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Feasibility of reconstructing the quintessential potential using type Ia supernova data

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We investigate the feasibility of the method for reconstructing the equation of state and the effective potential of the quintessence field from type Ia supernova (SNIa) data. We introduce a useful functional form to fit the luminosity distance with good accuracy (the relative error is less than 0.1%). We assess the ambiguity in reconstructing the equation of state and the effective potential, which originates from the uncertainty in Ω_M. We find that the equation of state is sensitive to the assumed Ω_M, while the shape of the effective potential is not. We also demonstrate the actual reconstruction procedure using the data created by Monte Carlo simulations. Future high precision measurements of distances to thousands of SNIa could reveal the shape of the quintessential potential.

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I. INTRODUCTION

Recent various observations [1,2], in particular distance measurements to type Ia supernova (SNIa) [3,4], strongly suggest that the universe is currently dominated by a positive vacuum energy density with negative pressure. The smallness of the vacuum energy density, $\sim(10^{-12}\text{ GeV})^4$, has revived the idea that the cosmological constant is not really a constant but rather decaying. The idea of quintessence [5] (see also [6,7], and references therein) is that the vacuum energy density is played by a scalar field rolling down the almost flat potential similar to cosmological inflation, and a lot of models have been proposed so far. However, there is currently no clear guidance from particle physics as to which quintessence models may be suitable. So it should be the observations that decide which model is correct or not. As a bottom-up approach, we have proposed that distance measurements to SNIa may allow one to reconstruct the equation of state of the dark energy or the effective potential of the quintessence field [8].

Future observational projects, such as SNAP (SuperNova/Acceleration Probe),[1] could gather 2000 SNIa in a single year, which could put significant constraint on the cosmological parameters (including the equation of state of the dark energy w). In view of the future prospect for a high-z SNIa search, we investigate in detail the feasibility of the method for reconstructing the equation of state and the effective potential of the quintessence field from SNIa data [8–10].[2]

II. PARAMETRIZING THE LUMINOSITY DISTANCE

Considering the future prospect for a high-z SNIa search, we believe it particularly useful to fit the observed luminosity distance $d_L(z)$ to a function of z which has the following properties: (1) good convergence (the relative error is, hopefully, less than 0.1% because the distance error expected from SNAP will be less than a few percent) for $0<z<10$; (2) the correct asymptotic behavior for $z\gg 1$ ($H(z)\propto(1+z)^{3/2}$). We present a fitting function for $d_L(z)$.

We restrict ourselves to a flat FRW universe henceforth, and assume Einstein gravity.[3] In a flat model, the luminosity distance $d_L(z)$ is written in terms of the coordinate distance $r(z)$ to an object at z as

$$\frac{d_L(z)}{(1+z)}=r(z)=\int_t^{t_0}\frac{dt'}{a(t')}=\int_0^z\frac{dz'}{H(z')}=\int_y^1\frac{2dy'}{H(y')y'^3}, \tag{1}$$

where t_0 is the present time and $y=1/\sqrt{1+z}$. It is interesting to note that in terms of y, $r(y)$ is a linear function of y if $\Omega_M=1$. Therefore, $d^2r(y)/dy^2$ contains the information of the nonzero pressure [see Eq. (5)]. We will elaborate on the advantage of using y and r in a separate paper.

In an analogy with Pen's powerful fitting formula for $r(z)$ for a flat FRW universe with a cosmological constant [14], we propose to fit $r(z)$ in the following functional form:[4]

$$H_0 r(z)=\eta(1)-\eta(y), \tag{2}$$

$$\eta(y)=2\alpha[y^{-8}+\beta y^{-6}+\gamma y^{-4}+\delta y^{-2}+\sigma]^{-1/8}. \tag{3}$$

The requirement $H(z)/H_0\to 1$ for $z\to 0$ implies that σ is found to be a dependent parameter: $\sigma=(\alpha(1+3\beta/4+\gamma/2+\delta/4))^{8/9}-1-\beta-\gamma-\delta$. However, we shall treat σ as a

[1] http://snap.lbl.gov

[2] While our work was being completed, we became aware of related work [11] which focuses on the uncertainty in the reconstruction of the equation of state of dark energy.

[3] The reconstruction equation for the so called extended quintessence [12] is presented in [13].

[4] An extension to an open or closed model is immediate once Ω_K is known: $H_0r(z)=|\Omega_k|^{-1/2}\sin_K(|\Omega_k|^{1/2}(\eta(1)-\eta(y)))$, where $\sin_K(x)=\sin(x)(\sinh(x))$ if $K=1(-1)$.

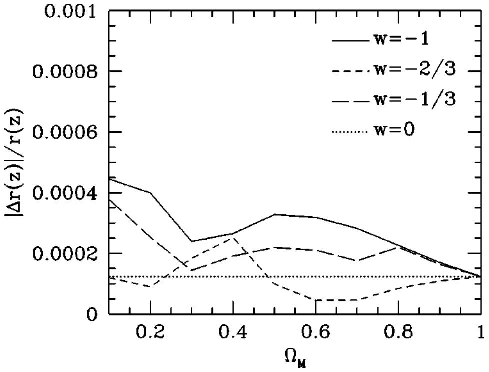

FIG. 1. The maximum deviation in the range $0.3<y<1$ $(10>z>0)$, between the actual value and that calculated from our fitting function, is shown as a function of Ω_M.

free parameter, for simplicity, in numerical calculations. For $y\to 0$ $(z\to\infty)$, we have $H(z)/H_0\to 1/(\alpha y^3)$. For the Einstein–de Sitter universe, $\alpha=1$ and $\beta=\gamma=\delta=0$.

We shall demonstrate that the fitting function, Eq. (3), does a good job indeed. For this purpose, we calculate the maximum relative error in $r(y)$ between the actual value and that calculated from the fitting function, Eq. (3) in the range $0.3<y<1$ $(10>z>0)$. We consider cosmological models consisting of matter and dark energy of constant equation of state $w\equiv p_X/\rho_X=0,-1/3,-2/3,-1$ with $0.1\leqslant\Omega_M\leqslant 1$. We fit each template luminosity distance, r_i, by the functional form, Eq. (3) using the Davidon-Fletcher-Powell method. We minimize $\Sigma_{y_i}(r_i-r(y_i))^2/r_i^2$ with $N=30$ data points. The result is shown in Fig. 1. We find the maximum relative error is less than 0.05%. Our fitting function thus seems more powerful by order of magnitude than the one proposed in [10].

III. RECONSTRUCTING THE EQUATION OF STATE AND THE EFFECTIVE POTENTIAL

One of the prime interests in the reconstruction issue is whether the effective equation of state of the *x*-component $w\equiv p_X/\rho_X$ is different from -1, which is the unique signature of dynamical vacuum energy. If observations would suggest $w\neq -1$, then the next urgent project would be the real reconstruction of the effective potential, which should have profound implications for both particle physics and cosmology.

In this section, we reconstruct the equation of state of dark energy and the effective potential of the quintessence field. Regarding the energy density of the Universe, we assume two components: nonrelativistic matter with its present density parameter Ω_M, and the *x*-component with $\Omega_X=1-\Omega_M$. An extension to include the radiation component is immediate, but with negligible effect.

A. Reconstructing the equation of state

In terms of a dimensionless variable $\hat{r}\equiv H_0 r$, the equation of state of the *x*-component w is written as

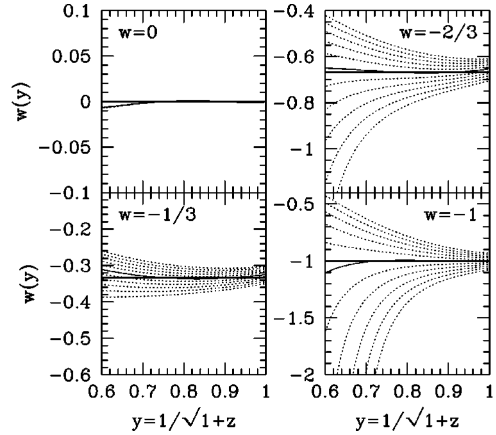

FIG. 2. The reconstructed equation of state, assuming $\Omega_M=0.30$ (solid curves). The curves assuming $\Omega_M=0.25,0.26,\ldots 0.35$ (from top to bottom) are shown by dotted curves.

$$w(z)=\frac{3d\hat{r}(z)/dz+2(1+z)d^2\hat{r}(z)/dz^2}{3(d\hat{r}(z)/dz)(\Omega_M(d\hat{r}(z)/dz)^2(1+z)^3-1)}, \quad (4)$$

$$=\frac{-4yd^2\hat{r}/dy^2}{3(d\hat{r}/dy)(\Omega_M(d\hat{r}/dy)^2-4)}. \quad (5)$$

Thus, w depends on the second derivative of the luminosity function. Put another way, the luminosity distance depends on w through a multiple integral relation [11]. Whether $w=-1$ or not will clearly signify whether the dark energy is constant in time or not.

In Fig. 2, we show the reconstructed $w(y)$ for cosmological models with constant equations of state for $w=0,-1/3,-2/3,-1$, to examine the effect of the ambiguity in Ω_M. The template $\hat{r}_i$ is constructed by assuming $\Omega_M=0.30$ for $0.60<y<1$ $(1.78>z>0)$. The template is fitted using the ansatz, Eq. (3), by minimizing $\Sigma_{y_i}(\hat{r}_i-\hat{r}(y_i))^2/\hat{r}_i^2$ with $N=10$ data points. We also plot the reconstructed $w(y)$ for $\Omega_M=0.25,0.26,\ldots 0.35$ (from top to bottom) as dotted curves. We find that the uncertainty in Ω_M would enlarge an error in $w(y)$ for small y (large z). In particular, when we overestimate Ω_M, $w(y)$ may diverge at some y where the denominator in Eq. (5) may vanish. However, an error in w remains relatively small near $y=1$. 10% uncertainty in Ω_M results in, at most, 19% error in w for $y>0.80$ $(z<0.56)$. The error is largest for the cosmological constant (19% error), and becomes less significant for larger w. For example, for a $w=-2/3$ model, the error is less than 12% for $y>0.80$. Hence, it might be possible to discriminate between $w=-1$ and $w\neq -1$. The combination of SNIa measurements and high precision measurements of the power spectrum expected from the Microwave Anisotropy Probe (MAP) and Planck satellites could make a clear distinction [15].

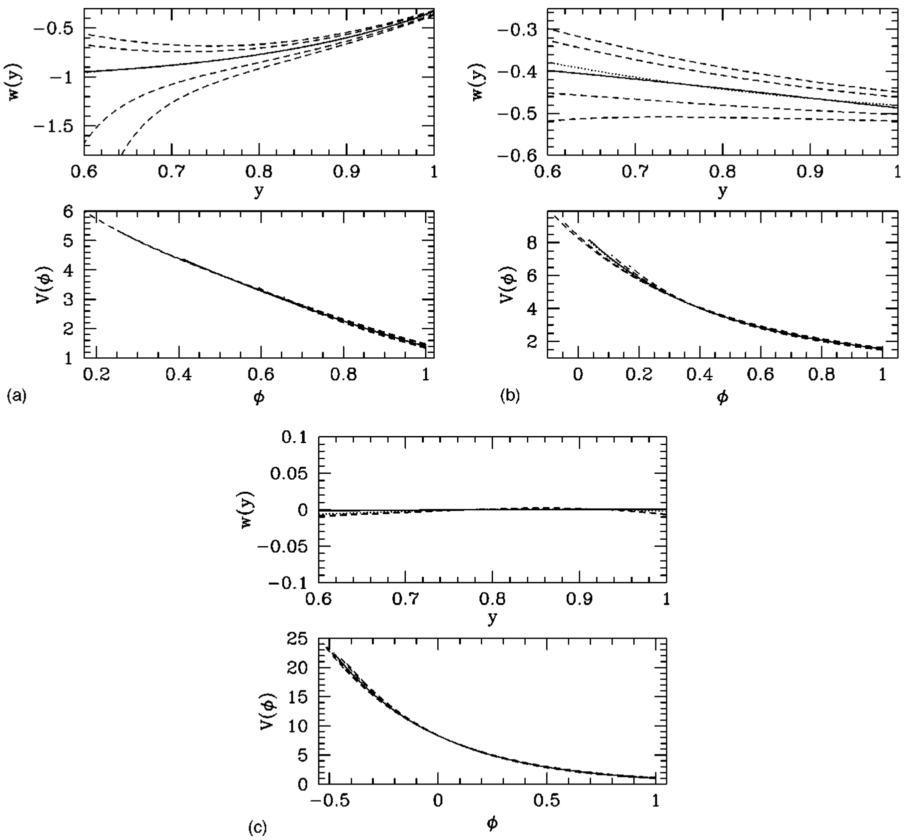

FIG. 3. The reconstructed equation of state and the effective potential: (a) is the cosine potential, (b) is the inverse power law potential, and (c) is the exponential potential. Solid curves are the original equations of state or effective potentials. Dotted curves are for $\Omega_M=0.30$. The curves assuming $\Omega_M=0.25,0.27,0.33,0.35$ [from top to bottom for $w(y)$, or from left to right for V] are shown by dashed curves.

B. Reconstructing the effective potential

The reconstruction equations for the effective potential are

$$\hat{V}(z)=\frac{3}{(d\hat{r}(z)/dz)^2}+(1+z)\frac{d^2\hat{r}(z)/dz^2}{(dr(z)/dz)^3}-\frac{3}{2}\Omega_M(1+z)^3, \quad (6)$$

$$\left(\frac{d\hat{\phi}(z)}{dz}\right)^2=\frac{(d\hat{r}(z)/dz)^2}{(1+z)^2}\left[-2(1+z)\frac{d^2\hat{r}(z)/dz^2}{(d\hat{r}(z)/dz)^3}-3\Omega_M(1+z)^3\right], \quad (7)$$

where $\hat{\phi}\equiv\kappa\phi$ and $\hat{V}\equiv\kappa^2V/H_0^2$ with $\kappa^2=8\pi G$. Alternatively, in terms of $y=1/\sqrt{1+z}$,

$$\hat{V}(y)=\frac{6}{y^6(d\hat{r}/dy)^2}-\frac{2d^2\hat{r}/dy^2}{y^5(d\hat{r}/dy)^3}-\frac{3\Omega_M}{2y^6}, \quad (8)$$

$$\left(\frac{d\hat{\phi}(y)}{dy}\right)^2=\frac{12}{y^2}+\frac{4d^2\hat{r}/dy^2}{y(d\hat{r}/dy)}-3\Omega_M\frac{(d\hat{r}/dy)^2}{y^2}. \quad (9)$$

In order to demonstrate the effectiveness of the fitting function, Eq. (3), we reconstruct the effective potential of the quintessence field. We consider three kinds of potentials which have some theoretical backgrounds: (a) cosine type [16], $\hat{V}(\hat{\phi})=M^4(\cos(\hat{\phi})+1)$; (b) inverse power law type [17], $\hat{V}(\hat{\phi})=M^4\hat{\phi}^{-\alpha}$; and (c) exponential type [18], $\hat{V}(\hat{\phi})=M^4\exp(-\lambda\hat{\phi})$. M and λ are fixed to give $\Omega_X=1-\Omega_M$. $\alpha=4$ is assumed hereafter.

The results are shown in Fig. 3. The dotted curves are numerically reconstructed $w(y)$ and $\hat{V}(\hat{\phi})$, with $\Omega_M=0.30$ being assumed, while the solid curves are the original ones up to $y=0.6$. We fix the present value of $\hat{\phi}$ to unity.

We also plot the dashed curves assuming $\Omega_M=0.25,0.27,0.33,0.35$. Since the smaller the y, the larger the error in $w(y)$, the range of $\hat{\phi}$ significantly depends on the assumed Ω_M. We note that for $\Omega_M>0.30$, w becomes less than -1, and thus, the right-hand side of Eq. (9) turns negative at some y. So we stop the reconstruction of $\hat{V}(\hat{\phi})$ there. The range of $\hat{\phi}$ is larger (smaller) for smaller (larger) Ω_M. For example, in the case of cosine potential, the true range of $\hat{\phi}$ is $0.41<\hat{\phi}\leqslant 1.00$, while the reconstructed range is $0.41<\hat{\phi}\leqslant 1.00$ for $\Omega_M=0.30$, $0.17<\hat{\phi}\leqslant 1.00$ for $\Omega_M=0.25$, $0.25<\hat{\phi}\leqslant 1.00$ for $\Omega_M=0.27$, $0.59<\hat{\phi}\leqslant 1.00$ for $\Omega_M=0.33$, and $0.66<\hat{\phi}\leqslant 1.00$ for $\Omega_M=0.35$. However, it is interesting that the whole shape of $\hat{V}(\hat{\phi})$ is less sensitive to the uncertainty in Ω_M, although w and the range of $\hat{\phi}$ are dependent on the assumed value of Ω_M. If we assume smaller Ω_M, the resulting potential energy is larger via Eq. (8). On the other hand, $\hat{\phi}$ decreases more rapidly via Eq. (9). The opposite is the case for larger Ω_M; both effects make the reconstructed shape of $\hat{V}(\hat{\phi})$ converge to the true one.

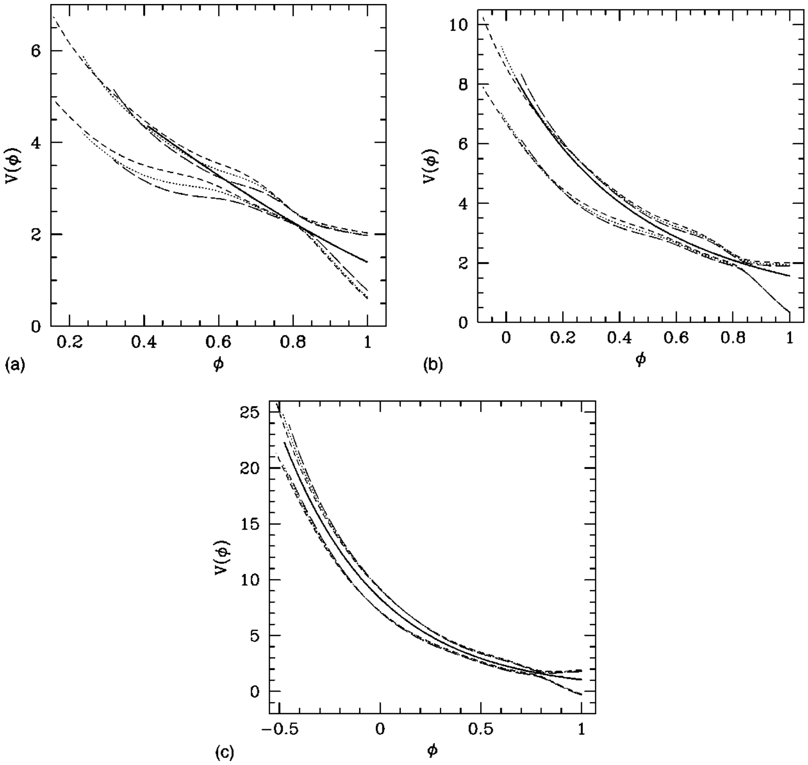

FIG. 4. One sigma intervals for the reconstructed quintessential potential assuming a luminosity distance error of 3% with $N=30$ data: (a) is the cosine potential, (b) is the inverse power law potential, and (c) is the exponential potential. Solid curves are the original potentials. Dotted curves are for $\Omega_M=0.30$, short dashed curves are for $\Omega_M=0.27$, and long dashed curves are for $\Omega_M=0.33$.

IV. RECONSTRUCTING THE EQUATION OF STATE AND THE EFFECTIVE POTENTIAL FROM SIMULATED DATA

We simulate the actual reconstruction procedure using numerically generated data $r_i=r(y_i)+\delta r_i$ $(i=1,\ldots N)$ with δr_i being Gaussian distributed [zero mean and a variance of $\sigma r(y_i)$]. The simulated data assumes a cosmological model with $\Omega_M=0.30$. We consider the $N=30$ data and take $\sigma=0.03$ or $\sigma=0.005$. The former error is the distance error of the binned data expected from the observations of 200 supernovae by SNAP, while the latter is for 6000 supernovae. We only consider statistical uncertainties.

We distribute the data uniformly in y from $y=0.95$ $(z=0.11)$ to $y=0.60$ $(z=1.78)$. We perform thousands of

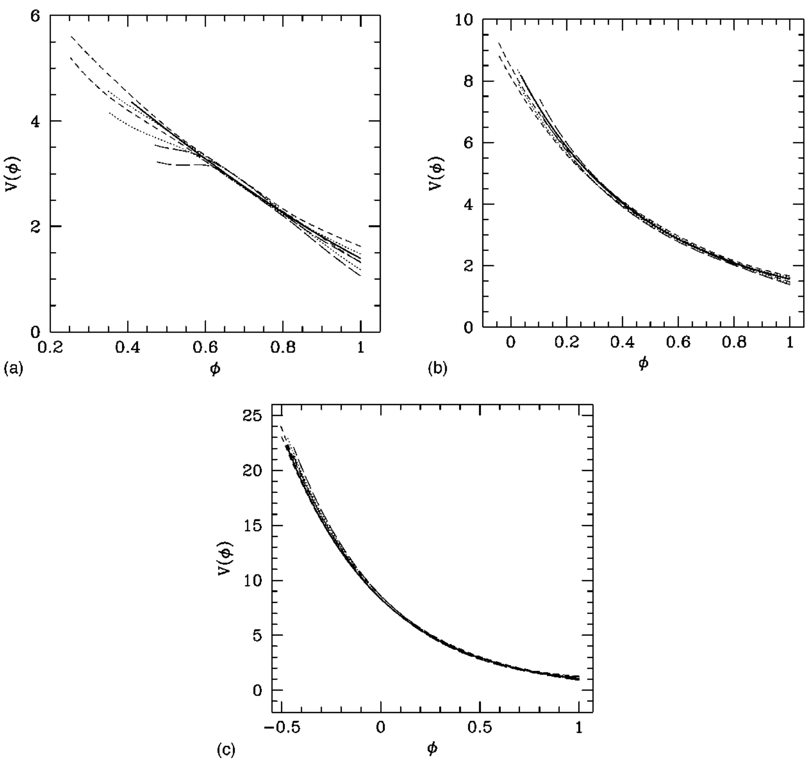

FIG. 5. One sigma intervals for the reconstructed quintessential potential assuming a luminosity distance error of 0.5%. The meaning of the curves is the same as in Fig. 4.

Monte Carlo realizations. The 68% confidence intervals of the reconstructed potentials are shown in Fig. 4 and Fig. 5. The horizontal axis is the averaged value of $\hat{\phi}$, while the vertical axis is the averaged value of $\hat{V}$. The effect of the reduced error may be dramatic. We allow for 10% error in Ω_M to assess the ambiguity in the reconstruction of the potential. Obervations of very distant supernovae at $z \geqslant 3$ by the Next Generation Space Telescope (NGST) and/or observations of galaxy cluster abundance by the Sunyaev–Zel'dovich Effect (SZE) survey will determine Ω_M to a few percent [20].

It should be noted that the fitted $\hat{r}(y)$ does not necessarily satisfy the positivity of the right-hand side of Eq. (9); that is, the weak energy condition. We perform the reconstruction using only the data which satisfies the weak energy condition.[5] That induces the bias towards larger $\dot{\phi}^2$. This is why the range of $\hat{\phi}$ is larger than that of the true one, and why the intervals of the reconstructed potential are distributed downward. Such an effect is particularly significant for the cosine type potential because in this model, the equation of state is almost $w=-1$ at higher redshift. Therefore, good estimates of the upper bound of Ω_M are crucial for the success of the reconstruction.

[5]Conversely, Eq. (9) might provide an upper bound on Ω_M, if we assume that the dark energy respects the weak energy condition, although that is not always the case [19].

V. SUMMARY

We have studied the feasibility of reconstructing the equation of state of dark energy and the effective potential of the quintessence field from SNIa data by taking into account the uncertainty in Ω_M and the error in the luminosity distance. We have found that w and the range of $\hat{\phi}$ are dependent on the assumed value of Ω_M, while the whole shape of $\hat{V}(\hat{\phi})$ is less sensitive to the uncertainty in Ω_M. If Ω_M could be constrained to some 10% accuracy by other observations, which may not be an unrealistic expectation [20], then future high precision measurements of distances to thousands of SNIa could reveal the shape of the quintessential potential.

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