Massive field contributions to the QCD vacuum tunneling amplitude

O-Kab Kwon*
Physics Department and Center for Theoretical Physics, Seoul National University, Seoul 151-742, Korea

Choonkyu Lee†
Physics Department and Center for Theoretical Physics, Seoul National University, Seoul 151-742, Korea
and School of Physics, Korea Institute for Advanced Study, 207-43 Cheongryangryi-Dong, Dongdaemun-Gu, Seoul 130-012, Korea

Hyunsoo Min‡
Physics Department, University of Seoul, Seoul 130-743, Korea
(Received 6 August 2000; published 9 November 2000)

For the one-loop contribution to the QCD vacuum tunneling amplitude by quarks of generic mass value, we make use of a calculational scheme exploiting a large mass expansion together with a small mass expansion. The large mass expansion for the effective action is given by a series involving higher-order Seeley-DeWitt coefficients, and we carry this expansion up to order $1/(m\rho)^8$, where m denotes mass of the quark and ρ the instanton size parameter. For the small mass expansion, we use the known exact expression for the particle propagation functions in an instanton background and evaluate explicitly the effective action to order $(m\rho)^2$. A smooth interpolation of the results from both expansions suggests that the quark contribution to the instanton tunneling amplitude have a relatively simple $m\rho$-dependent behavior.

PACS number(s): 12.38.Aw, 11.15.Tk

I. INTRODUCTION

Instantons [1,2], as localized finite-action solutions of the Euclidean Yang-Mills field equations, describe vacuum tunneling and are believed to have important nonperturbative roles in low-energy QCD. For an excellent review on instantons in QCD and general gauge theories, see Refs. [3,4]. For actual instanton calculations, one needs to know above all the one-loop tunneling amplitude or the Euclidean one-loop effective action in the background field of a single (anti-) instanton. The latter quantity is thus of fundamental importance in instanton physics, and in the zero mass limit of scalar or quark fields 't Hooft [5] was able to calculate the appropriate one-loop contribution exactly. But, with finite quark mass, such an exact calculation does not look feasible and one has to be satisfied with approximate results. (Note that, aside from up and down quarks, all other quarks possess sizable mass). In this paper we shall describe our approach to determine the quark mass dependence in the one-loop vacuum tunneling amplitude, and report some new results from this analysis.

By studying the field-theoretic effective action one can take the quantum nature of the fields into account systematically, and already at the one-loop level it has provided us with certain relevant information on various physically significant effects [6]. In particular, the leading-order renormalization group coefficients in field theories are encoded in the divergences of the corresponding bare one-loop effective action. These divergent terms can be found most simply with the help of the background field method [7,8] and the Schwinger-DeWitt proper-time algorithm [6,7]; they are entirely given by the second Seeley-DeWitt coefficient $\tilde{a}_2$ [7,9,10] in four-dimensional space-time. (For recent literature discussing this method, see Ref. [11].) But the evaluation of the full finite part of the one-loop effective action in any nontrivial background field corresponds to a formidable mathematical problem in general. For an approximate calculation (in a slowly varying background) the so-called derivative expansion of the effective action has been utilized by various authors [12,13].

As for the contribution to the instanton one-loop effective action by spin-0 or spin-1/2 matter fields of, say, mass m, we shall consider both the approximation applicable for relatively large $m\rho$ (ρ is the instanton size), i.e., large mass expansion and the mass perturbation scheme useful for relatively small $m\rho$. Note that the nature of the approximation is governed by the dimensionless parameter $m\rho$. (Dependence on the renormalization mass scale μ can be treated separately.) The large mass expansion is essentially a series involving higher-order Seeley-DeWitt coefficients, for which a simple computer algorithm has been developed recently [14–16]. We then make a smooth interpolation of the results found in those two different regimes, with the expectation that some general pattern, which is meaningful over a wide range of mass values, may emerge. This information should be valuable in phenomenological studies related to instanton effects. To connect the amplitude given for different mass scales, one should be careful about possible large finite-renormalization effects and renormalization schemes used. In this paper we treat various issues related to this general idea in a reasonably self-contained manner.

In Sec. II we present a concise review on the Schwinger proper-time representation of the effective action, various renormalization schemes, and the large mass expansion. Also discussed are finite renormalization effects specific to renor-

*Electronic address: kok@phya.snu.ac.kr
†Electronic address: cklee@phya.snu.ac.kr
‡Electronic address: hsmin@dirac.uos.ac.kr

malization prescriptions chosen, since they can introduce additional mass (as well as renormalization scale) dependences into the effective action. This understanding will become important when one has to change the results obtained in one renormalization prescription to that in another prescription.

In Sec. III the one-loop effective action for a massive scalar field in a constant Yang-Mills field background is considered to see how our general scheme would fit in for this simple case. Here we make a detailed comparison between the known, exact, effective action (given in a single integral form) and the corresponding result based on the large mass expansion.

In Sec. IV the spin-0 one-loop effective action in a Yang-Mills instanton background is studied on the basis of the large mass expansion. (Contributions due to fields of different spin can be related to this spin-0 amplitude.) We consider up to the sixth Seeley-DeWitt coefficient. Here our finding is that, for $m\rho \gtrsim 1.8$, the large mass expansion appears to give a good approximation to the effective action.

In Sec. V we study the spin-0 instanton effective action for small $m\rho$, utilizing the known expressions for the massless propagators [17] in an instanton background and the mass perturbation. Since the naive mass perturbation leads to a logarithmically divergent integral [18], a suitably modified perturbation method must be employed to obtain a well-defined small-mass correction term. We here reconfirm the $\mathcal{O}[(m\rho)^2 \ln m\rho]$ term previously found in Ref. [18], and provide for the first time the full $\mathcal{O}[(m\rho)^2]$ contribution to the instanton effective action.

In Sec. VI we consider an interpolation of our amplitude to intermediate values of $m\rho$, given the results of the previous two sections. Here we also make appropriate changes in our results so that they may describe the spin-1/2 instanton effective action; this result is directly relevant for quarks with nonzero mass. Note that, due to the hidden supersymmetry in an instanton background, one can utilize the result for the spin-0 case to find the contribution due to spin-1/2 fields [5].

In Sec. VII we conclude with some remarks. In Appendix A some explicit expressions for higher-order Seeley-DeWitt coefficients can be found. Appendix B contains an analysis of a certain function which figures in our small-mass expansion of Sec. V.

II. THE ONE-LOOP EFFECTIVE ACTION, RENORMALIZATION, AND THE LARGE MASS EXPANSION

To be definite, we will consider a four-dimensional, Euclidean, Yang-Mills theory with matter described by *complex* scalar or Dirac spinor fields of mass m. Then, in any given Yang-Mills background fields $A_\mu^a(x)$, one may represent the (Pauli-Villars regularized) one-loop effective action due to matter fields by

$$\Gamma(A)=\lambda \ln\left[\frac{\mathrm{Det}(G^{-1}+m^2)}{\mathrm{Det}(G_0^{-1}+m^2)}\frac{\mathrm{Det}(G_0^{-1}+\Lambda^2)}{\mathrm{Det}(G^{-1}+\Lambda^2)}\right]. \quad (2.1)$$

Here, $\lambda=1$ $(-\frac{1}{2})$ for scalar (spinor) fields, Λ is the large regulator mass, G^{-1} stands for the appropriate quadratic differential operator, viz.,

$$G^{-1}=\begin{cases} -D^2 & \text{(for scalar)},\\ (\gamma D)^2 & \text{(for spinor)}, \end{cases} \quad (2.2)$$

and $G_0^{-1}=G^{-1}|_{A_\mu=0}=-\partial^2$. (Also, $D^2=D_\mu D_\mu$ and $\gamma D=\gamma_\mu D_\mu$ with the covariant derivative $D_\mu=\partial_\mu - iA_\mu^a T^a \equiv \partial_\mu - iA_\mu$ (T^a denote the group generators in the matter representation satisfying the commutation relations $[T^a,T^b]=if_{abc}T^c$), and our γ matrices, which are antihermitian, satisfy the relations $\{\gamma_\mu,\gamma_\nu\}=-2\delta_{\mu\nu}$.

In the proper-time representation [6,7], one represents $\Gamma(A)$ by

$$\Gamma(A)=-\lambda\int_0^\infty \frac{ds}{s}(e^{-m^2 s}-e^{-\Lambda^2 s})\mathrm{Tr}[e^{-sG^{-1}}-e^{-sG_0^{-1}}], \quad (2.3)$$

where Tr denotes the trace over space-time coordinates and all other discrete indices. More explicitly, writing $\mathrm{Tr}=\int d^4x\,\mathrm{tr}$ and introducing the proper-time Green function $\langle xs|y\rangle=\langle x|e^{-sG^{-1}}|y\rangle$, $\Gamma(A)$ can be expressed as

$$\Gamma(A)=-\lambda\int_0^\infty \frac{ds}{s}(e^{-m^2 s}-e^{-\Lambda^2 s}) \times\int d^4x\,\mathrm{tr}[\langle xs|x\rangle-\langle xs|x\rangle|_{A_\mu=0}]. \quad (2.4)$$

The full effective action is thus determined if the coincidence limit (i.e., $y=x$) of the proper-time Green function is known. The expression (2.4) diverges logarithmically as we let $\Lambda\to\infty$; to isolate such divergent pieces, we may exploit the asymptotic expansion [7,9]

$$s\to 0+: \quad \langle xs|y\rangle=\frac{1}{(4\pi s)^2}e^{-(x-y)^2/4s}\left\{\sum_{n=0}^\infty s^n a_n(x,y)\right\}, \quad (2.5)$$

where the leading coefficient has the coincidence limit $a_0(x,x)=1$. Using this expansion in Eq. (2.4), we then see that the divergences in $\Gamma(A)$ as $\Lambda\to\infty$ are related to the coincidence limits $\tilde{a}_1(x)\equiv\mathrm{tr}\,a_1(x,x)$ and $\tilde{a}_2(x)\equiv\mathrm{tr}\,a_2(x,x)$, which correspond to the first and second Seeley-DeWitt coefficients, respectively. Simple calculations yield

$$\tilde{a}_1(x)=0 \quad \text{(for both scalar and spinor)}, \quad (2.6)$$

$$\tilde{a}_2(x)=\begin{cases} -\frac{1}{12}\mathrm{tr}[F_{\mu\nu}(x)F_{\mu\nu}(x)] & \text{(for scalar)},\\ \frac{2}{3}\mathrm{tr}[F_{\mu\nu}(x)F_{\mu\nu}(x)] & \text{(for spinor)}, \end{cases} \quad (2.7)$$

where $F_{\mu\nu}\equiv F^a_{\mu\nu}T^a=i[D_\mu,D_\nu]$. (The tr here refers to the trace over gauge group representation indices only.) Based on these, we may now write the above effective action for $m^2\neq 0$ as

$$\Gamma(A)=Y\left(\ln\frac{\Lambda^2}{m^2}\right)\int d^4x\, F^a_{\mu\nu}F^a_{\mu\nu}+\overline{\Gamma}(A) \tag{2.8}$$

with

$$Y=\begin{cases}\dfrac{1}{12}\dfrac{C}{(4\pi)^2} & \text{(for scalar)},\\[2ex] \dfrac{1}{3}\dfrac{C}{(4\pi)^2} & \text{(for spinor)},\end{cases} \tag{2.9}$$

(C is defined by $\mathrm{tr}(T^aT^b)=\delta_{ab}C$), and then the contribution

$$\overline{\Gamma}(A)=-\lambda\int_0^\infty\frac{ds}{s^3}e^{-m^2s}\int d^4x\left[1-\left(1+s\frac{\partial}{\partial s}+\frac{1}{2}s^2\frac{\partial^2}{\partial s^2}\right)\Big|_{s=0}\right]\mathrm{tr}(s^2\langle xs|x\rangle) \tag{2.10}$$

becomes well defined as long as m^2 is nonzero. (In Eq. (2.10), $[1+s(\partial/\partial s)+\frac{1}{2}s^2(\partial^2/\partial s^2)]|_{s=0}f(s)\equiv f(0)+sf'(0)+\frac{1}{2}s^2f''(0)$.)

The logarithmic divergence in $\Gamma(A)$ is canceled by the renormalization counterterm associated with the coupling constant renormalization of the classical (bare) action $(1/4g_0^2)\int d^4xF^a_{\mu\nu}F^a_{\mu\nu}$. But the resulting renormalized one-loop amplitude depends on the renormalization prescription chosen. From the very structure exhibited in Eq. (2.8), our amplitude $\overline{\Gamma}(A)$ can be considered as defining *a* renormalized one-loop effective action; but, this prescription cannot be used for the strictly massless case. Instead, one may here consider adding to $\Gamma(A)$ the counterterm

$$\Delta\Gamma(A)=-Y\left(\ln\frac{\Lambda^2}{\mu^2}\right)\int d^4x\, F^a_{\mu\nu}F^a_{\mu\nu} \tag{2.11}$$

(μ is an arbitrarily introduced renormalization mass) to obtain the renormalized one-loop effective action

$$\Gamma_{\mathrm{ren}}(A)=-Y\left(\ln\frac{m^2}{\mu^2}\right)\int d^4x\, F^a_{\mu\nu}F^a_{\mu\nu}+\overline{\Gamma}(A), \tag{2.12}$$

where $\overline{\Gamma}(A)$ is defined by Eq. (2.10). It should be remarked that $\Gamma_{\mathrm{ren}}(A)$, given by Eq. (2.12), is expected to have a well-defined limit for $m^2\to 0$ (i.e., does not exhibit infrared singularities), if the operator G^{-1} does not allow any normalizable zero eigenmode.

Other renormalization prescriptions may also be chosen. Let $\Gamma_{\mathrm{MS}}(A)$ denote the renormalized amplitude in the so-called minimal subtraction scheme [19] associated with the dimensional regularization. Then, to obtain the expression for $\Gamma_{\mathrm{MS}}(A)$ in the scalar case, one should add to that of $\Gamma_{\mathrm{ren}}(A)$ the following finite renormalization term:

$$\frac{C}{(4\pi)^2\times 12}(\ln 4\pi-\gamma)\int d^4x\, F^a_{\mu\nu}F^a_{\mu\nu}, \tag{2.13}$$

where $\gamma=0.5772\cdots$ is the Euler constant. In the thus found expression of $\Gamma_{\mathrm{MS}}(A)$ the mass parameter μ, which enters the expression through $\Gamma_{\mathrm{ren}}(A)$, describes the normalization mass conventionally introduced in the minimal subtraction scheme. As for the expression of $\Gamma_{\mathrm{MS}}(A)$ in the spinor case, the finite renormalization term to be added to that of $\Gamma_{\mathrm{ren}}(A)$ turns out to be

$$\frac{C}{(4\pi)^2\times 3}(\ln 4\pi-\gamma)\int d^4x\, F^a_{\mu\nu}F^a_{\mu\nu}. \tag{2.14}$$

(This is the case when the spinor trace of 1 is taken to be 4.) In another often-used prescription, one specifies the renormalization counterterm via the momentum-space subtraction scheme, i.e., by imposing a normalization condition at certain external momentum value $p^2=\mu^2$. Then the corresponding renormalized expression $\Gamma_{\mathrm{mom}}(A)$ is given by that of $\Gamma_{\mathrm{ren}}(A)$ plus the finite renormalization term

$$\frac{C}{(4\pi)^2\times 12}\left[-\frac{8}{3}+\ln\frac{m^2}{\mu^2}-8\frac{m^2}{\mu^2}+\left(1+4\frac{m^2}{\mu^2}\right)^{3/2}\times\ln\left(\frac{\sqrt{1+4\dfrac{m^2}{\mu^2}}+1}{\sqrt{1+4\dfrac{m^2}{\mu^2}}-1}\right)\right]\times\int d^4x\, F^a_{\mu\nu}F^a_{\mu\nu}\quad\text{(for scalar)} \tag{2.15}$$

or

$$\frac{C}{(4\pi)^2\times 3}\left[-\frac{5}{3}+\ln\frac{m^2}{\mu^2}+4\frac{m^2}{\mu^2}+\left(1-2\frac{m^2}{\mu^2}\right)\sqrt{1+4\frac{m^2}{\mu^2}}\times\ln\left(\frac{\sqrt{1+4\dfrac{m^2}{\mu^2}}+1}{\sqrt{1+4\dfrac{m^2}{\mu^2}}-1}\right)\right]\int d^4x\, F^a_{\mu\nu}F^a_{\mu\nu}\quad\text{(for spinor)} \tag{2.16}$$

with the corresponding reinterpretation of the parameter μ. These renormalization-prescription dependences of the one-loop effective action are of course explained by the fact that the tree-level contribution involves, as a multiplicative fac-

tor, the renormalized coupling $1/g_R^2$ (whose value may vary with renormalization prescriptions).

Thanks to the exact connection formulas we have described above, knowledge on the one-loop effective action in one renormalization prescription can immediately be changed into that in another prescription. In fact, in theories containing several matter fields of different mass scales (e.g., QCD with quarks of very different masses), one may well adopt different renormalization prescriptions for different matter field loops. We here note that use of the minimal subtraction for a heavy-quark loop is rather unnatural, due to the lack of manifest decoupling [20]. But this is not an issue in our discussions.

The next task is to find the actual full expression for the one-loop effective action—at present, this is possible only with a background field of very special character. But, if the mass parameter is sufficiently large, it can be studied for generic smooth background fields by utilizing a systematic large-mass expansion, which is obtained by inserting the asymptotic expansion (2.5), say, into the formula (2.10) for $\overline{\Gamma}(A)$. This assumes the form

$$\overline{\Gamma}(A) = -\frac{\lambda}{(4\pi)^2}\sum_{n=3}^{\infty}\frac{(n-3)!}{(m^2)^{n-2}}\int d^4x\,\tilde{a}_n(x),$$
$$[\tilde{a}_n(x) \equiv \mathrm{tr}\, a_n(x,x)]. \tag{2.17}$$

That is, for large enough mass, we have the one-loop effective action (in any renormalization prescription) expressed by a series involving higher-order Seeley-DeWitt coefficients $\tilde{a}_n(x)$ $(n \geqslant 3)$, the calculation of which may be performed using a computer [14–16]. If only the leading term is kept with the series (2.17), one finds, explicitly,

$$m\to\infty: \quad \overline{\Gamma}(A) = \begin{cases} -\dfrac{1}{16\pi^2}\dfrac{1}{m^2}\dfrac{1}{180}\displaystyle\int d^4x\,\mathrm{tr}\left[\frac{3}{2}(D_\mu F_{\nu\lambda})(D_\mu F_{\nu\lambda}) - 4iF_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}\right] & \text{(for scalar)}, \\ \dfrac{1}{32\pi^2}\dfrac{1}{m^2}\dfrac{2}{45}\displaystyle\int d^4x\,\mathrm{tr}[-3(D_\mu F_{\nu\lambda})(D_\mu F_{\nu\lambda}) + 13iF_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}] & \text{(for spinor)}, \end{cases} \tag{2.18}$$

where $D_\mu F_{\nu\lambda} \equiv [D_\mu, F_{\nu\lambda}]$. If the background fields under consideration satisfy the classical Yang-Mills field equations, one can show using the Bianchi identities and the property of trace that $\int d^4x\,\mathrm{tr}[(D_\mu F_{\nu\lambda})(D_\mu F_{\nu\lambda})] = 4i\int d^4x\,\mathrm{tr}[F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}]$. Hence, for the *on-shell* effective action, Eq. (2.18) can be further simplified as

$$m\to\infty: \quad \overline{\Gamma}(A) = \begin{cases} -\dfrac{1}{16\pi^2}\dfrac{1}{m^2}\dfrac{i}{90}\displaystyle\int d^4x\,\mathrm{tr}(F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}) & \text{(for scalar)} \\ \dfrac{1}{16\pi^2}\dfrac{1}{m^2}\dfrac{i}{45}\displaystyle\int d^4x\,\mathrm{tr}(F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}) & \text{(for spinor)}. \end{cases} \tag{2.19}$$

For some explicit expressions of the higher-order Seeley-DeWitt coefficients, see Sec. IV and Appendix A. Also note that the large mass expansion for the effective action in other renormalization schemes can be obtained from the expansion (2.17) for $\overline{\Gamma}(A)$ and the exact connection formulas.

The large-mass expansion is only an asymptotic series, and the useful range of the series (2.17) (as regards the magnitude of m) will depend much on the nature of the background field and also on some characteristic scale(s) entering the background. For a sufficiently smooth background, this large-mass expansion may be used to obtain a reliable approximation to the effective action even for moderately large values of m. But the series (2.17) is bound to lose the predictive power for ''small'' values of m, and for the small-m effective action one should employ a totally different strategy, such as the small-mass expansion if its exact expression in the massless limit has been found by some other methods. In the next section, we shall first see how good the large-mass expansion can be for the much studied case of the one-loop effective action in the constant Yang-Mills field background. Also considered is its small-mass expansion which may serve, together with the result of the large mass expansion, as a basis to infer the behavior of the effective action for arbitrary mass.

III. THE SPIN-0 EFFECTIVE ACTION IN A CONSTANT SELF-DUAL YANG-MILLS FIELD BACKGROUND

In this section, various approximation schemes to be used later will be tested against the exact result, choosing a rather

simple background field. In non-Abelian gauge theories, a constant field strength is realized either by an Abelian vector potential which varies linearly with x^μ or by a constant vector potential whose components do not commute [21]. In this paper we only consider the case of the Abelian vector potential. Assuming the SU(2) gauge group, an Abelian vector potential can be written as $A_\mu = -\frac{1}{4} f_{\mu\nu} x_\nu \tau^3$ (with the field strength tensor $F_{\mu\nu} = f_{\mu\nu}\tau^3/2$), where τ^3 is the third Pauli matrix. If we further restrict our attention to that with the self-dual field strength (i.e., $F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\delta} F_{\lambda\delta}$), its nonzero components may be specified by setting $f_{12} = f_{34} = H$ with the constant ''magnetic'' field H.

In this Abelian constant self-dual field, let us consider the one-loop effective action due to an isospin-1/2, spin-0 (complex-valued) matter field, taking the mass m of our spin-0 field to be relatively large so that the large mass expansion (2.17) may be used. For this case, some leading Seeley-DeWitt coefficients are easily evaluated (using the formulas given in Appendix A, for instance),

$$\tilde{a}_2 = -\frac{2}{3}(H/2)^2, \quad \tilde{a}_4 = \frac{2}{15}(H/2)^4, \tag{3.1}$$

$$\tilde{a}_6 = -\frac{4}{189}(H/2)^6, \quad \tilde{a}_8 = \frac{2}{675}(H/2)^8.$$

Note that we get zero for all odd coefficients here. Using these values, we then find that, for relatively large m, the effective action is given by the series

$$\bar{\Gamma}(H;m) = \frac{VH^2}{16\pi^2}\left[-\frac{1}{120}\left(\frac{H}{m^2}\right)^2 + \frac{1}{504}\left(\frac{H}{m^2}\right)^4 - \frac{1}{720}\left(\frac{H}{m^2}\right)^6 + \cdots \right], \tag{3.2}$$

where V denotes the four-dimensional Euclidean volume.

For this case, it is actually not difficult to find the *exact* expression for the one-loop effective action, following closely Schwinger's original analysis in QED [6]. After some algebras, one finds the trace of the proper-time Green function to be given by [22]

$$\mathrm{tr}\langle xs|x\rangle = \frac{2}{(4\pi s)^2}\left[\frac{(Hs/2)^2}{\sinh^2(Hs/2)}\right]. \tag{3.3}$$

One can easily check that the expressions given in Eq. (3.1) are correct ones by considering a small-s series of this exact expression. Inserting Eq. (3.3) into the formula (2.10) then yields the exact expression

$$\bar{\Gamma}(H;m) = -2V\int_0^\infty \frac{ds}{s} e^{-m^2 s}\frac{1}{(4\pi s)^2}\left[\frac{(Hs/2)^2}{\sinh^2(Hs/2)} - 1 + \frac{1}{3}(Hs/2)^2\right]. \tag{3.4}$$

Comparing the result of large mass expansion in Eq. (3.2) against this exact result, we can investigate the validity range of the former. From the plots in Fig. 1, it should be evident that for mass values in the range $m/\sqrt{H} \gtrsim 1$, summing only a few leading terms in the series (3.2) already produces the results which are very close to the exact one. The large mass expansion is useful if $m \gtrsim \sqrt{H}$.

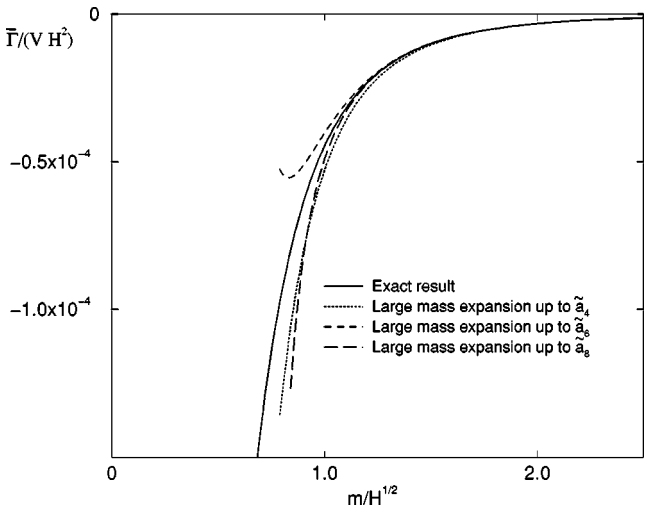

FIG. 1. Plot of the effective action $\bar{\Gamma}(H;m)$.

Now suppose that the exact expression (3.4) were not available to us. For mass value not larger than $\sqrt{H}$, the large mass expansion (3.2) fails to give useful information. Nevertheless, if one happens to know the one-loop effective action for *small* mass, this additional information and the large mass expansion might be used to infer the behavior of the effective action for general, small or large, mass. (Note that, in an instanton background, this becomes a real issue since the full m-dependence of the effective action is not known there.) In exhibiting this, $\bar{\Gamma}(H;m)$ will not be convenient since it becomes ill defined as $m \to 0$. So, based on the relation (2.12), we may consider the renormalized action $\Gamma_{\mathrm{ren}}(H;m,\mu)$ given by

$$\Gamma_{\mathrm{ren}}(H;m,\mu) = -\frac{VH^2}{(4\pi)^2 \times 6}\ln\left(\frac{m^2}{\mu^2}\right) + \bar{\Gamma}(H;m), \tag{3.5}$$

which is well behaved for small m. Large mass expansion for $\Gamma_{\mathrm{ren}}(H;m,\mu)$ results once if the expansion (3.2) is substituted in the right hand side of Eq. (3.5).

To find the small-m expansion, we find it convenient to consider the quantity

$$Q(H;m) \equiv \int_0^{m^2} d\bar{m}^2 \frac{\partial}{\partial \bar{m}^2}\Gamma_{\mathrm{ren}}(H;\bar{m},\mu) = \Gamma_{\mathrm{ren}}(H;m,\mu) - \Gamma_{\mathrm{ren}}(H;m=0,\mu). \tag{3.6}$$

In Eq. (3.6), from Eqs. (3.4) and (3.5),

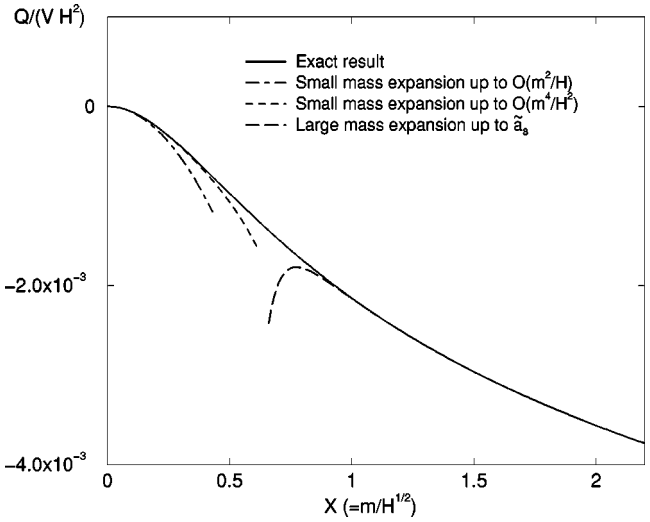

FIG. 2. Plot of $Q(H;m)$.

$$\Gamma_{\rm ren}(H;m=0,\mu)=\frac{VH^2}{16\pi^2}\left[\frac{1}{6}\ln\left(\frac{\mu^2}{H}\right)-2\zeta'(-1)\right], \quad (3.7)$$

where $\zeta'(s)$ is the first derivative of Riemann zeta function and $\zeta'(-1)\approx -0.165421$. Notice that $Q(H;m)$ is independent of the normalization mass μ and is well behaved in the small mass limit. Explicitly, it is given by the expression

$$Q(H;m)=2V\int_0^{m^2} d\bar{m}^2\int_0^\infty ds\,\frac{e^{-\bar{m}^2 s}}{(4\pi s)^2}\left[\frac{(Hs/2)^2}{\sinh^2(Hs/2)}-1\right], \quad (3.8)$$

and in the small mass limit, this leads to

$$\begin{aligned} Q(H;m)&=\frac{VH^2}{8\pi^2}\int_0^{m^2/H} d\bar{m}^2[-1/2-\bar{m}^2(\log\bar{m}^2+\gamma)+\cdots]\\ &=\frac{VH^2}{16\pi^2}[-m^2/H-(m^2/H)^2(\log(m^2/H)-1/2+\gamma)\\ &\quad+\cdots]. \end{aligned} \quad (3.9)$$

In Fig. 2, graphs for $Q(H;m)$, the exact one and those based on approximation schemes, are given as functions of $X\equiv m/\sqrt{H}$. The exact result, i.e., that based on the expression (3.8) is represented by a solid line, which exhibits a monotonically decreasing behavior starting from the maximum at $X=0$. Clearly the small mass expansion up to $\mathcal{O}(m^4/H^2)$ provides a reliable approximation for $X\lesssim 0.4$, while the large mass expansion for $Q(H;m)$,

$$\begin{aligned} Q(H;m)=\frac{VH^2}{16\pi^2}\Bigg[&-\frac{1}{6}\ln\left(\frac{m^2}{H}\right)+2\zeta'(-1)-\frac{1}{120}\left(\frac{H}{m^2}\right)^2\\ &+\frac{1}{504}\left(\frac{H}{m^2}\right)^4-\frac{1}{720}\left(\frac{H}{m^2}\right)^6+\cdots\Bigg] \end{aligned} \quad (3.10)$$

[this formula is obtained from Eqs. (3.2), (3.5), and (3.6)] can be trusted in the range $X\gtrsim 1$. In the intermediate region $0.4\lesssim X\lesssim 1$ the large-mass expansion curve (a long dashed line in Fig. 2) may then be smoothly connected to that given from the small-m expansion (3.9), assuming a monotonic behavior (as should be reasonable for a simple background field). Evidently, with this interpolation, one could have acquired a nice overall fit over the entire mass range even if the exact curve were not known. We also see from Fig. 2 the typical behaviors which are shown by the small-mass or large-mass expansion curves.

IV. LARGE MASS EXPANSION FOR THE SPIN-0 INSTANTON EFFECTIVE ACTION

We now turn to the case of a BPST instanton background [1], i.e., a self-dual solution of Yang-Mills field equations given by

$$A_\mu(x)\equiv A_\mu^a(x)\frac{\tau^a}{2}=\frac{\eta_{\mu\nu a}\tau^a x_\nu}{x^2+\rho^2}, \quad (4.1)$$

where $\eta_{\mu\nu a}$ $(a=1,2,3)$ are the so-called 't Hooft symbols [5] and ρ denotes the size of the instanton. The associated field strength $F_{\mu\nu}$ is

$$F_{\mu\nu}=-2\frac{\rho^2\eta_{\mu\nu a}\tau^a}{(x^2+\rho^2)^2}. \quad (4.2)$$

In this instanton background, the exact expression for the one-loop effective action due to a spin-0 or spin-1/2 matter field of nonzero mass is not known; only the result in the massless limit is known [5]. This quantity will be studied with the help of approximation schemes in this paper. Specifically, taking the matter field to be that of an isospin-1/2, spin-0 particle, the corresponding effective action is studied using the large mass expansion in this section and by the small mass expansion in the next section. In Sec. VI, we then use these results for a spin-0 matter field to obtain the corresponding results appropriate to a spin-1/2 matter field (i.e., quark). Note that, in the case of a spin-1/2 matter field, a *direct* application of the small mass expansion can be very subtle due to the presence of normalizable zero modes for the massless Dirac equation [23].

The large mass expansion for the spin-0 effective action is described by our formula (2.17). To use this formula, one needs to know some higher-order coefficients in the series (2.5), with $G^{-1}=-D^2$ and the instanton background given above. Calculations of these higher-order Seeley-DeWitt coefficients are straightforward in principle, but get very involved as the order increases. Fortunately, thanks to the rapidly growing computer capacity to handle a large number of terms in the symbolic calculations, the explicit expressions for the Seeley-DeWitt coefficients in general background fields have been found recently up to the sixth order [14–16]. We will utilize these results for our calculations below.

In the instanton background (4.1) the renormalized one-loop effective action $\bar{\Gamma}(A)$, defined by Eq. (2.10), will be a function of $m\rho$ only. Hence our large mass expansion is really an expansion in $1/m^2\rho^2$. Also the expressions for the Seeley-DeWitt coefficients are simplified considerably if we take into account the fact that our background field satisfies the classical Yang-Mills equations of motion. For such on-shell background fields, the space-time integral of the Seeley-DeWitt coefficients $\tilde{a}_n(x)$, $n=3,4,5$ (for a spin-0 matter field) are given as [16]

$$\int d^4x\,\tilde{a}_3(x)=\frac{i}{90}\int d^4x\,\mathrm{tr}[F_{\mu\nu}F_{\nu\rho}F_{\rho\mu}], \tag{4.3}$$

$$\int d^4x\,\tilde{a}_4(x)=\frac{1}{24}\int d^4x\,\mathrm{tr}\Big[\frac{17}{210}F_{\mu\nu}F_{\mu\nu}F_{\lambda\kappa}F_{\lambda\kappa}+\frac{2}{35}F_{\mu\nu}F_{\nu\rho}F_{\mu\lambda}F_{\lambda\rho}+\frac{1}{105}F_{\mu\nu}F_{\nu\rho}F_{\rho\sigma}F_{\sigma\mu}+\frac{1}{420}F_{\mu\nu}F_{\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\Big], \tag{4.4}$$

$$\begin{aligned}\int d^4x\,\tilde{a}_5(x)=\frac{1}{120}\int d^4x\,\mathrm{tr}\Big[&i\frac{1}{945}F_{\mu\nu}F_{\rho\sigma}F_{\tau\mu}F_{\nu\rho}F_{\sigma\tau}-i\frac{47}{126}F_{\mu\nu}F_{\mu\nu}F_{\rho\sigma}F_{\sigma\tau}F_{\tau\rho}+i\frac{1}{126}F_{\mu\nu}F_{\rho\sigma}F_{\mu\nu}F_{\sigma\tau}F_{\tau\rho}\\&+i\frac{1}{63}F_{\mu\nu}F_{\nu\rho}F_{\mu\sigma}F_{\sigma\tau}F_{\tau\rho}-i\frac{11}{189}F_{\mu\nu}F_{\rho\sigma}F_{\sigma\nu}F_{\mu\tau}F_{\tau\rho}+i\frac{37}{945}F_{\mu\nu}F_{\nu\rho}F_{\rho\sigma}F_{\sigma\tau}F_{\tau\mu}+\frac{4}{189}F_{\nu\tau}F_{\tau\sigma}(D_\mu F_{\nu\rho})\\&\times(D_\mu F_{\rho\sigma})-\frac{2}{63}F_{\lambda\kappa}(D_\mu F_{\nu\rho})F_{\nu\rho}(D_\mu F_{\lambda\kappa})-\frac{2}{189}F_{\lambda\sigma}(D_\mu F_{\nu\rho})F_{\rho\sigma}(D_\mu F_{\nu\lambda})+\frac{4}{63}F_{\sigma\tau}F_{\sigma\tau}(D_\mu F_{\nu\rho})(D_\mu F_{\nu\rho})\\&+\frac{2}{63}F_{\mu\tau}F_{\tau\sigma}(D_\mu F_{\nu\rho})(D_\sigma F_{\nu\rho})+\frac{4}{189}F_{\sigma\tau}F_{\tau\nu}(D_\mu F_{\nu\rho})(D_\mu F_{\rho\sigma})\Big].\end{aligned} \tag{4.5}$$

Note that the on-shell expressions for the space-time integral of $\tilde{a}_3(x)$ and $\tilde{a}_4(x)$ involve only the field strength, while that for $\tilde{a}_5(x)$ involves the derivatives of the field strength also. For the expression of $\tilde{a}_6(x)$, which occupies more than a page, see Ref. [15]. In Appendix A, the expressions valid without using the classical equations of motion (and before the space-time integration) can also be found.

Inserting the expression (4.2) for the field strength into the formulas (4.3) and (4.4) and carrying out tensor algebra and trace calculations, we find

$$\int d^4x\,\tilde{a}_3(x)=\int d^4x\,\frac{64\rho^6}{15(x^2+\rho^2)^6}=\frac{16}{75}\frac{\pi^2}{\rho^2}, \tag{4.6}$$

$$\int d^4x\,\tilde{a}_4(x)=\int d^4x\,\frac{544\rho^8}{35(x^2+\rho^2)^8}=\frac{272}{735}\frac{\pi^2}{\rho^4}. \tag{4.7}$$

The next coefficient $\tilde{a}_5$ involves the covariant derivative of field strength

$$D_\lambda F_{\mu\nu}=\frac{4\rho^2\tau^a}{(x^2+\rho^2)^3}[2\eta_{\mu\nu a}x_\lambda-\eta_{\lambda\mu a}x_\nu+\eta_{\lambda\nu a}x_\mu+\delta_{\lambda\mu}\eta_{\nu\sigma a}x_\sigma-\delta_{\lambda\nu}\eta_{\mu\sigma a}x_\sigma]. \tag{4.8}$$

Calculations of higher-order Seeley-DeWitt coefficients with the instanton background can be very laborious. Together with the formulas given above and that in Ref. [15] for $\tilde{a}_6(x)$, we have thus used the ''MATHEMATICA'' program to do the necessary trace calculations as well as tensor algebra. From the expression for the $\tilde{a}_5$ coefficient, we obtain the result

$$\int d^4x\,\tilde{a}_5(x)=\int d^4x\,\frac{512(35x^2\rho^8-39\rho^{10})}{315(x^2+\rho^2)^{10}}=-\frac{1856}{2835}\frac{\pi^2}{\rho^6}, \tag{4.9}$$

while, for the $\tilde{a}_6(x)$ term,

$$\begin{aligned}\int d^4x\,\tilde{a}_6(x)&=-\int d^4x\,\frac{256}{51975(\rho^2+x^2)^{12}}\\&\quad\times[397710x^8\rho^4-765270x^6\rho^6+404961x^4\rho^8-86418x^2\rho^{10}+2876\rho^{12}]\\&=\frac{63328}{444675}\frac{\pi^2}{\rho^8}.\end{aligned} \tag{4.10}$$

Based on the explicit calculations given above, we obtain the following large-mass expansion for $\bar{\Gamma}(m\rho)$:

$$\bar{\Gamma}(m\rho)=-\frac{1}{75}\frac{1}{m^2\rho^2}-\frac{17}{735}\frac{1}{m^4\rho^4}+\frac{232}{2835}\frac{1}{m^6\rho^6}-\frac{7916}{148225}\frac{1}{m^8\rho^8}+\cdots. \tag{4.11}$$

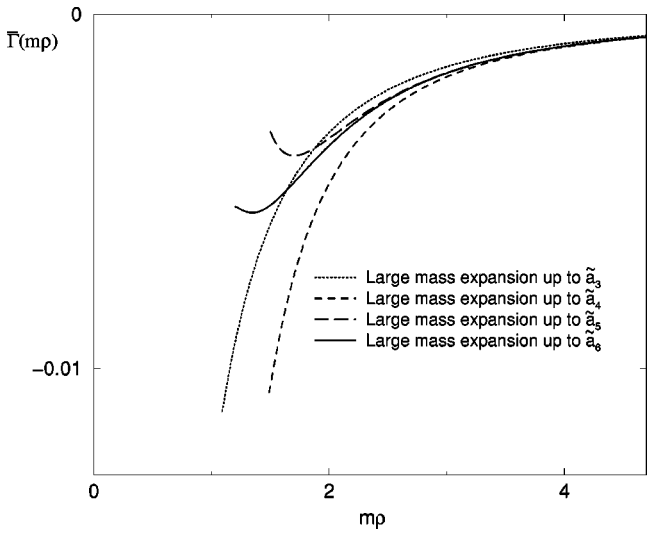

FIG. 3. Plot of $\bar{\Gamma}(m\rho)$ for the instanton background.

In Fig. 3 we have given the plots based on this expression (first keeping only the $\tilde{a}_3$ term, then including the $\tilde{a}_4$ term also, etc). This is a useful approximation when $m\rho$ is large, say, $m\rho \gtrsim K$. What would be the lower-end value K here ? In the absence of the exact expression for $\bar{\Gamma}(m\rho)$, a possible criterion for telling the validity range of the series (4.11) will be as follows. If A_l denotes the $\mathcal{O}[1/(m\rho)^{2l}]$ term in the series and $\Gamma_l \equiv \Sigma_{n=1}^{l} A_n$, we may demand that the series (4.11) remain *stable* in the sense that the relative importance of each newly added term decreases, i.e., $1=|A_1/\Gamma_1| > |A_2/\Gamma_2| > |A_3/\Gamma_3| > |A_4/\Gamma_4| > \cdots$. As this criterion is used, we obtain the (conservative) value $K \simeq 1.8$. The result of large mass expansion may thus be trusted in the range given by $m\rho \gtrsim 1.8$.

V. SMALL MASS EXPANSION FOR THE SPIN-0 INSTANTON EFFECTIVE ACTION

For small $m\rho$, that is, $m\rho$ significantly below 1, the one-loop effective action in the instanton background (4.1) can be studied with the help of the small mass expansion or mass perturbation, since its exact expression in the massless limit is known. Here we shall denote the corresponding spin-0 effective action $\Gamma_{\rm ren}$, which is defined by Eq. (2.12), as $\Gamma_{\rm ren}(m,\rho,\mu)$. For $m=0$ we have, from the computations of 't Hooft [5],

$$\Gamma_{\rm ren}(m=0,\rho,\mu) = \frac{1}{6}\ln \mu\rho + \alpha(1/2) \tag{5.1}$$

with $\alpha(1/2) = \frac{1}{6}\gamma + \frac{1}{6}\ln\pi - (1/\pi^2)\zeta'(2) - \frac{17}{72} \simeq 0.145873$. Our goal in this section is to compute explicitly the $\mathcal{O}(m^2)$ term of $\Gamma_{\rm ren}(m,\rho,\mu)$. Note that this small-$m$ approximation for $\Gamma_{\rm ren}(m,\rho,\mu)$ contains a nonanalytic piece in m and so it is not a trivial task to extract the desired term.

Our first task is to develop a small mass expansion for $\Gamma_{\rm ren}(m,\rho,\mu)$, which is finite at every order. For the purpose it is convenient to consider its derivative with respective to m^2, i.e., $\partial\Gamma_{\rm ren}/\partial m^2$, which is independent of the normalization mass μ. The latter, being equal to the m^2 derivative of our regularized effective action $\Gamma(A)$, will have the proper-time representation

$$\frac{\partial\Gamma_{\rm ren}(m,\rho,\mu)}{\partial m^2} = \lim_{\eta\to 0^+} \int_\eta^\infty ds\, e^{-m^2 s} \times \int d^4x\, {\rm tr}[\langle xs|x\rangle - \langle xs|x\rangle_{A_\mu=0}], \tag{5.2}$$

where we have used Eq. (2.4). Here note that $\langle xs|x\rangle \equiv \lim_{y\to x}\langle xs|y\rangle$ is nonsingular as long as $s>0$. Paying due attention to various (singular) limits involved, it is possible to recast the formula (5.2) into that involving ordinary spin-0 propagators

$$G(x,y;m^2) \equiv \langle x|\frac{1}{-D^2+m^2}|y\rangle,$$

$$G_0(x,y;m^2) \equiv \langle x|\frac{1}{-\partial^2+m^2}|y\rangle. \tag{5.3}$$

The explicit formula, which can be derived from Eq. (5.2), reads [24]

$$\frac{\partial\Gamma_{\rm ren}(m,\rho,\mu)}{\partial m^2} = \int d^4x \lim_{y\to x} {\rm tr}[G(x,y;m^2) - a_0(x,y)G_0(x,y;m^2)], \tag{5.4}$$

where $a_0(x,y)$ is the zeroth order coefficient in the asymptotic series (2.5). For small $(x-y)_\mu$, $a_0(x,y)$ has the following expression:

$$a_0(x,y) = I + i(x-y)_\mu A_\mu(y) + \frac{i}{4}(x-y)_\mu(x-y)_\nu \times [\partial_\mu A_\nu(y) + \partial_\nu A_\mu(y) + i\{A_\mu(y), A_\nu(y)\}] + \mathcal{O}[(x-y)^3]. \tag{5.5}$$

The presence of the $a_0 G_0$ term in Eq. (5.4) guarantees a finite result for $\partial\Gamma_{\rm ren}(m,\rho,\mu)/\partial m^2$.

For small m, one may then try to evaluate the right-hand side of Eq. (5.4) by exploiting the appropriate expansion of the propagators in m^2 and the known exact massless propagation function in the instanton background. We shall denote the latter by $\bar{G}(x,y) \equiv \langle x|1/-(D^2)|y\rangle$. But a naive expansion of the form

$$G(x,y;m^2) = \bar{G}(x,y) - m^2\int d^4z\, \bar{G}(x,z)\bar{G}(z,y) + m^4\int d^4z\, d^4w\, \bar{G}(x,z)\bar{G}(z,w)\bar{G}(w,y) + \cdots \tag{5.6}$$

is not valid since, aside from the leading term, all other terms of this series involve logarithmically divergent integrals. [Note that $\bar{G}(x,z)=\mathcal{O}(1/|z|^2)$ as $|z|\to\infty$.] Moreover, the $m^2=0$ limit of $\partial\Gamma_{\rm ren}/\partial m^2$ does not exist in the instanton background since, according to explicit calculations [see the comment after Eq. (5.22) below], the integral in the right-hand side of Eq. (5.4) for $m^2=0$ diverges logarithmically. This indicates that, as $m^2\to 0$, $\Gamma_{\rm ren}(m,\rho,\mu)$ approaches the 't Hooft result (5.1) in a nonanalytic manner. To resolve this problem, we shall below describe an alternative expansion scheme (which utilizes the idea of Carlitz and Creamer [18] in a suitable form).

The expansion we shall use has the form

$$\frac{1}{-D^2+m^2}=\frac{1}{(-D^2)}(-\partial^2)\frac{1}{-\partial^2+m^2}\left[\sum_{r=0}^{\infty}\left(-m^2\left\{\frac{1}{(-D^2)}-\frac{1}{(-\partial^2)}\right\}(-\partial^2)\frac{1}{-\partial^2+m^2}\right)^r\right]. \tag{5.7}$$

This can be derived in the following way. First observe that

$$\frac{1}{-D^2+m^2}=\frac{1}{-\partial^2+m^2}+(-D^2)\frac{1}{-D^2+m^2}\times\left\{\frac{1}{(-D^2)}-\frac{1}{(-\partial^2)}\right\}(-\partial^2)\frac{1}{-\partial^2+m^2}. \tag{5.8}$$

Then, using the identity

$$(-D^2)\frac{1}{-D^2+m^2}=1-m^2\frac{1}{-D^2+m^2}, \tag{5.9}$$

it is not difficult to see that Eq. (5.8) can be rewritten as

$$\frac{1}{-D^2+m^2}=\frac{1}{(-D^2)}(-\partial^2)\frac{1}{-\partial^2+m^2}-m^2\frac{1}{-D^2+m^2}\left\{\frac{1}{(-D^2)}-\frac{1}{(-\partial^2)}\right\}\times(-\partial^2)\frac{1}{-\partial^2+m^2}. \tag{5.10}$$

This last equation may be solved for $1/(-D^2+m^2)$ in an iterative manner, and the result is the expansion (5.7). Evidently, Eq. (5.7) is an expansion in powers of $m^2\{1/(-D^2)-1/(-\partial^2)\}(-\partial^2)1/(-\partial^2+m^2)$, and we expect that this will yield a convergent series for small m if the background field is such that

$$\langle x|\left\{\frac{1}{(-D^2)}-\frac{1}{(-\partial^2)}\right\}(-\partial^2)|y\rangle \to 0 \ \text{(sufficiently fast), as } |x-y|\to\infty. \tag{5.11}$$

In the case of the instanton background, this means that we have to work with the expression given in the singular gauge, i.e.,

$$A_\mu(x)=\frac{\rho^2\bar{\eta}_{\mu\nu a}\tau_a x_\nu}{x^2(x^2+\rho^2)}. \tag{5.12}$$

(Here, $\bar{\eta}_{\mu\nu a}$ differs from $\eta_{\mu\nu a}$ only by the sign in the components with μ or ν equal to 4.) This is allowed by the gauge invariance of the effective action.

If only the leading term of Eq. (5.7) is used in Eq. (5.4), we now find that

$$\frac{\partial\Gamma_{\rm ren}(m,\rho,\mu)}{\partial m^2}=\int d^4x\lim_{y\to x}\mathrm{tr}\left[\langle x|\frac{1}{(-D^2)}\times(-\partial^2)\frac{1}{-\partial^2+m^2}|y\rangle-a_0(x,y)\langle x|\frac{1}{-\partial^2+m^2}|y\rangle\right]+\mathcal{O}(m^2). \tag{5.13}$$

This is not yet in the convenient form for actual computations. So, based on the following observations:

$$\langle x|\frac{1}{(-D^2)}(-\partial^2)\frac{1}{-\partial^2+m^2}|y\rangle=\langle x|\frac{1}{(-D^2)}|y\rangle+\langle x|\frac{1}{(-D^2)}(-\partial^2)\times\left\{\frac{1}{-\partial^2+m^2}-\frac{1}{(-\partial^2)}\right\}|y\rangle, \tag{5.14}$$

$$a_0(x,y)\langle x|\frac{1}{-\partial^2+m^2}|y\rangle=a_0(x,y)\langle x|\frac{1}{(-\partial^2)}|y\rangle+\langle x|\left\{\frac{1}{-\partial^2+m^2}-\frac{1}{(-\partial^2)}\right\}|y\rangle+(\text{terms vanishing as } y\to x), \tag{5.15}$$

we make suitable rearrangements in the right-hand side of Eq. (5.13) to obtain the following formula (to be used for computations):

$$\frac{\partial\Gamma_{\rm ren}(m,\rho,\mu)}{\partial m^2}=\int d^4x\left[\lim_{y\to x}\mathrm{tr}\left\{\langle x|\frac{1}{(-D^2)}|y\rangle - a_0(x,y)\langle x|\frac{1}{(-\partial^2)}|y\rangle\right\}-J(x)\right]+\mathcal{O}(m^2), \tag{5.16}$$

where the function $J(x)$ is given by

$$J(x)=-\int d^4z\,\mathrm{tr}\left\{\langle x|\left(\frac{1}{(-D^2)}-\frac{1}{(-\partial^2)}\right)|z\rangle \times(-\overleftarrow{\partial}_z^{\,2})\langle z|\left(\frac{1}{-\partial^2+m^2}-\frac{1}{(-\partial^2)}\right)|x\rangle\right\}. \tag{5.17}$$

But for the $J(x)$ term, what we have in the right-hand side of Eq. (5.16) is just the (logarithmically divergent) expression representing $\partial\Gamma_{\rm ren}(m,\rho,\mu)/\partial m^2|_{m^2=0}$ [see Eq. (5.4)]. As we shall see below, this divergence is tamed by the additional term $J(x)$. The very structure of $J(x)$ given in Eq. (5.17) also ensures that it is free of any short-distance divergence.

The first term inside the integrand of Eq. (5.16) is evaluated as follows. The spin-0 (and isospin-1/2) massless propagator in the instanton background (5.12) is given by [17]

$$\overline{G}(x,y)\equiv\langle x|\frac{1}{-D^2}|y\rangle = \frac{1}{4\pi^2(x-y)^2}\frac{1+\dfrac{\rho^2(x\cdot y+i\overline{\eta}_{\mu\nu a}x_\mu y_\nu\tau_a)}{x^2y^2}}{\sqrt{1+\dfrac{\rho^2}{x^2}}\sqrt{1+\dfrac{\rho^2}{y^2}}}. \tag{5.18}$$

Then, writing $x=y+\epsilon$, we find after some straightforward calculations

$$\langle x|\frac{1}{-D^2}|y\rangle=\frac{1}{4\pi^2\epsilon^2}\left[\left(1+\frac{\rho^4(y\cdot\epsilon)^2-\rho^2y^2(y^2+\rho^2)\epsilon^2}{2(y^2)^2(y^2+\rho^2)^2}\right)+\left(1-\frac{(2y^2+\rho^2)(y\cdot\epsilon)}{y^2(y^2+\rho^2)}\right)\frac{i\rho^2\overline{\eta}_{\mu\nu a}\tau_a\epsilon_\mu y_\nu}{y^2(y^2+\rho^2)}\right]+\mathcal{O}(\epsilon). \tag{5.19}$$

On the other hand, if Eq. (5.12) is inserted into the expression (5.5), we have

$$a_0(x,y)=I+i(x-y)_\mu\frac{\rho^2\overline{\eta}_{\mu\nu a}\tau_a y_\nu}{y^2(y^2+\rho^2)}-\frac{1}{2}(x-y)_\mu(x-y)_\nu\frac{\rho^4(y^2\delta_{\mu\nu}-y_\mu y_\nu)+2i\rho^2(2y^2+\rho^2)\overline{\eta}_{\mu\lambda a}\tau_a y_\nu y_\lambda}{(y^2)^2(y^2+\rho^2)^2}+\mathcal{O}[(x-y)^3] \tag{5.20}$$

and therefore

$$\mathrm{tr}\left\{a_0(x,y)\langle x|\frac{1}{(-\partial^2)}|y\rangle\right\}=\frac{1}{2\pi^2\epsilon^2}\left(1-\frac{\rho^4y^2\epsilon^2-\rho^4(y\cdot\epsilon)^2}{2(y^2)^2(y^2+\rho^2)}\right)+\mathcal{O}(\epsilon). \tag{5.21}$$

From Eqs. (5.19) and (5.21), we thus obtain the following expression:

$$\lim_{y\to x}\mathrm{tr}\left\{\langle x|\frac{1}{(-D^2)}|y\rangle-a_0(x,y)\langle x|\frac{1}{(-\partial^2)}|y\rangle\right\}=-\frac{\rho^2}{4\pi^2(x^2+\rho^2)^2}. \tag{5.22}$$

[We here remark that the result (5.22) is unchanged even if one takes the regular-gauge instanton solution (4.1) as the background field.] Clearly, with this term alone, the remaining x integration would yield a logarithmically divergent result.

We now turn to the evaluation of $J(x)$. Noting that

$$\langle z|\left(\frac{1}{-\partial^2+m^2}-\frac{1}{(-\partial^2)}\right)|x\rangle=-m^2\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(z-x)}}{p^2(p^2+m^2)}, \tag{5.23}$$

Eq. (5.17) may be rewritten as

$$J(x)=m^2\int\frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot x}}{p^2(p^2+m^2)}F(x,p) \tag{5.24}$$

with $F(x,p)$ given by

$$F(x,p)=\int d^4z\,e^{-ip\cdot z}\,\mathrm{tr}\left\{\langle x|\left(\frac{1}{(-D^2)}-\frac{1}{(-\partial^2)}\right)|z\rangle(-\overleftarrow{\partial}_z^{\,2})\right\} =2\int d^4z\,e^{-ip\cdot z}(-\vec{\partial}_z^{\,2}) \times\left[\frac{1}{4\pi^2(x-z)^2}\left(\frac{1+\dfrac{\rho^2x\cdot z}{x^2z^2}}{\sqrt{1+\rho^2/x^2}\sqrt{1+\rho^2/z^2}}-1\right)\right]. \tag{5.25}$$

In Eq. (5.25) we have used the expression (5.18) and the factor 2 at front arose from the isospin trace. We are here interested in $\mathcal{O}(1)$ or $\mathcal{O}(\log m^2)$ contribution to the right hand side of Eq. (5.16). Let us see when and where such contribution can arise, based on our formulas (5.24) and (5.25). For any finite x value, the function $F(x,p)$ is well behaved for all p. Due to the overall multiplicative factor m^2 in Eq. (5.24), $J(x)$ for finite x (or, more precisely, for x^μ satisfying the condition $|x| \ll 1/m$) would then be $\mathcal{O}(m^2)$ and hence no desired contribution. It is thus sufficient to study $J(x)$ for large x, i.e., x^μ in the region $|x|>L$ with $\rho \ll L \ll m^{-1}$. Now, due to the factor $e^{ip\cdot x}/p^2(p^2+m^2)$ within the integrand of Eq. (5.24), we further conclude that the small-p region of $F(x,p)$, with $|x|>L$, can be the source for the desired contribution; if the contribution from the region $|p| \lesssim m$ is excluded from the right hand side of Eq. (5.24), $J(x)$ becomes $\mathcal{O}(m^2)$.

To study the function $F(x,p)$ for $|x|>L$ (with $\rho \ll L \ll m^{-1}$) and $|p| \lesssim m$, we write $F(x,p)$ as the sum of its value at $p=0$ plus the correction term, viz.,

$$F(x,p)=F(x,p=0)+C(x,p). \tag{5.26}$$

Then, from Eq. (5.25),

$$\begin{aligned} F(x,p=0) &= 2\int d^4z(-\vec{\partial}_z^2)\left[\frac{1}{4\pi^2(x-z)^2}\right. \\ &\quad \left.\times\left(\frac{1+\dfrac{\rho^2 x\cdot z}{x^2z^2}}{\sqrt{1+\rho^2/x^2}\sqrt{1+\rho^2/z^2}}-1\right)\right] \\ &= -\lim_{R\to\infty}\frac{1}{2\pi^2}\oint_{|z|=R} d^3\Omega\, R^3\frac{z_\mu}{R} \\ &\quad \times\left[\frac{2(x-z)_\mu}{[(x-z)^2]^2}\left(\frac{1+\dfrac{\rho^2 x\cdot z}{x^2z^2}}{\sqrt{1+\rho^2/x^2}\sqrt{1+\rho^2/z^2}}-1\right)\right. \end{aligned} \tag{5.27a}$$

$$\left. +\frac{1}{(x-z)^2}\frac{\partial}{\partial z_\mu}\left(\frac{1+\dfrac{\rho^2 x\cdot z}{x^2z^2}}{\sqrt{1+\rho^2/x^2}\sqrt{1+\rho^2/z^2}}\right)\right], \tag{5.27b}$$

where we used Gauss's law. [Note that, for very large $|z|$, the integrand in Eq. (5.27a) behaves as $\mathcal{O}(1/|z|^5)$]. Evaluating the surface integral in Eq. (5.27b) immediately gives

$$F(x,p=0)=2\left(\frac{1}{\sqrt{1+\rho^2/x^2}}-1\right), \tag{5.28}$$

and hence

$$F(x,p=0)=-\frac{\rho^2}{x^2}+\mathcal{O}\left(\frac{\rho^4}{x^4}\right), \quad \text{for } |x|>L. \tag{5.29}$$

Now, inserting the thus evaluated $F(x,p=0)$ for $F(x,p)$ in Eq. (5.24), we obtain the following contribution to $J(x)$:

$$\begin{aligned} J(x) &= \theta(|x|-L)\int d^4p\left(\frac{1}{p^2}-\frac{1}{p^2+m^2}\right)e^{ip\cdot x}\left(-\frac{\rho^2}{x^2}\right) \\ &\quad +\mathcal{O}(m^2) \\ &= -\frac{\rho^2}{4\pi^2(x^2)^2}[1-m|x|K_1(m|x|)] \\ &\quad \times\theta(|x|-L)+\mathcal{O}(m^2). \end{aligned} \tag{5.30}$$

(Note that we have assumed $mL \ll 1$.) On the other hand, it is possible to show (see Appendix B) that $C(x,p)$ in Eq. (5.26) is at most $\mathcal{O}[|p|L(\rho^2/x^2)]$ or $\mathcal{O}(\rho^4/x^3L)$ or $\mathcal{O}(\rho^4/x^4)$, when $|x|>L$ and $|p| \lesssim m$. With this finding used in Eq. (5.24), it is easy to see that no $\mathcal{O}(1)$ or $\mathcal{O}(\log m)$ contribution results from the $C(x,p)$ part of $F(x,p)$. Thus, to the order we want, our formula (5.30) has no further correction.

Evidently, if the contribution in Eq. (5.30) is considered together with that in Eq. (5.22), the x integration in Eq. (5.16) will give a finite result. Furthermore, since the function $F(x,p)$ does not involve mass m at all, the scale L we introduced can be chosen, for $m\rho\to 0$, such that $\rho \ll L \ll 1/m$. With this understanding, we may now perform the integral in the right-hand side of Eq. (5.16) to secure the unambiguous result

$$\frac{\partial\Gamma_{\rm ren}(m,\rho,\mu)}{\partial m^2}=\frac{\rho^2}{2}\ln(m\rho)+\frac{\rho^2}{2}\left(\gamma+\frac{1}{2}-\ln 2\right)+\mathcal{O}(m^2\rho^4). \tag{5.31}$$

Then, based on this formula and the 't Hooft result (5.1), we immediately obtain the desired small-mass expansion for $\Gamma_{\rm ren}(m,\rho,\mu)$:

$$\begin{aligned} \Gamma_{\rm ren}(m,\rho,\mu) &= \Gamma_{\rm ren}(m=0,\rho,\mu)+\int_0^{m^2} d\bar{m}^2\frac{\partial\Gamma_{\rm ren}(\bar{m},\rho,\mu)}{\partial\bar{m}^2} \\ &= \frac{1}{6}\ln(\mu\rho)+\alpha(1/2)+\frac{(m\rho)^2}{2}[\ln(m\rho)+\gamma \\ &\quad -\ln 2]+\mathcal{O}[(m\rho)^4]. \end{aligned} \tag{5.32}$$

The $\mathcal{O}[(m\rho)^2\ln(m\rho)]$ term in this formula was first found in Ref. [18], while the $\mathcal{O}[(m\rho)^2]$ term without the $\ln(m\rho)$ factor is new.

VI. MASS INTERPOLATION AND THE SPIN-1/2 INSTANTON EFFECTIVE ACTION

In the previous two sections the spin-0 instanton effective action were computed for relatively large $m\rho$ and for small $m\rho$. The result can be summarized by

$$\Gamma_{\rm ren}(m,\rho,\mu)=\frac{1}{6}\ln(\mu\rho)+\alpha(1/2)+Q(m\rho) \tag{6.1}$$

with the quantity Q, a function of $m\rho(\equiv X)$ only, behaving as

$$Q(X)=\begin{cases}\dfrac{1}{2}X^2\ln X+\dfrac{1}{2}(\gamma-\ln 2)X^2+\cdots & (X\lesssim 0.5),\\ -\dfrac{1}{6}\ln X-\alpha\left(\dfrac{1}{2}\right)-\dfrac{1}{75}\dfrac{1}{X^2}-\dfrac{17}{735}\dfrac{1}{X^4}+\dfrac{232}{2835}\dfrac{1}{X^6}-\dfrac{7916}{148225}\dfrac{1}{X^8}+\cdots & (X\gtrsim 1.8).\end{cases} \tag{6.2}$$

[Note that, in the instanton background, Eq. (2.12) implies that $\Gamma_{\rm ren}(m,\rho,\mu)=\frac{1}{6}\ln(\mu/m)+\bar{\Gamma}(m\rho)$.] In the indicated validity ranges of X, the function $Q(X)$ is plotted in Fig. 4. We have here assumed that our small mass expansion in Eq. (5.32) can be used reliably for $X\lesssim 0.5$; this estimate is based on measuring the effect of the $(m\rho)^4$ term [with the numerical coefficient taken to be $\mathcal{O}(1)$] against the terms which appear explicitly in Eq. (5.32).

Now, what could be said on the behavior of the function $Q(X)$ in the intermediate region $0.5\lesssim X\lesssim 1.8$? Since the background field under consideration has a smooth profile, one naturally expects that $Q(X)$ also be a smooth function of X; that is, $Q(X)$ would be represented by a smooth interpolating curve connecting the known forms of the curve in the regions $X\gtrsim 1.8$ and $X\lesssim 0.5$. Let us further assume that the region for interpolation, $0.5\lesssim X\lesssim 1.8$, can be viewed as being reasonably small. Then, looking at how $Q(X)$ actually behaves for $X\gtrsim 1.8$ and $X\lesssim 0.5$ (see Fig. 4), it appears to be quite plausible to suppose that $Q(X)$ is a *monotonically decreasing* function of X for all $X>0$. But, since we have in no way proved this monotonic behavior in the presence of the instanton background, one may regard this as a conjecture. [For instance, the possibility that $Q(X)$ may develop a local maximum or minimum within the range $0.5\lesssim X\lesssim 1.8$ is not excluded. Incidentally, such monotonic behavior was also observed in the case of a self-dual constant field strength (see Fig. 2).] Accepting the conjecture, it might be useful (especially for phenomenological analysis of instanton effects) to have a certain smooth function $Q(X)$ in the entire range $X>0$ which meets this requirement. With the plausible curve for $Q(X)$ taken by that given in Fig. 4, we have found (after some trial and errors) that it may be described by the function of the form

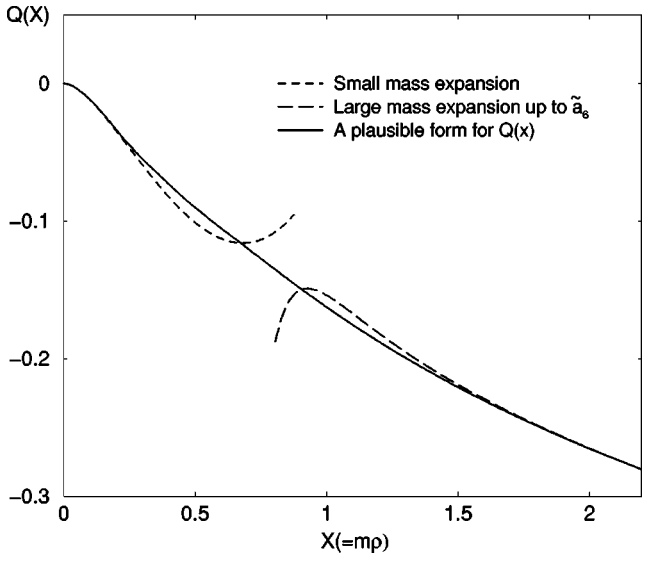

FIG. 4. Plot of $Q(m\rho)$.

$$Q(X)\sim -\frac{1}{6}\ln X-\alpha+\frac{\frac{1}{6}\ln X+\alpha-(3\alpha+\beta)X^2-\frac{1}{5}X^4}{1-3X^2+20X^4+15X^6}\quad(\text{for all } X>0), \tag{6.3}$$

with $\alpha\equiv\alpha(1/2)\simeq 0.145873$ and $\beta=\frac{1}{2}(\ln 2-\gamma)\simeq 0.05797$. This form incorporates correctly the small-X and leading large-X behaviors shown in Eq. (6.2). For the tunneling amplitude which is more directly related to $e^{-\Gamma_{\rm ren}}$, this amounts to using the expression (6.3) with

$$e^{-\Gamma_{\rm ren}(m,\rho,\mu)}=(\mu\rho)^{-1/6}e^{-\alpha(1/2)-Q(m\rho)}\quad(\text{for arbitrary mass } m). \tag{6.4}$$

Various results obtained for the spin-0 field case can be used to derive the corresponding results appropriate to the spin-1/2 one-loop instanton effective action. The latter will be needed if one wishes to consider the loop correction to the vacuum tunneling amplitude in QCD due to quark fields. In a self-dual Yang-Mills background, the hidden supersymmetry of the system allows one to express the spin-1/2 proper-time Green function $\langle xs|y\rangle^{(1/2)}\equiv\langle x|e^{-s(\gamma D)^2}|y\rangle$ in terms of the corresponding spin-0 function $\langle xs|y\rangle^{(0)}\equiv\langle x|e^{-s(-D^2)}|y\rangle$ (with the same isospin representation assumed). Explicitly, this is described by the operator relation [24]

$$e^{-s(\gamma D)^2}=e^{-s(-D^2)}\frac{1+\gamma_5}{2}+\gamma D\frac{1}{-D^2}\times e^{-s(-D^2)}\gamma D\frac{1-\gamma_5}{2}+P, \tag{6.5}$$

where P is the projection operator into the zero mode subspace of γD and can be expressed by $P=[1-\gamma D(1/-D^2)\gamma D][(1-\gamma_5)/2]$ [17]. Using the relation (6.5) with the definition of $\Gamma_{\rm ren}(A)$ (see Sec. II), it is then possible to derive a simple relationship between the spin-1/2 and spin-0 one-loop effective actions. If $\Gamma^{(1/2)}_{\rm ren}(A)$ $[\Gamma^{(0)}_{\rm ren}(A)]$ denotes the one-loop effective action as defined by Eq. (2.12) for a spin-1/2 (complex spin-0) field of mass m in a self-dual Yang-Mills background, we have in fact

$$\Gamma^{(1/2)}_{\rm ren}(A) = -\frac{1}{2} n_F \ln\left(\frac{m^2}{\mu^2}\right) - 2\Gamma^{(0)}_{\rm ren}(A) \tag{6.6}$$

or, for the respective contributions to the tunneling amplitude

$$e^{-\Gamma^{(1/2)}_{\rm ren}(A)} = \left(\frac{m}{\mu}\right)^{n_F} e^{2\Gamma^{(0)}_{\rm ren}(A)}, \tag{6.7}$$

where n_F is the number of normalizable spinor zero modes in the given background [23]. Now, using the result (6.1) for $\Gamma^{(0)}_{\rm ren}(A)$, we have the spin-1/2 instanton effective action expressed as (with $n_F=1$)

$$\Gamma^{(1/2)}_{\rm ren}(m,\rho,\mu) = -\ln\frac{m}{\mu} - \frac{1}{3}\ln\mu\rho - 2\alpha(1/2) - 2Q(m\rho), \tag{6.8}$$

or, for the tunneling amplitude,

$$e^{-\Gamma^{(1/2)}_{\rm ren}(m,\rho,\mu)} = \frac{m}{\mu}(\mu\rho)^{1/3} e^{2\alpha(1/2)+2Q(m\rho)}, \tag{6.9}$$

where $Q(m\rho)$ is the function specified in Eq. (6.2) (and represented in Fig. 4).

The expression in Eq. (6.8) or (6.9) describes the one-loop contribution to the vacuum tunneling by an isospin-1/2 quark field of mass m. If one accepts our conjecture, the function $Q(m\rho)$ may be taken as a monotonically decreasing function of $m\rho$ which has the limiting behaviors as given in Eq. (6.2). The renormalization prescription appropriate to the expression (6.8) is that specified by Eq. (2.12). If one wishes to obtain the corresponding amplitude in the minimal subtraction in the dimensional regularization scheme, the finite renormalization counterterm [see Eq. (2.14)]

$$\frac{1}{(4\pi)^2\times 6}(\ln 4\pi - \gamma)\int d^4x\, F^a_{\mu\nu}F^a_{\mu\nu} = \frac{1}{3}(\ln 4\pi - \gamma) \tag{6.10}$$

must be added to the expression (6.8). Thus, in the minimal subtraction scheme, the amplitude due to a spin-1/2 quark of mass m reads

$$e^{-\Gamma^{(1/2)}_{\rm MS}(m,\rho,\mu)} = \frac{m}{\mu}\left(\frac{\mu\rho}{4\pi e^{-\gamma}}\right)^{1/3} e^{2\alpha(1/2)+2Q(m\rho)}. \tag{6.11}$$

With $Q(m\rho)$ set to zero, this reduces to the result of 't Hooft [5]. For applications to the real QCD with the SU(3) gauge group, one must also take into account the well-known group theoretical factor associated with various ways of embedding the SU(2) instanton solution [25].

VII. DISCUSSIONS

In this work we studied the massive quark contribution to the one-loop instanton effective action in QCD. For this purpose, we made use of the approximation scheme valid for relatively large mass as well as the small-mass expansion. These considerations provide a reliable approximation to the one-loop effective action if the magnitude of $m\rho$ is such that $m\rho \gtrsim 1.8$ or $m\rho \lesssim 0.5$. The expression for the effective action contains a function $Q(m\rho)$, the magnitude of which is uncertain in the range $0.5 \lesssim m\rho \lesssim 1.8$. Based on the known behaviors of $Q(m\rho)$ in the ranges $m\rho \lesssim 1.8$ and $m\rho \lesssim 0.5$, we suggested that $Q(m\rho)$ be a smooth, monotonically decreasing function of $m\rho$. If the latter turns out to be true, a simple interpolation formula for $Q(m\rho)$ [as we considered in Eq. (6.3)] suffice for considerations in most phenomenological analyses.

What can be done to reduce the uncertainty in the function $Q(m\rho)$ for $0.5 \lesssim m\rho \lesssim 1.8$? With the explicit calculation of the $\mathcal{O}[(m\rho)^4]$ term in the small mass expansion, it should be possible to push the lower end of the uncertain range to a slightly higher value. On the other hand, we expect that including the next higher Seeley-DeWitt coefficient in the large-mass expansion would not bring a significant new information. More useful direction might be to try a direct numerical evaluation of the functional determinant, with the help of the scattering theory in a radially symmetric background field. (Some related techniques are discussed in Ref. [26].) Perhaps, by some mathematical argument, it might also be possible to actually prove that the function $Q(m\rho)$, which is equal to $\Gamma_{\rm ren}(m,\rho,\mu) - \Gamma_{\rm ren}(m=0,\rho,\mu)$ (for a spin-0 field) in an instanton background, is a monotonically decreasing function of $m\rho$. These are left for further study.

ACKNOWLEDGMENTS

One of us (C.L.) wishes to thank the Korea Institute for Advanced Study for hospitality while part of this work was undertaken. This work was supported in part by the BK 21 project of the Ministry of Education, Korea (O.K. and C.L.), by Korea Research Foundation Grant 2000-015-DP0072 (C.L.), and by the research supporting program of the University of Seoul (H.M.).

APPENDIX A

In the literature [14–16], the Seeley-DeWitt coefficients $\tilde{a}_n(x)$ for a spin-0 or spin-1/2 matter field have been calculated up to $n=6$. Here, for the case of a (complex) spin-0 field, we shall give the explicit expressions for $\tilde{a}_n(x)$ up to $n=5$ in a general off-shell background field. They read

$$\tilde{a}_3(x) = -\frac{1}{6}{\rm tr}\left[i\frac{2}{15}F_{\kappa\lambda}F_{\lambda\mu}F_{\mu\kappa} - \frac{1}{20}(D_\kappa F_{\lambda\mu})(D_\kappa F_{\lambda\mu})\right], \tag{A1}$$

$$\tilde{a}_4(x)=\frac{1}{24}\mathrm{tr}\Big[-\frac{1}{21}F_{\kappa\lambda}F_{\lambda\mu}F_{\mu\nu}F_{\nu\kappa}+\frac{11}{420}F_{\kappa\lambda}F_{\mu\nu}F_{\lambda\kappa}F_{\nu\mu}+\frac{2}{35}F_{\kappa\lambda}F_{\lambda\kappa}F_{\mu\nu}F_{\nu\mu}+\frac{4}{35}F_{\kappa\lambda}F_{\lambda\mu}F_{\kappa\nu}F_{\nu\mu}+i\frac{6}{35}F_{\kappa\lambda}(D_\mu F_{\lambda\nu})$$
$$\times(D_\mu F_{\nu\kappa})+i\frac{8}{105}F_{\kappa\lambda}(D_\lambda F_{\mu\nu})(D_\kappa F_{\nu\mu})+\frac{1}{70}(D_\kappa D_\lambda F_{\mu\nu})(D_\lambda D_\kappa F_{\nu\mu})\Big], \tag{A2}$$

$$\tilde{a}_5(x)=-\frac{1}{120}\mathrm{tr}\Big[-i\frac{2}{945}F_{\kappa\lambda}F_{\lambda\mu}F_{\mu\nu}F_{\nu\rho}F_{\rho\kappa}-i\frac{8}{63}F_{\kappa\lambda}F_{\lambda\mu}F_{\kappa\nu}F_{\mu\rho}F_{\rho\nu}+i\frac{16}{945}F_{\kappa\lambda}F_{\mu\nu}F_{\lambda\rho}F_{\nu\kappa}F_{\rho\mu}$$
$$+i\frac{22}{189}F_{\kappa\lambda}F_{\lambda\mu}F_{\kappa\nu}F_{\nu\rho}F_{\rho\mu}+i\frac{31}{378}F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho}F_{\mu\kappa}F_{\rho\nu}+i\frac{53}{378}F_{\kappa\lambda}F_{\lambda\kappa}F_{\mu\nu}F_{\nu\rho}F_{\rho\mu}+\frac{1}{9}F_{\kappa\lambda}F_{\lambda\mu}(D_\nu F_{\kappa\rho})(D_\nu F_{\rho\mu})$$
$$+i\frac{1}{18}F_{\kappa\lambda}(D_\lambda D_\mu F_{\nu\rho})(D_\mu D_\kappa F_{\rho\nu})-\frac{1}{189}F_{\kappa\lambda}F_{\lambda\mu}(D_\kappa F_{\nu\rho})(D_\mu F_{\rho\nu})+\frac{1}{252}(D_\kappa D_\lambda D_\mu F_{\nu\rho})(D_\mu D_\lambda D_\kappa F_{\rho\nu})$$
$$+\frac{1}{378}F_{\mu\kappa}F_{\rho\nu}F_{\kappa\lambda}(D_\lambda D_\mu F_{\nu\rho})+i\frac{2}{21}(D_\kappa F_{\lambda\mu})(D_\kappa D_\nu F_{\mu\rho})(D_\nu F_{\rho\lambda})+\frac{2}{63}F_{\kappa\lambda}(D_\mu F_{\nu\rho})F_{\lambda\kappa}(D_\mu F_{\rho\nu})$$
$$+i\frac{4}{63}F_{\kappa\lambda}(D_\mu D_\nu F_{\lambda\rho})(D_\nu D_\mu F_{\rho\kappa})-\frac{5}{63}F_{\kappa\lambda}F_{\lambda\mu}(D_\nu F_{\mu\rho})(D_\nu F_{\rho\kappa})+\frac{5}{63}F_{\kappa\lambda}(D_\mu F_{\lambda\nu})F_{\kappa\rho}(D_\mu F_{\rho\nu})$$
$$+\frac{5}{63}F_{\kappa\lambda}(D_\mu F_{\nu\rho})F_{\rho\lambda}(D_\mu F_{\nu\kappa})+i\frac{5}{126}(D_\lambda F_{\rho\nu})(D_\kappa F_{\lambda\mu})(D_\mu D_\kappa F_{\nu\rho})-\frac{10}{189}F_{\kappa\lambda}(D_\lambda F_{\mu\nu})F_{\nu\rho}(D_\rho F_{\mu\kappa})$$
$$-\frac{8}{189}F_{\rho\mu}F_{\kappa\lambda}(D_\lambda F_{\mu\nu})(D_\nu F_{\kappa\rho})+i\frac{5}{126}(D_\kappa F_{\lambda\mu})(D_\mu F_{\nu\rho})(D_\lambda D_\kappa F_{\rho\nu})-\frac{10}{189}F_{\kappa\lambda}(D_\mu F_{\lambda\nu})F_{\mu\rho}(D_\kappa F_{\rho\nu})$$
$$+\frac{11}{189}F_{\kappa\lambda}(D_\mu F_{\lambda\kappa})F_{\nu\rho}(D_\mu F_{\rho\nu})+\frac{11}{189}F_{\kappa\lambda}(D_\mu F_{\nu\rho})F_{\rho\nu}(D_\mu F_{\lambda\kappa})-\frac{11}{378}F_{\rho\nu}F_{\kappa\lambda}F_{\lambda\mu}(D_\mu D_\kappa F_{\nu\rho})$$
$$+\frac{13}{252}F_{\kappa\lambda}F_{\lambda\kappa}(D_\mu F_{\nu\rho})(D_\mu F_{\rho\nu})-\frac{16}{63}F_{\kappa\lambda}(D_\lambda F_{\mu\nu})F_{\nu\rho}(D_\kappa F_{\rho\mu})-\frac{16}{189}F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho}(D_\mu D_\kappa F_{\rho\nu})$$
$$-\frac{19}{756}F_{\kappa\lambda}(D_\lambda F_{\mu\nu})F_{\kappa\rho}(D_\rho F_{\nu\mu})-\frac{19}{756}F_{\kappa\lambda}(D_\mu F_{\nu\rho})F_{\mu\lambda}(D_\kappa F_{\rho\nu})+\frac{25}{189}F_{\rho\nu}F_{\kappa\lambda}(D_\mu F_{\lambda\nu})(D_\mu F_{\kappa\rho})$$
$$-\frac{26}{189}F_{\rho\nu}F_{\kappa\lambda}(D_\mu F_{\lambda\nu})(D_\rho F_{\mu\kappa})-\frac{34}{189}F_{\rho\mu}F_{\kappa\lambda}(D_\lambda F_{\mu\nu})(D_\rho F_{\nu\kappa})-\frac{41}{378}F_{\kappa\lambda}F_{\lambda\mu}(D_\mu F_{\nu\rho})(D_\kappa F_{\rho\nu})$$
$$+\frac{61}{756}F_{\rho\nu}F_{\kappa\lambda}(D_\mu F_{\lambda\kappa})(D_\mu F_{\nu\rho})+\frac{61}{756}F_{\kappa\lambda}F_{\mu\nu}(D_\rho F_{\lambda\kappa})(D_\rho F_{\nu\mu})\Big], \tag{A3}$$

where $D_\lambda F_{\mu\nu}\equiv[D_\lambda,F_{\mu\nu}]$ and $D_\kappa D_\lambda F_{\mu\nu}\equiv[D_\kappa,[D_\lambda,F_{\mu\nu}]]$, etc.

APPENDIX B

The function $C(x,p)$ in Eq. (5.26) is given by

$$C(x,p)=2\int d^4z(e^{-ip\cdot z}-1)(-\vec{\partial}_z^2)\left[\frac{1}{4\pi^2(x-z)^2}\left(\frac{1+\dfrac{\rho^2 x\cdot z}{x^2z^2}}{\sqrt{1+\rho^2/x^2}\sqrt{1+\rho^2/z^2}}-1\right)\right], \tag{B1}$$

and we are here interested in its behavior for $|x|>L$ (with $\rho\ll L\ll m^{-1}$) and $p\lesssim m$. We divide this quantity into two parts, i.e., $C(x,p)=C_<(x,p)+C_>(x,p)$, where $C_<(x,p)$ denotes the contribution with the region of integration restricted to $|z|\leqslant L_1$ (with $\rho\ll L_1<L$) and $C_>(x,p)$ that from the region $|z|>L_1$. (We take L_1 to be of the same order as L.) Then, for $C_<(x,p)$, it will be safe to make an approximation $e^{-ip\cdot z}-1\simeq -ip\cdot z$ [i.e., $\mathcal{O}(|p|L)$ at most] inside the integrand of Eq. (B1) and so we find immediately

$$C_<(x,p)=\mathcal{O}\left(|p|L\frac{\rho^2}{x^2}\right) \quad (\text{for } |x|>L). \tag{B2}$$

On the other hand, for $C_>(x,p)$, we may expand the factor $1/\sqrt{(1+\rho^2/x^2)}\sqrt{(1+\rho^2/z^2)}$ in the integrand of (B1) as a power series in $\rho/|x|$ and $\rho/|z|$, and then

$$\frac{1}{4\pi^2(x-z)^2}\left(\frac{1+\frac{\rho^2 x\cdot z}{x^2z^2}}{\sqrt{1+\rho^2/x^2}\sqrt{1+\rho^2/z^2}}-1\right)\to-\frac{\rho^2}{8\pi^2x^2z^2}+\frac{\rho^4}{4\pi^2(x-z)^2}\frac{3(x^2)^2+3(z^2)^2+2x^2z^2-4x\cdot z(x^2+z^2)}{8(x^2)^2(z^2)^2}+\cdots. \tag{B3}$$

As the differential operator $-\vec{\partial}_z^2$ acts on this expression, the first term in the right hand side of Eq. (B3) can be dropped. Thus, we may write

$$C_>(x,p)=2\int_{|z|>L_1}d^4z(e^{-ip\cdot z}-1)(-\vec{\partial}_z^2)\left[\frac{\rho^4}{4\pi^2(x-z)^2}\frac{3(x^2)^2+3(z^2)^2+2x^2z^2-4x\times z(x^2+z^2)}{8(x^2)^2(z^2)^2}+\cdots\right], \tag{B4}$$

and, for $|x|>L$ and $|p|\lesssim m$, it is not difficult to show that this can only lead to terms of $\mathcal{O}[|p|L(\rho^4/x^2L^2)]$ or $\mathcal{O}(\rho^4/x^3L)$ or $\mathcal{O}(\rho^4/x^4)$. Hence, $C(x,p)$ is at most $\mathcal{O}[|p|L(\rho^2/x^2)]$ or $\mathcal{O}[\rho^4/x^3L]$ or $\mathcal{O}(\rho^4/x^4)$.

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