

$$\left(\frac{\partial^3 Q}{\partial r_+^3}\right)_T\Bigg|_c \neq 0$$

and these imply

$$Q - Q_c = \left(\frac{\partial^3 Q}{\partial r_+^3}\right)_T\Bigg|_c r_c^3 \Delta^3 + O(\Delta^4). \tag{4.17}$$

Equations (4.13) and (4.17) together then give

$$C_Q \sim (Q - Q_c)^{-2/3}$$

and therefore

$$\phi = \frac{2}{3}. \tag{4.18}$$

6. Calculation of ψ

Because $S = \pi r_+^2$, in a sufficiently small neighborhood of r_c,

$$S - S_c \approx 2\pi r_c^2 \Delta. \tag{4.19}$$

Equations (4.19) and (4.11) imply that, along an isocharge curve at which $\Delta Q = 0$,

$$S - S_c \sim \epsilon^{1/3} \tag{4.20}$$

which is consistent with Eq. (4.14). From Eqs. (4.17) and (4.19), we also get

$$S - S_c \sim (Q - Q_c)^{1/3}$$

and therefore

$$\psi = \frac{1}{3}. \tag{4.21}$$

For a summary of the critical exponents obtained from the above calculations, see Table I in Sec. V. To conclude this section, we remark that the numerical values of the critical exponents calculated above are all multiples of $\frac{1}{3}$. There is a rational explanation for the numerical coincidence, which may be described as follows. The critical exponents are obtained by means of Taylor expanding a thermodynamical quantity in terms of a variable whose choice is dictated by the relevant critical exponent we want to calculate. It turns out that in the Taylor expansions, both the first and second order terms vanish near the van der Waals like critical point r_c and the lowest order non-trivial contributions come from the cubic term in the expansions.

D. Scaling symmetry for the free energy near criticality

In the case of a van der Waals liquid gas system, scaling symmetry exists for the singular part of the Gibbs free energy near the critical point located at the critical isotherm and the critical exponents may all be expressed in terms of the two independent homogenity degrees of the Gibbs energy [9]. In this subsection, we shall show that for a RNAdS black hole similiar scaling symmetry also exists for the singular part of the free energy in the critical regime near r_c from which scaling laws for the critical exponents may be derived. However, the similiarity holds only at a qualitative level as the two independent degree of homogenity for the free energy of a RNAdS black hole are different from that of the Gibbs energy of a van der Waals liquid gas system. Scaling symmetry in the black hole critical phenomena was first discussed in [10] in the context of of Kerr Newman black holes.

Sufficiently close to r_c, the free energy for a RNAdS black hole may be written as $F=F_r+F_s$. Here F_r is the regular part of the free energy whose second order partial derivatives are well behaved at the critical point at r_c, and F_s is the part of the free energy responsible for the singular thermodynamic behavior of a RNAdS black hole near r_c. With the help of Eqs. (4.13) and (4.15) together with Eq. (4.11), F_s may further worked out to be

$$\begin{aligned} F_s &= C_Q\epsilon^3+\left(\frac{\partial Q}{\partial \Phi}\right)_T \Pi^3 \\ &= a\epsilon^{4/3}+b\Pi^{4/3} \end{aligned} \tag{4.22}$$

for some constant a,b dependent on λ. From Eq. (4.22), we find

$$F(\Lambda^p\epsilon,\Lambda^q\Pi)=\Lambda F(\epsilon,\Pi) \tag{4.23}$$

with $p=q=\frac{3}{4}$ and Λ a real constant. As in the case of a van der Waals liquid-gas system (see e.g. [9]), the critical exponents derived in the previous section may be expressed in terms of p,q as

$$\begin{aligned} \alpha &= 2-\frac{1}{p} \\ \beta &= \frac{1-q}{p} \\ \gamma &= \frac{2q-1}{p} \\ \delta &= \frac{q}{1-q} \\ \psi &= \frac{1-p}{q} \\ \phi &= \frac{1-2p}{q}. \end{aligned} \tag{4.24}$$

From Eqs. (4.24), it may also be seen that the critical exponents in the critical regime of r_c are not independent. They are related by following equations (see [9,11]):

$$\begin{aligned} &\alpha+2\beta+\gamma=2 \\ &\alpha+\beta(\delta+1)=2 \\ &\gamma(\delta+1)=(2-\alpha)(\delta-1) \\ &\gamma=\beta(\delta-1) \\ &(2-\alpha)(\delta\psi-1)+1=(1-\alpha)\delta \\ &\phi+2\psi-\frac{1}{\delta}=1. \end{aligned} \tag{4.25}$$

Apart from obtaining the algebraic relations among the critical exponents, Eqs. (4.24) or (4.25) also enable us to give a consistency check of the validity of the critical exponents obtained in Sec. IV C.

E. Thermodynamic critical fluctuation

As in the cases of the critical points at r_1 and r_2, we may also ask whether the thermal fluctuation of certain thermodynamical variables near r_c will behave anomalously in a canonical ensemble. The arguments presented in Sec. III D are also applicable when we come to consider the critical thermal fluctuation near r_c, with however the following minor differences. (i) As the temperature of the black hole is no longer bounded above or below near the critical point, the range of fluctuation of the temperature near r_c may be chosen to be $|\Delta\epsilon|<c$ [compared with Eq. (3.17)] where c is a sufficiently small positive constant. This in fact makes the calculations simpler. (ii) The critical equation of state near r_c given in Eq. (4.11) is used in place of Eq. (4.1).

Since the calculations are similiar to that presented in Sec. III D, we shall only state the results of our calculations which are

$$\begin{aligned} \langle(\Delta M)^2\rangle,\langle(\Delta S)^2\rangle &\sim \frac{1}{|\epsilon|^{4/3}} \\ \langle(\Delta\eta)^2\rangle &\sim \frac{1}{|\epsilon|^{4/3}}. \end{aligned} \tag{4.26}$$

TABLE I. Summary of the critical exponents obtained in this work.

	van der Waals's system	RNAdS black hole
Correspondence of thermodynamical variables	(P,V,T)	(Φ,Q,T)
Main critical exponents		
α	0	2/3
β	1/2	1/3
γ	1	2/3
δ	3	3
Scaling symmetry	For Gibbs energy $G(\Lambda^{1/2}\epsilon,\Lambda^{3/4}\Pi)$ $=\Lambda G(\epsilon,\Pi)$	For free energy $F(\Lambda^{3/4}\epsilon,\Lambda^{3/4}\Pi)$ $=\Lambda F(\epsilon,\Pi)$
Order parameter	$\Delta\rho=\rho_l-\rho_g$ ρ_l, density of liquid ρ_g, density of gas	$\Delta\Phi=\Phi_a-\Phi_b$
Phase structure		
$T>T_c$	Only gas state exists	Two thermally stable states separated by an unstable phase (see Fig. 5)
$T<T_c$	Possibility of the existence of both liquid and gas states	Only thermally stable phase exists

We see from Eq. (4.26) that, unlike that in the case of the critical points at r_1 and r_2, the mean square fluctuation of the order parameter also exhibits anomalous behavior. Further, the rate at which $\langle(\Delta M)^2\rangle$ or $\langle(\Delta S)^2\rangle$ becomes divergent near r_c is faster than that near r_1 or r_2.

F. RNAdS black hole and van der Waals liquid gas system: A comparsion

We first summarize the similiarities and differences between the thermodynamic behavior of a RNAdS black hole and a van der Waals's liquid gas system in Table I.

From Table I, we see that, despite certain qualitative similiar features exhibited by both a RNAdS black hole and a van der Waals liquid gas system in the corresponding phase diagrams, the critical behavior of a RNAdS black hole is essentially different from that of a van der Waals liquid gas system.

In the first place, the temperature of the subcritical isotherms of a RNAdS black hole is above T_c, while that of a van der Waals liquid gas system is below the critical temperature. Moreover, from a thermodynamic perspective, the formal correspondence $(\Phi,Q)\leftrightarrow(V,P)$ is not very natural because instead of Φ, Q now plays the very odd role of a chemical potential in the analogy. If we write down the differential form of the Gibbs energy of a van der Waals liquid gas system,

$$dG=TdS+VdP,$$

and the differential form of the free energy of a RNAdS black hole,

$$dF=TdS+\Phi dQ,$$

then we see that the unnatural correspondence arises because we try to compare the Gibbs energy of a van der Waals liquid gas system (pertained to a grand canonical ensemble) with the free energy of a RNAdS black hole (pertained to a canonical ensemble). So the physical meaning of this thermdynamic analogy remains very obscure at the present stage. From the numerical value of the critical exponents, it may also be seen that a RNAdS black hole and a van der Waals liquid gas system are actually not in the same universality class as far as critical behavior is concerned.

In view of these disparities, are we going to dismiss the thermodynamic analogy between a RNAdS black hole and a van der Waals liquid gas system altogether? Perhaps we should not, at least not in a hasty way. As may be seen from the preceding subsection, the analogy serves as a very useful guide in our study of the critical thermodynamic behavior of a RNAdS black hole. The choice of the order parameter as well as the appropiate thermodynamic variables to study along the critical isotherm (or the critical isocharge curve) are all suggested by the analogy. In the future, if we try to probe deeper into the microscopic structure of black holes, it is conceivable that we may also learn something useful from the statistical mechanics of a van der Waals liquid gas system. At the same time, the quantitative differences spelled out in this work (for instance the critical exponents computed) will serve as a guide in looking for an appropiate statistical model which deviates from that of a van der Waals liquid gas system.

V. DISCUSSIONS

In the present work, we have presented some results which may be regarded as a preliminary step to gain a better understanding of the multi-critical phenomena of RNAdS black holes at the thermodynamical level. Many questions remain to be addressed. Perhaps the most straighforward one is whether it is possible to generalize the present work to the Kerr Newman AdS black holes. Preliminary calculations indicate that modulo a constant dependent on charge as well as angular momentum, the numerical value of the critical exponents calculated for a RNAdS black hole may be carried over to the more general context of a Kerr Newman AdS black hole. The generalization to higher dimensional AdS black holes also does not seem to present any serious obstacles.

More interesting problems come up when we try to understand the critical behavior of a RNAdS black hole in the context of the AdS-CFT duality. For instance, it is worth trying to understand the phase transition from the dual CFT picture [12]. Further, we may also ask whether it is possible to exploit the AdS-CFT duality in order to gain better insight into the phase structure of a RNAdS black hole. For many statistical systems which undergo second order phase transition, notably the 2D Ising model, the scaling relation satisfied by the free energy near criticality is a familiar characteristic which follows from the renormalization group analysis near the infrared fixed point. So it is a natural to ask if there is a renormalization group scheme underlying the critical phenomena of black holes from which the scaling relation follows. The problem is worth looking at not only from the viewpoint of black hole physics. Hopefully, it will also contribute to our understanding of the behavior of a gravitational field at different energy scales when general relativity is looked on as an effective field theory. This question is certainly more tractable in the dual CFT picture as renormalization is much better understood in that context. Of course, the most challenging question will be how we may use the knowledge of the phase structure and possibly the van der Waals analogy to build up the statistical mechanics of black holes, as discussed in the previous section. Much work remains to be done to address these open questions.

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[1] P.C.W. Davis, Proc. R. Soc. London **A353**, 499 (1977); P. Hut, Mon. Not. R. Astron. Soc. **180**, 379 (1997).
[2] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000).
[3] A. Chamblin, R. Emparan, C.V. Johnson, and R.C. Myers, Phys. Rev. D **60**, 104026 (1999).
[4] S.W. Hawking and D.N. Page, Commun. Math. Phys. **81**, 577 (1983).
[5] Y.K. Lau, Phys. Lett. A **186**, 41 (1994); **188**, 245 (1994).
[6] Z.X. Huang and Y.K. Lau, Phys. Lett. A **259**, 1 (1999); erratum (to be published).
[7] J. Louko and S.N. Winters-Hilt, Phys. Rev. D **54**, 2647 (1996).
[8] L.D. Landau and E.M. Lifshitz, *Statistical Physics* (Pergamon, New York, 1980), Vol. 1.
[9] H. E. Stanley, *Introduction to Phase Transitions and Critical Phemonena* (Oxford University Press, New York, 1987).
[10] C.O. Lousto, Nucl. Phys. **B410**, 155 (1993).
[11] R. E. Reichl, *A Modern Course in Statistical Physics* (University of Texas Press, Austin, TX, 1980).
[12] E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998).