$O(\alpha^3 \ln \alpha)$ corrections to positronium decay rates

Kirill Melnikov*
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94309

Alexander Yelkhovsky †
Budker Institute for Nuclear Physics, Novosibirsk 630090, Russia
(Received 24 August 2000; published 30 October 2000)

We compute $O(\alpha^3 \ln \alpha)$ corrections to the decay rates of para- and orthopositronium into two and three photons, respectively. For this calculation we employ the nonrelativistic QED regularized dimensionally and we explain how in this framework the logarithms of the fine-structure constant can be extracted.

PACS number(s): 11.10.St, 12.20.Ds

I. INTRODUCTION

Positronium decays into two and three photons provide an interesting test of bound state quantum electrodynamics (QED). While the parapositronium (p-Ps) decay rate is well described by QED, it is known that the decay of orthopositronium (o-Ps) into three photons is still a controversial issue [1]. However, on the theoretical side there has been a major breakthrough recently and both decay rates are now known with $O(\alpha^2)$ accuracy [2,3]. The next level of precision, i.e., $O(\alpha^3)$ correction, is currently beyond reach, although parts of the $O(\alpha^3)$ correction that are enhanced by the logarithms of the fine-structure constant, can be computed.

Knowledge of these corrections is not of significant phenomenological importance at present since they are much smaller than the experimental accuracy for both p-Ps and o-Ps decays. Nevertheless, from a theoretical viewpoint, it is an interesting problem because at order α^3 there are both leading $O(\alpha^3 \ln^2\alpha)$ and subleading $O(\alpha^3 \ln \alpha)$ corrections. The calculation of the leading $O(\alpha^3 \ln^2\alpha)$ corrections is a fairly simple enterprise; it has been done quite some time ago [4]. Here we are interested in subleading logarithmic corrections. In order to compute them, we utilize the nonrelativistic QED in dimensional regularization and we explain how in this framework the logarithmic corrections can be computed using a limited amount of information.

Before diving into a description of the calculation, let us summarize our results. For the $O(\alpha^3 \ln \alpha)$ corrections to para- and orthopositronium decay rates we find

$$\Delta\Gamma_{\rm p} = \frac{\alpha^3}{\pi} \ln \alpha \left\{ -\frac{367}{90} + 10 \ln 2 - 2A_{\rm p} \right\} \Gamma_{\rm p}^{(0)}, \qquad (1)$$

$$\Delta\Gamma_{\rm o} = \frac{\alpha^3}{\pi} \ln \alpha \left\{ -\frac{229}{30} + 8 \ln 2 + \frac{A_{\rm o}}{3} \right\} \Gamma_{\rm o}^{(0)}. \qquad (2)$$

The coefficients $A_{\rm p,o}$ describe $O(\alpha/\pi)$ corrections to the lowest order decay widths $\Gamma_{\rm p,o}^{(0)}$. They are [5,6]

$$A_{\rm p} = \frac{\pi^2}{4} - 5, \quad A_{\rm o} = -10.286\,606(10). \qquad (3)$$

Numerically, the $O(\alpha^3 \ln \alpha)$ corrections, Eqs. (1) and (2), evaluate to $7.7919\alpha^3/\pi \ln \alpha \Gamma_{\rm p}^{(0)}$ for p-Ps and to $-5.517\alpha^3/\pi \ln \alpha \Gamma_{\rm o}^{(0)}$ for o-Ps. These corrections are therefore quite comparable with the ''leading'' $O(\alpha^3 \log^2\alpha)$ corrections computed in [4].

The paper is organized as follows. In Sec. II we set up the framework of the calculation. We then continue with detailed discussion of how various contributions to Ps decays at $O(\alpha^3 \ln \alpha)$ are computed. In the last section we present our conclusions.

II. FRAMEWORK OF THE CALCULATION

Let us first discuss the framework of our calculation. We work in nonrelativistic QED regularized dimensionally; $d = 3 - 2\epsilon$ is the number of spatial dimensions and ϵ is the regularization parameter. General features of this technique have been described at length in our previous paper [9]. Here we would like to discuss a new issue which was not considered in [9]: how logarithmic $\ln\alpha$ corrections can be extracted.

In order to extract logarithms of the fine-structure constant in a self-consistent way, we use the fact that the matrix element of any operator in dimensional regularization is a *uniform* function of the fine-structure constant. This implies that all the dependence on the fine-structure constant can be scaled out of any dimensionally regularized matrix element.[1] The scaling, however, should be done in d dimensions.

In order to establish the scaling rules, we need to know how different quantities involved in bound-state calculations scale with α. To do that, we rewrite the d-dimensional Schrödinger equation in ''atomic units'' familiar from the standard treatment of hydrogen atom in three-dimensional quantum mechanics.

Consider the Schrödinger equation for positronium in d dimensions:

*Email address: melnikov@slac.stanford.edu
†Email address: yelkhovsky@inp.nsk.su

[1]We stress that this is the feature of dimensional regularization and it is not valid in other regularization schemes.

$$\left(\frac{p^2}{m}-\frac{c(d)\alpha}{r^{d-2}}\right)\Psi=E\Psi, \tag{4}$$

where $c(d)=\Gamma(d/2-1)/\pi^{d/2-1}$ (see [9]). Let us rescale $p\to p\gamma$ and $r\to r/\gamma$ and choose $\gamma=(m\alpha/2)^{1/(1+2\epsilon)}$. Then both the Coulomb Hamiltonian $H=p^2/m-c(d)\alpha/r^{d-2}$ and its eigenvalue E scale as γ^2/m. The wave function of a bound state, being normalized to unity, scales as $\gamma^{d/2}$. Hence, when expressed in atomic units, the energy of a bound state and its properly normalized wave function depend on d only. The scaling rules above provide sufficient information to write the matrix element of any operator in atomic units and therefore scale out all the dependence on the fine-structure constant.

There is another important point that makes the extraction of the $\ln\alpha$ corrections possible using limited information. In order to explain it, we remind the reader that in bound-state calculations different contributions to the final result can be distinguished. In particular, there are so-called hard contributions. The $O(\alpha^3)$ hard corrections to the Ps decay rates are described by the three-loop Feynman diagrams for the process $e^+e^-\to 2(3)\gamma$, which have to be computed exactly at the threshold. Schematically, such diagrams generate the correction

$$V_{\rm hard}=\left(\frac{\alpha}{\pi}\right)^3\left[\frac{t_1}{\epsilon^2}+\frac{t_2}{\epsilon}+t_3\right]V_{\rm Born}, \tag{5}$$

(t_{1-3} are some constants) to the annihilation kernel $V_{\rm Born}$ that is responsible for the lowest order decay rate:

$$\Gamma^{(0)}_{\rm p,o}=\langle\Psi|V_{\rm Born}|\Psi\rangle\propto\Psi_0^2. \tag{6}$$

Here Ψ_0 stands for the positronium wave function at the origin. The effective potential (5), in turn, generates the correction to the decay rate,

$$\delta_{\rm hard}\Gamma=\langle\Psi|V_{\rm hard}|\Psi\rangle\propto\left[\frac{t_1}{\epsilon^2}+\frac{t_2}{\epsilon}+t_3\right]\Psi_0^2. \tag{7}$$

If we rewrite Ψ_0 in atomic units, Eq. (7) generates logarithms of the fine-structure constant. These logarithms are artificial, since we anticipate that other, soft scale contributions also generate divergences which exactly match and cancel all the divergences in $V_{\rm hard}$. This implies that the logarithms associated with the rescaling of the wave function at the origin get cancelled as well. Therefore, the easiest way to avoid considering $V_{\rm hard}$ (which is not available at present) is to work with relative, rather than absolute, corrections to the decay width. This automatically discourages Eq. (7) as the source of logarithms of the fine-structure constant since in this case there is simply nothing to rescale.

In order to illustrate how these arguments help to compute the $\ln\alpha$ corrections, let us consider the matrix element of a nonrelativistic operator O that delivers the $O(\alpha^3)$ correction to the lowest order annihilation kernel $V_{\rm Born}$. In accordance with the above comment we consider a relative correction to the decay rate:

$$\Delta_O=\frac{\delta_O\Gamma}{\Gamma^{(0)}_{\rm p,o}}=\frac{\langle\Psi|O|\Psi\rangle}{\langle\Psi|V_{\rm Born}|\Psi\rangle}. \tag{8}$$

The operator O is a function of the coordinate and momentum operators that act on the positronium wave function. After rescaling of all the quantities on the right-hand side of Eq. (8) as described above, we end up with the following equation:

$$\Delta_O=\alpha^n\frac{\gamma^{3-n+l\epsilon}}{m^{3-n+k\epsilon}}\cdot\frac{\langle\Psi|O|\Psi\rangle}{\langle\Psi|V_{\rm Born}|\Psi\rangle}\Big|_{\gamma=1}, \tag{9}$$

where $n=1,2$ is a power of α that explicitly enters O and l,k are some integer numbers. If the matrix element is finite, we can safely set $\epsilon=0$ and then the relative correction to the decay width is α^3 times the α-independent ratio and hence no logarithms of α appear. Therefore, after the rescaling, the *only* place where $\ln\alpha$ can come from is the expansion of the factor $\gamma^{3-n+l\epsilon}$ in powers of ϵ; this implies that in order to generate the $\ln\alpha$ corrections, the nonrelativistic matrix elements should diverge and only divergent pieces of the matrix elements have been known to determine logarithmic in α corrections to the decay rate. Note also, that because of the relation between γ and α,

$$\gamma=\left(\frac{m\alpha}{2}\right)^{1/(1+2\epsilon)},$$

even the operators that scale as *integer* powers of γ can generate $\ln\alpha$ corrections.

Let us note that the extraction of the $\ln\alpha$ corrections in dimensional regularization can lead to some counterintuitive results; for example, logarithms of the fine-structure constant are generated by the operators, which in more ''physical'' regularization schemes, such as, e.g., the schemes that use either the photon mass or the momentum cutoff λ, can only lead to the logarithms of m/λ but not to $\ln\alpha$. Therefore, it appears that individual contributions to the final result are scheme dependent. Nevertheless, we would like to stress that once dimensional regularization and clear rules for extracting $\ln\alpha$ are adopted, there is no other way as to consider all possible contributions; none of them can be disregarded by invoking the fact that in a different regularization scheme a particular operator cannot generate the $O(\ln\alpha)$ correction.

The calculation of the nonrelativistic contributions that are relevant at order $O(\alpha^3\ln\alpha)$ is described in the following sections. Some useful integrals that we need in the calculation, are summarized in Appendix. We give intermediate formulas for the relative logarithmic corrections to the positronium decay rate expressed in units of $(\alpha^3/\pi)\Gamma(1+\epsilon)^3(4\pi)^{3\epsilon}$.

III. IRREDUCIBLE CONTRIBUTION

This particular contribution arises as the average value of a local operator that comes from the Taylor expansion of the one-loop corrections to the annihilation kernel in spatial momenta $\boldsymbol{p}$ of electron and positron. For our purpose we need

the $O(\alpha p^2)$ correction to $V_{\rm Born}$. Since this operator is constructed by Taylor expanding in external momentum, the non-analytic dependence on p^2 cannot appear. Then, from the rescaling argument we know that only the divergent piece of the Wilson coefficient of this operator is required. It turns out that this divergent piece can easily be computed using rather general arguments. According to the rules of nonrelativistic QED, we extract Wilson coefficients of various effective operators from the corresponding on-shell scattering amplitudes. The key observation is that the divergence in the Wilson coefficient of the $O(\alpha p^2)$ operator causes the divergence in the *on-shell* annihilation process $e^+e^- \to n\gamma$, $n=2,3$, and that it is in fact a ''true'' infrared divergence that should be compensated by the real emission of an additional soft photon in the same process.

In order to avoid confusion we stress that the mechanism of canceling this divergence by real radiation does not apply to the bound state because of C-parity conservation. There is no contradiction, however. The real infrared divergences in the bound-state calculations are absent because electron and positron in positronium are off-shell. The divergence appears only when we put them on mass shell in order to extract the Wilson coefficient of the relevant operator. The crucial observation is that considering the process in a different kinematic regime (on-shell annihilation), we easily find a divergent piece of the appropriate Wilson coefficient from the known amplitude of the real soft photon emission.

Requiring that virtual and real corrections to the on-shell annihilation process $e^+e^- \to n\gamma$ add up to a finite quantity, we get the following correction to the annihilation kernel $V_{\rm Born}$:

$$V_{\rm irr} = \frac{2\alpha}{3\pi\epsilon}\,\frac{p^2+p'^2}{m^2}\,V_{\rm Born}\,. \tag{10}$$

Applying the rescaling arguments, we end up with the following correction induced by the irreducible operator (10):

$$\Delta_{\rm irr} = \frac{4}{3}\ln\alpha. \tag{11}$$

IV. ''HARD LOOP'' CONTRIBUTIONS

These contributions arise in the second order of the nonrelativistic perturbation theory. This means that the corresponding nonrelativistic operators O are of the form VGV', where G is the reduced Green function of the Coulomb Hamiltonian from Eq. (4) and V,V' are some local operators with one of the two originating from a hard one-loop correction.

There are two sources of the ''hard loop'' contributions. The first one is the one-loop renormalization of the annihilation kernel:

$$V_{\rm p,o} = \frac{\alpha}{\pi} A_{\rm p,o} V_{\rm Born}\,. \tag{12}$$

The corresponding $O(\alpha^3 \ln\alpha)$ correction to the decay rate is then easily related to the $O(\alpha^2\ln\alpha)$ correction computed in [10,11]. We find

$$\Delta_{\rm p,o} = \left(\frac{7}{6}\boldsymbol{S}^2 - 2\right) A_{\rm p,o}\ln\alpha. \tag{13}$$

The second ''hard loop'' contribution corresponds to the ''hard'' piece in $O(m\alpha^5)$ effective potential. It reads [12]

$$V_{\rm hl} = -\frac{\alpha^2}{3m^2}\,\frac{\Gamma(1+\epsilon)}{(4\pi)^{-\epsilon}}\left[\frac{1}{\epsilon} + \frac{39}{5} - 12\ln 2 + \boldsymbol{S}^2\left(\frac{32}{3} + 6\ln 2\right) - 2\ln m\right]\delta(\boldsymbol{r}), \tag{14}$$

where $\boldsymbol{S}$ is the operator of the total spin. There is no problem with defining the total spin operator here, since it multiplies explicitly finite quantity.

The $O(\alpha^3)$ correction to the decay rate generated by the potential from Eq. (14) then reads

$$\delta_{\rm hl}\Gamma = 2\langle\Psi|V_{\rm Born}\,G\,V_{\rm hl}|\Psi\rangle, \tag{15}$$

and is proportional to the Green function at the origin, $G(0,0)$. All necessary results for this Green function can be found in [9]. Finally, we obtain

$$\Delta_{\rm hl} = \frac{\ln^2\alpha}{3} + \left[-\frac{1}{6\epsilon} + 2\ln 2 - \frac{59}{30} - \boldsymbol{S}^2\left(\frac{16}{9} + \ln 2\right) + \ln m\right]\ln\alpha. \tag{16}$$

Let us note that since there is an explicitly divergent term in Eq. (16), one may wonder whether or not the difference in the energy E of the bound state in d and three dimensions should be taken into account. We have checked that the cancellation of all divergent terms in the final result for the $O(\alpha^3\ln\alpha)$ correction to the decay rate occurs for arbitrary E and for this reason we use the three-dimensional expression for this (rescaled) quantity to present individual contributions as well.

V. SEAGULL CONTRIBUTION

The correction to the decay rate,

$$\delta_{\rm s}\Gamma = 2\langle\Psi|V_{\rm Born}\,G\,V_{\rm s}|\Psi\rangle, \tag{17}$$

is induced by the double seagull effective potential $V_{\rm s}$,

$$V_{\rm s}(\boldsymbol{q}) = -\frac{4\pi^2\alpha^2}{m^2}\int\frac{d^dk}{(2\pi)^d}\,\frac{\mathcal{P}_{ij}(\boldsymbol{k})\mathcal{P}_{ij}(\boldsymbol{k}')}{kk'(k+k')}, \tag{18}$$

where $\boldsymbol{k}' = \boldsymbol{q} - \boldsymbol{k}$ and $\mathcal{P}_{ij}(\boldsymbol{k}) = \delta_{ij} - k_ik_j/k^2$.

To facilitate the calculation of this potential, we introduce auxiliary integration variable k_0 and rewrite $V_{\rm s}(\boldsymbol{q})$ as follows:

$$V_s(\boldsymbol{q}) = -\frac{\alpha^2}{m^2}\int_0^\infty \frac{dk_0}{2\pi}\int \frac{d^dk}{(2\pi)^d} T_{ij}(\boldsymbol{k}) T_{ij}(\boldsymbol{q}-\boldsymbol{k}), \quad (19)$$

with

$$T_{ij}(\boldsymbol{k}) = \frac{4\pi \mathcal{P}_{ij}(\boldsymbol{k})}{k^2+k_0^2}. \quad (20)$$

It is convenient to consider the Fourier transform of $V_s(\boldsymbol{q})$:

$$V_s(\boldsymbol{r}) = -\frac{\alpha^2}{m^2}\int_0^\infty \frac{dk_0}{2\pi} T_{ij}(\boldsymbol{r}) T_{ij}(\boldsymbol{r}), \quad (21)$$

where $T_{ij}(\boldsymbol{r})$ stands for

$$T_{ij}(\boldsymbol{r}) = \delta_{ij}\int \frac{d^dk}{(2\pi)^d} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \frac{4\pi}{k^2+k_0^2} + \frac{1}{k_0^2}\partial_i\partial_j \int \frac{d^dk}{(2\pi)^d} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\left(\frac{4\pi}{k^2} - \frac{4\pi}{k^2+k_0^2}\right). \quad (22)$$

To compute $T_{ij}(\boldsymbol{r})$ we use

$$\int \frac{d^dk}{(2\pi)^d} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \frac{4\pi}{k^2+k_0^2} = 2\left(\frac{k_0}{2\pi r}\right)^{d/2-1} K_{d/2-1}(k_0 r), \quad (23)$$

where $K_\nu(x)$ is the modified Bessel function of the second kind.

Inserting Eq. (23) into Eq. (22), and integrating over k_0 in Eq. (21), we obtain

$$V_s(\boldsymbol{r}) = -\frac{\alpha^2 r^{-3+4\epsilon}}{2\pi m^2}\frac{\Gamma(1-\epsilon)^2}{(4\pi)^{-2\epsilon}}\left[1-\epsilon\frac{17-8\ln 2}{3}+O(\epsilon^2)\right]. \quad (24)$$

We then rescale the relative correction to the decay rate and obtain

$$\frac{\delta_s\Gamma}{\Gamma^{(0)}_{p,o}} = -\frac{\alpha^2\gamma^{1-4\epsilon}}{2\pi m}\frac{\Gamma(1-\epsilon)^2}{(4\pi)^{-2\epsilon}}\left[1-\epsilon\frac{17-8\ln 2}{3}+O(\epsilon^2)\right] \times \int d^dr r^{-3+4\epsilon} G(\boldsymbol{r},0)\frac{\Psi(\boldsymbol{r})}{\Psi_0}. \quad (25)$$

As we explained previously, we need only a divergent part of the integral in Eq. (25). Since in three dimensions the Green function $G(\boldsymbol{r},0)$ behaves as $O(r^{-1})$ for small values of r, we may expand the wave function in series around $r=0$ and keep only two first terms in such an expansion:

$$\frac{\Psi(\boldsymbol{r})}{\Psi_0} = 1 - \frac{c(d) r^{4-d}}{4-d} + O(r^{8-2d}). \quad (26)$$

The Green function G can be written as a sum of three pieces $G = G_0 + G_1 + G_{\text{multi}}$ according to the number of Coulomb interactions.[2] In three dimensions, $G_0 \sim r^{-1}$, $G_1 \sim \ln r$, and $G_{\text{multi}} \sim r^0$ as $r \to 0$. Therefore, the first term from the expansion Eq. (26) is sufficient to extract the singularities caused by the contributions of G_1 and G_{multi}. For G_{multi} we derive

$$\int d^dr r^{-3+4\epsilon} G_{\text{multi}}(\boldsymbol{r},0) = 4\pi\int_0^{\sim 1} dr r^{-1+2\epsilon} G_{\text{multi}}(0,0) + O(1) = -\frac{3}{\epsilon} + O(1). \quad (27)$$

In order to analyze $G_{0,1}$ contributions, it is convenient to switch to the momentum space. We obtain

$$\int d^dr r^{-3+4\epsilon} G_1(\boldsymbol{r},0) = \frac{4^\epsilon \pi^{3/2-\epsilon}\Gamma(\epsilon)}{\Gamma(3/2-2\epsilon)}\int \frac{d^dp}{(2\pi)^d}\frac{G_1(\boldsymbol{p})}{p^{2\epsilon}}, \quad (28)$$

$$\int d^dr r^{-3+4\epsilon} G_0(\boldsymbol{r},0)\left(1-\frac{c(d) r^{1+2\epsilon}}{1+2\epsilon}\right) = \frac{2\pi}{\epsilon}\int \frac{d^dp}{(2\pi)^d}\frac{G_0(\boldsymbol{p})}{p^{2\epsilon}}\left(1-\frac{\pi\epsilon}{p^{1+2\epsilon}}\right) + O(1). \quad (29)$$

The integrals in Eqs. (28),(29) can be expressed through the integrals listed in Appendix:

$$\int \frac{d^dp}{(2\pi)^d}\frac{G_1(\boldsymbol{p})}{p^{2\epsilon}} = -16\pi[I_2(1+\epsilon,1,1) + I_1(1,1,1) - I_2(1,1,1)] + O(\epsilon), \quad (30)$$

$$\int \frac{d^dp}{(2\pi)^d}\frac{G_0(\boldsymbol{p})}{p^{2z}} = -2I_0(1,z). \quad (31)$$

Substituting all the relevant expressions into Eq. (25) and expanding in ϵ, we arrive at the final result for the seagull contribution:

$$\Delta_s = \frac{3\ln^2\alpha}{2} - \left(\frac{1}{2\epsilon}+\frac{4\ln 2}{3}+\frac{5}{3}-3\ln m\right)\ln\alpha. \quad (32)$$

VI. RETARDATION

In this section we discuss the retardation effect, caused by the exchange of a photon with a typical momentum of the order of the inverse Bohr radius $k \sim m\alpha$ between electron

[2] G_0 is the free Green function, G_1 is the single Coulomb correction to G_0, and G_{multi} accounts for two and more Coulomb interactions.

and positron in the bound state. The corresponding formula reads

$$\delta_{\rm ret}\Gamma = 2\left\langle \Psi\right| V_{\rm Born} G \int \frac{d^dk}{(2\pi)^d} j_i^e e^{i\mathbf{k}\cdot\mathbf{r}_e} \frac{4\pi\alpha}{2k} \frac{\mathcal{P}_{ij}(\mathbf{k})}{k+H-E} \times j_j^p e^{-i\mathbf{k}\cdot\mathbf{r}_p}\left|\Psi\right\rangle - \langle\Psi|V_{\rm Born}|\Psi\rangle \times \left\langle \Psi\right| \int \frac{d^dk}{(2\pi)^d} j_i^e e^{i\mathbf{k}\cdot\mathbf{r}_e} \frac{4\pi\alpha}{2k} \frac{\mathcal{P}_{ij}(\mathbf{k})}{(k+H-E)^2} \times j_j^p e^{-i\mathbf{k}\cdot\mathbf{r}_p}\left|\Psi\right\rangle + \text{H.c.}, \tag{33}$$

where $\boldsymbol{j}^e = \boldsymbol{p}^e/m - [\boldsymbol{\sigma}\boldsymbol{k},\boldsymbol{\sigma}]/(4m)$ and $\boldsymbol{j}^p = \boldsymbol{p}^p/m + [\boldsymbol{\sigma}'\boldsymbol{k},\boldsymbol{\sigma}']/(4m)$ are electron and positron currents, respectively.

The appearance of two matrix elements in Eq. (33) is related to the fact that we need the *second*-order correction to the wave function of a bound state. In contrast to the first order, at second order of perturbation theory one should be careful to maintain the normalization of the wave function. This is the reason why the second term in Eq. (33) appears. The general formula for the second-order correction to the wave function can be found in [13].

When writing Eq. (33), we have assumed that a magnetic photon with the momentum $\boldsymbol{k}$ is emitted by the electron at the point $\boldsymbol{r}_e$ and absorbed by the positron at the point $\boldsymbol{r}_p$. Using the fact that $H-E\sim m\alpha^2$ and $k\sim m\alpha$ for the retardation effects, we expand the right-hand side of Eq. (33) in powers of $(H-E)/k$ and obtain:

$$\delta_{\rm ret}\Gamma = 2\langle\Psi|V_{\rm Born} G V_{\rm ret}|\Psi\rangle - \langle\Psi|V_{\rm Born}|\Psi\rangle\langle\Psi|V'_{\rm ret}|\Psi\rangle, \tag{34}$$

where the ''retardation potential'' $V_{\rm ret}$ reads

$$V_{\rm ret} = -\int \frac{d^dk}{(2\pi)^d} \frac{4\pi\alpha\mathcal{P}_{ij}(\mathbf{k})}{2k^3} j_i^e e^{i\mathbf{k}\cdot\mathbf{r}_e} \times (H-E) j_j^p e^{-i\mathbf{k}\cdot\mathbf{r}_p} + \text{H.c.}, \tag{35}$$

and $V'_{\rm ret} = \partial V_{\rm ret}/\partial E$. Let us illustrate how one deals with such expressions using the spin-dependent parts of the currents as an example. The corresponding contribution to $V_{\rm ret}$ then reads

$$V_{\rm ret}^{\rm spin} = \frac{\pi\alpha}{16m^2}\int \frac{d^dk}{(2\pi)^d} \frac{[\boldsymbol{\sigma}\boldsymbol{k},\sigma_i][\boldsymbol{\sigma}'\boldsymbol{k},\sigma_i']}{k^3} \times \{(H-E)e^{i\mathbf{k}\cdot\mathbf{r}} + e^{i\mathbf{k}\cdot\mathbf{r}}(H-E)\} + \text{H.c.}, \tag{36}$$

where the relative coordinate $\boldsymbol{r} = \boldsymbol{r}_e - \boldsymbol{r}_p$ has been introduced. It is now easy to see that if we insert this result into Eq. (34) and use the Schrödinger equation both for the wave function Ψ and for the reduced Green function G, we obtain a finite correction to the decay rate. As we explained in the Introduction, finite contributions cannot generate logarithms of the fine-structure constant.

The analysis of the spin-independent contribution to $V_{\rm ret}$ is only slightly more cumbersome. We again pull out $H-E$ both to the right and to the left by commuting it with either electron or positron current and obtain the following expression for the retardation potential:

$$V_{\rm ret} = -\frac{\pi\alpha}{m^2}\int \frac{d^dk}{(2\pi)^d} \frac{\mathcal{P}_{ij}(\mathbf{k})}{k^3} (\{H-E, e^{i\mathbf{k}\cdot\mathbf{r}} p_i p_j\} + [p_i,[H,p_j]]e^{i\mathbf{k}\cdot\mathbf{r}}) + \text{H.c.},$$

where $\{H-E, e^{i\mathbf{k}\cdot\mathbf{r}} p_i p_j\}$ denotes the anticommutator of the two operators. Using this expression for the retardation potential in Eq. (34) and applying equations of motion, we arrive at the following correction to the decay rate:

$$\delta_{\rm ret}\Gamma = -\frac{4\pi\alpha}{m^2}\langle\Psi|V_{\rm Born} G[U_{ij}[C,p_i],p_j]|\Psi\rangle, \tag{37}$$

where $C = -c(d)\alpha/r^{d-2}$ is the Coulomb potential, and $U_{ij}(\boldsymbol{r})$ is defined as

$$U_{ij}(\boldsymbol{r}) = \int \frac{d^dk}{(2\pi)^d} \frac{\mathcal{P}_{ij}(\mathbf{k})}{k^3} e^{i\mathbf{k}\cdot\mathbf{r}} = \frac{\Gamma(-\epsilon) r^{2\epsilon}}{6\pi^{2-\epsilon}} (\delta_{ij} - \epsilon n_i n_j). \tag{38}$$

Performing rescaling and computing the double commutator in Eq. (37), we get the result for the relative correction to the decay rate,

$$\frac{\delta_{\rm ret}\Gamma}{\Gamma^{(0)}_{\rm p,o}} = -\frac{8\alpha\gamma^{2-2\epsilon}}{3m^2} \frac{\Gamma(2-\epsilon)\Gamma(3/2-\epsilon)}{\pi^{3/2-\epsilon}} \times \int d^d r r^{-3+4\epsilon} G(\boldsymbol{r},0) \frac{\Psi(\boldsymbol{r})}{\Psi_0}, \tag{39}$$

which is very similar to the correction induced by the seagull potential, Eq. (25). We can, therefore, borrow much of the analysis from the previous section. We finally obtain

$$\Delta_{\rm ret} = 2\ln^2\alpha - \left(\frac{2}{3\epsilon} - \frac{4\ln 2}{3} + 4 - 4\ln m\right)\ln\alpha. \tag{40}$$

VII. ULTRASOFT CONTRIBUTION

By definition, the ultrasoft contribution is due to the photons with energy and momentum of order $m\alpha^2$. Such soft photons cannot resolve the structure of the bound state and for this reason they interact directly with the positronium as a whole. Since the positronium is chargeless, the interaction is necessarily of the dipole nature.

A general formula for the correction to the decay rate caused by ultrasoft contributions is

$$\delta_{\rm us}\Gamma = 2\left\langle \Psi\left| V_{\rm Born}\, G \frac{2p_i}{m}\int \frac{d^dk}{(2\pi)^d}\,\frac{4\pi\alpha}{2k}\,\frac{\mathcal{P}_{ij}(\boldsymbol{k})}{E-H-k}\,\frac{2p_j}{m}\right|\Psi\right\rangle - \langle\Psi|V_{\rm Born}|\Psi\rangle \times\left\langle \Psi\left|\frac{2p_i}{m}\int\frac{d^dk}{(2\pi)^d}\,\frac{4\pi\alpha}{2k}\,\frac{\mathcal{P}_{ij}(\boldsymbol{k})}{(E-H-k)^2}\,\frac{2p_j}{m}\right|\Psi\right\rangle. \tag{41}$$

After integration over directions of $\boldsymbol{k}$ and performing the rescaling we get

$$\frac{\delta_{\rm us}\Gamma}{\Gamma^{(0)}_{\rm p,o}} = \frac{4^{2-\epsilon}\pi\alpha\gamma^{2-4\epsilon}}{m^{2-2\epsilon}}\,\frac{\Omega_d}{(2\pi)^d}\,\frac{d-1}{d} \times\left(\left\langle 0\left| G\boldsymbol{p}\int_0^\infty \frac{dk k^{-2\epsilon}}{k+H-E}[H,\boldsymbol{p}]\right|\tilde\Psi\right\rangle - \left\langle\Psi\left|\boldsymbol{p}\int_0^\infty\frac{dk k^{1-2\epsilon}}{(k+H-E)^2}\boldsymbol{p}\right|\Psi\right\rangle\right). \tag{42}$$

Both matrix elements in Eq. (42) must be computed in atomic units; also $\tilde\Psi=\Psi/\Psi_0$, and $\Omega_d=2\pi^{3/2-\epsilon}/\Gamma(3/2-\epsilon)$ is the d-dimensional angular volume.

The second matrix element in Eq. (42) is easy to compute:[3]

$$-\left\langle\Psi\left|\boldsymbol{p}\int_0^\infty\frac{dk k^{1-2\epsilon}}{(k+H-E)^2}\,\boldsymbol{p}\right|\Psi\right\rangle = -\frac{\langle\Psi|\boldsymbol{p}(H-E)^{-2\epsilon}\boldsymbol{p}|\Psi\rangle}{2\epsilon} = -\frac{1}{2\epsilon}. \tag{43}$$

Then, consider the first matrix element from Eq. (42). It is convenient to write it as a sum:

$$\left\langle 0\left| G\boldsymbol{p}\int_0^\infty\frac{dk k^{-2\epsilon}}{k+H-E}[H,\boldsymbol{p}]\right|\tilde\Psi\right\rangle = M_0+M_1, \tag{44}$$

where the two terms correspond to the number of Coulomb interactions between the moments of emission and absorption of the ultrasoft photon.[4]

Let us consider M_0. In this case, integrating over k we obtain

$$M_0=\frac{1}{2\epsilon}\langle 0|G(H_0-E)^{-2\epsilon}\boldsymbol{p}[H,\boldsymbol{p}]|\tilde\Psi\rangle, \tag{45}$$

where $H_0=p^2/2$ is the free Hamiltonian in atomic units. Using

$$\boldsymbol{p}[H,\boldsymbol{p}]=[\boldsymbol{p},[H,\boldsymbol{p}]]+[H,\boldsymbol{p}]\boldsymbol{p}=4\pi\delta(\boldsymbol{r})+[H,\boldsymbol{p}]\boldsymbol{p}, \tag{46}$$

we represent M_0 as the sum of two terms,

$$M_0=M_\delta+M_\psi, \tag{47}$$

where

$$M_\delta=\frac{4\pi}{2\epsilon}\langle 0|G(H_0-E)^{-2\epsilon}|0\rangle, \tag{48}$$

$$M_\psi=\frac{1}{2\epsilon}\langle 0|G(H_0-E)^{-2\epsilon}[H,\boldsymbol{p}]\boldsymbol{p}|\tilde\Psi\rangle. \tag{49}$$

We now analyze the two terms in Eq. (47) separately. In order to compute M_δ we use the fact that only two first terms in the expansion of the Green function in the number of Coulomb exchanges, G_0 and G_1, diverge at zero spatial separation. Since $G_{\rm multi}(0,0)$ is finite, we can compute it for $d=3$. On the other hand, both G_0 and G_1 are known explicitly [see, e.g., [9]] and hence the corresponding integrals can easily be computed. Using expressions for integrals summarized in Appendix, we arrive at the following expression for M_δ:

$$M_\delta=-\frac{2^{2+2\epsilon}\pi}{\epsilon}I_0(1+2\epsilon,0)-\frac{2^{5+2\epsilon}\pi^2}{\epsilon}I_1(1+2\epsilon,1,1)-\frac{3}{\epsilon}. \tag{50}$$

Consider the second matrix element M_ψ. In this case, it is sufficient to separate $G=G_0+(G-G_0)$. Since there is an overall factor ϵ^{-1} in Eq. (49), the finite matrix element that contains $G-G_0$ can be calculated in three dimensions. Using the expression for $G-G_0$ from [9], we derive

$$M^{\rm f}_\psi=\frac{1}{2\epsilon}\langle 0|(G-G_0)[H,\boldsymbol{p}]\boldsymbol{p}|\tilde\Psi\rangle=\frac{2}{\epsilon}. \tag{51}$$

The matrix element containing G_0 can be rewritten in momentum space as

$$M^{\rm i}_\psi=\frac{1}{2\epsilon}\langle 0|G_0(H_0-E)^{-2\epsilon}[H,\boldsymbol{p}]\boldsymbol{p}|\tilde\Psi\rangle = -\frac{1}{2\epsilon}\int\frac{d^dp'd^dp}{(2\pi)^{2d}}\left(\frac{2}{p'^2+1}\right)^{1+2\epsilon}\frac{4\pi(\boldsymbol{p}'-\boldsymbol{p})\boldsymbol{p}}{(\boldsymbol{p}'-\boldsymbol{p})^2}\,\frac{\Psi(\boldsymbol{p})}{\Psi_0}, \tag{52}$$

where $\Psi(\boldsymbol{p})$ is the Fourier transform of $\Psi(\boldsymbol{r})$. Since $M^{\rm i}_\psi$ has an overall divergence and hence we need the integral up to a constant only, we can use the three-dimensional expression for the wave function $\Psi(\boldsymbol{p})$:

$$M^{\rm i}_\psi=-\frac{4\pi^2}{\epsilon}\int\frac{d^dp'd^dp}{(2\pi)^{2d}}\left(\frac{2}{p'^2+1}\right)^{1+2\epsilon}\frac{(\boldsymbol{p}'-\boldsymbol{p})\boldsymbol{p}}{(\boldsymbol{p}'-\boldsymbol{p})^2}\left(\frac{2}{p^2+1}\right)^2. \tag{53}$$

[3]Recall that only the terms that are singular for $\epsilon\to 0$ are needed.

[4]If two or more Coulomb photons are exchanged, the resulting matrix element becomes finite and, in accordance with the argument given in Sec. II, it cannot generate the $O(\ln\alpha)$ correction.

We now rewrite the scalar product in the numerator as a linear combination of denominators and obtain our final expression for $M_\psi = M_\psi^{\rm f} + M_\psi^{\rm i}$:

$$M_\psi = -\frac{4^{2+\epsilon}\pi^2}{\epsilon}[I_1(2\epsilon,1,2) - I_1(1+2\epsilon,1,1) - I_1(1+2\epsilon,0,2)] + \frac{2}{\epsilon}. \tag{54}$$

Consider next the matrix element M_1. We start with the following expression:

$$M_1 = -\left\langle 0 \left| G\boldsymbol{p}\int_0^\infty dk k^{-2\epsilon} \frac{1}{k+H_0-E} C \frac{1}{k+H_0-E}[H,\boldsymbol{p}] \right| \tilde{\Psi} \right\rangle. \tag{55}$$

Simple power counting shows that we can safely take $\tilde{\Psi}$ at the origin, $\tilde{\Psi}\to 1$, and also replace the Green function G by its high-momentum asymptotics, $G(p)\to -2/p^2$. We then obtain symmetric and uniform expression

$$M_1 = -8\pi \int_0^\infty dk k^{-2\epsilon} \int \frac{d^d p' d^d p}{(2\pi)^{2d}} \frac{2}{p'^2(p'^2+1+2k)} \times \frac{4\pi \boldsymbol{p}'\boldsymbol{p}}{(\boldsymbol{p}'-\boldsymbol{p})^2} \frac{2}{p^2(p^2+1+2k)}. \tag{56}$$

If we now rescale both $\boldsymbol{p}$ and $\boldsymbol{p}'$ as $\boldsymbol{p}\to\sqrt{2k+1}\,\boldsymbol{p}$, the integration over k factorizes and we obtain

$$M_1 = -\frac{8\pi^2}{\epsilon}[2I_2(1,1,1) - 2I_1(1,1,1) - I_0(1,1)^2]. \tag{57}$$

Finally, using explicit expressions for the integrals from the Appendix, we arrive at our final result for the ultrasoft correction to the decay rate:

$$\Delta_{\rm us} = -\frac{16\ln\alpha^2}{3} + \left(\frac{4}{3\epsilon} + 8\ln 2 + \frac{20}{9} - 8\ln m\right)\ln\alpha. \tag{58}$$

VIII. CONCLUSIONS

The sum of all the contributions from Eqs. (11), (13), (16), (32), (40), and (58) gives the final result for the $O(\alpha^3 \ln \alpha)$ corrections to the Ps decay rate:

$$\frac{\Delta\Gamma_{\rm p}}{\Gamma_{\rm p}^{(0)}} = \frac{\alpha^3}{\pi}\left[-\frac{3}{2}\ln^2\alpha + \ln\alpha\left\{-\frac{367}{90} + 10\ln 2 - 2A_{\rm p}\right\}\right] = \frac{\alpha^3}{\pi}\left[-\frac{3}{2}\ln^2\alpha + 7.919\ln\alpha\right], \tag{59}$$

$$\frac{\Delta\Gamma_{\rm o}}{\Gamma_{\rm o}^{(0)}} = \frac{\alpha^3}{\pi}\left[-\frac{3}{2}\ln^2\alpha + \ln\alpha\left\{-\frac{229}{30} + 8\ln 2 + \frac{A_{\rm o}}{3}\right\}\right] = \frac{\alpha^3}{\pi}\left[-\frac{3}{2}\ln^2\alpha - 5.517\ln\alpha\right]. \tag{60}$$

Numerically, these corrections cause a negligible change in the theoretical prediction for p-Ps and o-Ps lifetimes at the current level of precision. It is interesting to note, however, that the magnitude of the leading $O(\alpha^3 \ln^2\alpha)$ and the subleading $O(\alpha^3 \ln \alpha)$ corrections is comparable; in the case of o-Ps they almost cancel each other.

Our results Eqs. (59) and (60) are in agreement with two recent calculations of $O(\alpha^3 \ln \alpha)$ corrections [7,8]. In Ref. [7] the result for the $O(\alpha^3 \ln \alpha)$ correction to the o-Ps decay rate has been obtained numerically, where as in Ref. [8] analytical methods similar to ours have been employed. We believe that the achieved agreement between three independent calculations ensures that the results, Eqs. (59) and (60), are correct.

As we mentioned, the $O(\alpha^3 \ln \alpha)$ correction to Ps decay rates at present is not very interesting phenomenologically. A more important question, which we think we fully addressed in this paper, is how the logarithms of the fine-structure constant can be efficiently extracted in the bound-state calculation when the dimensional regularization is used to regulate the nonrelativistic dynamics. It is true that dimensional regularization offers many technical advantages in the calculation. This does not go without a price, however, since one has to be extremely careful in defining basic objects of the nonrelativistic theory, e.g., the wave functions and energies. If this is not done, one is left guessing whether or not the calculation is correct.

Our key observation, which we think cures such problems and makes our calculation unambiguous, is the fact that the matrix elements in d dimensions are the uniform functions of the fine-structure constant, and that the corresponding power of α can be determined by expressing the matrix elements in ''d-dimensional'' atomic units. We think that these arguments have not been spelled out before in the literature on one hand, and that they are necessary to make a convincing case, on the other.

Finally, let us note that the technique discussed in this paper can obviously be used in other bound-state QED problems, as well as for the heavy quarkonium states in QCD.

ACKNOWLEDGMENTS

This work was supported in part by DOE under Grant No. DE-AC03-76SF00515, and by the Russian Foundation for Basic Research Grant No. 00-02-17646. We are grateful to the BNL High Energy Theory Group, where part of this work has been done, for the hospitality extended to us.

APPENDIX

We give the definitions of the integrals that were used in the derivation.

$$I_0(a,b)=\int\frac{d^dp}{(2\pi)^d}\left(\frac{1}{p^2+1}\right)^a\left(\frac{1}{p^2}\right)^b=(4\pi)^{-d/2}\frac{\Gamma(a+b-d/2)\Gamma(d/2-b)}{\Gamma(a)\Gamma(d/2)}, \tag{A1}$$

$$I_1(a,b,c)=\int\frac{d^dp'd^dp}{(2\pi)^{2d}}\left(\frac{1}{p'^2+1}\right)^a\left(\frac{1}{(\boldsymbol{p}'-\boldsymbol{p})^2}\right)^b\left(\frac{1}{p^2+1}\right)^c$$
$$=(4\pi)^{-d}\frac{\Gamma(a+b+c-d)\Gamma(a+b-d/2)\Gamma(b+c-d/2)\Gamma(d/2-b)}{\Gamma(a)\Gamma(c)\Gamma(d/2)\Gamma(a+2b+c-d)}, \tag{A2}$$

$$I_2(a,b,c)=\int\frac{d^dp'd^dp}{(2\pi)^{2d}}\left(\frac{1}{p'^2}\right)^a\left(\frac{1}{(\boldsymbol{p}'-\boldsymbol{p})^2}\right)^b\left(\frac{1}{p^2+1}\right)^c$$
$$=(4\pi)^{-d}\frac{\Gamma(a+b+c-d)\Gamma(a+b-d/2)\Gamma(d/2-a)\Gamma(d/2-b)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d/2)}. \tag{A3}$$

[1] R. Conti, ''Experimental Tests of QED in Positronium: Recent Advances,'' contribution to the Conference *Hydrogen Atom II*, Castiglione della Pescaia, Italy, 2000.

[2] A. Czarnecki, K. Melnikov, and A. Yelkhovsky, Phys. Rev. A **61**, 052502 (2000); **62**, 059902(E) (2000).

[3] G. S. Adkins, R. N. Fell, and J. Sapirstein, Phys. Rev. Lett. **84**, 5086 (2000).

[4] S. G. Karshenboim, Zh. Éksp. Teor. Fiz. **103**, 1105 (1993) [JETP Lett. **76**, 541 (1993)].

[5] I. Harris and L. M. Brown, Phys. Rev. **105**, 1656 (1957).

[6] G. S. Adkins, Phys. Rev. Lett. **76**, 4903 (1996).

[7] R. Hill and G. P. Lepage, Phys. Rev. D **62**, 111301(R) (2000).

[8] B. Kniehl and A. Penin, Phys. Rev. Lett. **85**, 1210 (2000); **85**, 3065(E) (2000).

[9] A. Czarnecki, K. Melnikov, and A. Yelkhovsky, Phys. Rev. A **59**, 4316 (1999).

[10] I. B. Khriplovich and A. S. Yelkhovsky, Phys. Lett. B **246**, 520 (1990).

[11] W. E. Caswell and G. P. Lepage, Phys. Rev. A **20**, 36 (1979).

[12] A. Pineda and J. Soto, Phys. Rev. D **59**, 016005 (1999).

[13] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-relativistic Theory* (Pergamon, Oxford, 1976).