Post-Newtonian gravitational radiation and equations of motion via direct integration of the relaxed Einstein equations: Foundations

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We present a self-contained framework called direct integration of the relaxed Einstein equations for calculating equations of motion and gravitational radiation emission for isolated gravitating systems based on the post-Newtonian approximation. We cast the Einstein equations into their ''relaxed'' form of a flat-spacetime wave equation together with a harmonic gauge condition, and solve the equations formally as a retarded integral over the past null cone of the field point (chosen to be within the near zone when calculating equations of motion and in the far zone when calculating gravitational radiation). The ''inner'' part of this integral (within a sphere of radius $\mathcal{R}\sim$ one gravitational wavelength) is approximated in a slow-motion expansion using standard techniques; the ''outer'' part, extending over the radiation zone, is evaluated using a null integration variable. We show generally and explicitly that all contributions to the inner integrals that depend on $\mathcal{R}$ cancel corresponding terms from the outer integrals, and that the outer integrals converge at infinity, subject only to reasonable assumptions about the past behavior of the source. The method cures defects that plagued previous ''brute-force'' slow-motion approaches to motion and gravitational radiation for isolated systems. We detail the procedure for iterating the solutions in a weak-field, slow-motion approximation, and derive expressions for the near-zone field through 3.5 post-Newtonian order in terms of Poisson-like potentials.

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I. INTRODUCTION

The motion of multiple, isolated bodies under their mutual gravitational attraction and the resulting emission of gravitational radiation is a long-standing problem that dates back to the first years following the publication of general relativity (GR). It has at times been controversial (for a thorough review see [1]). In 1916–1918 Einstein calculated the gravitational radiation emitted by a laboratory-scale object using the linearized version of GR [2]. Some of his assumptions were questionable and his answer for the energy flux was off by a factor of 2 (an error pointed out by Eddington [3]). In 1916, de Sitter [4] derived *N*-body equations of motion in what later would be termed the post-Newtonian (PN) approximation. However, his equations contained an error that was discovered in the course of a disputed claim by Levi-Cività [5] that the center of mass of a binary star system would suffer a ''self-acceleration.'' Eddington and Clark [6] corrected the error, and found no self-acceleration. Einstein, Infeld and Hoffman (EIH) [7] attempted to demonstrate explicitly that the Einstein equations alone imply equations of motion, by matching solutions of the vacuum equations, expanded in a weak-field, slow motion approximation, to fields representing the near-zone fields of ''point'' masses, working to first PN order. The result was the well-known EIH *N*-body equations of motion. Other highlights in this early history of the problem of motion include the development of the post-Newtonian approximation for fluid sytems by Fock [8] and Chandrasekhar [9], its extension by Chandrasekhar and later workers to 2.5PN order [10,11], and the development of equations of motion for spinning bodies by Papapetrou [12].

Gravitational theory presents one problem essentially identical to that of electromagnetic theory: how to mesh the natural solution of the field equations in the near zone where the bodies reside, which involves slow-motion expansions and instantaneous fields, with the solution in the far zone, which involves retarded fields. Such a meshing is needed if one is to calculate the effects of the gravitational radiation reaction that results from the emission of energy and angular momentum to infinity. One approach to resolving this problem was that of matched asymptotic expansions. Although well rooted in applied mathematics, it was first expounded in 1971 as a powerful technique for electromagnetic and gravitational problems by Burke [13]. Another, related approach is the ''post-Minkowskian'' framework, elaborated and developed most fully by Blanchet and Damour and their collaborators [14–19].

A second important problem of gravitation, which distinguishes it from electromagnetism, is the non-linearity of Einstein's equations. Gravitation itself acts as a source of gravitation. Consequently this source extends over all space, resulting in the possibility of divergent or ill-defined integrals. In many ways, this has been the most serious difficulty to overcome. Techniques for resolving it have ranged from sweeping the difficulties under the rug, to the sophisticated analytic regularization methods of the post-Minkowskian program. A central thrust of this paper is to present a straightforward method for resolving this difficulty.

A third ''problem,'' which is less a problem for gravitation than it is for electromagnetism, is that of ''point'' sources. In electromagnetic theory, where there is a belief that fundamental charges like the electron are pointlike, the singular nature of the fields at the source has led to problems of mass regularization, especially in deriving equations of electromagnetic radiation reaction; it also raises issues of the boundary between classical and quantum electrodynamics. In gravitation theory, this is less of an issue of principle, because the primary interest is in the motion of and radiation by astrophysical systems, whose members are clearly not point masses. Instead, the use of ''point,'' i.e., delta-function, sources is meant as an efficient means of approximating the mass distribution of bodies that are nearly spherical and that are small compared to the typical separation between them, so that tidal effects, which depend on the finite size of the bodies, can be ignored. Here the issue is how to make use of a point mass approximation (which simplifies many calculations) in a way that captures all the physics without introducing spurious effects.

A fourth problem is of a technical nature: in electromagnetic theory, radiation damping in the equations of motion occurs at order $(v/c)^3$ beyond the simple Coulomb forces between charges, and is relatively easy to compute in a systematic approximation method, modulo the other problems listed above. By contrast, gravitational radiation damping occurs at order $(v/c)^5$ beyond Newtonian gravity, and requires a higher order of approximation that captures all relevant contributions. Over the years, numerous inequivalent results have been quoted for the leading gravitational radiation reaction effects. One finds published papers in which the coefficient in the relevant formula has ranged from $-21/16$ to the correct coefficient of unity; a study by Walker and Will [20] showed that the divergent results were all the simple consequence of missing one or more terms that contribute to the final answer.

These four ''problems'' were the origin of the so-called ''quadrupole controversy,'' which arose from a critique by Ehlers and colleagues [21] of the foundations of the quadrupole formula for the leading-order gravitational radiation energy flux and orbital damping. This critique had the beneficial effect of spurring new research on those foundations, including a study of the systematic structure of the approximation sequence of Einstein's equations in a slow-motion, weak-field approach; analysis of energy balance as an argument for connecting the far-zone energy flux to the near-zone damping forces and elaboration of the post-Minkowskian approach, among others (see [1] for a review). The work inspired by the critique of Ehlers *et al.* served to confirm the quadrupole formula and to strengthen its foundations. The ultimate test, of course, came in 1979 with the announcement of the measurement of orbital damping of the binary pulsar PSR 1913+16 in agreement with the quadrupole formula [22]; current results agree to better than 0.5% [23].

The problem of motion and radiation has received renewed interest since 1990, with the proposal for the Laser Interferometric Gravitational Wave Observatory (LIGO) in the U.S. (and similar observatories abroad), and the realization that a leading candidate source of detectable waves would be the radiation-reaction driven inspiral of a binary system of compact objects (neutron stars or black holes) [24]. Furthermore, it was noted [25] that the leading method for data analysis of signals from such systems, optimal matched filtering, would require theoretical template waveforms that are accurate (primarily in the evolution of the orbital frequency or phase) well beyond the leading-order prediction of the quadrupole formula, possibly as high as corrections of order $(v/c)^6$.

This presented a major theoretical challenge: to calculate the motion and radiation to very high PN order, a formidable algebraic task, while addressing each of the problems listed above sufficiently well to ensure that the results were physically meaningful. This challenge was taken up by three groups of workers.

One group, headed by Blanchet, Damour and Iyer [14–19], used the post-Minkowskian (PM) approach to derive the gravitational waveform, equations of motion and energy flux explicitly to 2PN order $[O(v/c)^4]$ and beyond. The idea is to solve the vacuum Einstein equations in the radiation zone in an expansion in powers of Newton's constant G and to express the asymptotic solutions in terms of a set of formal, time-dependent, symmetric and trace-free (STF) multipole moments [26]. Then, in a near zone within one characteristic wavelength of the radiation, the equations including the material source are solved in a slow-motion approximation (expansion in powers of $1/c$) that yields both equations of motion for the source bodies, as well as a set of STF source multipole moments expressed as integrals over the ''effective'' source, including both matter and gravitational field contributions. The solutions involving the two sets of moments are then matched in an intermediate overlap zone, resulting in a connection between the formal radiative moments and the source moments. The matching also provides a natural way, using analytic continuation, to regularize integrals involving the non-compact contributions of gravitational stress-energy, which might otherwise be divergent.

The second group of Will, Wiseman and Pati use the approach described in the present paper, direct integration of the relaxed Einstein equations (DIRE), which builds upon earlier work by Epstein, Wagoner, Will and Wiseman [27–32]. Like the PM approach, it involves rewriting the Einstein equations in their ''relaxed'' form, namely as an inhomogeneous, flat-spacetime wave equation for a field $h^{\alpha\beta}$, whose source consists of both the material stress-energy and a ''gravitational stress-energy'' made up of all the terms nonlinear in $h^{\alpha\beta}$. The wave equation is accompanied by a harmonic or deDonder gauge condition on $h^{\alpha\beta}$, which serves to specify a coordinate system and also imposes equations of motion on the sources. Unlike the post-Minkowskian approach, a *single* formal solution is written down, valid everywhere in spacetime. This formal solution, based on the flat-spacetime retarded Green function, is a retarded integral equation for $h^{\alpha\beta}$, which is then iterated in a slow-motion $(v/c<1)$, weak-field $(||h^{\alpha\beta}||<1)$ approximation that is very similar to the corresponding procedure in electromagnetism. However, because the integrand of this retarded integral is not compact by virtue of the non-linear field contributions, one quickly runs up against integrals that are not well defined or, worse, are divergent. Although at the lowest quadrupole and first PN order various arguments were given to justify sweeping such problems under the rug [27,28], they were not very rigorous, and provided no guarantee that the divergences would not become insurmountable at higher PN orders. Indeed it is straightforward to demonstrate that at second post-Newtonian (2PN) order, the rug is indeed pulled out from under such arguments.

DIRE resolves these problems. The solution of the relaxed Einstein equation is a retarded integral over the past null cone of the field point. The part of the integral that extends over the intersection between the past null cone and the material source and the near zone is approximated by a slow-motion expansion involving spatial integrals of moments of the source, including the non-compact gravitational contributions, just as in the post-Minkowskian and Epstein-Wagoner frameworks. But instead of extending the spatial integrals to infinity as was implicit in earlier procedures, we terminate the integrals at the boundary of the near zone, chosen to be at a radius $\mathcal{R}$ given roughly by one wavelength of the gravitational radiation. For the integral over the rest of the past null cone exterior to the near zone (''radiation zone''), we use a change of integration variables to convert the integral into a convenient, easy-to-calculate form that is manifestly convergent, subject only to reasonable assumptions about the past behavior of the source, which fully accounts for the retardation of the fields comprising the source stress-energy and which does not involve an explicit slow-motion expansion. This transformation was suggested by our earlier work on a non-linear gravitational-wave phenomenon called the Christodoulou memory [30] (it is also implicit in Appendix D of [14]). Not only are all integrations now explicitly finite and convergent, we can show explicitly that all contributions from the near-zone spatial integrals that depend upon the radius $\mathcal{R}$ are actually *canceled* by corresponding terms from the radiation-zone integrals, for all powers of $\mathcal{R}$ (including $\ln \mathcal{R}$) and for any order in the PN expansion. Thus the procedure, as expected, has no dependence on the arbitrarily chosen boundary radius $\mathcal{R}$ of the near zone, and provides a simple practical method for regularizing integrals over non-compact sources.

The ultimate products of this work will consist of equations of motion, gravitational waveforms, and energy flux expressions, in reasonably ready-to-use forms. The equations of motion for a binary system will have the schematic form

$$d^2\mathbf{x}/dt^2 = -(Gm\mathbf{x}/r^3)[1+O(\epsilon)+O(\epsilon^2) + O(\epsilon^{5/2})+O(\epsilon^3)+O(\epsilon^{7/2})+\cdots], \quad (1.1)$$

where m is the total mass of the binary system, $\mathbf{x}=\mathbf{x}_1-\mathbf{x}_2$ is the separation vector and $r=|\mathbf{x}|$. The expansion parameter ϵ is related to the orbital variables by $\epsilon \sim Gm/rc^2 \sim (v/c)^2$, where v is the relative velocity. The leading term is Newtonian gravity. The next term $O(\epsilon)$ is the first post-Newtonian correction, which gives rise to such effects as the advance of the periastron. The terms of $O(\epsilon^2)$ and $O(\epsilon^3)$ are non-dissipative 2PN and 3PN corrections. The $O(\epsilon^{5/2})$ and $O(\epsilon^{7/2})$ terms are the leading 2.5PN and post-Newtonian corrected 3.5PN gravitational radiation-reaction terms. (We do not include in this discussion contributions from spin, whose ordering in the PN hierarchy for compact bodies follows a special convention.) Explicit formulas for terms through various orders have been calculated by various authors: non-radiative terms through 2PN order [1,33–36], radiation reaction terms at 2.5PN and 3.5PN order [37–39], and non-radiatve 3PN terms [40–43].

In order to derive equations of motion to the 3.5PN order shown, one must derive the near-zone metric $g_{\alpha\beta}$ as a function of spacetime and a functional of the source variables to 3.5PN order, which implies the following specific PN orders: g_{00} through $O(\epsilon^{9/2})$, g_{0i} through $O(\epsilon^4)$, g_{ij} through $O(\epsilon^{7/2})$. In this paper we provide the required expressions in the form of (a) Poisson-like integrals of source densities, $\int_{\mathcal{M}} f(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^p d^3x'$, where $f(t,\mathbf{x}')$ could be proportional to source stress-energy densities, and thus have compact support, or could be a function of other potentials, and thus extend over the entire near-zone region of integration $\mathcal{M}$, and (b) expressions involving time derivatives of source multipole moments $M^{ijk\cdots}$ contracted with spatial vectors $x^i x^j x^k \ldots$. These expressions can be simplified, iterated, and evaluated more explicitly, depending on the application envisioned (''point'' mass binary system, spinning masses, perfect fluid distributions, etc.).

The second product will be expressions for the gravitational waveform, given schematically by

$$h^{ij} = \frac{G\mu}{Rc^4}\{v^2[1+O(\epsilon^{1/2})+O(\epsilon)+O(\epsilon^{3/2}) + O(\epsilon^2)+O(\epsilon^{5/2})+O(\epsilon^3)\ \ldots]\}_{TT}, \quad (1.2)$$

where μ is the reduced mass, and the subscript TT denotes the ''transverse-traceless'' part. The leading contribution $G\mu v^2/Rc^4 \sim G\ddot{I}^{ij}/Rc^4$ is the standard quadrupole formula. Explicit formulas for all terms through 2.5PN order have been derived by various authors [28,29,32,44–48].

From the waveform, one can also derive expressions for fluxes of energy, angular momentum and linear momentum; the energy flux can be written in the schematic form

$$dE/dt = (dE/dt)_Q[1+O(\epsilon)+O(\epsilon^{3/2})+O(\epsilon^2)+O(\epsilon^{5/2}) + O(\epsilon^3)+\cdots], \quad (1.3)$$

where $(dE/dt)_Q$ denotes the lowest-order quadrupole contribution.

A third approach focuses on the limit in which one body is much less massive than the other, and employs black-hole perturbation theory to derive the gravitational waveform and energy flux, for particles orbiting both rotating and non-rotating holes. This method yields both numerically accurate results as well as analytic PN expansions up to orders as high as $(v/c)^{11}$ [44,49–53]. Work is currently in progress to extend these methods beyond the test-mass approximation, in an effort to compute corrections to first order in μ/M, the ratio of the mass of the particle to that of the black hole [54–56].

This is the first in a series of papers that will treat the problem of motion and gravitational radiation systematically using the DIRE approach. This paper lays out the foundations of the method and derives formal solutions to the near-zone fields through 3.5PN order [order $(v/c)^7$ beyond Newtonian gravity] in a form useful for future applications. Subsequent papers in the series will derive the explicit equations of motion and near-zone gravitational fields for binary systems of compact objects through 2PN order, and deal with radiation reaction at 2.5PN and 3.5PN order.

Our conventions and notation generally follow those of [57,26]. Henceforth we use units in which $G=c=1$. Greek indices run over four spacetime values 0, 1, 2, 3, while Latin indices run over three spatial values 1, 2, 3; commas denote partial derivatives with respect to a chosen coordinate system, while semicolons denote covariant derivatives; repeated indices are summed over; $\eta^{\mu\nu}=\eta_{\mu\nu}=\mathrm{diag}(-1,1,1,1)$; $g \equiv \det(g_{\mu\nu})$; $a^{(ij)}\equiv(a^{ij}+a^{ji})/2$; $a^{[ij]}\equiv(a^{ij}-a^{ji})/2$; ϵ^{ijk} is the totally antisymmetric Levi-Civita symbol ($\epsilon^{123}=+1$). We use a multi-index notation for products of vector components and partial derivatives, and for multiple spatial indices: $x^{ij\cdots k}\equiv x^i x^j \ldots x^k$, $\partial_{ij\ldots k}\equiv\partial_i\partial_j \ldots \partial_k$, with a capital letter superscript denoting an abstract product of that dimensionality: $x^Q\equiv x^{i_1}x^{i_2}\ldots x^{i_q}$ and $\partial_Q\equiv\partial_{i_1}\partial_{i_2}\ldots\partial_{i_q}$. Also, for a tensor of rank Q, $f^Q\equiv f^{i_1 i_2\cdots i_q}$. Angular brackets around indices denote STF combinations (see Appendix A for definitions). Spatial indices are freely raised and lowered with δ^{ij} and δ_{ij}.

II. FOUNDATIONS OF DIRE

A. Relaxed Einstein equations

We begin by reviewing the method for recasting the Einstein equations

$$R^{\alpha\beta}-\frac{1}{2}g^{\alpha\beta}R=8\pi T^{\alpha\beta} \tag{2.1}$$

into their ''relaxed'' form. Here $R^{\alpha\beta}$ and R are the Ricci tensor and scalar, respectively, $g^{\alpha\beta}$ is the spacetime metric and $T^{\alpha\beta}$ is the stress-energy tensor of the matter. We define the potential

$$h^{\alpha\beta}\equiv\eta^{\alpha\beta}-(-g)^{1/2}g^{\alpha\beta} \tag{2.2}$$

(see e.g. [26]) and choose a particular coordinate system defined by the deDonder or harmonic gauge condition

$$h^{\alpha\beta}{}_{,\beta}=0. \tag{2.3}$$

With these definitions the Einstein equations (2.1) take the form

$$\Box h^{\alpha\beta}=-16\pi\tau^{\alpha\beta}, \tag{2.4}$$

where $\Box\equiv-\partial^2/\partial t^2+\nabla^2$ is the flat-spacetime wave operator. The source on the right-hand side is given by the ''effective'' stress-energy pseudotensor

$$\tau^{\alpha\beta}=(-g)T^{\alpha\beta}+(16\pi)^{-1}\Lambda^{\alpha\beta}, \tag{2.5}$$

where $\Lambda^{\alpha\beta}$ is the non-linear ''field'' contribution given by

$$\Lambda^{\alpha\beta}=16\pi(-g)t^{\alpha\beta}_{LL}+(h^{\alpha\mu}{}_{,\nu}h^{\beta\nu}{}_{,\mu}-h^{\alpha\beta}{}_{,\mu\nu}h^{\mu\nu}), \tag{2.6}$$

and $t^{\alpha\beta}_{LL}$ is the ''Landau-Lifshitz'' pseudotensor given by

$$\begin{aligned}16\pi(-g)t^{\alpha\beta}_{LL}\equiv{}& g_{\lambda\mu}g^{\nu\rho}h^{\alpha\lambda}{}_{,\nu}h^{\beta\mu}{}_{,\rho}\\&+\frac{1}{2}g_{\lambda\mu}g^{\alpha\beta}h^{\lambda\nu}{}_{,\rho}h^{\rho\mu}{}_{,\nu}-2g_{\mu\nu}g^{\lambda(\alpha}h^{\beta)\nu}{}_{,\rho}h^{\rho\mu}{}_{,\lambda}\\&+\frac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu}-g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau}\\&-g_{\rho\sigma}g_{\nu\tau})h^{\nu\tau}{}_{,\lambda}h^{\rho\sigma}{}_{,\mu}.\end{aligned} \tag{2.7}$$

By virtue of the gauge condition (2.3), this source term satisfies the conservation law

$$\tau^{\alpha\beta}{}_{,\beta}=0, \tag{2.8}$$

which is equivalent to the equation of motion of the matter

$$T^{\alpha\beta}{}_{;\beta}=0. \tag{2.9}$$

Equation (2.4) is exact, and relies only on the assumption that spacetime can be covered by harmonic coordinates. It is called ''relaxed'' because it can be solved formally as a functional of source variables without specifying the motion of the source. Then, the harmonic gauge condition, Eq. (2.3), or the equations of motion, are imposed to determine the metric as a function of spacetime.

Notice that the ''source'' in Eq. (2.4) contains a gravitational part that depends explicitly on $h^{\alpha\beta}$, the very quantity for which we are trying to solve. Also, we can expect $\tau^{\alpha\beta}$, which depends on the fields $h^{\alpha\beta}$, to have infinite spatial extent. Indeed the very outgoing radiation that we hope to calculate will, at some level of approximation, serve as a contribution to the source, thus generating an additional component of the radiation.

Another complication in Eq. (2.4) is that the second derivative term $h^{\alpha\beta}{}_{,\mu\nu}h^{\mu\nu}$ in the source really ''belongs'' on the left-hand side with the other second derivative terms in the wave operator. This term modifies the propagation characteristics of the field from the flat-spacetime characteristics represented by the d'Alembertian operator to those of the true null cones of the curved spacetime around the source, which deviate from the flat null cones of the harmonic coordinates. Nevertheless, the DIRE technique automatically recovers the leading manifestations of this effect, commonly known as ''tails.''

The material will be modeled as perfect fluid, having stress-energy tensor

$$T^{\alpha\beta}\equiv(\rho+p)u^\alpha u^\beta+pg^{\alpha\beta}, \tag{2.10}$$

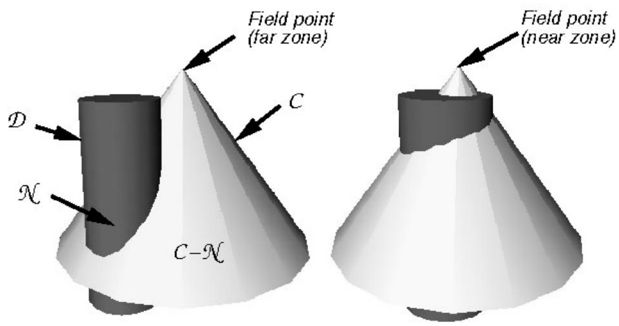

FIG. 1. Past harmonic null cone $\mathcal{C}$ of the field point intersects the near zone $\mathcal{D}$ in the hypersurface $\mathcal{N}$. Left: field point in the far zone. Right: field point in the near zone. Inner integrals are over the hypersurface $\mathcal{N}$, and outer integrals are over the remainder $\mathcal{C}-\mathcal{N}$ of the null cone.

where ρ and p are the locally measured energy density and pressure, respectively, and u^α is the four-velocity of an element of fluid. Until we begin to apply our results to specific physical situations, such as compact binary stars, we will have no need to specialize $T^{\alpha\beta}$ further.

B. Near zone and radiation zone

We consider the material source to consist of a bound system of characteristic size $\mathcal{S}$, with a suitably defined center of mass chosen to be at the origin of coordinates, $\mathbf{X}=0$. The *source zone* then consists of the world tube $\mathcal{T}=\{x^\alpha | r<\mathcal{S}, -\infty<t<\infty\}$. Outside $\mathcal{T}$, $T^{\alpha\beta}=0$.

The fluid is assumed to move with characteristic velocity $v \ll 1$. The characteristic reduced wavelength of gravitational radiation, $\lambda\!\!\!^{-}=\lambda/2\pi \sim \mathcal{S}/v \equiv \mathcal{R}$ serves to define the boundary of the *near zone*, defined to be the world tube $\mathcal{D}=\{x^\alpha | r<\mathcal{R}, -\infty<t<\infty\}$. Within the near zone, the gravitational fields can be treated as almost instantaneous functions of the source variables; i.e., retardation can be ignored or treated as a small perturbation of instantaneous solutions. For physical situations of interest, up to the point where the post-Newtonian approximation breaks down, $\mathcal{R} \gg \mathcal{S}$. The region exterior to the near zone is the *radiation zone*, $r>\mathcal{R}$.

The formal ''solution'' to Eq. (2.4) with an outgoing wave boundary condition can be written down in terms of the retarded, flat-space Green function

$$h^{\alpha\beta}(t,\mathbf{x})=4\int \frac{\tau^{\alpha\beta}(t',\mathbf{x}')\,\delta(t'-t+|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}d^4x', \tag{2.11}$$

but is really just a conversion of the differential equation (2.4) to an integral equation. It represents an integration of $\tau^{\alpha\beta}/|\mathbf{x}-\mathbf{x}'|$ over the past harmonic null cone $\mathcal{C}$ emanating from the field point $(t,\mathbf{x})$ (see Fig. 1). This past null cone intersects the world tube $\mathcal{D}$ enclosing the near zone at the three-dimensional hypersurface $\mathcal{N}$. Thus the integral of Eq. (2.11) consists of two pieces, an integration over the hypersurface $\mathcal{N}$ and an integration over the rest of the past null cone $\mathcal{C}-\mathcal{N}$. Each of these integrations will be treated differently. We will also treat differently the two cases in which (a) the field point is outside the near zone and (b) the field point is within the near zone (Fig. 1). The former case will be relevant for calculating the gravitational-wave signal, while the latter will be important for calculating field contributions to $\tau^{\alpha\beta}$ that must be integrated over the near zone, as well as for calculating fields that enter the equations of motion for the source.

C. Radiation-zone field point, inner integration

For a field point in the radiation zone and integration over the near zone (inner integral), we first carry out the t' integration in Eq. (2.11), to obtain

$$h^{\alpha\beta}_{\mathcal{N}}(t,\mathbf{x})=4\int_{\mathcal{N}} \frac{\tau^{\alpha\beta}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'. \tag{2.12}$$

Within the near zone, the spatial integration variable $\mathbf{x}'$ satisfies $|\mathbf{x}'|\le\mathcal{R}<r$, where the distance to the field point $r=|\mathbf{x}|$. Expanding the x' dependence in both occurrences of $|\mathbf{x}-\mathbf{x}'|$ in the integrand in powers of $|\mathbf{x}'|/r$, it is straightforward to show that

$$h^{\alpha\beta}_{\mathcal{N}}(t,\mathbf{x})=4\sum_{q=0}^{\infty}\frac{(-1)^q}{q!}\partial_Q\left(\frac{1}{r}M^{\alpha\beta Q}(u)\right), \tag{2.13}$$

where

$$M^{\alpha\beta Q}(u)\equiv\int_{\mathcal{M}}\tau^{\alpha\beta}(u,\mathbf{x}')x'^Q d^3x'. \tag{2.14}$$

In Eqs. (2.13) and (2.14), the index Q is a multi-index, such that $\partial_Q \equiv \partial_{i_1}\partial_{i_2}\ldots\partial_{i_q}$, and the superscript Q in $M^{\alpha\beta Q}$ denotes $i_1 i_2 \ldots i_q$, with summation over repeated indices assumed. The integrations in Eq. (2.14) are now over the hypersurface $\mathcal{M}$, which is the intersection of the near-zone world tube with the constant-time hypersurface $t_{\mathcal{M}}=u=t-r$. Roughly speaking, each term in the Taylor series is smaller than its predecessor by a factor of order $v\ll 1$, provided we restrict our attention to slow-motion sources.

Note that the field and source variables appearing in the integrand $\tau^{\alpha\beta}$ are evaluated at the single retarded time u; however, because the field contributions to $\tau^{\alpha\beta}$ fall off as some power of r, one can expect to encounter integrals that depend on positive powers of the radius $\mathcal{R}$ of the boundary of integration, especially in some of the higher-order moments. If this boundary were to be formally taken to ∞ (as has been the conventional approach in the past), these integrals would diverge. Instead we shall demonstrate (Sec. II I and Appendix B) that such $\mathcal{R}$-dependent effects are *precisely* canceled by contributions from the ''outer'' integral.

For the gravitational-wave signal, we need only to focus on the spatial components of $h^{\alpha\beta}$ and on the leading component in $1/R$, where R is the distance to the detector. Using the fact that $u_{,i}=-\hat{N}^i$, where $\hat{\mathbf{N}}\equiv\mathbf{x}/R$ denotes the observation direction, we obtain

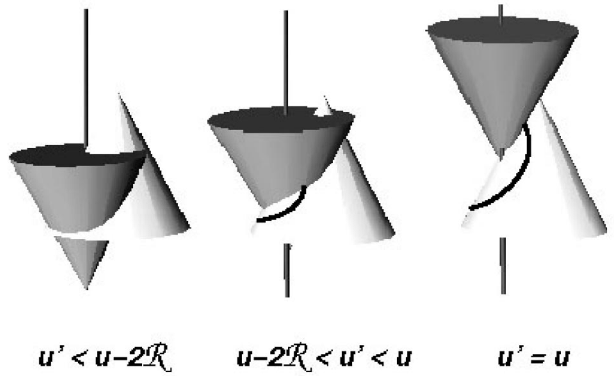

FIG. 2. Change of variables for the outer integrals. The vertical line represents the material source world line. The variable u' is constant on the two-dimensional intersection between the past null cone of the field point and a future null cone from the center of mass of the system. Left: $u'<u-2\mathcal{R}$, the two cones intersect fully outside the near zone, so the angular integrations are complete. Middle: $u-2\mathcal{R}<u'<u$, angular integration terminates where the intersection between the two cones meets the boundary of the near zone. Right: $u'=u$, the upper limit of integration; the two cones are tangent to one another.

$$h^{ij}_{\mathcal{N}}(t,\mathbf{x})=\frac{4}{R}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m}{\partial t^m}\int_{\mathcal{M}}\tau^{ij}(u,\mathbf{x}')(\hat{\mathbf{N}}\cdot\mathbf{x}')^m d^3x' + O(R^{-2}). \tag{2.15}$$

D. Radiation-zone field point, outer integration

By making a change of integration variable from (r',θ',ϕ') to (u',θ',ϕ'), where

$$t-u'=r'+|\mathbf{x}-\mathbf{x}'|, \tag{2.16}$$

we can write the integral over the rest of the past null cone $\mathcal{C}-\mathcal{N}$ in the form

$$h^{\alpha\beta}_{\mathcal{C}-\mathcal{N}}(t,\mathbf{x})=4\int_{-\infty}^{u}du'\oint_{\mathcal{C}-\mathcal{N}}\frac{\tau^{\alpha\beta}(u'+r',\mathbf{x}')}{t-u'-\hat{\mathbf{n}}'\cdot\mathbf{x}}\times[r'(u',\Omega')]^2 d^2\Omega', \tag{2.17}$$

where, from Eq. (2.16),

$$r'(u',\Omega')=[(t-u')^2-r^2]/[2(t-u'-\hat{\mathbf{n}}'\cdot\mathbf{x})]. \tag{2.18}$$

This change of variables represents an integration first over the two-dimensional intersection of the past null cone $\mathcal{C}$ with the future null cone $t'=u'+r'$ emanating from the center of mass of the system at $t_{\mathrm{c.m.}}=u'$ (Fig. 2), followed by the u' integration over all such future-directed cones, starting from the infinite past and terminating in the cone emanating from the center of mass at time u, which is tangent to the past null cone of the observation point.

For explicit calculations, it is useful to choose the field point $\mathbf{x}$ to be in the z direction, so that $\hat{\mathbf{n}}'\cdot\mathbf{x}=r\cos\theta'$, and to write the outer integral in the form

$$\begin{aligned}h^{\alpha\beta}_{\mathcal{C}-\mathcal{N}}(t,\mathbf{x})=&4\int_{u-2\mathcal{R}}^{u}du'\int_0^{2\pi}d\phi'\int_{1-\alpha}^{1}\frac{\tau^{\alpha\beta}(u'+r',\mathbf{x}')}{t-u'-\hat{\mathbf{n}}'\cdot\mathbf{x}}\times[r'(u',\Omega')]^2 d\cos\theta'\\&+4\int_{-\infty}^{u-2\mathcal{R}}du'\oint\frac{\tau^{\alpha\beta}(u'+r',\mathbf{x}')}{t-u'-\hat{\mathbf{n}}'\cdot\mathbf{x}}\times[r'(u',\Omega')]^2 d^2\Omega',\end{aligned} \tag{2.19}$$

where

$$\alpha(u')=(u-u')(2r-2\mathcal{R}+u-u')/2r\mathcal{R}. \tag{2.20}$$

The incomplete angular integration in the first integral of Eq. (2.19) reflects the fact that for $u\geqslant u'\geqslant u-2\mathcal{R}$, the two-dimensional intersections meet the boundary of the near zone. For $u'<u-2\mathcal{R}$, the angular integration covers the full 4π. Note that $\tau^{\alpha\beta}$ contains only field contributions evaluated in the radiation zone; because they are themselves retarded, the ''time dependence'' $u'+r'=t-|\mathbf{x}-\mathbf{x}'|$ is approximately constant over each angular integration, since it follows the hypersurface $t-|\mathbf{x}|=u=$const, and the dominant contribution to the fields comes from $|\mathbf{x}'|<\mathcal{R}$. This allows a kind of slow-motion, multipole expansion to be exploited in evaluating these integrals, despite their range well outside the near zone.

E. Near-zone field point, inner integration

In this case, in Eq. (2.11), both $\mathbf{x}$ and $\mathbf{x}'$ are within the near zone, and hence $|\mathbf{x}-\mathbf{x}'|\leqslant 2\mathcal{R}$. Consequently, the variation in retarded time can be treated as a small perturbation, since $\tau^{\alpha\beta}$ varies on a time scale $\sim\mathcal{R}$. We therefore expand the retardation in powers of $|\mathbf{x}-\mathbf{x}'|$, to obtain

$$h^{\alpha\beta}_{\mathcal{N}}(t,\mathbf{x})=4\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{\partial^m}{\partial t^m}\int_{\mathcal{M}}\tau^{\alpha\beta}(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^{m-1}d^3x', \tag{2.21}$$

where $\mathcal{M}$ here denotes the intersection of the hypersurface $t=$const with the near-zone world tube. This version will be used for explicit calculations of the near-zone metric for use in the equations of motion. However, an alternative formulation will be useful for studying the $\mathcal{R}$ dependence of the inner integrals; substituting the general Taylor expansion $|\mathbf{x}-\mathbf{x}'|^{m-1}=\Sigma_{q=0}^{\infty}(-1)^q(q!)^{-1}x^Q_<\partial_{>Q}(r^{m-1}_>)$, where $<(>)$ denotes the lesser (greater) of $|\mathbf{x}|$ and $|\mathbf{x}'|$, we obtain

$$h^{\alpha\beta}_{\mathcal{N}}(t,\mathbf{x})=4\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{\partial^m}{\partial t^m}\sum_{q=0}^{\infty}\frac{(-1)^q}{q!}\times\int_{\mathcal{M}}\tau^{\alpha\beta}(t,\mathbf{x}')x^Q_<\partial_{>Q}(r^{m-1}_>)d^3x'. \tag{2.22}$$

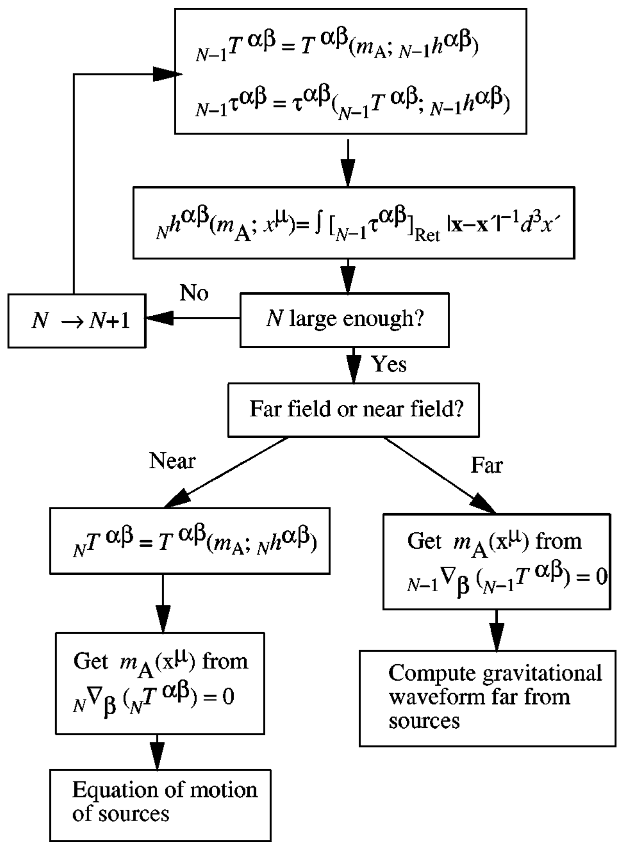

FIG. 3. Structure of iteration procedure.

F. Near-zone field point, outer integration

The formulas from Sec. II D, such as Eqs. (2.18) and (2.20), carry over to this case with the result

$$h^{\alpha\beta}_{\mathcal{C}-\mathcal{N}}(t,\mathbf{x}) = 4\int_{u-2\mathcal{R}}^{u-2\mathcal{R}+2r} du' \int_0^{2\pi} d\phi' \int_{1-\alpha}^{1} \frac{\tau^{\alpha\beta}(u'+r',\mathbf{x}')}{t-u'-\hat{\mathbf{n}}'\cdot\mathbf{x}} \times [r'(u',\Omega')]^2 d\cos\theta' + 4\int_{-\infty}^{u-2\mathcal{R}} du' \oint \frac{\tau^{\alpha\beta}(u'+r',\mathbf{x}')}{t-u'-\hat{\mathbf{n}}'\cdot\mathbf{x}} \times [r'(u',\Omega')]^2 d^2\Omega'. \tag{2.23}$$

Notice that the u' integration ends at $u-2\mathcal{R}+2r$ rather than u because that corresponds to the last future null cone that intersects points in the far zone.

G. Iteration of the relaxed Einstein equations

Because the field $h^{\alpha\beta}$ appears in the source of the field equation, the usual method of solution is to iterate: substitute $h^{\alpha\beta}=0$ on the right-hand side of Eq. (2.11) and solve for the first-iterated $_1h^{\alpha\beta}$; substitute that into Eq. (2.11) and solve for the second-iterated $_2h^{\alpha\beta}$, and so on [imposing the gauge condition Eq. (2.3) consistently at each order]. The general sequence of iterations is shown shematically in Fig. 3. The matter variables m_A and the $(N-1)$-iterated field $_{N-1}h^{\alpha\beta}$ are used to determine $_{N-1}T^{\alpha\beta}$ and $_{N-1}\Lambda^{\alpha\beta}$. Equation (2.11) then yields $_Nh^{\alpha\beta}$ as a function of spacetime and a functional of the matter variables. Then, if one wishes to determine the motion of the source, one substitutes $_Nh^{\alpha\beta}$ into the matter stress-energy tensor and obtains the equations of motion from $_N\nabla_\beta({}_NT^{\alpha\beta})=0$ where $_N\nabla_\beta$ denotes the covariant derivative using the Nth iterated field. If one wishes to determine the Nth iterated gravitational field as a function of spacetime (i.e. with the matter variables determined as functions of spacetime to a consistent order), then one only needs to solve the equations of motion $_{N-1}\nabla_\beta({}_{N-1}T^{\alpha\beta})=0$, which are equivalent to the Nth iterated gauge condition $_Nh^{\alpha\beta}{}_{,\beta}=0$.

H. General structure of the outer integrals

At the first iteration, the solution is simply linearized general relativity. With $_0h^{\alpha\beta}=0$ substituted into the right-hand side of Eq. (2.11), the outer integrals vanish, and the inner integrals over the special relativistic $T^{\alpha\beta}$ have compact support. There is no $\mathcal{R}$ dependence in the integrals, trivially. For field points outside the source ($|\mathbf{x}|>|\mathbf{x}'|$), within both the near and far zones, the first-iterated $_1h^{\alpha\beta}$ takes the form of Eq. (2.13). Since $M^{\alpha\beta Q}$ is a function only of $u=t-r$, the spatial gradients ∂_Q produce only unit radial vectors $\hat{n}^i$, powers of r and retarded time derivatives of $M^{\alpha\beta Q}$. Products of $\hat{n}^i$ can be grouped into STF products $\hat{n}^{\langle L\rangle}$, which are analogous to Y_{LM} (see Appendix A for useful formulas related to STF products). Thus, outside the source, $_1h^{\alpha\beta}$ can be written as a sequence of terms of the form

$$_1h^{\alpha\beta}{}_{B,L}(t,\mathbf{x}) = f_{B,L}(u)\hat{n}^{\langle L\rangle} r^{-B}. \tag{2.24}$$

At the second iteration, in the far zone, $T^{\alpha\beta}=0$, and $_1\Lambda^{\alpha\beta}(u'+r',\mathbf{x}')$ consists of products of spatial and temporal derivatives of $_1h^{\alpha\beta}(u'+r',\mathbf{x}')$. It therefore can also be expressed as a sequence of terms of the form

$$\Lambda^{\alpha\beta}(u'+r',\mathbf{x}') \sim f_{B,L}(u')\hat{n}'^{\langle L\rangle} r'^{-B}. \tag{2.25}$$

Whenever the source at a given $(N-1)$ iteration takes this form, it is straightforward to evaluate the general form of the outer integrals for the Nth iterate. Defining the new variables $\zeta \equiv (t-u')/r = 1+(u-u')/r$, $y=\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}'=\cos\theta$, we find, from Eq. (2.18),

$$r' = r(\zeta^2-1)/2(\zeta-y). \tag{2.26}$$

Substituting Eqs. (2.25) and (2.26) into Eq. (2.19), and changing to integration variables ζ, y and ϕ, we obtain

$$_Nh^{\alpha\beta}_{\mathcal{C}-\mathcal{N}B,L} = \frac{1}{2}\left(\frac{2}{r}\right)^{B-2} \hat{n}^{\langle L\rangle}\int_{-1}^{1} P_L(y)dy \times \int_{\zeta(y)}^{\infty} \frac{(\zeta-y)^{B-3}}{(\zeta^2-1)^{B-2}} f_{B,L}[u-r(\zeta-1)]d\zeta, \tag{2.27}$$

where $\zeta(y)=z+\sqrt{z^2-2zy+1}$, $z=\mathcal{R}/r$, and $P_L(y)$ is the Legendre polynomial.

For far-zone field points, $z<1$; Taylor expanding $f_{B,L}[u-r(\zeta-1)]$ about u, we obtain, for $B>2$,

$$
{}_N h^{\alpha\beta}_{\mathcal{C}-\mathcal{N}B,L} = \left(\frac{2}{r}\right)^{B-2} \hat{n}^{\langle L\rangle} \sum_{q=0}^{\infty} \mathcal{D}^q_{B,L}(z) r^q \frac{d^q f_{B,L}(u)}{du^q}, \tag{2.28}
$$

where the coefficients $\mathcal{D}^q_{B,L}(z)$ are given by

$$
\mathcal{D}^q_{B,L}(z) = \frac{(-)^q}{q!} \int_1^{1+2z} \frac{(\zeta-1)^q}{(\zeta^2-1)^{B-2}} A_{B,L}(\zeta,\alpha) d\zeta - \sum_{p=0}^{q} k^{(q-p+1)}_{B,L}(1+2z) \frac{(-2z)^p}{p!}, \tag{2.29}
$$

where

$$
A_{B,L}(\zeta,\alpha) \equiv \frac{1}{2} \int_{1-\alpha}^{1} P_L(y)(\zeta-y)^{B-3} dy, \tag{2.30a}
$$

$$
\alpha \equiv (\zeta-1)(\zeta+1-2z)/2z, \tag{2.30b}
$$

$$
dk^{(m)}_{B,L}(\zeta)/d\zeta \equiv k^{(m-1)}_{B,L}(\zeta), \quad m \geqslant 1, \tag{2.30c}
$$

$$
k^{(0)}_{B,L}(\zeta) \equiv A_{B,L}(\zeta,2)/(\zeta^2-1)^{B-2}. \tag{2.30d}
$$

The case $B=2$ is special, and leads to the result

$$
{}_N h^{\alpha\beta}_{\mathcal{C}-\mathcal{N}2,L} = \frac{\hat{n}^{\langle L\rangle}}{r} \int_0^{\infty} f_{2,L}(u-s) Q_L\left(1+\frac{s}{r}\right) ds + \hat{n}^{\langle L\rangle} \sum_{q=0}^{\infty} \mathcal{D}^q_{2,L}(z) r^q \frac{d^q f_{2,L}(u)}{du^q}, \tag{2.31}
$$

where

$$
\mathcal{D}^q_{2,L}(z) = \frac{(-)^{q+1}}{2q!} \int_1^{1+2z} (\zeta-1)^q d\zeta \int_{-1}^{1-\alpha} \frac{P_L(y)}{(\zeta-y)} dy, \tag{2.32}
$$

where $Q_L(y)$ is the Legendre function.

Notice that, for $B\neq 2$, the outer integral returns a result of the same generic form as the input function. The case $B=2$ returns terms with a logarithmic dependence on r (via the Q_L's); terms of this form are called ‘‘tails.’’

Similarly, for field points in the near zone, $z>1$, we Taylor expand $f_{B,L}[u-r(\zeta-1)]$ about $u+r=t$, and obtain, for $B>2$,

$$
{}_N h^{\alpha\beta}_{\mathcal{C}-\mathcal{N}B,L} = \left(\frac{2}{r}\right)^{B-2} \hat{n}^{\langle L\rangle} \sum_{q=0}^{\infty} \mathcal{E}^q_{B,L}(z) r^q \frac{d^q f_{B,L}(t)}{dt^q}, \tag{2.33}
$$

where the coefficients $\mathcal{E}^q_{B,L}(z)$ are given by

$$
\mathcal{E}^q_{B,L}(z) = \frac{(-)^q}{q!} \int_{2z-1}^{2z+1} \frac{\zeta^q}{(\zeta^2-1)^{B-2}} A_{B,L}(\zeta,\alpha) d\zeta - \sum_{p=0}^{q} k^{(q-p+1)}_{B,L}(1+2z) \frac{(-1-2z)^p}{p!} \tag{2.34}
$$

and, for $B=2$,

$$
{}_N h^{\alpha\beta}_{\mathcal{C}-\mathcal{N}2,L} = \frac{\hat{n}^{\langle L\rangle}}{r} \int_0^{\infty} f_{2,L}(u-s) Q_L\left(1+\frac{s}{r}\right) ds + \hat{n}^{\langle L\rangle} \sum_{q=0}^{\infty} \mathcal{E}^q_{2,L}(z) r^q \frac{d^q f_{2,L}(t)}{dt^q}, \tag{2.35}
$$

where

$$
\mathcal{E}^q_{2,L}(z) = \frac{(-)^{q+1}}{q!} \left\{ \int_1^{2z-1} \zeta^q Q_L(\zeta) d\zeta + \frac{1}{2} \int_{2z-1}^{2z+1} \zeta^q d\zeta \int_{-1}^{1-\alpha} \frac{P_L(y)}{(\zeta-y)} dy \right\}. \tag{2.36}
$$

Notice that, for near-zone field points, the functions $f_{B,L}$ are evaluated at the local time t, not retarded time u.

I. Cancellation of $\mathcal{R}$ dependence

It is evident that the inner integrals and outer integrals for the field $h^{\alpha\beta}$ will separately depend upon the radius $\mathcal{R}$ of the boundary between the near zone and the far zone. But since each integral was simply a rewriting of a piece of the original integral, Eq. (2.11), which had no $\mathcal{R}$ dependence, it is equally evident that the separate $\mathcal{R}$ dependences must cancel between the inner and outer integrals. In [32], referred to hereafter as Will and Wiseman (WW), we demonstrated such a cancellation explicitly for contributions to the gravitational waveform at 2PN order that depended on positive powers of $\mathcal{R}$. Here we demonstrate the cancellation generally, for both near-zone and far-zone field points, for arbitrary powers of $\mathcal{R}$ (including $\ln \mathcal{R}$) and to an order of iteration sufficient for our purposes.

The proof proceeds by induction. First, as we pointed out above, the first-iterated field ${}_1h^{\alpha\beta}$ is trivially independent of $\mathcal{R}$.

Second, we assume that the $(N-1)$-iterated field does not depend on $\mathcal{R}$, i.e. that all $\mathcal{R}$ dependence cancels at this order of iteration. We wish to demonstrate that this implies cancellation of the $\mathcal{R}$ dependence in the N-iterated field. The proof consists of considering the limiting behavior of the inner and outer integrals for ${}_Nh^{\alpha\beta}$ in the vicinity of $|\mathbf{x}'|\to\mathcal{R}$. Here $T^{\alpha\beta}$ vanishes, and we only need to consider ${}_{N-1}\Lambda^{\alpha\beta}$, which is a functional of ${}_{N-1}h^{\alpha\beta}$. We have already seen that, in the far zone, ${}_{N-1}\Lambda^{\alpha\beta}$ can be decomposed into terms of the form $f_{B,L}(u)\hat{n}^{\langle L\rangle}r^{-B}$. (We consider tail contributions with $\ln r$ dependence separately.) Since the $(N-1)$-iterated field does not depend on $\mathcal{R}$ by assumption, continuity of the fields means that ${}_{N-1}\Lambda^{\alpha\beta}$ will have this same form just inside the near zone. Thus we will calculate the limiting behavior of the inner integral of a term of this form as the integration variable $r'\to\mathcal{R}$ from below, and compare its $\mathcal{R}$ dependence with that of the outer integral of the same term.

For far-zone field points, we must calculate the $\mathcal{R}$ dependence of the moments $M^{\alpha\beta Q}$, and substitute into Eq. (2.13); after considerable algebra (see Appendix B), we obtain, for the limiting behavior of ${}_N h^{\alpha\beta}_{\mathcal{N}}$ as the integration variable approaches $\mathcal{R}$ from inside,

$$ {}_N h^{\alpha\beta}_{\mathcal{N}\,B,L}\to\left(\frac{2}{r}\right)^{B-2}\hat{n}^{\langle L\rangle}\sum_{q=0}^{\infty}\mathcal{D}^{\mathrm{in},q}_{B,L}(z)r^q\frac{d^q f_{B,L}(u)}{du^q}, \tag{2.37}$$

where

$$\mathcal{D}^{\mathrm{in},q}_{B,L}(z)=\sum_{m=0}^{q}\sum_{j=0}^{j_{max}}\frac{(-)^m 2^{2+j-B}}{m!(q-m-j+2L+1)!}\frac{\left[\frac{1}{2}(q-m-j+2L)\right]!}{\left[\frac{1}{2}(q-m-j)\right]!}\frac{(2L-j)!}{j!(L-j)!}$$
$$\times\begin{cases} z^{3+L-B+q-j}/(3+L-B+q-j), & 3+L-B+q-j\neq 0,\\ \ln\mathcal{R}, & 3+L-B+q-j=0,\end{cases} \tag{2.38}$$

where $j_{max}=$lesser of $\{q-m,L\}$, and $q-m-j$ $=$even integer$\geqslant 0$. Equation (2.37) is of the same form as the outer integral for far-zone field points, Eq. (2.28). The coefficients $\mathcal{D}^q_{B,L}(z)$ from the outer integrals are most easily evaluated using computer algebra methods (we calculated the coefficients using independent MAPLE and MATHEMATICA programs); the result is, for each B, L and q,

$$\mathcal{D}^{\mathrm{in},q}_{B,L}(z)+\{z\text{-dependent part of }\ \mathcal{D}^q_{B,L}(z)\}=0. \tag{2.39}$$

Thus the $\mathcal{R}$ dependence cancels term by term. A similar cancellation occurs for near-zone field points, as well as for the case where the integrand has $r^{-B}\ln r$ dependence. Details are given in Appendix B.

This cancellation, while inevitable, has practical consequences, in the following sense. In calculating the inner contributions to the fields, we must integrate over a finite hypersurface, $\mathcal{M}$, sources that extend throughout $\mathcal{M}$. Consequently, any such integral will have terms that are independent of $\mathcal{R}$, as well as terms that depend on $\mathcal{R}^q$ or $\ln\mathcal{R}$. Because we know that all terms of the latter form cancel with contributions from the outer integrals in the final expression for the field, we can drop them in any individual result. Similarly, we can drop all $\mathcal{R}$-dependent terms that arise in any individual outer integral. This provides a kind of regularization of integrals, which cures the problem of divergent integrals that haunted earlier slow-motion methods. In fact, one can show that there is a close connection between this method of regularization and the method of analytic continuation used by Blanchet [59].

Thus, our procedure for determining the field is to determine separately $h^{\alpha\beta}_{\mathcal{N}}$ and $h^{\alpha\beta}_{\mathcal{C}-\mathcal{N}}$ to a given PN order, keeping only $\mathcal{R}$-independent terms in each expression, then sum them to obtain

$$h^{\alpha\beta}=h^{\alpha\beta}_{\mathcal{N}}+h^{\alpha\beta}_{\mathcal{C}-\mathcal{N}}. \tag{2.40}$$

III. WEAK FIELD, SLOW-MOTION APPROXIMATION

We now turn to a discussion of the number N of iterations needed to derive equations of motion or gravitational waveforms of a desired accuracy, in a weak-field, slow-motion approximation.

We assume that, for the fluid source,

$$v^2\sim m/\mathcal{S}\sim p/\rho\sim\epsilon\ll 1, \tag{3.1}$$

where ϵ will be used as an expansion parameter. But from the nature of the iteration procedure, it is evident that each iteration of the field introduces corrections of order $m/\mathcal{S}$. In terms of ϵ, m and $\mathcal{S}$ the equations of motion (1.1) can be rewritten schematically as

$$d\mathbf{v}/dt\sim(m/\mathcal{S}^2)[1+O(\epsilon)+O(\epsilon^2)+O(\epsilon^{5/2})+O(\epsilon^3)+O(\epsilon^{7/2})+\cdots], \tag{3.2}$$

where the terms inside the square brackets represent the Newtonian, post-Newtonian, 2PN, 2.5PN (radiation-reaction), 3PN, and 3.5PN (radiation-reaction) terms respectively. For a term of order ϵ^N, the largest number of powers of $m/\mathcal{S}$ that can appear in it (including one power from the

$m/\mathcal{S}^2$ prefactor) is $N+1$. The radiation reaction terms of order $\epsilon^{N+1/2}$ must contain an odd number of velocities (in order to be odd under time reversal); thus the maximum number of powers of $m/\mathcal{S}$ for them is also $N+1$. Since one iteration gives the Newtonian potential, which yields the Newtonian equations of motion $(N=1)$, then, to obtain the 1PN terms $(N=2)$, one must have the second iterated field, and to obtain the 2PN and 2.5PN terms $(N=3)$, one must have the third iterated field, while to obtain the 3PN and 3.5PN terms $(N=4)$, one must have the fourth iterated field.

Similarly, to obtain a result for the waveform accurate to the order of the quadrupole formula, $h\sim\ddot{\mathcal{I}}^{ij}/R\sim(m/R)(v^2+m/\mathcal{S})\sim\epsilon^2$ $(N=2)$, the second-iterated field is needed. Note that the term $m/\mathcal{S}$ in $\ddot{\mathcal{I}}^{ij}$ arises through the use of the Newtonian equation of motion. Then, to obtain the 1PN, 2PN and 3PN corrections to the quadrupole approximation, the third-, fourth-, and fifth-iterated fields are needed, respectively. This would be an impossible task if it were not for the judicious use of the conservation law, Eq. (2.8). Consider for example, the source ${}_{N-1}\tau^{ij}$ of the Nth iterated gravitational-wave field ${}_Nh^{ij}_{\mathcal{N}}$, Eq. (2.15), specifically the leading, $m=0$ term. The conservation law, Eq. (2.8), converts ${}_{N-1}\tau^{ij}$ into two time derivatives of ${}_{N-1}\tau^{00}x^ix^j$ (modulo total divergences). Because of the slow-motion approximation, two time derivatives increase the order by ϵ, and thus, to sufficient accuracy, only ${}_{N-2}\tau^{00}$ is needed in practice. An important caveat to this is that the surface terms that arise from the total divergences and the outer integrals must formally be evaluated using the $N-1$ expressions. However, in practice, these terms contribute at sufficiently high order that they can be treated without resort to explicit $N-1$ expressions. Effectively, the burden of accuracy has been shifted from the Nth iteration of the field, to the $N-1$-iterated equations of motion, which enter via the two time derivatives and which are needed anyway to evaluate the field as a function of spacetime. Thus, for $N=2$, the leading quadrupole approximation, only ${}_0\tau^{00}=\rho$ is needed, together with the Newtonian equations of motion. This circumstance is responsible for the prevalent, but erroneous view that linearized gravity (one iteration) suffices to derive the quadrupole formula. The formula so derived turns out to be "correct," but its foundation is not (see [58] for discussion).

Thus, in WW, to evaluate the 2PN waveforms (fourth iteration), only second-iterated fields were needed in the source terms. For 3PN waveforms, only third-iterated fields will be needed.

IV. FORMAL STRUCTURE OF NEAR-ZONE FIELDS

A. Metric and stress-energy pseudotensor in terms of the fields

We begin by defining a simplified notation for the field $h^{\alpha\beta}$:

$$N\equiv h^{00}\sim O(\epsilon),$$

$$K^i\equiv h^{0i}\sim O(\epsilon^{3/2}),$$

$$B^{ij}\equiv h^{ij}\sim O(\epsilon^2),$$

$$B\equiv h^{ii}\equiv\sum_i h^{ii}\sim O(\epsilon^2), \tag{4.1}$$

where we show the leading order dependence on ϵ in the near zone. To obtain the equations of motion to 3.5PN order, we need to determine the components of the physical metric to the following orders: g_{00} to $O(\epsilon^{9/2})$, g_{0i} to $O(\epsilon^4)$, and g_{ij} to $O(\epsilon^{7/2})$. From the definition (2.2), one can invert to find $g_{\alpha\beta}$ in terms of $h^{\alpha\beta}$. Expanding to the required order, we find

$$\begin{aligned} g_{00}=&-\left(1-\frac{1}{2}N+\frac{3}{8}N^2-\frac{5}{16}N^3+\frac{35}{128}N^4\right)\\ &+\frac{1}{2}B\left(1-\frac{1}{2}N+\frac{3}{8}N^2\right)+\frac{1}{4}\left(B^{ij}B^{ij}-\frac{1}{2}B^2\right)\\ &+\frac{1}{2}K^jK^j-\frac{3}{4}NK^jK^j+O(\epsilon^5), \end{aligned}\tag{4.2a}$$

$$g_{0i}=-K^i\left(1-\frac{1}{2}N-\frac{1}{2}B+\frac{3}{8}N^2\right)-K^jB^{ij}+O(\epsilon^{9/2}),\tag{4.2b}$$

$$\begin{aligned} g_{ij}=&\delta^{ij}\left(1+\frac{1}{2}N-\frac{1}{8}N^2+\frac{1}{16}N^3-\frac{1}{4}NB+\frac{1}{2}K^kK^k\right)\\ &+B^{ij}-\frac{1}{2}B\delta^{ij}-K^iK^j+\frac{1}{2}NB^{ij}+O(\epsilon^4), \end{aligned}\tag{4.2c}$$

$$(-g)=1+N-B-NB+K^iK^i+O(\epsilon^4).\tag{4.2d}$$

Notice that, in order to find the metric $g_{\alpha\beta}$ to the desired order, we must obtain N to $O(\epsilon^{9/2})$, K^i to $O(\epsilon^4)$, B^{ij} to $O(\epsilon^{7/2})$ and B to $O(\epsilon^{9/2})$. In fact, because B contributes linearly to g_{00}, we will treat B differently from B^{ij}.

Using Eq. (4.2), we can express the matter stress-energy tensor $T^{\alpha\beta}$, Eq. (2.10), as a PN expansion. However, the details of such an expansion will depend on the basic variables used to characterize the matter. For example, to discuss the structure of a star in a PN expansion, it is convenient to use the mass-energy density ρ and pressure p, together with an equation of state. However, to discuss the motion of compact bodies in an effective "point-mass" limit, it is more convenient to ignore the pressure totally and to use the so-called "conserved," or baryon density, $\rho^*\equiv\rho\sqrt{-g}u^0$. For now, we follow the convention of Blanchet and Damour [16], and define the quantities

$$\sigma\equiv T^{00}+T^{ii},$$

$$\sigma^i\equiv T^{0i},$$

$$\sigma^{ij}\equiv T^{ij}.\tag{4.3}$$

We will express various potentials formally in terms of these densities, and later make a PN expansion of them in terms of the densities most appropriate to the application.

Substituting the formulas for $h^{\alpha\beta}$ and $g_{\alpha\beta}$ into Eqs. (2.6) and (2.7) for $\Lambda^{\alpha\beta}$, we obtain, to the required order,

$$\begin{aligned}\Lambda^{00}=&-\frac{7}{8}(\nabla N)^2+\left\{\frac{5}{8}\dot{N}^2-\ddot{N}N-2\dot{N}^{,k}K^k+\frac{1}{2}K^{i,j}(3K^{j,i}+K^{i,j})+\dot{K}^jN^{,j}-B^{ij}N^{,ij}+\frac{1}{4}\nabla N\cdot\nabla B+\frac{7}{8}N(\nabla N)^2\right\}\\&+\left\{K^{k,j}\dot{B}^{jk}+\frac{1}{4}B^{jk,l}(B^{jk,l}-2B^{kl,j})+\frac{1}{4}\dot{N}\dot{B}-\frac{1}{8}(\nabla B)^2+\frac{1}{4}\dot{N}N^{,j}K^j\right.\\&\left.+\frac{7}{8}N^{,j}N^{,k}B^{jk}-\frac{1}{2}K^jN^{,k}(3K^{j,k}+4K^{k,j})-\frac{7}{8}N^2(\nabla N)^2\right\}+O(\rho\epsilon^4),\end{aligned}\tag{4.4a}$$

$$\begin{aligned}\Lambda^{0i}=&\left\{N^{,k}(K^{k,i}-K^{i,k})+\frac{3}{4}\dot{N}N^{,i}\right\}+\left\{\dot{N}\dot{K}^i-N\ddot{K}^i-2K^k\dot{K}^{i,k}-B^{lm}K^{i,lm}+K^{k,l}(B^{il,k}+B^{ik,l}-B^{kl,i})+N^{,k}\dot{B}^{ik}-\frac{1}{4}\dot{N}B^{,i}\right.\\&\left.-\frac{1}{4}N^{,i}\dot{B}-NN^{,k}(K^{k,i}-K^{i,k})-\frac{3}{4}N\dot{N}N^{,i}+\frac{1}{8}K^i(\nabla N)^2-\frac{1}{4}K^kN^{,k}N^{,i}\right\}+O(\rho\epsilon^{7/2}),\end{aligned}\tag{4.4b}$$

$$\begin{aligned}\Lambda^{ij}=&\frac{1}{4}\left\{N^{,i}N^{,j}-\frac{1}{2}\delta^{ij}(\nabla N)^2\right\}+\left\{2K^{k,(i}K^{j),k}-K^{k,i}K^{k,j}-K^{i,k}K^{j,k}+2N^{,(i}\dot{K}^{j)}+\frac{1}{2}N^{,(i}B^{,j)}\right.\\&\left.-\frac{1}{2}N\left(N^{,i}N^{,j}-\frac{1}{2}\delta^{ij}(\nabla N)^2\right)-\delta^{ij}\left(K^{l,k}K^{[k,l]}+N^{,k}\dot{K}^k+\frac{3}{8}\dot{N}^2+\frac{1}{4}\nabla N\cdot\nabla B\right)\right\}\\&+\left\{2\dot{K}^i\dot{K}^j+\dot{B}^{k(i}(K^{j),k}-K^{k,j)})-2\dot{B}^{ij,k}K^k-N\ddot{B}^{ij}-B^{ij,lm}B^{lm}+B^{ik,l}(B^{jl,k}+B^{jk,l})-2B^{kl,(i}B^{j)k,l}+\frac{1}{2}B^{kl,i}B^{kl,j}\right.\\&-\frac{1}{4}B^{,i}B^{,j}-N(2K^{k,(i}K^{j),k}-K^{k,i}K^{k,j}-K^{i,k}K^{j,k})+K^kK^{k,(i}N^{,j)}-2NN^{,(i}\dot{K}^{j)}-\frac{1}{2}\dot{N}N^{,(i}K^{j)}-\frac{1}{2}N^{,k}N^{,(i}B^{j)k}\\&-\frac{1}{2}NN^{,(i}B^{,j)}+\frac{1}{8}(\nabla N)^2B^{ij}+\frac{3}{4}N^2\left(N^{,i}N^{,j}-\frac{1}{2}\delta^{ij}(\nabla N)^2\right)+\frac{1}{8}\delta^{ij}[(\nabla B)^2+2\dot{N}\dot{B}+8K^{k,l}\dot{B}^{kl}+4B^{kl,m}B^{km,l}\\&\left.-2B^{kl,m}B^{kl,m}+3N\dot{N}^2+8NN^{,k}\dot{K}^k+8NK^{l,k}K^{[k,l]}-4K^kN^{,l}K^{k,l}+2\dot{N}K^kN^{,k}+N^{,k}N^{,l}B^{kl}+2N\nabla N\cdot\nabla B]\right\}\\&+O(\rho\epsilon^4),\end{aligned}\tag{4.4c}$$

$$\begin{aligned}\Lambda^{ii}=&-\frac{1}{8}(\nabla N)^2+\left\{K^{l,k}K^{[k,l]}-N^{,k}\dot{K}^k-\frac{1}{4}\nabla N\cdot\nabla B-\frac{9}{8}\dot{N}^2+\frac{1}{4}N(\nabla N)^2\right\}\\&+\left\{2\dot{K}^k\dot{K}^k-2\dot{B}^{,k}K^k+3\dot{B}^{kl}K^{k,l}-N\ddot{B}+\frac{3}{4}\dot{N}\dot{B}+\frac{1}{8}(\nabla B)^2-B^{,lm}B^{lm}+\frac{3}{4}B^{kl,m}B^{kl,m}+\frac{1}{2}B^{kl,m}B^{km,l}-NK^{l,k}K^{[k,l]}\right.\\&\left.-\frac{1}{2}N^{,l}K^kK^{k,l}+NN^{,k}\dot{K}^k+\frac{1}{4}\dot{N}N^{,k}K^k-\frac{1}{8}N^{,k}N^{,l}B^{kl}+\frac{1}{4}N\nabla N\cdot\nabla B+\frac{1}{8}(\nabla N)^2B+\frac{9}{8}N\dot{N}^2-\frac{3}{8}N^2(\nabla N)^2\right\}\\&+O(\rho\epsilon^4),\end{aligned}\tag{4.4d}$$

where an overdot denotes $\partial/\partial t$. In the above expressions, terms grouped within braces make leading contributions of the same order. For example, in Λ^{00}, the three groupings correspond to $O(\rho\epsilon)$, $O(\rho\epsilon^2)$, and $O(\rho\epsilon^3)$, respectively.

B. Source moments and other integral quantities

Throughout our calculations, a number of integrals of the source stress-energy pseudotensor occur, for example, in the multipole expansions of Eq. (2.14). It is useful to define and collect these quantities and to discuss their properties. All integrals are carried out over a constant time (or constant retarded time) hypersurface $\mathcal{M}$, within the near-zone. In general, these integrals will have $\mathcal{R}$ dependence, but, in line with the foregoing discussion, we shall consistently drop such terms. The relevant integrals are

$$P^\mu \equiv M^{\mu 0} = \int_{\mathcal{M}} \tau^{\mu 0} d^3x, \tag{4.5a}$$

$$\mathcal{I}^Q \equiv M^{00Q} = \int_{\mathcal{M}} \tau^{00} x^Q d^3x, \tag{4.5b}$$

$$\mathcal{J}^{iQ} \equiv \epsilon^{iab} M^{0baQ} = \epsilon^{iab} \int_{\mathcal{M}} \tau^{0b} x^{aQ} d^3x, \tag{4.5c}$$

$$\mathcal{P}^{ijabQ} \equiv \int_{\mathcal{M}} x^{[a} \tau^{i][j} x^{b]Q} d^3x. \tag{4.5d}$$

By making use of the equations of motion $\tau^{\alpha\beta}{}_{,\beta}=0$, we can transform some of these integrals into other forms, modulo surface integrals at the boundary $\partial\mathcal{M}$ of the near zone. For example,

$$\dot{P}^\mu = -\oint_{\partial\mathcal{M}} \tau^{\mu j} d^2S_j\,,$$

$$\dot{\mathcal{J}}^i = -\epsilon^{iab} \oint_{\partial\mathcal{M}} \tau^{jb} x^a d^2S_j\,,$$

$$\dot{\mathcal{I}}^i = P^i - \oint_{\partial\mathcal{M}} \tau^{0j} x^i d^2S_j\,. \tag{4.6}$$

These identities express the conservation of total energy, momentum and angular momentum, and uniform center-of-mass motion, modulo a flux of gravitational radiation from the system. In calculations, the surface terms must be checked carefully to see if they make $\mathcal{R}$-independent contributions to the order considered. For the most part, such surface terms turn out to make no contribution.

Henceforth, we shall set $\mathcal{I}^i = \dot{\mathcal{I}}^i = 0$, which amounts to attaching the origin of coordinates to the center of mass of the system.

Other useful identities include

$$M^{ij} = \frac{1}{2}\ddot{\mathcal{I}}^{ij} + \frac{1}{2}\oint_{\partial\mathcal{M}} [\tau^{lm}(x^{ij})_{,l} + \dot{\tau}^{m0} x^{ij}] d^2S_m\,, \tag{4.7a}$$

$$M^{ijk} = \frac{1}{6}\dddot{\mathcal{I}}^{ijk} + \frac{2}{3}\epsilon^{lk(i}\dot{\mathcal{J}}^{l|j)} + \frac{1}{6}\oint_{\partial\mathcal{M}} [\tau^{lm}(x^{ijk})_{,l} + \dot{\tau}^{m0} x^{ijk}] d^2S_m - \frac{2}{3}\oint_{\partial\mathcal{M}} [\tau^{l[k} x^{i]j} + \tau^{l[k} x^{j]i}] d^2S_l\,, \tag{4.7b}$$

$$\begin{aligned} M^{ijQ} = {} & \frac{1}{(q+1)(q+2)}\ddot{\mathcal{I}}^{ijQ} + \frac{2}{(q+2)}\epsilon^{mk_1(i}\dot{\mathcal{J}}^{m|j)k_2 \cdots k_q}(\mathrm{sym}\,k{:}Q) + \frac{8(q-1)}{(q+1)}\mathcal{P}^{ij(k_1k_2 \cdots k_q)} \\ & + \frac{1}{(q+1)(q+2)}\oint_{\partial\mathcal{M}} [\tau^{lm}(x^{ijQ})_{,l} + \dot{\tau}^{m0} x^{ijQ}] d^2S_m - \frac{2}{(q+2)}\oint_{\partial\mathcal{M}} [\tau^{l[k_1} x^{i]jk_2 \cdots k_q} + \tau^{l[k_1} x^{j]ik_2 \cdots k_q}] d^2S_l (\mathrm{sym}\,k{:}Q), \end{aligned} \tag{4.7c}$$

$$M^{0jQ} = \frac{1}{q+1}\dot{\mathcal{I}}^{jQ} - \frac{q}{q+1}\epsilon^{mj(k_1}\mathcal{J}^{m|k_2 \cdots k_q)} + \frac{1}{q+1}\oint_{\partial\mathcal{M}} \tau^{0m} x^{jQ} d^2S_m\,, \tag{4.7d}$$

where the notation (sym $k{:}Q$) means symmetrize on the indices k_1 through k_q, and the superscript notation ${}^{m|k\cdots)}$ means that only the indices following the vertical line are involved in symmetrization.

C. Near-zone field expanded to 3.5 PN order

We now carry out the explicit expansion of the near-zone field through 3.5PN order, beginning with the inner integral, Eq. (2.21), applying the above identities where possible. Inserting powers of ϵ to indicate the leading order of each term, we obtain the result

$$N_{\mathcal{N}}=4\epsilon\int_{\mathcal{M}}\frac{\tau^{00}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'+2\epsilon^2\partial_t^2\int_{\mathcal{M}}\tau^{00}(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|d^3x'-\frac{2}{3}\epsilon^{5/2}\overset{(3)}{\mathcal{I}}{}^{kk}(t)+\frac{1}{6}\epsilon^3\partial_t^4\int_{\mathcal{M}}\tau^{00}(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^3d^3x'$$
$$-\frac{1}{30}\epsilon^{7/2}\Big\{(4x^{kl}+2r^2\delta^{kl})\overset{(5)}{\mathcal{I}}{}^{kl}(t)-4x^k\overset{(5)}{\mathcal{I}}{}^{kll}(t)+\overset{(5)}{\mathcal{I}}{}^{kkll}(t)\Big\}+\frac{1}{180}\epsilon^4\partial_t^6\int_{\mathcal{M}}\tau^{00}(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^5d^3x'-\frac{1}{1260}\epsilon^{9/2}\Big\{3r^4\overset{(7)}{\mathcal{I}}{}^{kk}(t)$$
$$+12r^2x^{ij}\overset{(7)}{\mathcal{I}}{}^{ij}(t)-12r^2x^i\overset{(7)}{\mathcal{I}}{}^{ikk}(t)-8x^{ijk}\overset{(7)}{\mathcal{I}}{}^{ijk}(t)+3r^2\overset{(7)}{\mathcal{I}}{}^{iikk}(t)+12x^{ij}\overset{(7)}{\mathcal{I}}{}^{ijkk}(t)-6x^i\overset{(7)}{\mathcal{I}}{}^{ikkll}(t)+\overset{(7)}{\mathcal{I}}{}^{iikkll}(t)\Big\}+N_{\partial\mathcal{M}}+O(\epsilon^5),\tag{4.8a}$$

$$K^i_{\mathcal{N}}=4\epsilon^{3/2}\int_{\mathcal{M}}\frac{\tau^{0i}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'+2\epsilon^{5/2}\partial_t^2\int_{\mathcal{M}}\tau^{0i}(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|d^3x'+\frac{2}{9}\epsilon^3\Big\{3x^k\overset{(4)}{\mathcal{I}}{}^{ik}(t)-\overset{(4)}{\mathcal{I}}{}^{ikk}(t)+2\epsilon^{mik}\overset{(3)}{\mathcal{J}}{}^{mk}(t)\Big\}$$
$$+\frac{1}{6}\epsilon^{7/2}\partial_t^4\int_{\mathcal{M}}\tau^{0i}(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^3d^3x'+\frac{1}{450}\epsilon^4\Big\{30r^2x^k\overset{(6)}{\mathcal{I}}{}^{ik}(t)-10r^2\overset{(6)}{\mathcal{I}}{}^{ikk}(t)-20x^{kl}\overset{(6)}{\mathcal{I}}{}^{ikl}(t)+15x^k\overset{(6)}{\mathcal{I}}{}^{ikll}(t)-3\overset{(6)}{\mathcal{I}}{}^{ikkll}(t)$$
$$+\epsilon^{mil}\Big[20r^2\overset{(5)}{\mathcal{J}}{}^{ml}(t)+40x^{kl}\overset{(5)}{\mathcal{J}}{}^{mk}(t)-15x^l\overset{(5)}{\mathcal{J}}{}^{mkk}(t)-30x^k\overset{(5)}{\mathcal{J}}{}^{mkl}(t)+12\overset{(5)}{\mathcal{J}}{}^{mlkk}(t)\Big]\Big\}+K^i_{\partial\mathcal{M}}+O(\epsilon^{9/2}),\tag{4.8b}$$

$$B^{ij}_{\mathcal{N}}=4\epsilon^2\int_{\mathcal{M}}\frac{\tau^{ij}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'-2\epsilon^{5/2}\overset{(3)}{\mathcal{I}}{}^{ij}(t)+2\epsilon^3\partial_t^2\int_{\mathcal{M}}\tau^{ij}(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|d^3x'-\frac{1}{9}\epsilon^{7/2}\Big\{3r^2\overset{(5)}{\mathcal{I}}{}^{ij}(t)-2x^k\overset{(5)}{\mathcal{I}}{}^{ijk}(t)-8x^k\epsilon^{mk(i}\overset{(4)}{\mathcal{J}}{}^{m|j)}(t)$$
$$+6\overset{(3)}{M}{}^{ijkk}(t)\Big\}+\frac{1}{6}\epsilon^4\partial_t^4\int_{\mathcal{M}}\tau^{ij}(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^3d^3x'-\frac{1}{180}\epsilon^{9/2}\Big\{3r^4\overset{(7)}{\mathcal{I}}{}^{ij}(t)-4r^2x^k\overset{(7)}{\mathcal{I}}{}^{ijk}(t)-16r^2x^k\epsilon^{mk(i}\overset{(6)}{\mathcal{J}}{}^{m|j)}(t)$$
$$+12r^2\overset{(5)}{M}{}^{ijkk}(t)+24x^{kl}\overset{(5)}{M}{}^{ijkl}(t)-24x^k\overset{(5)}{M}{}^{ijkll}(t)+6\overset{(5)}{M}{}^{ijkkll}(t)\Big\}+B^{ij}_{\partial\mathcal{M}}+O(\epsilon^5).\tag{4.8c}$$

Explicit formulas for the boundary terms $N_{\partial\mathcal{M}}$, $K^i_{\partial\mathcal{M}}$ and $B^{ij}_{\partial\mathcal{M}}$ are given in Appendix C. Through 3.5PN order, the terms in Eq. (4.8) divide naturally into two types: *even* terms, i.e. terms of integer powers in ϵ in N and B^{ij} and odd-half integer powers in K^i, and *odd* terms, of odd-half integer powers in N and B^{ij} and integer powers in K^i. The even terms produce the leading Newtonian, PN, 2PN and 3PN contributions to the equations of motion, while the odd terms produce the gravitational radiation reaction forces. (Note that the even terms have odd contributions embedded within them, via contributions of the metric itself to $\tau^{\alpha\beta}$.) Through 3.5PN order, there is a clean division between even and odd terms, in the sense that even terms produce non-dissipative contributions to the equations of motion, while odd terms produce radiation reaction effects. At 4PN order this separation fails, because of the presence of tails—these are $O(\epsilon^{3/2})$ modifications of the leading 2.5PN radiation-reaction terms, which result in disspative effects at 4PN order. We derive the leading contributions of these 4PN tail terms in Sec. VI C.

The outer integrals for near-zone field points turn out to contribute only beginning at 3PN order (and, as we will see, do not contribute observable effects until 4PN order). This can be seen schematically as follows: for a source term of the form $f_{B,L}(u)\hat{n}^{\langle L\rangle}r^{-B}$, the outer integral has the form

$$\int f(t-r')(\hat{n}')^{\langle L\rangle}(r')^{-B}\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$
$$\sim\int_{\mathcal{R}}^{\infty}\frac{d^qf(t)}{dt^q}r'^{\,q+1-B}dr'\sim\frac{d^qf(t)}{dt^q}\mathcal{R}^{\,q+2-B},\tag{4.9}$$

where we have used the fact that $|\mathbf{x}|\ll|\mathbf{x}'|$. The only possible $\mathcal{R}$-independent terms come from the case $q=B-2$. Thus the outer integral gives a schematic contribution $h^{\alpha\beta}_{\mathcal{C}-\mathcal{N}}\sim f^{(B-2)}(t)$ where the superscript $(B-2)$ denotes $B-2$ time derivatives. From Eq. (4.4a), the leading contribution to the source comes from $(\nabla N)^2$, where, from Eq. (2.13), N has the far-zone form $N\approx4\mathcal{I}/r+2(3\hat{n}^{\langle kl\rangle}\mathcal{I}^{kl}/r^3+3\hat{n}^{\langle kl\rangle}\dot{\mathcal{I}}^{kl}/r^2+\hat{n}^{kl}\ddot{\mathcal{I}}^{kl}/r)+\cdots$. Taking the gradient of this expression and squaring, we get, schematically, $(\nabla N)^2\sim\mathcal{I}^2/r^4+\mathcal{I}(\mathcal{I}^{kl}/r^6+\dot{\mathcal{I}}^{kl}/r^5+\ddot{\mathcal{I}}^{kl}/r^4+\cdots)$. The first term $(B=4)$ gives no contribution, since $\mathcal{I}$ is constant to the order considered ($\mathcal{I}$ varies only via gravitational radiation energy loss). The second, third and fourth terms $(B=6,5,4)$ together give $h\sim\mathcal{I}\mathcal{I}^{kk(4)}(t)$. Since $\mathcal{I}^{kk}\sim mr^2$, we find $h\sim(m/r)^2v^4\sim O(\epsilon^4)$, which is a 3PN contribution. Thus, for near zone field points, the outer integrals can be ignored until 3PN order. A similar argument for far-zone field points reveals that outer integrals begin to contribute only at 2PN order, as was found by WW [32].

D. Compendium of useful post-Newtonian near-zone potentials

The even terms in Eq. (4.8) have the form of ordinary Poisson-like potentials and their generalizations, sometimes called superpotentials. For a source *f*, we define the Poisson potential, superpotential, and superduperpotential to be

$$P(f)\equiv\frac{1}{4\pi}\int_{\mathcal{M}}\frac{f(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x', \quad \nabla^2P(f)=-f, \tag{4.10a}$$

$$S(f)\equiv\frac{1}{4\pi}\int_{\mathcal{M}}f(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|d^3x', \quad \nabla^2S(f)=2P(f), \tag{4.10b}$$

$$SD(f)\equiv\frac{1}{4\pi}\int_{\mathcal{M}}f(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^3d^3x',$$

$$\nabla^2SD(f)=12S(f). \tag{4.10c}$$

We also define potentials based on the ''densities'' σ, σ^i and σ^{ij} constructed from $T^{\alpha\beta}$,

$$\Sigma(f)\equiv\int_{\mathcal{M}}\frac{\sigma(t,\mathbf{x}')f(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'=P(4\pi\sigma f), \tag{4.11a}$$

$$\Sigma^i(f)\equiv\int_{\mathcal{M}}\frac{\sigma^i(t,\mathbf{x}')f(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'=P(4\pi\sigma^i f), \tag{4.11b}$$

$$\Sigma^{ij}(f)\equiv\int_{\mathcal{M}}\frac{\sigma^{ij}(t,\mathbf{x}')f(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'=P(4\pi\sigma^{ij} f), \tag{4.11c}$$

along with the superpotentials

$$X(f)\equiv\int_{\mathcal{M}}\sigma(t,\mathbf{x}')f(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|d^3x'=S(4\pi\sigma f), \tag{4.12a}$$

$$Y(f)\equiv\int_{\mathcal{M}}\sigma(t,\mathbf{x}')f(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^3d^3x'=SD(4\pi\sigma f), \tag{4.12b}$$

$$Z(f)\equiv\int_{\mathcal{M}}\sigma(t,\mathbf{x}')f(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^5d^3x', \tag{4.12c}$$

and their obvious counterparts X^i, X^{ij}, Y^i, Y^{ij}, and so on. A number of potentials occur sufficiently frequently in the PN expansion that is it useful to define them specifically. First and foremost is the ''Newtonian'' potential

$$U\equiv\int_{\mathcal{M}}\frac{\sigma(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'=P(4\pi\sigma)=\Sigma(1). \tag{4.13}$$

The potentials needed for the post-Newtonian limit are

$$V^i\equiv\Sigma^i(1), \quad \Phi_1^{ij}\equiv\Sigma^{ij}(1), \quad \Phi_1\equiv\Sigma^{ii}(1),$$

$$\Phi_2\equiv\Sigma(U), \quad X\equiv X(1). \tag{4.14}$$

Useful 2PN potentials include

$$V_2^i\equiv\Sigma^i(U), \quad \Phi_2^i\equiv\Sigma(V^i),$$

$$X^i\equiv X^i(1), \quad X^{ij}\equiv X^{ij}(1),$$

$$X_1\equiv X^{ii}, \quad X_2\equiv X(U),$$

$$P_2^{ij}\equiv P(U^{,i}U^{,j}), \quad P_2\equiv P_2^{ii}=\Phi_2-\frac{1}{2}U^2,$$

$$G_1\equiv P(\dot{U}^2), \quad G_2\equiv P(U\ddot{U}),$$

$$G_3\equiv-P(\dot{U}^{,k}V^k), \quad G_4\equiv P(V^{i,j}V^{j,i}),$$

$$G_5\equiv-P(\dot{V}^kU^{,k}), \quad G_6\equiv P(U^{,ij}\Phi_1^{ij}),$$

$$G_7^i\equiv P(U^{,k}V^{k,i})+\frac{3}{4}P(U^{,i}\dot{U}), \quad H\equiv P(U^{,ij}P_2^{ij}). \tag{4.15}$$

At 3PN order, the following potentials are useful:

$$Y_1\equiv Y^{ii}, \quad Y_2\equiv Y(U), \quad Z\equiv Z(1). \tag{4.16}$$

A variety of properties of these and general Poisson potentials are described in Appendix D. Note that, in evaluating Poisson potentials and superpotentials of sources that do not have compact support, our rule is to evaluate them on the finite, constant time hypersurface $\mathcal{M}$, and to discard all terms that depend on $\mathcal{R}$.

V. EXPANSION OF NEAR-ZONE FIELDS TO 2.5PN ORDER

We now turn to explicit evaluation of the near-zone fields and the metric to higher PN order, in terms of Poisson potentials and multipole moments. In addition to evaluating the inner integrals shown above, we must evaluate the outer integrals consistently at each PN order, to check whether any finite, $\mathcal{R}$-independent contributions result.

In evaluating the contributions at each order, we shall use the following notation:

$$N=\epsilon(N_0+\epsilon N_1+\epsilon^{3/2}N_{1.5}+\epsilon^2N_2+\epsilon^{5/2}N_{2.5}$$
$$+\epsilon^3N_3+\epsilon^{7/2}N_{3.5})+O(\epsilon^5), \tag{5.1a}$$

$$K^i=\epsilon^{3/2}(K_1^i+\epsilon K_2^i+\epsilon^{3/2}K_{2.5}^i+\epsilon^2K_3^i$$
$$+\epsilon^{5/2}K_{3.5}^i)+O(\epsilon^{9/2}), \tag{5.1b}$$

$$B=\epsilon^2(B_1+\epsilon^{1/2}B_{1.5}+\epsilon B_2+\epsilon^{3/2}B_{2.5}$$
$$+\epsilon^2B_3+\epsilon^{5/2}B_{3.5})+O(\epsilon^5), \tag{5.1c}$$

$$B^{ij}=\epsilon^2(B_2^{ij}+\epsilon^{1/2}B_{2.5}^{ij}+\epsilon B_3^{ij}+\epsilon^{3/2}B_{3.5}^{ij})+O(\epsilon^4), \quad (5.1d)$$

where the subscript on each term indicates the level (1PN, 2PN, 2.5PN, etc.) of its leading contribution to the equations of motion. Notice that our separate treatment of B and B^{ij} leads to the slightly awkward notational circumstance that, for example, $B_2^{ii}=B_1$.

A. Newtonian and 1.5PN solution

At lowest order in the PN expansion, we only need to evaluate $\tau^{00}=(-g)T^{00}+O(\rho\epsilon)=\sigma+O(\rho\epsilon)$ (recall that $\sigma^{ii}\sim\epsilon\sigma$). Since this has compact support, the outer integrals vanish, and we find

$$N_0=4U. \quad (5.2)$$

To this order, $(-g)=1+4U+O(\epsilon^2)$.

To the next PN order, we obtain, from Eqs. (2.5), (4.4) and (5.2),

$$\tau^{00}=\sigma-\sigma^{ii}+4\sigma U-\frac{7}{8\pi}\nabla U^2+O(\rho\epsilon^2),$$

$$\tau^{0i}=\sigma^i+O(\rho\epsilon^{3/2}),$$

$$\tau^{ii}=\sigma^{ii}-\frac{1}{8\pi}\nabla U^2+O(\rho\epsilon^2),$$

$$\tau^{ij}=O(\rho\epsilon). \quad (5.3)$$

Substituting into Eqs. (4.8), and calculating terms through 1.5PN order [e.g. $O(\epsilon^{5/2})$ in N], we obtain

$$N_1=7U^2-4\Phi_1+2\Phi_2+2\ddot{X}, \quad (5.4a)$$

$$K_1^i=4V^i, \quad (5.4b)$$

$$B_1=U^2+4\Phi_1-2\Phi_2, \quad (5.4c)$$

$$N_{1.5}=-\frac{2}{3}\overset{(3)}{\mathcal{I}^{kk}}(t), \quad (5.4d)$$

$$B_{1.5}=-2\overset{(3)}{\mathcal{I}^{kk}}(t). \quad (5.4e)$$

It is straightforward in this case to show that the outer integrals and surface terms give no $\mathcal{R}$-independent terms. It is useful to illustrate the cancellation of an $\mathcal{R}$-dependent term in this simple case. In the far zone to Newtonian order, the field, from Eq. (2.13), is given by $N\approx 4\mathcal{I}/r$, where we focus on the monopole contribution. This contributes to Λ^{00} in the far zone a term of the form $\Lambda^{00}=-14\mathcal{I}^2/r^4$. To evaluate the near-zone contribution of the outer integral of this term, we must evaluate the coefficient $\mathcal{E}_{B,L}^q$ in Eq. (2.33) with $q=0$ (no time derivative, since $\mathcal{I}$ is constant, to lowest order), $B=4$, $L=0$. From Eqs. (2.30) and (2.34), this yields a contribution to N given by $N_{\mathcal{C}-\mathcal{N}}=-7\mathcal{I}^2/\mathcal{R}^2$. However, in evaluating $N_{\mathcal{N}}$, we encounter the Poisson potential $-14P(\nabla U^2)=-14P_2$ [see Eq. (4.15)]. Upon integrating by parts and keeping the surface term at $\mathcal{R}$ [see Eq. (D3a)], this gives a contribution $7U^2-14\Phi_2+7\mathcal{I}^2/\mathcal{R}^2$, whose $\mathcal{R}$-dependent term cancels that from the outer integral.

The physical metric to 1.5PN order is then

$$g_{00}=-1+2U-2U^2+\ddot{X}-\frac{4}{3}\overset{(3)}{\mathcal{I}^{kk}}(t)+O(\epsilon^3), \quad (5.5a)$$

$$g_{0i}=-4V^i+O(\epsilon^{5/2}), \quad (5.5b)$$

$$g_{ij}=\delta_{ij}(1+2U)+O(\epsilon^2). \quad (5.5c)$$

Notice that, in our formulation, the potential U is *not* a retarded potential; the retardation is expressed by the PN potential $\ddot{X}$ and the 1.5PN term $-\frac{4}{3}\mathcal{I}^{kk(3)}(t)$. This contrasts with the PM approach, where retarded, rather than Poisson potentials are used, and the retardation is expanded only much later in the computation. The apparently 1.5PN term $-\frac{4}{3}\mathcal{I}^{kk(3)}(t)$ in g_{00} actually does not contribute to the equations of motion at this order because it is purely a function of time, and the leading contribution is through a spatial gradient. As a consequence, the lowest-order observable contribution to radiation reaction is at 2.5PN order. (An alternative way to treat this 1.5PN term would be to absorb it in a redefinition of the time coordinate.)

We note here the useful identity, which follows from Eqs. (5.3), $\sigma=\tau^{00}+\tau^{ii}-(1/2\pi)\nabla^2(U^2)+O(\rho\epsilon^2)$, whose consequence is

$$\int_{\mathcal{M}}\sigma(t,\mathbf{x})d^3x=\mathcal{I}+\frac{1}{2}\ddot{\mathcal{I}}^{ii}+O(\mathcal{I}\epsilon^2), \quad (5.6)$$

where surface terms make no $\mathcal{R}$-independent contribution.

B. 2.5PN solution

At 2PN and 2.5PN order, we obtain, from Eqs. (2.5), (4.4b), (4.4c), (5.2), and (5.4),

$$\tau^{ij}=\sigma^{ij}+\frac{1}{4\pi}\left(U^{,i}U^{,j}-\frac{1}{2}\delta^{ij}\nabla U^2\right)+O(\rho\epsilon^2), \quad (5.7a)$$

$$\tau^{0i}=\sigma^i+4\sigma^iU+\frac{2}{\pi}U^{,j}V^{[j,i]}+\frac{3}{4\pi}\dot{U}U^{,i}+O(\rho\epsilon^{5/2}). \quad (5.7b)$$

Including outer integrals and boundary terms (which contribute nothing), we obtain, from Eq. (4.8c),

$$B_2^{ij}=4\Phi_1^{ij}+4P_2^{ij}-\delta^{ij}(2\Phi_2-U^2), \tag{5.8a}$$

$$K_2^i=8V_2^i-8\Phi_2^i+8UV^i+16G_7^i+2\ddot{X}^i, \tag{5.8b}$$

$$B_{2.5}^{ij}=-2\overset{(3)}{\mathcal{I}^{ij}}(t), \tag{5.8c}$$

$$K_{2.5}^i=\frac{2}{3}x^k\overset{(4)}{\mathcal{I}^{ik}}(t)-\frac{2}{9}\overset{(4)}{\mathcal{I}^{ikk}}(t)+\frac{4}{9}\epsilon^{mik}\overset{(3)}{\mathcal{J}^{mk}}(t). \tag{5.8d}$$

The solutions for B_2^{ij} and $B_{2.5}^{ij}$, along with the earlier 1.5PN solutions, must now be substituted into $(-g)T^{\alpha\beta}$ and Eqs. (4.4a) (4.4d), with the result

$$\begin{aligned}\tau^{00}=&\sigma-\sigma^{ii}+4\sigma U-\frac{7}{8\pi}\nabla U^2\\&+\sigma(7U^2-8\Phi_1+2\Phi_2+2\ddot{X})-4\sigma^{ii}U\\&+\frac{1}{4\pi}\left\{\frac{5}{2}\dot{U}^2-4U\ddot{U}-8\dot{U}^{,k}V^k+2V^{i,j}(3V^{j,i}+V^{i,j})+4\dot{V}^jU^{,j}-4U^{,ij}\Phi_1^{ij}\right.\\&\left.+8\nabla U\cdot\nabla\Phi_1-4\nabla U\cdot\nabla\Phi_2-\frac{7}{2}\nabla U\cdot\nabla\ddot{X}-10U\nabla U^2-4U^{,ij}P_2^{ij}\right\}\\&+\frac{4}{3}\sigma\overset{(3)}{\mathcal{I}^{kk}}(t)+\frac{1}{2\pi}U^{,ij}\overset{(3)}{\mathcal{I}^{ij}}(t),\end{aligned} \tag{5.9a}$$

$$\tau^{ii}=\sigma^{ii}-\frac{1}{8\pi}\nabla U^2+4\sigma^{ii}U-\frac{1}{4\pi}\left\{\frac{9}{2}\dot{U}^2+4V^{i,j}V^{[i,j]}+4\dot{V}^jU^{,j}+\frac{1}{2}\nabla U\cdot\nabla\ddot{X}\right\}. \tag{5.9b}$$

Substituting into Eqs. (4.8a) and (4.8c) and evaluating terms through $O(\epsilon^{7/2})$, and verifying that the outer integrals and surface terms make no $\mathcal{R}$-independent contributions, we obtain

$$\begin{aligned}N_2=&-16U\Phi_1+8U\Phi_2+7U\ddot{X}+\frac{20}{3}U^3-4V^iV^i-16\Sigma(\Phi_1)+\Sigma(\ddot{X})+8\Sigma^i(V^i)\\&-2\ddot{X}_1+\ddot{X}_2+\frac{1}{6}\overset{(4)}{Y}-4G_1-16G_2+32G_3+24G_4-16G_5-16G_6-16H,\end{aligned} \tag{5.10a}$$

$$B_2=U\ddot{X}+4V^iV^i-\Sigma(\ddot{X})-8\Sigma^i(V^i)+16\Sigma^{ii}(U)+2\ddot{X}_1-\ddot{X}_2-20G_1+8G_4+16G_5, \tag{5.10b}$$

$$N_{2.5}=-\frac{1}{15}(2x^{kl}+r^2\delta^{kl})\overset{(5)}{\mathcal{I}^{kl}}(t)+\frac{2}{15}x^k\overset{(5)}{\mathcal{I}^{kll}}(t)-\frac{1}{30}\overset{(5)}{\mathcal{I}^{kkll}}(t)+\frac{16}{3}U\overset{(3)}{\mathcal{I}^{kk}}(t)-4X^{,kl}\overset{(3)}{\mathcal{I}^{kl}}(t), \tag{5.10c}$$

$$B_{2.5}=-\frac{1}{3}r^2\overset{(5)}{\mathcal{I}^{ii}}(t)+\frac{2}{9}x^k\overset{(5)}{\mathcal{I}^{iik}}(t)+\frac{8}{9}x^k\epsilon^{mki}\overset{(4)}{\mathcal{J}^{mi}}(t)-\frac{2}{3}\overset{(3)}{M^{iikk}}(t). \tag{5.10d}$$

C. Far-zone field to 1.5PN order

In anticipation of finding non-zero outer-integral contributions to the near-zone field at 3PN order, we must determine the far-zone field to an order needed for the source $\Lambda^{\alpha\beta}$. Our foregoing discussion indicates that counting PN orders for outer integrals is different than the standard method, because the inverse radial variable $r^{-1}<\mathcal{R}^{-1}\sim v/\mathcal{S}$; in other words, when considering contributions to the outer integrals, additional powers of r in a term in the far-zone field can be regarded as increasing the effective order of that term by half a power in ϵ. For example, expanding $h_{\mathcal{N}}^{00}=N_{\mathcal{N}}$ in the far-zone, Eq. (2.13), we obtain

$$N_{\mathcal{N}}=4\left\{\underset{\epsilon}{\frac{\mathcal{I}}{r}}+\underset{\epsilon^2}{\frac{1}{2}\partial_{kl}\left(\frac{\mathcal{I}^{kl}(u)}{r}\right)}-\underset{\epsilon^{5/2}}{\frac{1}{6}\partial_{klm}\left(\frac{\mathcal{I}^{klm}(u)}{r}\right)}+\cdots,\right. \tag{5.11}$$

where the effective PN order of each term is indicated. In ordinary applications, the second potential in Eq. (5.11) would contribute a term of order ϵ of the form $\hat{n}^{\langle kl\rangle}\mathcal{I}^{kl}/r^3$, which is simply the Newtonian quadrupole potential. But in the outer integral, this term contributes an $\mathcal{R}$-independent term only through several time derivatives, and thus its effective contribution is higher order, in fact of the same order as that of the term $\hat{n}^{kl}\ddot{\mathcal{I}}^{kl}/r$, which also comes from the second potential.

At this order, we must also be careful to include any outer integral and boundary contributions to the far-zone field. From the lowest-order far-zone field, we find, to the order needed, that $\Lambda^{00}=-14(\mathcal{I}/r^2)^2$, $\Lambda^{ij}=4(\mathcal{I}/r^2)^2(\hat{n}^{\langle ij\rangle}-\delta^{ij}/6)$. Evaluating the coefficients $\mathcal{D}^0_{4,2}$ and $\mathcal{D}^0_{4,0}$, Eq. (2.29), we obtain, in the far zone, $N_{\mathcal{C}-\mathcal{N}}=7(\mathcal{I}/r)^2$ and $B^{ij}_{\mathcal{C}-\mathcal{N}}=(\mathcal{I}/r)^2\hat{n}^{ij}$. Combining the multipole expansions of Eq. (2.13) with the outer integral contributions, we obtain in the far-zone, to the order needed,

$$N=4\frac{\mathcal{I}}{r}+2\partial_{kl}\left(\frac{\mathcal{I}^{kl}(u)}{r}\right)-\frac{2}{3}\partial_{klm}\left(\frac{\mathcal{I}^{klm}(u)}{r}\right)+7\frac{\mathcal{I}^2}{r^2}+O(\epsilon^3), \tag{5.12a}$$

$$K^i=-2\partial_k\left(\frac{\dot{\mathcal{I}}^{ik}(u)}{r}\right)+2\epsilon^{aib}\frac{\hat{n}^a\mathcal{J}^b}{r^2}+\frac{2}{3}\partial_{kl}\left(\frac{\dot{\mathcal{I}}^{ikl}(u)}{r}\right)+\frac{4}{3}\epsilon^{aib}\partial_{ak}\left(\frac{\mathcal{J}^{bk}(u)}{r}\right)+O(\epsilon^3), \tag{5.12b}$$

$$B^{ij}=2\frac{\ddot{\mathcal{I}}^{ij}(u)}{r}+\frac{2}{3}\partial_k\left(\frac{\ddot{\mathcal{I}}^{ijk}(u)}{r}\right)+\frac{8}{3}\epsilon^{ak(i}\partial_k\left(\frac{\dot{\mathcal{J}}^{a|j)}(u)}{r}\right)+\frac{\mathcal{I}^2}{r^2}\hat{n}^{ij}+O(\epsilon^3). \tag{5.12c}$$

It will turn out, however, that, despite the formal possibility of 3PN contributions from the outer integrals, the *actual* contributions will not begin until 4PN order (see Sec. VI C).

VI. EXPANSION OF NEAR-ZONE FIELDS TO 3.5PN ORDER

A. B^{ij} and K^j to 3PN and 3.5PN order

At 3PN and 3.5PN order, we obtain, from Eqs. (2.5), (4.4b), (4.4c), (5.2) and (5.4),

$$\tau^{ij}=\sigma^{ij}+\frac{1}{4\pi}\left(U^{,i}U^{,j}-\frac{1}{2}\delta^{ij}\nabla U^2\right)+4\sigma^{ij}U+\frac{1}{4\pi}\left\{U^{,(i}\ddot{X}^{,j)}-16V^{[i,k]}V^{[j,k]}+8U^{,(i}\dot{V}^{j)}-\delta^{ij}\left(\frac{1}{2}\nabla U\cdot\nabla\ddot{X}-4V^{[l,k]}V^{[l,k]}+4U^{,(k}\dot{V}^{k)}+\frac{3}{2}\dot{U}^2\right)\right\}+O(\rho\epsilon^3), \tag{6.1a}$$

$$\begin{aligned}\tau^{0i}=&\sigma^i+4\sigma^iU+\frac{2}{\pi}U^{,j}V^{[j,i]}+\frac{3}{4\pi}\dot{U}U^{,i}+\sigma^i(7U^2-8\Phi_1+2\Phi_2+2\ddot{X})+\frac{1}{16\pi}\{64U^{,k}(V_2^{[k,i]}-\Phi_2^{[k,i]})+32UU^{,k}V^{k,i}\\&-16UU^{,k}V^{i,k}+16U^{,i}U^{,k}V^k-24V^i(\nabla U)^2+16U^{,k}\ddot{X}^{[k,i]}+128U^{,k}G_7^{[k,i]}-32\Phi_1^{;k}V^{[k,i]}-16\Phi_2^{;k}V^{i,k}-16\ddot{X}^{,k}V^{[k,i]}-16\dot{U}\Phi_1^{;i}\\&+48U\dot{U}U^{,i}+6\dot{U}\ddot{X}^{,i}+6U^{,i}\overset{(3)}{X}-16U^{,i}\dot{\Phi}_1+16\dot{U}\dot{V}^i-16U\ddot{V}^i-32V^k\dot{V}^{i,k}-16V^{i,kl}(\Phi_1^{kl}+P_2^{kl})+16U^{,k}(\dot{\Phi}_1^{ik}+\dot{P}_2^{ik})\\&+16V^{k,l}(\Phi_1^{il,k}+\Phi_1^{ik,l}-\Phi_1^{kl,i})+16V^{k,l}(P_2^{il,k}+P_2^{ik,l}-P_2^{kl,i})\}+\frac{4}{3}\sigma^i\overset{(3)}{\mathcal{I}}{}^{kk}(t)+\frac{1}{2\pi}(V^{i,kl}\overset{(3)}{\mathcal{I}}{}^{kl}(t)-U^{,k}\overset{(4)}{\mathcal{I}}{}^{ik}(t)),\end{aligned} \tag{6.1b}$$

where the first line in each expression is the contribution through 2PN order obtained earlier. Substituting into Eqs. (4.8b), (4.8c), and keeping contributions through $O(\epsilon^{7/2})$, and checking that surface terms and outer integrals make no contribution to this order, we obtain

$$B_3^{ij}=16\Sigma^{ij}(U)+4P(U^{,(i}\ddot{X}^{,j)})-64P(V^{[i,k]}V^{[j,k]})+32P(U^{,(i}\dot{V}^{j)})+2\ddot{X}^{ij}+2\ddot{S}(U^{,i}U^{,j})+\delta^{ij}(U\ddot{X}-4V^kV^k-\Sigma(\ddot{X})+8\Sigma^k(V^k)-\ddot{X}_2-8G_1-8G_4+16G_5), \tag{6.2a}$$

$$\begin{aligned}K_3^i=&12U^2V^i+16UV_2^i-16U\Phi_2^i+4U\ddot{X}^i+32UG_7^i+4V^i\ddot{X}-8\Phi_1V^i+8\Phi_2V^i-8V^k\Phi_1^{ik}-8V^kP_2^{ik}-16\Sigma(V_2^i)+16\Sigma(\Phi_2^i)\\&-16\Sigma(UV^i)-4\Sigma(\ddot{X}^i)-32\Sigma(G_7^i)-24\Sigma^i(\Phi_1)+4\Sigma^i(\ddot{X})+8\Sigma^{kk}(V^i)+8\Sigma^k(\Phi_1^{ik})+8\Sigma^k(P_2^{ik})+8\Sigma^{ik}(V^k)\\&+24P(U\dot{U}U^{,i})+24P(U^{,k}U^{,i}V^k)+32P(U^{,k}V_2^{k,i})-32P(U^{,k}\Phi_2^{k,i})+64P(U^{,k}G_7^{k,i})+8P(U^{,k}\ddot{X}^{k,i})+16P(U^{,k}\dot{\Phi}_1^{ik})\\&+16P(U^{,k}\dot{P}_2^{ik})-16P(U^{,i}\dot{\Phi}_1)+6P(U^{,i}\overset{(3)}{X})-16P(\dot{U}\Phi_1^{;i})+6P(\dot{U}\ddot{X}^{,i})+32P(UU^{,k}V^{i,k})-16P(U\ddot{V}^i)\end{aligned}$$

$$+16P(V^{k,l}\Phi_1^{il,k})-16P(V^{k,l}\Phi_1^{kl,i})+16P(V^{k,l}P_2^{il,k})-32P(V^k\dot{V}^{i,k})-16P(V^{k,i}\Phi_1^{,k})+8P(V^{k,i}\ddot{X}^{,k})-16P(V^{k,l}P_2^{kl,i})$$

$$-16P(V^{i,lm}\Phi_1^{lm})-16P(V^{i,lm}P_2^{lm})+4\ddot{X}^i(U)-4\ddot{X}(V^i)+8\ddot{S}(U^{,k}V^{k,i})+6\ddot{S}(U^{,i}\dot{U})+\frac{1}{6}\overset{(3)}{Y^i}, \tag{6.2b}$$

$$B_{3.5}^{ij}=-\frac{1}{3}r^2\overset{(5)}{\mathcal{I}^{ij}}(t)+\frac{2}{9}x^k\overset{(5)}{\mathcal{I}^{ijk}}(t)+\frac{8}{9}x^k\epsilon^{mk(i}\overset{(4)}{\mathcal{J}^{m|j)}}(t)-\frac{2}{3}\overset{(3)}{M^{ijkk}}(t), \tag{6.2c}$$

$$K_{3.5}^i=\frac{1}{15}r^2x^k\overset{(6)}{\mathcal{I}^{ik}}(t)-\frac{1}{45}r^2\overset{(6)}{\mathcal{I}^{ikk}}(t)-\frac{2}{45}x^{kl}\overset{(6)}{\mathcal{I}^{ikl}}(t)+\frac{1}{30}x^k\overset{(6)}{\mathcal{I}^{ikll}}(t)-\frac{1}{150}\overset{(6)}{\mathcal{I}^{ikkll}}(t)$$
$$+\epsilon^{mil}\left[\frac{2}{45}r^2\overset{(5)}{\mathcal{J}^{ml}}(t)+\frac{4}{45}x^{kl}\overset{(5)}{\mathcal{J}^{mk}}(t)-\frac{1}{30}x^l\overset{(5)}{\mathcal{J}^{mkk}}(t)-\frac{1}{15}x^k\overset{(5)}{\mathcal{J}^{mkl}}(t)+\frac{2}{75}\overset{(5)}{\mathcal{J}^{mlkk}}(t)\right]$$
$$+\frac{16}{3}V^i\overset{(3)}{\mathcal{I}^{kk}}(t)-4X^{i,kl}\overset{(3)}{\mathcal{I}^{kl}}(t)+4X^{,k}\overset{(4)}{\mathcal{I}^{ik}}(t). \tag{6.2d}$$

B. N and B to 3PN and 3.5PN order

The expressions for τ^{00} and τ^{ii} to 3PN and 3.‘5PN order are too lengthy to be reproduced explicitly. Instead, by substituting the expansions (5.1) into Eqs. (2.5), (4.4b) and (4.4c), and keeping terms of $O(\rho\epsilon^3)$ and $O(\rho\epsilon^{3.5})$, we obtain the formal contributions

$$\tau_3^{00}=\sigma(N_2-B_2-N_0B_1+K_1^iK_1^i)-\sigma^{ii}(N_1-B_1)$$
$$+\frac{1}{16\pi}\Bigg\{-\frac{7}{8}[2\nabla N_0\cdot\nabla N_2+(\nabla N_1)^2]+\frac{5}{4}\dot{N}_0\dot{N}_1-\ddot{N}_0N_1-N_0\ddot{N}_1$$
$$-2\dot{N}_0^{,i}K_2^i-2\dot{N}_1^{,i}K_1^i+K_1^{i,j}(3K_2^{j,i}+K_2^{i,j})+N_0^{,i}\dot{K}_2^i+N_1^{,i}\dot{K}_1^i-N_0^{,ij}B_3^{ij}$$
$$-N_1^{,ij}B_2^{ij}+\frac{1}{4}(\nabla N_0\cdot\nabla B_2+\nabla N_1\cdot\nabla B_1)+\frac{7}{8}N_1(\nabla N_0)^2$$
$$+\frac{7}{4}N_0(\nabla N_0\cdot\nabla N_1)+K_1^{i,j}\dot{B}_2^{ij}+\frac{1}{4}B_2^{ij,k}(B_2^{ij,k}-2B_2^{jk,i})+\frac{1}{4}\dot{N}_0\dot{B}_1-\frac{1}{8}(\nabla B_1)^2$$
$$+\frac{1}{4}\dot{N}_0N_0^{,i}K_1^i+\frac{7}{8}N_0^{,i}N_0^{,j}B_2^{ij}-\frac{1}{2}N_0^{,i}K_1^j(4K_1^{i,j}+3K_1^{j,i})-\frac{7}{8}N_0^2(\nabla N_0)^2\Bigg\}, \tag{6.3a}$$

$$\tau_{3.5}^{00}=\sigma(N_{2.5}-B_{2.5}-N_0B_{1.5})-\sigma^{ii}(N_{1.5}-B_{1.5})$$
$$+\frac{1}{16\pi}\Bigg\{-\frac{7}{4}\nabla N_0\cdot\nabla N_{2.5}+\frac{5}{4}\dot{N}_0\dot{N}_{1.5}-\ddot{N}_0N_{1.5}-N_0\ddot{N}_{1.5}-2\dot{N}_0^{,i}K_{2.5}^i+N_0^{,i}\dot{K}_{2.5}^i$$
$$+K_1^{i,j}(3K_{2.5}^{j,i}+K_{2.5}^{i,j})-N_0^{,ij}B_{3.5}^{ij}-N_1^{,ij}B_{2.5}^{ij}+\frac{1}{4}\nabla N_0\cdot\nabla B_{2.5}+\frac{7}{8}N_{1.5}(\nabla N_0)^2+K_1^{i,j}\dot{B}_{2.5}^{ij}$$
$$+\frac{1}{4}\dot{N}_0\dot{B}_{1.5}+\frac{7}{8}N_0^{,i}N_0^{,j}B_{2.5}^{ij}\Bigg\}, \tag{6.3b}$$

$$\tau_3^{ii}=\sigma^{ii}(N_1-B_1)$$
$$+\frac{1}{16\pi}\Bigg\{-\frac{1}{8}[2\nabla N_0\cdot\nabla N_2+(\nabla N_1)^2]+2K_1^{i,j}K_2^{[j,i]}-N_0^{,i}\dot{K}_2^i-N_1^{,i}\dot{K}_1^i-\frac{1}{4}(\nabla N_0\cdot\nabla B_2+\nabla N_1\cdot\nabla B_1)$$
$$-\frac{9}{4}\dot{N}_0\dot{N}_1+\frac{1}{4}N_1(\nabla N_0)^2+\frac{1}{2}N_0(\nabla N_0\cdot\nabla N_1)+2\dot{K}_1^i\dot{K}_1^i-2\dot{B}_1^{,i}K_1^i+3\dot{B}_2^{ij}K_1^{i,j}-N_0\ddot{B}_1$$

$$
\begin{aligned}
&+\frac{3}{4}\dot{N}_0\dot{B}_1+\frac{1}{8}(\nabla B_1)^2-B_1^{,ij}B_2^{ij}+\frac{3}{4}B_2^{ij,k}B_2^{ij,k}+\frac{1}{2}B_2^{ij,k}B_2^{ik,j}-N_0K_1^{i,j}K_1^{[j,i]}-\frac{1}{2}N_0^{,i}K_1^jK_1^{j,i}+N_0N_0^{,i}\dot{K}_1^i\\
&+\frac{1}{4}\dot{N}_0N_0^{,i}K_1^i-\frac{1}{8}N_0^{,i}N_0^{,j}B_2^{ij}+\frac{1}{4}N_0\nabla N_0\cdot\nabla B_1+\frac{1}{8}(\nabla N_0)^2B_1+\frac{9}{8}N_0\dot{N}_0^2-\frac{3}{8}N_0^2(\nabla N_0)^2\Bigg\},
\end{aligned}
\tag{6.3c}
$$

$$
\begin{aligned}
\tau_{3.5}^{ii}=\sigma^{ii}(N_{1.5}-B_{1.5})+\frac{1}{16\pi}\Bigg\{&-\frac{1}{4}\nabla N_0\cdot\nabla N_{2.5}+2K_1^{i,j}K_{2.5}^{[j,i]}-N_0^{,i}\dot{K}_{2.5}^i-\frac{1}{4}\nabla N_0\cdot\nabla B_{2.5}-\frac{9}{4}\dot{N}_0\dot{N}_{1.5}\\
&+\frac{1}{4}N_{1.5}(\nabla N_0)^2+3\dot{B}_{2.5}^{ij}K_1^{i,j}-N_0\ddot{B}_{1.5}+\frac{3}{4}\dot{N}_0\dot{B}_{1.5}-B_1^{,ij}B_{2.5}^{ij}-\frac{1}{8}N_0^{,i}N_0^{,j}B_{2.5}^{ij}+\frac{1}{8}(\nabla N_0)^2B_{1.5}\Bigg\}.
\end{aligned}
\tag{6.3d}
$$

We have simplified the expressions slightly by taking into account the fact that $N_{1.5}$, $B_{1.5}$ and $B_{2.5}^{ij}$ are purely functions of time, so that spatial gradients of them vanish. To obtain the full expressions, one substitutes for N_0, N_1, B_1, K_1^i, etc., from Eqs. (5.2), (5.4), (5.8), (5.10), and (6.2). Substituting this into Eqs. (4.8a) and (4.8c) (the latter contracted on indices ij), and including surface terms and outer integrals, we obtain the final 3PN and 3.5PN results for N and B:

$$
\begin{aligned}
N_3=&\frac{19}{6}U^4-28U^2\Phi_1+14U^2\Phi_2+10U^2\ddot{X}-8U\ddot{X}_1+4U\ddot{X}_2+4U\Sigma(\ddot{X})-4UG_1-56UG_2+112UG_3+80UG_4-64UG_5\\
&-56UG_6-56UH+\frac{7}{12}U\overset{(4)}{Y}+32U\Sigma^i(V^i)-56U\Sigma(\Phi_1)-8U\Sigma^{ii}(U)+10\Phi_1^2-8\Phi_1\Phi_2+2\Phi_2^2-8\Phi_1\ddot{X}+4\Phi_2\ddot{X}-16V^iV_2^i\\
&+16V^i\Phi_2^i-32V^iG_7^i-4V^i\ddot{X}^i+\frac{7}{4}\ddot{X}^2-2\Phi_1^{ij}\Phi_1^{ij}-4\Phi_1^{ij}P_2^{ij}-2P_2^{ij}P_2^{ij}-8\Sigma(U\Phi_1)+36\Sigma(G_1)-8\Sigma(G_2)+16\Sigma(G_3)\\
&-48\Sigma(G_4)-8\Sigma(G_6)-8\Sigma(H)-8\Sigma(\ddot{X}_1)+\frac{1}{12}\Sigma(\overset{(4)}{Y})+16\Sigma^i(V_2^i)-16\Sigma^i(\Phi_2^i)-16\Sigma^i(UV^i)+32\Sigma^i(G_7^i)+4\Sigma^i(\ddot{X}^i)\\
&+8\Sigma^{ii}(U^2)+12\Sigma^{ii}(\Phi_1)+4\Sigma^{ij}(\Phi_1^{ij})+4\Sigma^{ij}(P_2^{ij})-8\Sigma(\Sigma(\Phi_1))+64\Sigma(\Sigma^i(V^i))-56\Sigma(\Sigma^{ii}(U))-32P(U^2\ddot{U})\\
&-28P(U\dot{U}^2)+16P(\ddot{U}\Phi_1)+16P(\dot{U}\dot{\Phi}_1)+16P(U\ddot{\Phi}_1)-8P(U\ddot{\Phi}_2)-8P(U\overset{(4)}{X})-4P(\dot{U}\overset{(3)}{X})-8P(\ddot{U}\ddot{X})+32P(UU^{,i}\dot{V}^i)\\
&-64P(U\dot{U}^{,i}V^i)-40P(\dot{U}U^{,i}V^i)+32P(UV^{i,j}V^{i,j})+16P(UV^{i,j}V^{j,i})+32P(U^{,i}\dot{V}_2^i)-32P(U^{,i}\dot{\Phi}_2^i)+64P(U^{,i}\dot{G}_7^i)\\
&+8P(U^{,i}\overset{(3)}{X}{}^i)+64P(\dot{U}^{,i}\Phi_2^i)-64P(\dot{U}^{,i}V_2^i)-128P(\dot{U}^{,i}G_7^i)-16P(\dot{U}^{,i}\ddot{X}^i)+4P(U^{,i}U^{,j}\Phi_1^{ij})+4P(U^{,i}U^{,j}P_2^{ij})\\
&-64P(U^{,ij}\Sigma^{ij}(U))-8P(U^{,ij}\ddot{S}(U^{,i}U^{,j}))-128P(U^{,ij}P(U^{,(i}\dot{V}^{j)}))-16P(U^{,ij}P(U^{,(i}\ddot{X}^{,j)}))+256P(U^{,ij}P(V^{[k,i]}V^{[k,j]}))\\
&-8P(U^{,ij}\ddot{X}^{ij})+32P(V^i\dot{\Phi}_1^{,i})-16P(V^i\dot{\Phi}_2^{,i})-16P(V^i\overset{(3)}{X}{}^{,i})+16P(V^{i,j}\dot{\Phi}_1^{ij})+16P(V^{i,j}\dot{P}_2^{ij})+96P(V^{i,j}V_2^{j,i})-96P(V^{i,j}\Phi_2^{j,i})\\
&+192P(V^{i,j}G_7^{j,i})+24P(V^{i,j}\ddot{X}^{j,i})+8P(\dot{V}^i\dot{V}^i)-16P(\dot{V}^i\Phi_1^{,i})+8P(\dot{V}^i\ddot{X}^{,i})-8P(\Phi_1^{ij,k}\Phi_1^{jk,i})-16P(\Phi_1^{ij,k}P_2^{jk,i})\\
&-8P(P_2^{ij,k}P_2^{jk,i})+16P(\Phi_1^{,ij}\Phi_1^{ij})+16P(\Phi_1^{,ij}P_2^{ij})-8P(\Phi_1^{ij}\ddot{X}^{,ij})-8P(\Phi_2^{,ij}\Phi_1^{ij})-8P(\Phi_2^{,ij}P_2^{ij})-8P(P_2^{ij}\ddot{X}^{,ij})\\
&-\frac{1}{6}\overset{(4)}{Y}_1+\frac{1}{12}\overset{(4)}{Y}_2+\frac{1}{180}\overset{(6)}{Z}-8\ddot{X}(\Phi_1)+\frac{1}{2}\ddot{X}(\ddot{X})+4\ddot{X}^i(V^i)-2\ddot{S}(\dot{U}^2)-8\ddot{S}(U\ddot{U})-8\ddot{S}(U^{,ij}\Phi_1^{ij})\\
&-8\ddot{S}(U^{,ij}P_2^{ij})+12\ddot{S}(V^{i,j}V^{j,i})+8\ddot{S}(U^{,i}\dot{V}^i)-16\ddot{S}(V^i\dot{U}^{,i}),
\end{aligned}
\tag{6.4a}
$$

$$
\begin{aligned}
B_3=&\frac{1}{2}U^4-4U^2\Phi_1-2U^2\Phi_2+16UV^iV^i-12UG_1-8UG_2+16UG_3+16UG_4-8UG_6-8UH+\frac{1}{12}U\overset{(4)}{Y}-8U\Sigma(\Phi_1)\\
&+8U\Sigma^{ii}(U)-2\Phi_1^2+8\Phi_1\Phi_2+2\Phi_2^2+16V^iV_2^i-16V^i\Phi_2^i+32V^iG_7^i+4V^i\ddot{X}^i+\frac{1}{4}\ddot{X}^2-6\Phi_1^{ij}\Phi_1^{ij}-12\Phi_1^{ij}P_2^{ij}-6P_2^{ij}P_2^{ij}\\
&+8\Sigma(U\Phi_1)+12\Sigma(G_1)+8\Sigma(G_2)-16\Sigma(G_3)-16\Sigma(G_4)+8\Sigma(G_6)+8\Sigma(H)-\frac{1}{12}\Sigma(\overset{(4)}{Y})-16\Sigma^i(V_2^i)+16\Sigma^i(\Phi_2^i)
\end{aligned}
$$

$$
\begin{aligned}
&-48\Sigma^i(UV^i)-32\Sigma^i(G_7^i)-4\Sigma^i(\ddot{X}^i)+24\Sigma^{ii}(U^2)-28\Sigma^{ii}(\Phi_1)+8\Sigma^{ii}(\ddot{X})+12\Sigma^{ij}(\Phi_1^{ij})+12\Sigma^{ij}(P_2^{ij})+8\Sigma(\Sigma(\Phi_1))\\
&-8\Sigma(\Sigma^{ii}(U))-68P(U\dot{U}^2)-16P(U\ddot{\Phi}_1)+8P(U\ddot{\Phi}_2)+48P(\dot{U}\dot{\Phi}_1)-20P(\dot{U}\overset{(3)}{X})-56P(\dot{U}U^{,i}V^i)-32P(UU^{,i}\dot{V}^i)\\
&+32P(UV^{i,j}V^{i,j})-16P(UV^{i,j}V^{j,i})-32P(U^{,i}\dot{V}_2^i)+32P(U^{,i}\dot{\Phi}_2^i)-64P(U^{,i}\dot{G}_7^i)-8P(U^{,i}\overset{(3)}{X^i})-4P(U^{,i}U^{,j}\Phi_1^{ij})\\
&-4P(U^{,i}U^{,j}P_2^{ij})-32P(V^i\dot{\Phi}_1^{;i})+16P(V^i\dot{\Phi}_2^{;i})+48P(V^{i,j}\dot{\Phi}_1^{ij})+48P(V^{i,j}\dot{P}_2^{ij})+32P(V^{i,j}V_2^{j,i})-32P(V^{i,j}\Phi_2^{j,i})\\
&+64P(V^{i,j}G_7^{j,i})+8P(V^{i,j}\ddot{X}^{j,i})+24P(\dot{V}^i\dot{V}^i)+16P(\dot{V}^i\Phi_1^{;i})-8P(\dot{V}^i\ddot{X}^{,i})+8P(\Phi_1^{ij,k}\Phi_1^{jk,i})+16P(\Phi_1^{ij,k}P_2^{jk,i})\\
&+8P(P_2^{ij,k}P_2^{jk,i})-16P(\Phi_1^{;ij}\Phi_1^{ij})-16P(\Phi_1^{;ij}P_2^{ij})+8P(\Phi_2^{;ij}\Phi_1^{ij})+8P(\Phi_2^{;ij}P_2^{ij})+\frac{1}{6}\overset{(4)}{Y}_1-\frac{1}{12}\overset{(4)}{Y}_2-\frac{1}{2}\ddot{X}(\ddot{X})-4\ddot{X}^i(V^i)\\
&+8\ddot{X}^{ii}(U)-10\ddot{S}(\dot{U}^2)+4\ddot{S}(V^{i,j}V^{j,i})-8\ddot{S}(U^{,i}\dot{V}^i)+\frac{4}{3}\mathcal{I}\overset{(4)}{\mathcal{I}}{}^{jj}(t),
\end{aligned}
\tag{6.4b}
$$

$$
\begin{aligned}
N_{3.5}=&-\frac{1}{420}r^4\overset{(7)}{\mathcal{I}}{}^{jj}(t)-\frac{1}{105}r^2x^{ij}\overset{(7)}{\mathcal{I}}{}^{ij}(t)+\frac{1}{105}r^2x^i\overset{(7)}{\mathcal{I}}{}^{ijj}(t)+\frac{2}{315}x^{ijk}\overset{(7)}{\mathcal{I}}{}^{ijk}(t)-\frac{1}{420}r^2\overset{(7)}{\mathcal{I}}{}^{iijj}(t)-\frac{1}{105}x^{ij}\overset{(7)}{\mathcal{I}}{}^{ijkk}(t)+\frac{1}{210}x^i\overset{(7)}{\mathcal{I}}{}^{ijjkk}(t)\\
&-\frac{1}{1260}\overset{(7)}{\mathcal{I}}{}^{iijjkk}(t)-\left(\frac{8}{15}x^{ij}U+\frac{6}{5}x^iX^{,j}+\frac{2}{3}r^2X^{,ij}-\frac{2}{9}x^kY^{,ijk}+\frac{11}{45}Y^{,ij}\right)\overset{(5)}{\mathcal{I}}{}^{ij}(t)+\left(\frac{16}{15}r^2U-\frac{34}{15}x^iX^{,i}+\frac{16}{5}X\right)\overset{(5)}{\mathcal{I}}{}^{jj}(t)\\
&-\frac{1}{45}(16x^iU-52X^{,i})\overset{(5)}{\mathcal{I}}{}^{ijj}(t)+\left(\frac{4}{9}x^kX^{,ij}-\frac{2}{27}Y^{,ijk}\right)\overset{(5)}{\mathcal{I}}{}^{ijk}(t)-\frac{2}{15}U\overset{(5)}{\mathcal{I}}{}^{iijj}(t)-\left(\frac{4}{3}X^{i,j}-\frac{8}{3}x^i\dot{X}^{,j}+\frac{10}{9}\dot{Y}^{,ij}\right)\overset{(4)}{\mathcal{I}}{}^{ij}(t)+8\dot{X}\overset{(4)}{\mathcal{I}}{}^{jj}(t)\\
&-\frac{8}{9}\dot{X}^{,i}\overset{(4)}{\mathcal{I}}{}^{ijj}(t)-\left(14UX^{,ij}+2\Sigma(X^{,ij})-4X_1^{;ij}+2X_2^{;ij}+\frac{2}{3}\ddot{Y}^{,ij}\right)\overset{(3)}{\mathcal{I}}{}^{ij}(t)+\frac{1}{3}(70U^2-16\Phi_1+20\Phi_2+4\ddot{X})\overset{(3)}{\mathcal{I}}{}^{jj}(t)\\
&+\frac{8}{3}U\overset{(3)}{M}{}^{jjkk}(t)-\frac{4}{3}X^{,ij}\overset{(3)}{M}{}^{ijkk}(t)+\frac{16}{9}x^kX^{,ij}\epsilon^{mk(i}\overset{(4)}{\mathcal{J}}{}^{m|j)}(t)-\frac{4}{9}(8x^kU-5X^{,k})\epsilon^{mkj}\overset{(4)}{\mathcal{J}}{}^{mj}(t)+\frac{16}{9}\dot{X}^{,k}\epsilon^{mkj}\overset{(3)}{\mathcal{J}}{}^{mj}(t),
\end{aligned}
\tag{6.4c}
$$

$$
\begin{aligned}
B_{3.5}=&-\frac{1}{60}r^4\overset{(7)}{\mathcal{I}}{}^{jj}(t)+\frac{1}{45}r^2x^i\overset{(7)}{\mathcal{I}}{}^{ijj}(t)-\frac{1}{5}(Y^{,ij}-6x^iX^{,j})\overset{(5)}{\mathcal{I}}{}^{ij}(t)-\frac{1}{5}(16X+2x^iX^{,i})\overset{(5)}{\mathcal{I}}{}^{jj}(t)-\frac{4}{15}X^{,i}\overset{(5)}{\mathcal{I}}{}^{ijj}(t)+12X^{i,j}\overset{(4)}{\mathcal{I}}{}^{ij}(t)\\
&+\left(6U^2+\frac{16}{3}\Phi_1-12\Phi_2\right)\overset{(3)}{\mathcal{I}}{}^{jj}(t)-(2UX^{,ij}-2\Sigma(X^{,ij})+4X_1^{;ij}-2X_2^{;ij}-8P_2^{ij})\overset{(3)}{\mathcal{I}}{}^{ij}(t)\\
&+\frac{4}{45}r^2x^k\epsilon^{mkj}\overset{(6)}{\mathcal{J}}{}^{mj}(t)+\frac{4}{3}X^{,k}\epsilon^{mkj}\overset{(4)}{\mathcal{J}}{}^{mj}(t)-\frac{1}{15}r^2\overset{(5)}{M}{}^{jjkk}(t)-\frac{2}{15}x^{jk}\overset{(5)}{M}{}^{iijk}(t)+\frac{2}{15}x^j\overset{(5)}{M}{}^{iijkk}(t)-\frac{1}{30}\overset{(5)}{M}{}^{iijjkk}(t).
\end{aligned}
\tag{6.4d}
$$

The final term in the expression for B_3 is purely a function of time, and as such does not affect the equations of motion through 3.5PN order. It comes in part from the surface terms Eqs. (C1) and in part from various integrations by parts of Poisson potentials to achieve the expressions shown. In N_3, all such terms cancel. Similarly, purely time-dependent terms which appear in $N_{3.5}$ and $B_{3.5}$ do not contribute to the equations of motion.

As expected, the outer integrals make their first formal contribution to the field at 3PN order; however, the *observable* contribution vanishes to this order, so we have not shown any such contributions explicitly in Eqs. (6.4). In the next subsection, we study the contributions of the outer integrals in more detail, and show that through 3.5PN order, all contributions from the outer integrals are pure gauge terms.

C. Outer integrals and the contributions of ''tails''

Our earlier qualitative discussion suggested that terms involving products of the monopole moment $\mathcal{I}$ and the quadrupole moment $\mathcal{I}^{ij}$ of the far-zone fields would contribute via the outer integrals at 3PN order. Because higher multipole moments involve higher powers of $1/r$ or higher time derivatives, they would be expected to contribute at even higher PN order. Thus working through 3.5PN order, we might expect at most that products of $\mathcal{I}$ with quadrupole $\mathcal{I}^{ij}$, octupole $\mathcal{I}^{ijk}$ or current quadrupole $\mathcal{J}^{ij}$ moments would contribute. Other terms, such as products of $\mathcal{I}$ with higher-order moments or products of higher-order moments, such as terms quadratic in $\mathcal{I}^{ij}$, will be 4PN order or higher. In studying the contribution of the outer integrals to the fields at 3.5PN order, therefore, it suffices to employ the far-zone field given in Eq. (5.12). However, to illustrate the first non-trivial ''tail'' contribution, we will evaluate certain pieces of the outer integrals through 4PN order.

We substitute Eqs. (5.12) into Eqs. (4.4) using the ''quick and dirty'' rule expressed by Eq. (4.9) to determine which terms to keep, and obtain, in the far zone:

$$\begin{aligned}\Lambda^{00}=&14\mathcal{I}r^{-2}\hat{n}^i\partial_{ijk}(\mathcal{I}^{jk}/r)-8\mathcal{I}r^{-1}\partial_{ij}(\ddot{\mathcal{I}}^{ij}/r)+8\mathcal{I}r^{-2}\hat{n}^i\partial_j(\ddot{\mathcal{I}}^{ij}/r)-24\mathcal{I}r^{-4}\hat{n}^{\langle ij\rangle}\ddot{\mathcal{I}}^{ij}\\&-2\mathcal{I}r^{-2}\hat{n}^i\partial_i(\ddot{\mathcal{I}}^{jj}/r)-\frac{14}{3}\mathcal{I}r^{-2}\hat{n}^i\partial_{ijkl}(\mathcal{I}^{jkl}/r)+\frac{8}{3}\mathcal{I}r^{-1}\partial_{ijk}(\ddot{\mathcal{I}}^{ijk}/r)-\frac{8}{3}\mathcal{I}r^{-2}\hat{n}^i\partial_{jk}(\ddot{\mathcal{I}}^{ijk}/r)-8\mathcal{I}r^{-3}\hat{n}^{\langle ij\rangle}\partial_k(\ddot{\mathcal{I}}^{ijk}/r)\\&-\frac{2}{3}\mathcal{I}r^{-2}\hat{n}^i\partial_{ij}(\ddot{\mathcal{I}}^{jkk}/r)+\frac{16}{3}\mathcal{I}r^{-2}\hat{n}^i\epsilon^{aib}\partial_{ak}(\dot{\mathcal{J}}^{bk}/r)-32\mathcal{I}r^{-3}\hat{n}^{\langle ij\rangle}\epsilon^{aki}\partial_k(\dot{\mathcal{J}}^{aj}/r)\\&-\frac{8}{3}\mathcal{I}r^{-2}\hat{n}^i\epsilon^{akj}\partial_{ik}(\dot{\mathcal{J}}^{aj}/r)+O(\rho\epsilon^4),\end{aligned}\tag{6.5a}$$

$$\begin{aligned}\Lambda^{0i}=&8\mathcal{I}r^{-2}\hat{n}^j\partial_{ik}(\dot{\mathcal{I}}^{jk}/r)-8\mathcal{I}r^{-2}\hat{n}^j\partial_{jk}(\dot{\mathcal{I}}^{ik}/r)-6\mathcal{I}r^{-2}\hat{n}^i\partial_{jk}(\dot{\mathcal{I}}^{jk}/r)+8\mathcal{I}r^{-1}\partial_j(\overset{(3)}{\mathcal{I}}{}^{ij}/r)\\&-\frac{8}{3}\mathcal{I}r^{-1}\partial_{jk}(\overset{(3)}{\mathcal{I}}{}^{ijk}/r)+\frac{16}{3}\mathcal{I}r^{-1}\epsilon^{iab}\partial_{ak}(\ddot{\mathcal{J}}^{bk}/r)-8\mathcal{I}r^{-2}\hat{n}^j(\overset{(3)}{\mathcal{I}}{}^{ij}/r)+2\mathcal{I}r^{-2}\hat{n}^i(\overset{(3)}{\mathcal{I}}{}^{jj}/r)+O(\rho\epsilon^{7/2}),\end{aligned}\tag{6.5b}$$

$$\begin{aligned}\Lambda^{ii}=&2\mathcal{I}r^{-2}\hat{n}^i\partial_{ijk}(\mathcal{I}^{jk}/r)-8\mathcal{I}r^{-2}\hat{n}^i\partial_j(\ddot{\mathcal{I}}^{ij}/r)+2\mathcal{I}r^{-2}\hat{n}^i\partial_i(\ddot{\mathcal{I}}^{jj}/r)-8\mathcal{I}r^{-2}\overset{(4)}{\mathcal{I}}{}^{ii}-\frac{2}{3}\mathcal{I}r^{-2}\hat{n}^i\partial_{ijkl}(\mathcal{I}^{jkl}/r)\\&+\frac{8}{3}\mathcal{I}r^{-2}\hat{n}^i\partial_{jk}(\ddot{\mathcal{I}}^{ijk}/r)+\frac{2}{3}\mathcal{I}r^{-2}\hat{n}^i\partial_{ij}(\ddot{\mathcal{I}}^{jkk}/r)-\frac{8}{3}\mathcal{I}r^{-1}\partial_i(\overset{(4)}{\mathcal{I}}{}^{ijj}/r)-\frac{16}{3}\mathcal{I}r^{-2}\hat{n}^i\epsilon^{aib}\partial_{ak}(\dot{\mathcal{J}}^{bk}/r)\\&+\frac{8}{3}\mathcal{I}r^{-2}\hat{n}^i\epsilon^{akj}\partial_{ik}(\dot{\mathcal{J}}^{aj}/r)-\frac{32}{3}\mathcal{I}r^{-1}\epsilon^{aki}\partial_k(\overset{(3)}{\mathcal{J}}{}^{ai}/r)+O(\rho\epsilon^4),\end{aligned}\tag{6.5c}$$

$$\Lambda^{ij}=-8\mathcal{I}r^{-2}\overset{(4)}{\mathcal{I}}{}^{ij}+O(\rho\epsilon^3).\tag{6.5d}$$

All moments $\mathcal{I}^{ij}$, $\mathcal{I}^{ijk}$, and $\mathcal{J}^{ij}$ in these expression are functions of retarded time $t-r$. Notice that the term kept in Λ^{ij} is actually of $O(\rho\epsilon^3)$ (4PN order) according to our scheme; however, because it has $1/r^2$ dependence, it will yield a 4PN tail contribution of a form which we wish to keep.

We expand the derivatives and evaluate the coefficients $\mathcal{E}^q_{B,L}$ and $\mathcal{E}^q_{2,L}$ [Eqs. (2.34) and (2.36)] for each term, throwing away all $\mathcal{R}$-dependent terms. Terms with $1/r^2$ falloff yield integrals over Legendre functions Q_L, as in Eq. (2.35). The result, through 3.5PN order (and keeping all formally 4PN terms involving integrals over Q_L), is

$$(N_3)_{\mathcal{C}-\mathcal{N}}=\mathcal{I}\left\{-8\hat{n}^{\langle ij\rangle}\int_1^\infty\overset{(4)}{\mathcal{I}}{}^{ij}(t-r\zeta)Q_2(\zeta)d\zeta-\frac{8}{3}\int_1^\infty\overset{(4)}{\mathcal{I}}{}^{jj}(t-r\zeta)Q_0(\zeta)d\zeta+\frac{4}{3}(\hat{n}^{\langle ij\rangle}-2\delta^{ij}-2\delta^{ij}\ln r)\overset{(4)}{\mathcal{I}}{}^{ij}(t)\right\},\tag{6.6a}$$

$$\begin{aligned}(N_{3.5})_{\mathcal{C}-\mathcal{N}}=\mathcal{I}\Bigg\{&-\frac{8}{3}\hat{n}^{\langle ijk\rangle}\int_1^\infty\overset{(5)}{\mathcal{I}}{}^{ijk}(t-r\zeta)Q_3(\zeta)d\zeta-\frac{8}{5}\hat{n}^i\int_1^\infty\overset{(5)}{\mathcal{I}}{}^{ijj}(t-r\zeta)Q_1(\zeta)d\zeta-\frac{2}{3}r(3\hat{n}^{\langle ij\rangle}-2\delta^{ij})\overset{(5)}{\mathcal{I}}{}^{ij}(t)\\&+\frac{2}{45}(5\hat{n}^{\langle ijk\rangle}+18\hat{n}^i\delta^{jk})\overset{(5)}{\mathcal{I}}{}^{ijk}(t)\Bigg\},\end{aligned}\tag{6.6b}$$

$$(K_{3.5})^i_{\mathcal{C}-\mathcal{N}}=\mathcal{I}\left\{-8\hat{n}^j\int_1^\infty\overset{(4)}{\mathcal{I}}{}^{ij}(t-r\zeta)Q_1(\zeta)d\zeta+4\hat{n}^j\overset{(4)}{\mathcal{I}}{}^{ij}(t)\right\},\tag{6.6c}$$

$$(K_4)^i_{\mathcal{C}-\mathcal{N}}=\mathcal{I}\Bigg\{-\frac{8}{3}\hat{n}^{\langle jk\rangle}\int_1^\infty \overset{(5)}{\mathcal{I}}{}^{ijk}(t-r\zeta)Q_2(\zeta)d\zeta-\frac{8}{9}\int_1^\infty \overset{(5)}{\mathcal{I}}{}^{ijj}(t-r\zeta)Q_0(\zeta)d\zeta$$
$$+\frac{16}{3}\hat{n}^{\langle ak\rangle}\epsilon^{iaj}\int_1^\infty \overset{(4)}{\mathcal{J}}{}^{jk}(t-r\zeta)Q_2(\zeta)d\zeta+\frac{16}{9}\epsilon^{ikj}\int_1^\infty \overset{(4)}{\mathcal{J}}{}^{jk}(t-r\zeta)Q_0(\zeta)d\zeta\Bigg\}, \tag{6.6d}$$

$$(B_3)_{\mathcal{C}-\mathcal{N}}=\mathcal{I}\Bigg\{-8\int_1^\infty \overset{(4)}{\mathcal{I}}{}^{ii}(t-r\zeta)Q_0(\zeta)d\zeta+8(1-\ln r)\overset{(4)}{\mathcal{I}}{}^{ii}(t)\Bigg\}, \tag{6.6e}$$

$$(B_{3.5})_{\mathcal{C}-\mathcal{N}}=\mathcal{I}\Bigg\{+\frac{8}{3}\hat{n}^i\int_1^\infty \overset{(5)}{\mathcal{I}}{}^{ijj}(t-r\zeta)Q_1(\zeta)d\zeta+\frac{32}{3}\epsilon^{aij}\hat{n}^i\int_1^\infty \overset{(4)}{\mathcal{J}}{}^{aj}(t-r\zeta)Q_1(\zeta)d\zeta+4r\overset{(5)}{\mathcal{I}}{}^{ii}(t)$$
$$-\frac{4}{3}\hat{n}^i\overset{(5)}{\mathcal{I}}{}^{ijj}(t)-\frac{16}{3}\epsilon^{aij}\hat{n}^i\overset{(4)}{\mathcal{J}}{}^{aj}(t)\Bigg\}, \tag{6.6f}$$

$$(B_4)^{ij}_{\mathcal{C}-\mathcal{N}}=-8\mathcal{I}\int_1^\infty \overset{(4)}{\mathcal{I}}{}^{ij}(t-r\zeta)Q_0(\zeta)d\zeta. \tag{6.6g}$$

Using the recursion relations satisfied by Legendre functions, we can establish the general formulas

$$\int_1^\infty X(t-r\zeta)Q_L(\zeta)d\zeta=\frac{1}{L(L+1)}X(t-r)-\frac{1}{2L+1}\int_1^\infty X'(t-r\zeta)[Q_{L+1}(\zeta)-Q_{L-1}(\zeta)]d\zeta,$$
$$\int_1^\infty X(t-r\zeta)Q_0(\zeta)d\zeta=X(t-r)-\int_1^\infty X'(t-r\zeta)[Q_1(\zeta)+Q_0(\zeta)]d\zeta+\int_0^\infty \dot{X}(t-r-s)\ln(s/2r)ds, \tag{6.7}$$

where a prime denotes $\partial/\partial\zeta$, $s=r(\zeta-1)$, X represents one of the multipole moments of the system ($\mathcal{I}^{ij}$ and higher), and we assume that, in the distant past, the system becomes sufficiently ''stationary'' that as $s\to\infty$, $X(t-r-s)\ln s\to 0$. Since for a binary system that becomes unbound ($r\to v_0 s$) in the infinite past (because of gravitational-radiation antidamping, looking backwards), X in the worst case is proportional to $(d/dt)^4\mathcal{I}^{ij}\sim mv^4/r^2\to mv_0^2/s^2$; then this boundary condition is satisfied (see [60] for a detailed discussion of the past behavior of binary systems whose evolution includes gravitational radiation reaction). Repeated use of these identities allows us to convert many of the integrals in Eqs. (6.6) into integrals of higher time derivatives of the expressions, which are thus of higher PN order, plus residual terms that cancel many of the non-integral terms in Eqs. (6.6). It is also useful to expand the retarded time $t-r-s$ about $t-s$, and to separate the $\ln r$ terms from the $\ln(s/2)$ terms in the integrals, leaving only terms proportional to $X^{(n)}(t)$ and $\int_0^\infty X^{(n)}(t-s)\ln(s/2)ds$. In the end, the only terms that remain at 3PN and 3.5PN order are

$$N_{\mathcal{C}-\mathcal{N}}=\mathcal{I}\Bigg\{-\frac{16}{3}\overset{(4)}{\mathcal{I}}{}^{ii}(t)-\frac{8}{3}\int_0^\infty \overset{(5)}{\mathcal{I}}{}^{ii}(t-s)\ln(s/2)ds\Bigg\}+O(\epsilon^5), \tag{6.8a}$$

$$K^i_{\mathcal{C}-\mathcal{N}}=O(\epsilon^{9/2}), \tag{6.8b}$$

$$B^{ij}_{\mathcal{C}-\mathcal{N}}=O(\epsilon^4), \tag{6.8c}$$

$$B_{\mathcal{C}-\mathcal{N}}=-8\mathcal{I}\int_0^\infty \overset{(5)}{\mathcal{I}}{}^{ii}(t-s)\ln(s/2)ds+O(\epsilon^5). \tag{6.8d}$$

As these are purely functions of time, they do not contribute to the equations of motion through 3.5PN order. Alternatively, one can show that the terms in Eqs. (6.6) turn out to be purely gauge terms through 3.5PN order. In fact, by making the gauge transformation $h^{\mu\nu}\to h^{\mu\nu}-\xi^{\mu,\nu}-\xi^{\nu,\mu}+\eta^{\mu\nu}\xi^\alpha_{,\alpha}$ (the linear transformation suffices to this order), with

$$\xi^0=\mathcal{I}\Bigg\{\frac{4}{3}\overset{(3)}{\mathcal{I}}{}^{ii}(t)+\frac{8}{3}\int_0^\infty \overset{(4)}{\mathcal{I}}{}^{ii}(t-s)\ln(s/2)ds$$
$$-\frac{2}{3}x^{ij}\int_0^\infty \overset{(6)}{\mathcal{I}}{}^{ij}(t-s)\ln(s/2)ds$$
$$+\frac{2}{3}r^2\int_0^\infty \overset{(6)}{\mathcal{I}}{}^{ii}(t-s)\ln(s/2)ds$$
$$+\frac{4}{45}x^i\int_0^\infty \overset{(6)}{\mathcal{I}}{}^{ijj}(t-s)\ln(s/2)ds$$
$$+\frac{8}{9}x^i\epsilon^{ikj}\int_0^\infty \overset{(5)}{\mathcal{J}}{}^{jk}(t-s)\ln(s/2)ds\Bigg\}, \tag{6.9}$$

$$\xi^i = \mathcal{I}\left\{ -4x^j \int_0^\infty \overset{(5)}{\mathcal{I}}{}^{ij}(t-s)\ln(s/2)ds + \frac{4}{3}x^i \int_0^\infty \overset{(5)}{\mathcal{I}}{}^{jj}(t-s)\ln(s/2)ds + \frac{44}{45}\int_0^\infty \overset{(5)}{\mathcal{I}}{}^{ijj}(t-s)\ln(s/2)ds - \frac{8}{9}\epsilon^{ikj}\int_0^\infty \overset{(4)}{\mathcal{J}}{}^{jk}(t-s)\ln(s/2)ds \right\}, \tag{6.10}$$

we can convert the outer integral contributions to $h^{\alpha\beta}$ in Eq. (6.6) to a form consisting of nothing but a 4PN tail term:

$$(N+B)_{\mathcal{C}-\mathcal{N}} = -\frac{16}{5}\mathcal{I}x^{ij}\int_0^\infty \overset{(7)}{\mathcal{I}}{}^{\langle ij\rangle}(t-s)\ln(s/2)ds + O(\epsilon^5),$$
$$(K^i)_{\mathcal{C}-\mathcal{N}} = O(\epsilon^{9/2}),$$
$$(B^{ij})_{\mathcal{C}-\mathcal{N}} = O(\epsilon^4). \tag{6.11}$$

Note that, to this order, $N+B=2g_{00}$, and only the gradient of the term in Eq. (6.11) contributes to the acceleration; hence this term can be thought of as a 4PN tail modification of the Newtonian gravitational potential or as a 1.5PN modification due to tails of the 2.5PN radiation-reaction potentials. This result is in complete agreement with the near-zone tail contribution derived by Blanchet and Damour [15] using matched asymptotic expansions within the post-Minkowskian formalism.

VII. DISCUSSION

We have presented a method for direct integration of the relaxed Einstein equations in a post-Newtonian expansion, applicable to equations of motion and gravitational radiation from isolated gravitating systems. As a foundation for future work, we presented a solution for the near-zone gravitational field through 3.5 post-Newtonian order in terms of Poisson potentials, together with a prescription for ensuring that no divergent or undefined integrals occur. In subsequent work, we will apply the near-zone results to the derivation of equations of motion for binary systems of compact objects through 2.5 PN order and including 3.5 PN radiation reaction terms. Work on the 3PN contributions to the equations of motion is in progress.

The results presented here can also be applied to the gravitational radiation waveform and energy flux from binary systems to as high as 3PN order beyond the quadrupole approximation. It can also be used to discuss equations of motion and radiation damping of systems containing spinning bodies, as well as the structure and evolution of fluid bodies. These will be the subject of future work.

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APPENDIX A: STF TENSORS AND THEIR PROPERTIES

Throughout this series of papers, we shall make frequent use of the properties of symmetric, trace-free (STF) products of unit vectors. The general formula for such STF products is

$$\hat{n}^{\langle L\rangle} \equiv \sum_{p=0}^{[l/2]} (-1)^p \frac{(2l-l-2p)!!}{(2l-1)!!}[\hat{n}^{L-2P}\delta^P + \mathrm{sym}(q)], \tag{A1}$$

where $[l/2]$ denotes the integer just less than or equal to $l/2$, the capitalized superscripts denote the dimensionality, $l-2p$ or p, of products of $\hat{n}^i$ or δ^{ij} respectively, and ''sym(q)'' denotes all distinct terms arising from permutations of indices, where $q=l!/[(2^p p!(l-2p)!]$ is the total number of such terms (see [14,26] for compendia of formulas). For convenience, we display the first several examples explicitly:

$$\hat{n}^{\langle ij\rangle} = \hat{n}^{ij} - \frac{1}{3}\delta^{ij}, \tag{A2a}$$

$$\hat{n}^{\langle ijk\rangle} = \hat{n}^{ijk} - \frac{1}{5}(\hat{n}^i\delta^{jk} + \hat{n}^j\delta^{ik} + \hat{n}^k\delta^{ij}), \tag{A2b}$$

$$\hat{n}^{\langle ijkl\rangle} = \hat{n}^{ijkl} - \frac{1}{7}[\hat{n}^{ij}\delta^{kl} + \mathrm{sym}(6)] + \frac{1}{35}(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}), \tag{A2c}$$

$$\hat{n}^{\langle ijklm>} = \hat{n}^{ijklm} - \frac{1}{9}[\hat{n}^{ijk}\delta^{kl} + \mathrm{sym}(10)] + \frac{1}{63}[\hat{n}^i\delta^{jk}\delta^{lm} + \mathrm{sym}(15)], \tag{A2d}$$

$$\hat{n}^{\langle ijklmn\rangle} = \hat{n}^{ijklmn} - \frac{1}{11}[\hat{n}^{ijkl}\delta^{mn} + \mathrm{sym}(15)] + \frac{1}{99}[\hat{n}^{ij}\delta^{kl}\delta^{mn} + \mathrm{sym}(45)] - \frac{1}{693}[\delta^{ij}\delta^{kl}\delta^{mn} + \mathrm{sym}(15)]. \tag{A2e}$$

There is a close connection between these STF tensors and spherical harmonics. For example, it is straightforward to show that, for any unit vector $\hat{\mathbf{N}}$, the contraction of $\hat{N}^L$ with $\hat{n}^{\langle L\rangle}$ is given by

$$\hat{N}^L \hat{n}^{\langle L\rangle} = \frac{l!}{(2l-1)!!} P_l(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}}), \tag{A3}$$

where P_l is a Legendre polynomial.

APPENDIX B: CANCELLATION OF THE $\mathcal{R}$ DEPENDENCE BETWEEN INNER AND OUTER INTEGRALS

Here we demonstrate explicitly the cancellation of $\mathcal{R}$-dependent terms between the inner and outer integrals. We assume that, at each iteration step, from just inside the boundary of the near zone out into the far zone, the source stress-energy tensor ${}_{N-1}\Lambda^{\alpha\beta}$ can be decomposed into terms of the form $f_{B,L}(u)\hat{n}^{\langle L\rangle} r^{-B}$, where $u=t-r$ is retarded time, and $\hat{n}^{\langle L\rangle}$ is a STF product of unit radial vectors. We calculate the behavior of the inner integral of such a term as the integration variable approaches $\mathcal{R}$ from below with the result obtained from the outer integral of the same term. We consider far-zone and near-zone field points separately.

1. Far-zone field points

The inner integral is given by Eq. (2.13), with the multipole moment given by Eq. (2.14). We want to examine the behavior of the moment, as $|\mathbf{x}'|\to\mathcal{R}$, that is

$$M^{\alpha\beta\bar{Q}}(u) \to \frac{1}{16\pi}\int^{\mathcal{R}} f_{B,L}(u-r')\frac{\hat{n}'^{\langle L\rangle}}{r'^B} x'^{\bar{Q}} r'^2 dr' d\Omega'$$

$$= \frac{1}{4}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!} f_{B,L}^{(m)}(u) G^m_{B,L,\bar{Q}}(\mathcal{R})\Delta^{L,\bar{Q}}, \tag{B1}$$

where the superscript (m) denotes m retarded time derivatives, and where

$$\Delta^{L,\bar{Q}} = \frac{1}{4\pi}\oint \hat{n}^{\langle L\rangle}\hat{n}^{\bar{Q}} d\Omega, \tag{B2a}$$

$$G^m_{B,L,\bar{Q}}(\mathcal{R}) = \int^{\mathcal{R}} r'^{2+\bar{q}-B+m} dr' = \begin{cases} \mathcal{R}^{3+\bar{q}-B+m}/(3+\bar{q}-B+m), & 3+\bar{q}-B+m \neq 0, \\ \ln\mathcal{R}, & 3+\bar{q}-B+m=0. \end{cases} \tag{B2b}$$

Then, from inside $\mathcal{R}$,

$$h^{\alpha\beta}_{\mathcal{N}\,B,L} \to \sum_{\bar{q}=0}^{\infty}\frac{(-1)^{\bar{q}}}{\bar{q}!}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\partial_{\bar{Q}} \times\left(\frac{1}{r}f_{B,L}^{(m)}(u)\right) G^m_{B,L,\bar{Q}}(\mathcal{R})\Delta^{L,\bar{Q}}. \tag{B3}$$

It is straightforward to show that the contraction of $\partial_{\bar{Q}}$ with $\Delta^{L,\bar{Q}}$ is given by

$$\Delta^{L,\bar{Q}}\partial_{\bar{Q}} = \begin{cases} 0 & \bar{q}<L, \\ 0 & L+\bar{q}=\text{odd}, \\ \dfrac{2^L \bar{q}!((\bar{q}+L)/2)!}{(\bar{q}+L+1)!((\bar{q}-L)/2)!}|\nabla^2|^{(\bar{q}-L)/2}\partial_{\langle L\rangle} & \bar{q}\geqslant L. \end{cases} \tag{B4}$$

Using the fact that

$$\nabla^2\left(\frac{f(u)}{r}\right) = \frac{\ddot{f}}{r}, \tag{B5a}$$

$$\partial_{\langle L\rangle}\left(\frac{f(u)}{r}\right) = (-1)^L \hat{n}^{\langle L\rangle}\sum_{k=0}^{L}\frac{(L+k)!}{2^k k!(L-k)!}\frac{f^{(L-k)}(u)}{r^{k+1}} \tag{B5b}$$

(see e.g. [14]) and redefining summation variables, $q=m+\bar{q}-k$, $j=L-k$, we obtain Eqs. (2.37) and (2.38). Evaluating the outer integral for the same term yields z-dependent or $\ln\mathcal{R}$-dependent terms that are precisely equal and opposite those of Eq. (2.38).

2. Near-zone field point

In the near zone, for $|\mathbf{x}'|>|\mathbf{x}|$, Eq. (2.22) together with the specific decomposition of $\Lambda^{\alpha\beta}$ gives

$$
{}_N h^{\alpha\beta}_{\mathcal{N}\,B,L} \to \frac{1}{4\pi} \sum_{\bar{q}=0}^{\infty} \frac{(-1)^{\bar{q}}}{\bar{q}!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} x^{\bar{Q}} \partial_t^n \times \int^{\mathcal{R}} f_{B,L}(t-r') \frac{\hat{n}'^{\langle L\rangle}}{r'^B} \partial'_{\bar{Q}}(r'^{m-1}) r'^2 dr' d\Omega'. \tag{B6}
$$

We use the fact that [14]

$$
\partial'_{\bar{Q}} r'^{m-1} = \sum_{k=0}^{k_m} \frac{(2\bar{q}-4k+1)!!}{(2\bar{q}-2k+1)!!} \frac{m!}{(m-2k)!} \times \left[\frac{(m-2k-1)!!}{(m-2\bar{q}+2k-1)!!} \right] \frac{\bar{q}!}{2^k k!(\bar{q}-2k)!} \times \delta^K \hat{n}'^{\langle \bar{Q}-2K\rangle} r'^{m-\bar{q}-1}, \tag{B7}
$$

where $k_m =$ lesser of$\{[\bar{q}/2],[m/2]\}$, δ^K denotes a product of K Kronecker deltas, the quantity in square brackets can be evaluated for negative or positive values of the arguments, and the expression $\delta^K \hat{n}'^{\langle \bar{Q}-2K\rangle}$ is to be symmetrized on all indices (since the expression ultimately is to be contracted on $x^{\bar{Q}}$ no explicit symmetrization is needed). It can then be shown that

$$
\hat{n}^{\bar{Q}} \frac{1}{4\pi} \oint \delta^K \hat{n}'^{\langle \bar{Q}-2K\rangle} \hat{n}'^{\langle L\rangle} d\Omega' = \frac{L!}{(2L+1)!!} \delta_{L,\bar{q}-2k} \hat{n}^{\langle L\rangle}. \tag{B8}
$$

We then expand $f(t-r') = \Sigma_{n=0}^{\infty} (-1)^n f^{(n)}(t) r'^n/n!$, integrate over r' toward $\mathcal{R}$, rearrange the summations, and define $r = (\bar{q}-L)/2$, and $q = m+n$, and obtain

$$
{}_N h^{\alpha\beta}_{\mathcal{N}\,B,L} \to \left(\frac{2}{r}\right)^{B-2} \hat{n}^{\langle L\rangle} \sum_{q=0}^{\infty} \mathcal{E}^{\text{in},q}_{B,L}(z) r^q \frac{d^q f_{B,L}(t)}{dt^q}, \tag{B9}
$$

with

$$
\mathcal{E}^{\text{in},q}_{B,L}(z) = \sum_{r=0}^{[q/2]} \sum_{m=2r}^{q} \frac{(-1)^{L+q}(2)^{2+L-B}(L+r)!}{(q-m)!(2L+2r+1)!(m-2r)!r!} \times \left[\frac{(m-1-2r)!!}{(m-1-2r-2L)!!} \right] \begin{cases} z^{q-L-2r-B+2}/(q-L-2r-B+2), & q-L-2r-B+2 \neq 0, \\ \ln \mathcal{R}, & q-L-2r-B+2 = 0. \end{cases} \tag{B10}
$$

Here, too, evaluating the outer integral for the same term, for each B, L and q yields z-dependent or $\ln\mathcal{R}$-dependent terms that are precisely equal and opposite those of Eq. (B10).

3. Source terms with ln *r* dependence

Until now we have assumed that the stress-energy source $\Lambda^{\alpha\beta}$ can be decomposed into terms of the form $f_{B,L}(u)\hat{n}^{\langle L\rangle} r^{-B}$. At sufficiently high PN order, tail contributions to the fields will arise, leading to the possibility of $\ln r$ dependence in $\Lambda^{\alpha\beta}$. To illustrate that cancellation of $\mathcal{R}$ dependence occurs in this event also, we consider source terms of the form $f_{B,L}(u')\hat{n}'^{\langle L\rangle} r'^{-B} \ln r'$. Noting that, from Eq. (2.26), $\ln r' = -\ln[2(\zeta-y)/r(\zeta^2-1)]$, and incorporating this logarithmic term into the outer integral, Eq. (2.27), we obtain

$$
{}_N h^{\alpha\beta(\ln)}_{\mathcal{C}-\mathcal{N}\,B,L} = -\frac{1}{2}\hat{n}^{\langle L\rangle} \int_{-1}^{1} P_L(y) dy \int_{\zeta(y)}^{\infty} \left(\frac{2(\zeta-y)}{r(\zeta^2-1)} \right)^{B-2} \times \ln\left(\frac{2(\zeta-y)}{r(\zeta^2-1)} \right) f_{B,L}[u - r(\zeta-1)] \frac{d\zeta}{\zeta-y} = -\frac{\partial}{\partial B} {}_N h^{\alpha\beta}_{\mathcal{C}-\mathcal{N} B,L}. \tag{B11}
$$

For the inner integral, the only difference which the logarithmic term makes is in the radial integral, now given by

$$
G^m_{B,L,\bar{Q}}(\mathcal{R})^{(\ln)} = \int^{\mathcal{R}} r'^{2+\bar{q}-B+m} \ln r' dr' = -\frac{\partial}{\partial B} G^m_{B,L,\bar{Q}}(\mathcal{R}). \tag{B12}
$$

Thus, if the original coefficients cancel for all B (and if we can treat B formally as a continuous parameter), then the coefficients generated by $\ln r$ terms cancel.

An alternative method is to show directly from the definitions [e.g. Eqs. (B11) and (B12)] that, for both the inner and outer integrals and for $z<1$ and $z>1$,

$$
{}_N h^{\alpha\beta(\ln)}_{B,L} = \ln \mathcal{R}\, {}_N h^{\alpha\beta}_{B,L} - \int_1^z {}_N h^{\alpha\beta}_{B,L} \overline{dz/z}, \tag{B13}
$$

modulo z- or $\mathcal{R}$-independent terms. Then, if the z-dependent parts of ${}_N h^{\alpha\beta}_{B,L}$ cancel between outer and inner integrals, so too do the z-dependent parts of ${}_N h^{\alpha\beta(\ln)}_{B,L}$.

APPENDIX C: BOUNDARY TERMS

The boundary terms in $h^{\alpha\beta}_{\mathcal{N}}$ that arise from integrating by parts various integrals over $\mathcal{M}$ are given by

$$N_{\partial\mathcal{M}}=4\oint_{\partial\mathcal{M}}\tau^{0j}(t,\mathbf{x}')d^2S'_j+\frac{2}{3}r^2\partial_t^2\oint_{\partial\mathcal{M}}\tau^{0j}(t,\mathbf{x}')d^2S'_j-\frac{4}{3}x^i\partial_t\oint_{\partial\mathcal{M}}\tau^{ij}(t,\mathbf{x}')d^2S'_j-\frac{4}{3}x^i\partial_t^2\oint_{\partial\mathcal{M}}\tau^{0j}(t,\mathbf{x}')x'^{ij}d^2S'_j$$
$$+\frac{1}{30}r^4\partial_t^4\oint_{\partial\mathcal{M}}\tau^{0j}(t,\mathbf{x}')d^2S'_j-\frac{2}{15}r^2x^i\partial_t^4\oint_{\partial\mathcal{M}}\tau^{0j}(t,\mathbf{x}')x'^{ij}d^2S'_j-\frac{2}{15}r^2x^i\partial_t^3\oint_{\partial\mathcal{M}}\tau^{ij}(t,\mathbf{x}')d^2S'_j\,, \tag{C1a}$$

$$K^i_{\partial\mathcal{M}}=4\oint_{\partial\mathcal{M}}\tau^{ij}(t,\mathbf{x}')d^2S'_j+\frac{2}{3}r^2\partial_t^2\oint_{\partial\mathcal{M}}\tau^{ij}(t,\mathbf{x}')d^2S'_j+\frac{2}{3}x^k\partial_t^3\oint_{\partial\mathcal{M}}\tau^{0j}(t,\mathbf{x}')x'^{ik}d^2S'_j$$
$$-\frac{4}{3}x^k\partial_t^2\oint_{\partial\mathcal{M}}\tau^{j[i}(t,\mathbf{x}')x'^{k]}d^2S'_j-\frac{2}{9}\partial_t^3\oint_{\partial\mathcal{M}}\tau^{0j}(t,\mathbf{x}')r'^2x'^i d^2S'_j\,, \tag{C1b}$$

$$B^{ij}_{\partial\mathcal{M}}=-4\partial_t\oint_{\partial\mathcal{M}}\tau^{k(i}(t,\mathbf{x}')x'^{j)}d^2S'_k-2\partial_t^2\oint_{\partial\mathcal{M}}\tau^{0k}(t,\mathbf{x}')x'^{ij}d^2S'_k-\frac{2}{3}r^2\partial_t^3\oint_{\partial\mathcal{M}}\tau^{k(i}(t,\mathbf{x}')x'^{j)}d^2S'_k$$
$$-\frac{1}{3}r^2\partial_t^4\oint_{\partial\mathcal{M}}\tau^{0k}(t,\mathbf{x}')x'^{ij}d^2S'_k+\frac{2}{3}x^l\partial_t^3\oint_{\partial\mathcal{M}}\tau^{k(i}(t,\mathbf{x}')x'^{jl)}d^2S'_k+\frac{2}{9}x^l\partial_t^4\oint_{\partial\mathcal{M}}\tau^{0k}(t,\mathbf{x}')x'^{ijl}d^2S'_k$$
$$+\frac{8}{9}x^l\partial_t^3\oint_{\partial\mathcal{M}}(\tau^{k[i}(t,\mathbf{x}')x'^{l]j}+\tau^{k[j}(t,\mathbf{x}')x'^{l]i})d^2S'_k-\frac{1}{18}\partial_t^3\oint_{\partial\mathcal{M}}[\tau^{lk}(t,\mathbf{x}')(r'^2x'^{ij})_{,l}+\dot{\tau}^{0k}(t,\mathbf{x}')r'^2x'^{ij}]d^2S'_k$$
$$+\frac{1}{3}\partial_t^3\oint_{\partial\mathcal{M}}(\tau^{k[l}(t,\mathbf{x}')x'^{i]jl}+\tau^{k[l}(t,\mathbf{x}')x'^{j]il})d^2S'_k-\frac{1}{30}r^4\partial_t^5\oint_{\partial\mathcal{M}}\tau^{k(i}(t,\mathbf{x}')x'^{j)}d^2S'_k$$
$$-\frac{1}{60}r^4\partial_t^6\oint_{\partial\mathcal{M}}\tau^{0k}(t,\mathbf{x}')x'^{ij}d^2S'_k+\frac{1}{15}r^2x^l\partial_t^5\oint_{\partial\mathcal{M}}\tau^{k(i}(t,\mathbf{x}')x'^{jl)}d^2S'_k+\frac{1}{45}r^2x^l\partial_t^6\oint_{\partial\mathcal{M}}\tau^{0k}(t,\mathbf{x}')x'^{ijl}d^2S'_k$$
$$+\frac{4}{45}r^2x^l\partial_t^5\oint_{\partial\mathcal{M}}(\tau^{k[i}(t,\mathbf{x}')x'^{l]j}+\tau^{k[j}(t,\mathbf{x}')x'^{l]i})d^2S'_k\,. \tag{C1c}$$

APPENDIX D: PROPERTIES OF POISSON POTENTIALS

Here we list some useful properties of Poisson potentials and superpotentials, given by Eqs. (4.10). These rely upon the general result, which can be obtained by integration by parts,

$$P(\nabla^2 g)=-g+\mathcal{B}_P(g), \tag{D1}$$

where $\mathcal{B}_P(g)$ denotes the boundary term, given by

$$\mathcal{B}_P(g)\equiv\frac{1}{4\pi}\times\oint_{\partial\mathcal{M}}\left[\frac{g(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\partial'_r\ln[g(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|]\right]_{r'=\mathcal{R}}\mathcal{R}^2d\Omega'. \tag{D2}$$

The boundary terms must be carefully evaluated case by case to determine if any $\mathcal{R}$-independent terms survive. All $\mathcal{R}$-*dependent* terms can be discarded. Some useful formulas that result from this include

$$P(|\nabla g|^2)=-\frac{1}{2}\{g^2+2P(g\nabla^2 g)-\mathcal{B}_P(g^2)\}, \tag{D3a}$$

$$P(\nabla g\cdot\nabla f)=-\frac{1}{2}\{fg+P(f\nabla^2 g)+P(g\nabla^2 f)-\mathcal{B}_P(fg)\}, \tag{D3b}$$

$$P(f|\nabla U|^2)=-\frac{1}{2}\{fU^2+P(U^2\nabla^2 f)-2\Sigma(fU)+4P(U\nabla U\cdot\nabla f)-\mathcal{B}_P(fU^2)\}. \tag{D3c}$$

In many specific cases, the boundary terms can be dropped:

$$P(U)=-\frac{1}{2}X, \tag{D4a}$$

$$P(X)=-\frac{1}{12}Y, \tag{D4b}$$

$$P(|\nabla U|^2)=-\frac{1}{2}U^2+\Phi_2, \tag{D4c}$$

$$P(x^iU^{,jk\cdots})=-\frac{1}{2}x^iX^{,jk\cdots}+\frac{1}{12}Y^{,ijk\cdots}, \tag{D4d}$$

$$P(r^2U^{,ij})=-\frac{1}{2}r^2X^{,ij}-\frac{1}{12}Y^{,ij}+\frac{1}{6}x^kY^{,ijk}, \tag{D4e}$$

while, in others, there are contributions from the boundary terms. For example, in the 2PN potential $P(\nabla U\cdot\nabla\ddot{X})$, the boundary term yields the term $\frac{1}{2}\int_{\mathcal{M}}\sigma(t,\mathbf{x})d^3x\,\partial_t^2\int_{\mathcal{M}}\sigma(t,\mathbf{y})d^3y$. Using Eq. (5.6), we obtain, to the necessary order,

$$P(\nabla U\cdot\nabla\ddot{X})=-\frac{1}{2}\left\{U\ddot{X}-\Sigma(\ddot{X})+2G_2-\frac{1}{2}\overset{(4)}{\mathcal{I}\mathcal{I}}{}^{ii}(t)\right\}+O(\epsilon^5). \tag{D5}$$

Similarly, we find for the 3PN potential,

$$P(\nabla U\cdot\nabla\overset{(4)}{Y})=-\frac{1}{2}\{U\overset{(4)}{Y}-\Sigma(\overset{(4)}{Y})+12P(U\overset{(4)}{X})-2\overset{(4)}{\mathcal{I}\mathcal{I}}{}^{ii}(t)\}+O(\epsilon^5). \tag{D6}$$

For the Poisson superpotential $S(f)$, we have

$$S(\nabla^2g)=2P(g)+\mathcal{B}_S(g), \tag{D7}$$

where

$$\mathcal{B}_S(g)\equiv\frac{1}{4\pi}\oint_{\partial\mathcal{M}}\left[g(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|\partial_r'\times\ln\left(\frac{g(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right)\right]_{r'=\mathcal{R}}\mathcal{R}^2d\Omega'. \tag{D8}$$

Thus, for example, in the superpotential $(\partial/\partial t)^2\int_{\mathcal{M}}\tau^{00}|\mathbf{x}-\mathbf{x}'|d^3x'$, we find the term

$$\ddot{S}(\nabla^2U^2)=2\ddot{P}(U^2)-3(d/dt)^2\left(\int_{\mathcal{M}}\sigma d^3x\right)^2+O(\epsilon^5)=4G_1+4G_2-3\overset{(4)}{\mathcal{I}\mathcal{I}}{}^{ii}(t)+O(\epsilon^5). \tag{D9}$$

Other useful identities include

$$\Sigma(x^i)=x^iU-X^{,i}, \tag{D10a}$$

$$\Sigma(x^{ij})=\frac{1}{3}Y^{,ij}-\delta^{ij}X+x^{ij}U-2x^{(i}X^{,j)}. \tag{D10b}$$

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