Renormalization of the lattice heavy quark effective theory Isgur-Wise function

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We compute the perturbative renormalization factors required to match to the continuum Isgur-Wise function, calculated using lattice heavy quark effective theory. The velocity, mass, wave function, and current renormalizations are calculated for both the forward-difference and backward-difference actions for a variety of velocities. Subtleties are clarified regarding tadpole improvement, regulating divergences, and variations of techniques used in these renormalizations.

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I. INTRODUCTION

The unitarity of the Cabibbo-Kobayashi-Maskawa (CKM) matrix is regarded as a crucial test of the standard model [1,2]; the precise determination of these matrix elements has received extensive experimental and theoretical scrutiny. The V_{cb} CKM matrix element can be extracted from the reaction $B \to D^* l \overline{\nu_l}$, if the theoretical factors in the decay rate can be reliably computed. The heavy quark effective field theory (HQET) formalism is well suited to the analysis of this decay. The differential decay rate of the above process is

$$\frac{d}{d(v \cdot v')} \Gamma(B \to D^* l \overline{\nu_l}) = \frac{G_F^2}{48\pi^2} k(m_B, m_D, v \cdot v') |V_{cb}|^2 \xi^2(v \cdot v'), \quad (1)$$

where $\xi(v \cdot v')$ is a universal form factor, the Isgur-Wise function. The function k can be calculated in perturbation theory using various approximations [1,3]. The Isgur-Wise function is a QCD matrix element that must be computed non-perturbatively. Previously and in a companion paper [4,5] we discussed the numerical calculation of the Isgur-Wise function using lattice HQET. In this paper we discuss the perturbative matching of lattice HQET to continuum HQET, which allows the conversion of the results from the numerical simulations into physical predictions. Specifically, we shall be matching from the lattice to the continuum matrix element,

$$\langle D, v | J_\mu^{b \to c}(0) | B, v' \rangle = \sqrt{M_D M_B}(v_\mu + v'_\mu) \xi(v \cdot v'), \quad (2)$$

where v and v' denote the 4-velocities of the c and b quarks, and

$$J_\mu^{b \to c}(x) = \overline{c}(x) \gamma_\mu (1 - \gamma_5) b(x) \quad (3)$$

is the weak current for the transition of a bottom to a charm quark [6].

The Isgur-Wise form factor describes the response of the quark-gluon sea surrounding the heavy quark due to a sudden change in velocity of the heavy quark when it decays. In HQET, the Isgur-Wise function is nonperturbatively equal to one at the point of zero recoil, $v = v'$; HQET does not constrain the Isgur-Wise function at non-zero recoil. Continuum perturbative corrections are required to obtain the zero recoil result in QCD; however, these are known to 2 loops [3]. Unfortunately, there are no experimental data at zero recoil, so the experimental data are extrapolated [7] to zero recoil in order to estimate V_{cb} using Eq. (1). Knowledge of the functional form of the Isgur-Wise function would greatly aid this extrapolation. The Isgur-Wise function can be calculated nonperturbatively, in principle, from QCD for arbitrary recoil. In our companion paper [5], we describe our simulations that use lattice HQET to calculate the Isgur-Wise function.

There have been previous calculations of the renormalization factors for lattice HQET. Unfortunately, not all of the perturbative factors required for our numerical simulations were calculated. After lattice HQET was introduced by Mandula and Ogilvie [8], there were a number of concerns about the validity of the lattice HQET formalism [9–11]. The consistency of lattice HQET was finally demonstrated by Aglietti [12] in perturbation theory. However, Aglietti used a form of lattice HQET action that is less convenient for numerical simulation than the one originally used by Mandula and Ogilvie. The difference between the HQET actions was in the use of a forward or backward finite difference in the time direction (see Sec. III). Also, Aglietti considered only a special kinematic limit with one quark at rest and the other quark at finite velocity. Mandula and Ogilvie [13] limited their work to the velocity renormalization factors for the forward-difference action (which we used in our simulations); they calculated neither the vertex function nor the wave function renormalization which are required to renormalize the lattice data.

In this paper, we calculate the perturbative factors required to renormalize the Isgur-Wise function obtained from a lattice HQET simulation. The calculation includes two HQET actions: one with the forward time derivative and one with the backward time derivative. We follow the formalism developed by Aglietti [12], but generalize Aglietti's expres-

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sion for the vertex function to arbitrary input and output velocities (as is required for the analysis of the simulation data). We also include the effects of tadpole improvement and discuss a subtlety in the calculation of the vertex function.

Section II will provide a sparse review of continuum HQET in order to put the lattice calculation into context. Section III will describe the details of the velocity, mass, wave function, and vertex renormalizations for the lattice actions, including a discussion of the ''reduced'' results and an evaluation at nonzero recoil. Section IV will describe how these are combined into a single renormalization for the lattice current to be matched to the continuum. Section V concludes with some remarks concerning the renormalization process.

There have also been a number of attempts to calculate some of the required renormalization factors numerically [13,14]. The renormalization factors computed from numerical simulations should agree with the perturbative calculations as the weak coupling limit is approached. This is an important test of the numerical renormalization techniques, which has not yet been attempted. The renormalization of the current has never been computed numerically.

II. CONTINUUM HQET

Heavy quark effective theory is a way of studying a single heavy quark in a hadron when the mass of the quark is much larger than Λ_{QCD}. See Neubert [15] for a nice review of HQET. Mannel *et al.* [16] make rigorous Georgi's [17] intuition that the heavy quarks at different velocities do not interact. They do so by showing that the QCD Green functions which involve two heavy quarks at different velocities go to zero in the infinite mass limit. So, there is a separate field for each heavy quark at each velocity. In HQET [15], the connection between the HQET fields and the quark fields, Q, in QCD is

$$h_v(x)=e^{iMv\cdot x}P_+Q(x),$$
$$H_v(x)=e^{iMv\cdot x}P_-Q(x), \tag{4}$$

where $P_\pm=\frac{1}{2}(1\pm\not{v})$. The new form of the QCD Lagrangian has h describing massless degrees of freedom and H describing fluctuations with twice the heavy quark mass. Further, explicit Gaussian integration of the H fields produces the effective, non-local Lagrangian. Upon integrating out the heavy degrees of freedom, the H term is replaced by a local term involving the light degrees of freedom h and the mass of the heavy quark M. The Lagrangian is then expanded in the reciprocal of the heavy quark mass; the zeroth order HQET Lagrangian is

$$\mathcal{L}_{\mathrm{eff}}=\overline{h_v}iv\cdot Dh_v\,, \tag{5}$$

with the additional terms treated perturbatively as higher order in the reciprocal of the heavy quark mass. At zeroth order, i.e., in the infinite mass limit, the theory is independent of the mass of the heavy quark, and the Isgur-Wise function is universal (flavor blind).

In HQET, the momentum of the heavy quark (Mv) is distinguished from the momentum of the light quarks and gluons (k, the ''residual momentum''):

$$M_{\mathrm{had}}=Mv+k.$$

The residual momentum is the difference between the momentum of the hadron $(M_{\mathrm{had}}v)$ and the momentum of the heavy quark. The velocity of the heavy quark becomes a parameter of the theory and it is the residual momentum which becomes conjugate to the position. In the infinite mass limit, the momentum of the hadron is due only to the heavy quark.

The matrix element in the continuum modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme is connected to the matrix element calculated on the lattice by [18]

$$\langle v|V_\mu|v'\rangle^{\overline{MS}}=\frac{Z_\xi^{\mathrm{c}}}{Z_\xi^{\mathrm{l}}}\langle v|V_\mu|v'\rangle^{\mathrm{latt}}=Z_\xi^{\mathrm{cl}}\langle v|V_\mu|v'\rangle^{\mathrm{latt}}, \tag{6}$$

where Z_ξ^{c} is a continuum perturbative factor, Z_ξ^{l} is the lattice perturbative factor, and Z_ξ^{cl} is the ratio of the two. Falk *et al.* [19] calculated the continuum renormalization factor

$$Z_\xi^{\mathrm{c}}=1+\frac{g^2}{12\pi^2}\{2[1-(v\cdot v')r(v\cdot v')]\ln(\mu/\lambda)^2+\delta_c\}, \tag{7}$$

where

$$r(w)=\frac{\ln(w+\sqrt{w^2-1})}{\sqrt{w^2-1}} \tag{8}$$

and λ is the gluon mass introduced as a infrared regulator. The dependence on λ must cancel in Z_ξ^{cl}, the ratio of Z_ξ's, of Eq. (6). In the $\overline{\mathrm{MS}}$ scheme, $\delta_c=0$ [20]. The calculation of the lattice renormalization factor, Z_ξ^{l}, is the subject of the next section. Z_ξ^{cl} will be discussed further in Sec. IV when we discuss the matching from the lattice to the continuum.

III. LATTICE HQET

The Euclidean formulation of the lattice HQET action was introduced by Mandula and Ogilvie [8]:

$$S=\sum_x\left\{v_0\psi^\dagger(x)\Delta_t\psi(x)-i\sum_j v_j\psi^\dagger(x)\frac{\Delta_j+\Delta_{-j}}{2}\psi(x)\right\}. \tag{9}$$

There is some freedom in the choice of which lattice derivatives are used in Eq. (9). The tadpole improved finite differences are defined by

$$\Delta_\mu\psi_{\vec{x}}=\frac{U_{\vec{x},\vec{x}+\hat{\mu}}}{u_0}\psi_{\vec{x}+\hat{\mu}}-\psi_{\vec{x}}, \tag{10}$$

$$\Delta_{-\mu}\psi_{\vec{x}}=\psi_{\vec{x}}-\frac{U^\dagger_{\vec{x},\vec{x}-\hat{\mu}}}{u_0}\psi_{\vec{x}-\hat{\mu}} \tag{11}$$

such that

$$\psi^{\dagger}_{\vec{x}t}\Delta_{\mu}\psi_{\vec{x}t} \quad \text{is a forward difference,}$$

$$\psi^{\dagger}_{\vec{x}t}\Delta_{-\mu}\psi_{\vec{x}t} \quad \text{is a backward difference,}$$

$$\tfrac{1}{2}(\psi^{\dagger}_{\vec{x}t}\Delta_{\mu}\psi_{\vec{x}t}+\psi^{\dagger}_{\vec{x}t}\Delta_{-\mu}\psi_{\vec{x}t}) \quad \text{is a centered difference,}$$

and u_0 is the tadpole improvement factor [21]. The tadpole renormalization of the lattice HQET action is subtle because of the constraint on the velocity; these subtleties are addressed in Appendix A.

The centered difference approximates the continuum derivative to $O(a^2)$ (where a is the lattice spacing); both the forward- and backward-difference derivatives have $O(a)$ corrections to the continuum. Therefore, it seems that the centered difference is the preferred type of derivative. This is true for the spatial derivative; however, Mandula and Ogilvie [13] emphasize that for consistency an asymmetric time difference must be employed, rather than a centered difference. If a centered difference is employed, then the propagator vanishes on alternate sites in the positive time direction and there is no continuum limit. The source of this problem is that the heavy quark fields are defined separately from the heavy antiquark fields and are distinct for each velocity [recall Eq. (4)]; thus, heavy quarks can only propagate in one temporal direction.

The lattice HQET action originally proposed by Mandula and Ogilvie [8] used a forward time derivative. The backward time derivative can be less convenient for use in simulations because a three-dimensional matrix must be inverted for each time step. The forward time derivative only requires a matrix multiplication at each time step, and so is computationally cheaper to simulate.

This choice of a forward time derivative has also been discussed by Davies and Thacker [22] in the context of nonrelativistic QCD (NRQCD). However, recent NRQCD calculations follow the prescription of Lepage *et al.* [23] who use a backward time derivative but avoid having to invert a large spatial matrix by splitting the spatial part of the action over two adjacent time slices. Their action, which can be $\mathcal{O}(a)$ improved, is symmetric with respect to time reversal, yet avoids the problems of the centered difference. Improved heavy-Wilson actions [24] also go over to the backward derivative in the static limit. Similarly, better choices for the HQET lattice actions can be made, and if we were to rerun the program with higher-order corrections, it would indeed be advantageous to use the backwards time derivative as is done for heavy-Wilson and modern NRQCD actions. But for our present purposes, the zeroth-order action suffices, and at this order the forward difference provides a technical advantage in computation.

Since Aglietti's [12] perturbative calculation used the backward-difference time derivative, we do the perturbative calculations for both types of time derivative. We can check our results against Aglietti's, against the results from the static theory [25], and also the static limit of NRQCD [22]. Comparison in perturbation theory between the forward- and backward-difference actions for the static case has led to the introduction of the "reduced wave function renormalization" discussed in Sec. III D and summarized in Appendix B. (Please see Appendix C for a comparison of the notation between the groups.)

We introduce the notation

$$\sigma=\begin{cases}+1 & \text{forward difference,}\\ -1 & \text{backward difference}\end{cases} \tag{12}$$

in order to compare the forward- versus backward-difference actions. Both the forward- and backward-difference actions can be represented simultaneously by replacing Δ_t by $\Delta_{\sigma t}$, where $\Delta_{\sigma t}$ is either a forward time difference or a backward time difference, depending on the choice of action.

Feynman rules can be derived from the action

$$\text{quark propagator}\quad \left[v_0\sigma\left(\frac{1}{u_0}e^{i\sigma p_4}-1\right)+\sum_j \frac{v_j}{u_0}\sin(p_j)\right]^{-1}, \tag{13}$$

$$\text{gluon propagator}\quad \Delta(k)=\left[\sum_{\mu} 4\sin^2\frac{k_\mu}{2}+\lambda^2a^2\right]^{-1}, \tag{14}$$

$$\text{vertex}\quad \left[\delta_{\mu,0}\left(ig(T^a)^{bc}\frac{v_0}{u_0}e^{i\sigma(2p_4+k_4)/2}\right)\right.$$
$$\left.+\sum_j \delta_{\mu,j}\left(g(T^a)^{bc}\frac{v_j}{u_0}\cos\frac{2p_j+k_j}{2}\right)\right], \tag{15}$$

$$\text{tadpole vertex}\quad \left[\delta_{\mu,0}\left(\sigma\frac{g^2}{2}\frac{v_0}{u_0}(T^a)^{bd}(T^a)^{dc}e^{i\sigma p_4}\right)\right.$$
$$\left.+\sum_j \delta_{\mu,j}\left(\frac{g^2}{2}\frac{v_j}{u_0}(T^a)^{bd}(T^a)^{dc}\sin p_j\right)\right], \tag{16}$$

$$\text{internal integrations}\quad \int_{-\pi/a}^{\pi/a}\frac{d^4k}{(2\pi)^4}. \tag{17}$$

The T^a are the color generators and $C_F=\frac{4}{3}$ is the Casimir invariant. λ is a gluon mass, which is needed to regulate the infrared divergences (as is done in the continuum) and which will be taken to zero at the end of the calculation.

From the Feynman rules, it is straightforward to derive the usual self-energy $[\Sigma(p)]$, tadpole $[\Sigma^{\rm tad}(p)]$, and vertex $[V(p,p')]$ corrections [the self-energy is $\Sigma(p)+\Sigma^{\rm tad}(p)$]:

$$\Sigma(p)=g^2C_F\int\frac{d^4k}{(2\pi)^4}\frac{1}{\Delta(-k)}\frac{-(v_0^2/u_0^2)e^{i\sigma(2p_4+k_4)}+\sum_j (v_j^2/u_0^2)\cos^2[(2p_j+k_j)/2]}{\left\{v_0\sigma[(1/u_0)e^{i\sigma(p_4+k_4)}-1]+\sum_j (v_j/u_0)\sin(p_j+k_j)\right\}}, \tag{18}$$

$$\Sigma^{\rm tad}(p)=-\frac{g^2C_F}{2}\left(-\sigma\frac{v_0}{u_0}e^{i\sigma p_4}-\sum_j \frac{v_j}{u_0}\sin(p_j)\right)\int\frac{d^4k}{(2\pi)^4}\frac{1}{\Delta(k)}, \tag{19}$$

$$V(0,0)=g^2C_F\int\frac{d^4k}{(2\pi)^4}\frac{1}{\Delta(-k)}\left\{-\frac{v_0v_0'}{u_0^2}e^{i\sigma k_4/2}e^{i\sigma k_4/2}+\sum_j\frac{v_jv_j'}{u_0^2}\cos\frac{k_j}{2}\cos\frac{k_j}{2}\right\}\Bigg/\left\{\left[v_0\sigma\left(\frac{e^{i\sigma k_4}}{u_0}-1\right)+\sum_j\frac{v_j}{u_0}\sin(k_j)\right]\right.$$
$$\left.\times\left[v_0'\sigma\left(\frac{e^{i\sigma k_4}}{u_0}-1\right)+\sum_j\frac{v_j'}{u_0}\sin(k_j)\right]\right\}. \tag{20}$$

It is sufficient for our purposes to evaluate the vertex function with zero external momentum. The explicit p dependence is kept in the self-energy since the derivative will be considered. The integral which appears in the tadpole correction is standard and has the value $[1/(2\pi)^4]\int d^4k/\Delta(k) = 0.154933$.

The evaluation of the integrals is nonstandard because of the problems caused by the spectrum of the Euclidean HQET action not being bounded from below. We follow the formalism developed by Aglietti [12] and by Mandula and Ogilvie [13], in which we must first perform the k_4 integration analytically and do so by transforming to z space ($z=e^{\pm ik_4}$). A contour is chosen that enforces the forward propagation of the HQET quarks [13] as described below. (The connection to Minkowski space via a Wick contraction is discussed by Aglietti [12].) The resulting three-dimensional integrals are then calculated numerically. (All of the numeric integrations were computed with the VEGAS routine [26].)

The analytic k_4 integration of Eqs. (18) through (20) reduces the four-dimensional integration to a three-dimensional integration. It is, however, more convenient to do this as a contour integration in z space [13] after an action-dependent change of variables

$$z=e^{i\sigma k_4}. \tag{21}$$

For this change of variables, the gluon propagator is written

$$\Delta(k_\mu)=2\sum_\mu (1-\cos k_\mu)+(a\lambda)^2$$
$$= 2\ (1-\cos k_4)+\Delta_3(\vec{k}) \tag{22}$$

which defines $\Delta_3(\vec{k})$.

The k_4 contour (along the real axis) transforms into the unit circle in complex z space. A subtlety arises when deciding which poles to enclose by the contour. The quark propagator pole appears as

$$z_Q=e^{-i\sigma p_4}\left(u_0-\sigma\sum_j \tilde{v}_j\sin(p_j+k_j)\right). \tag{23}$$

The gluon propagator poles appear at

$$z_\pm=1+\frac{\Delta_3}{2}\pm\frac{1}{2}\sqrt{\Delta_3^2+4\Delta_3}, \tag{24}$$

where Δ_3 is defined by Eq. (22). The contour separates the gluon poles. The contour should enclose the quark pole and one of the gluon poles. The subtlety is in choosing which gluon pole. Because the energy-momentum relation from the quark propagator, Eq. (B2), can be negative, we split k space (or z space) into a positive-energy region and a negative-energy region and enclose the gluon pole which lies in the positive-energy region of the space. For negligible external momentum with a quark momentum $p+k$, the upper k_4 half-plane is positive energy and, using Eq. (12) to distinguish the actions, it is convenient to define z via Eq. (21) such that $z=e^{+ik_4}$ for the forward-difference action and $z=e^{-ik_4}$ for the backward-difference action. (For $p-k$, the lower k_4 half-plane is positive energy and it is convenient to use $z=e^{-i\sigma k_4}$.) With either of these choices, the backward difference action will have the positive-energy region outside of the z-space unit circle and the forward-difference action will have the positive-energy region inside the z-space unit circle.

The quark pole

$$z_Q\sim\left(1-\sigma\frac{\vec{v}\cdot\vec{k}}{v_0}\right), \tag{25}$$

with positive-energy [using Eq. (B2), $z_Q\approx 1-\sigma\varepsilon$] is just inside (outside) of the unit circle for the forward (backward) difference action. Since $\sqrt{\Delta_3^2+4\Delta_3}\geqslant\Delta_3$, we find z_+ outside (and z_- inside) of the unit circle. z_σ (which is equal to z_+ for the forward difference and z_- for the backward difference) is therefore in the negative-energy region. Since the k_+ (z_+) gluon pole is always in the positive-energy region for the backward-difference action and always in the negative-energy region for the forward-difference action, we can write

$$\begin{array}{ll} \text{Backward difference} & \left\{\begin{array}{ll} z_+ & \text{positive-energy pole} \\ z_- & \text{negative-energy pole} \end{array}\right. \\ \text{Forward difference} & \left\{\begin{array}{ll} z_+ & \text{negative-energy pole} \\ z_- & \text{positive-energy pole} \end{array}\right. \end{array} \quad \begin{array}{ll} z_\sigma & \text{negative-energy pole,} \\ z_{-\sigma} & \text{positive-energy pole.} \end{array} \tag{26}$$

In both cases, it is the quark and the positive-energy gluon poles which are enclosed by the contour regardless of where the quark pole actually appears. When the quark pole moves into the negative-energy region, it is necessary to deform the contour to keep the quark pole enclosed. (This is discussed for k space by Aglietti [12] and for z space by Mandula and Ogilvie [13].) However, to simplify, one can equate this to the negative of the contour integral which encloses only the negative-energy gluon pole. The three-dimensional integrals resulting from the contour integration have an action-dependent form due to the appearance of the negative-energy gluon pole (z_σ). This pole is a function of $\vec{k}$.

In order to compute the renormalizations, the unrenormalized propagator is compared to the renormalized propagator. (We include the mass term in order to calculate the mass renormalization.) The renormalized propagator has the form

$$iH^{\rm r}(k)=\frac{Z}{\left[iv_0^{\rm r}k_4+\sum_j v_j^{\rm r}k_j+M^{\rm r}+O(k^2)\right]}. \tag{27}$$

The renormalization factors are obtained by Taylor series expanding the unrenormalized propagator

$$iH(k)=\Bigg[v_0\sigma\left(\frac{1}{u_0}e^{i\sigma k_4}-1\right)+\sum_j \frac{v_j}{u_0}\sin(k_j) +M_0-\Sigma(k,v)\Bigg]^{-1}. \tag{28}$$

We used

$$\Sigma(k,v)=\Sigma(0,v)+k_4X_4+\sum_j k_jX_j+O(k^2) \tag{29}$$

and

$$\frac{1}{u_0}e^{i\sigma k_4}=e^{i\sigma k_4-\ln u_0}=1+i\sigma k_4-\ln u_0+O(k^2), \tag{30}$$

where

$$X_\mu=\left.\frac{\partial\Sigma(k)}{\partial k_\mu}\right|_{k=0}. \tag{31}$$

Equation (30) was used for the static case [27,28] to elucidate that the tadpole factor u_0 results in mass renormalization rather than wave function renormalization. Notice that u_0 has the perturbative expansion $\{1-[g^2C_F/(4\pi)^2]\pi^2+O(g^4)\}$, so $\ln u_0 \sim O(g^2)$; the higher order terms are neglected. After a little algebra which involves the addition and subtraction of some deducible terms, one can write the propagator in the form

$$iH(k)=\left\{[1-\delta Z]\left[i(v_0+\delta v_0)k_4+\sum_j \left(\frac{v_j}{u_0}+\delta\left(\frac{v_j}{u_0}\right)\right)k_j +(M_0+\delta M)+O(k^2)\right]\right\}^{-1}, \tag{32}$$

which implies the expressions for the renormalizations

$$\delta M=M^{\rm r}-M_0=-\Sigma(0,v)-\sigma v_0\ln u_0, \tag{33}$$

$$\delta Z=Z-1=-iv_0X_4-u_0\sum_j v_jX_j, \tag{34}$$

$$\delta\left(\frac{v_i}{u_0}\right)=v_i^{\rm r}-\frac{v_i}{u_0}=-iv_0\frac{v_i}{u_0}X_4-(1+v_i^2)X_i-v_i\sum_{j\neq i} v_jX_j, \tag{35}$$

$$\delta v_0=v_0^{\rm r}-v_0=-i(v_0^2-1)X_4-v_0u_0\sum_j v_jX_j. \tag{36}$$

We make the following points regarding these expressions: First, in the HQET formalism, the residual momentum is conjugate to the position, leaving the velocity as a free parameter. As discussed by Aglietti [12], the velocity is renormalized on the lattice. In the continuum, the four-vector X_μ is proportional to v_μ, the only available four-vector; this implies that there is no velocity renormalization. On the lattice, with reduced rotational symmetry, this is not the case. Secondly, if u_0 is set to unity and the special case of $\vec{v}=v_z\hat{z}$ is taken, then these reduce, for the backward-difference case, to Aglietti's result [12]. Thirdly, $\delta(v_j/u_0)$ is a notation to remind the reader that this quantity renormalizes v_j/u_0 rather than v_j as can be seen in Eq. (35). For $u_0=1$, our $\delta(v_j/u_0)$ corresponds to Aglietti's δv_j. Further, the velocity renormalization can be written as follows:

$$v_j^{r,\rm tad}=v_j^{b,\rm tad}Z_{v_j}^{\rm tad}, \quad v_0^{r,\rm tad}=v_0^{b,\rm tad}Z_{v_0}^{\rm tad},$$

$$Z_{v_j}^{\rm tad}=\frac{1}{u_0}\left(1+\frac{\delta(v_j/u_0)}{v_j/u_0}\right), \quad Z_{v_0}^{\rm tad}=1+\frac{\delta v_0}{v_0}. \tag{37}$$

Finally, the u_0 that appears in these expressions is the perturbatively expanded $u_0=1-(g^2C_F/16\pi^2)\pi^2$. It is taken at lowest order (unity) and the terms higher order in g^2 are ignored because $X_\mu \sim O(g^2)$. The result is that the wave function renormalization and the first term of the mass renormalization, $\Sigma(0,v)$, are the same to $O(g^2)$ whether or not one uses tadpole improvement. The velocity renormalization is affected by tadpole improvement as

$$Z_{v_j}^{\rm tad} = \left[1 + \frac{\delta(v_j/u_0)}{v_j/u_0} - \frac{g^2 C_F}{16\pi^2}(-\pi^2)\right],$$

$$= \left[1 + \frac{g^2 C_F}{16\pi^2}(c(\tilde{v}) + \pi^2)\right], \tag{38}$$

where the $(-\pi^2)$ is from the perturbative expansion of u_0, $v_j \equiv v_j/u_o$, and $c(\tilde{v})$ is the same [to $O(g^2)$] regardless of tadpole improvement.

Of the renormalization factors [Eqs. (33) to (36)], only the mass renormalization, Eq. (33), depends explicitly on the choice of forward or backward time difference (the σ parameter). However, all the renormalization factors implicitly depend on σ via the X_μ functions. The explicit dependence of the mass renormalization on σ is zero when tadpole improvement is not used; this is discussed further in Appendix B, above Eq. (B8).

A. Velocity renormalization

Mandula and Ogilvie [13] renormalize $\tilde{v}_j \equiv v_j/v_0$ rather than v. We will not be using their notation, rather we will be renormalizing v, and calculating $c(\tilde{v})$ defined by Eqs. (35), (37), and (38):

$$\frac{\delta(v_j/u_0)}{v_j/u_0} = \frac{g^2 C_F}{16\pi^2} c(\tilde{v}). \tag{39}$$

This parallels the notation of Aglietti [12]. (See Appendix C for a comparison.) Recall that this is the perturbative renormalization to the tadpole-improved velocity. Neither Mandula and Ogilvie nor Aglietti use a tadpole-improved action.

The expression for $c(\tilde{v})$ is found from the self-energy Feynman diagrams as expressed through Eq. (35). Continuing to use the $\sigma = \pm 1$ to distinguish between the actions, we find

$$c(\tilde{v}) = \frac{2v_0^2}{\pi}\int \frac{d^3k}{\sqrt{\Delta_3^2 + 4\Delta_3}}\left(\frac{-2\sigma z_\sigma(k) + (\tilde{v}_i/u_0)[(1/v_0^2) + \tilde{v}_i^2]\sin(k_i) + \sum_{j\neq i}(\tilde{v}_j^3/u_0)\sin(k_j)}{\sigma[(1/u_0)z_\sigma(k) - 1] + \sum_j (\tilde{v}_j/u_0)\sin(k_j)} + \frac{\left[z_\sigma(k) - \sum_j (\tilde{v}_j^2/u_0^2)\cos^2(k_j/2)\right]\left[z_\sigma(k) - [(1/v_0^2) + \tilde{v}_i^2]\cos(k_i) - \sum_{j\neq i}\tilde{v}_j^2\cos(k_j)\right]}{[\sigma[(1/u_0)z_\sigma(k) - 1] + \sum_j (\tilde{v}_j/u_0)\sin(k_j)]^2}\right). \tag{40}$$

The u_0 are perturbatively expanded such that at this order in g^2, they can be replaced with unity. (They are included as a reminder that in the next order there will be an effect.) Note that $z_\sigma(k)$ is the negative-energy gluon pole, defined by Eqs. (24) and (26), introduced from the residue of the contour.

Mandula and Ogilvie [13] perform an expansion in small velocity and present the velocity renormalization as coefficients to powers of the velocity. (This is convenient in that whenever a calculation at a new velocity is desired, the value for the velocity has precalculated coefficients so that the calculation need not be done repeatedly.) While this is straightforward for the velocity renormalization, the divergences in the wave function renormalization and the vertex correction make this technique more complicated for these other calculations. However, if we consider the expansion for the velocity renormalization, then we get consistent results at $O(\tilde{v}^6)$ (notice that our format is slightly different because c renormalizes v rather than $\tilde{v}$)

$$\begin{aligned} c(\tilde{v}) = {} & c_{000} + c_{200}\tilde{v}_i^2 + c_{020}\sum_{j\neq i}\tilde{v}_j^2 + c_{400}\tilde{v}_i^4 + c_{220}\tilde{v}_i^2\sum_{j\neq i}\tilde{v}_j^2 \\ & + c_{040}\sum_{j\neq i}\tilde{v}_j^4 + c_{022}\sum_{k\neq j,i}\sum_{j\neq i}\tilde{v}_j^2\tilde{v}_k^2 + c_{600}\tilde{v}_i^6 \\ & + c_{420}\tilde{v}_i^4\sum_{j\neq i}\tilde{v}_j^2 + c_{240}\tilde{v}_i^2\sum_{j\neq i}\tilde{v}_j^4 + c_{222}\tilde{v}_i^2\sum_{k\neq j,i}\sum_{j\neq i}\tilde{v}_j^2\tilde{v}_k^2 \\ & + c_{060}\sum_{j\neq i}\tilde{v}_j^6 + c_{042}\sum_{k\neq j,i}\sum_{j\neq i}\tilde{v}_j^4\tilde{v}_k^2 + \cdots . \end{aligned} \tag{41}$$

The forward and backward difference results of this expansion are listed in Table I. Mandula's and Ogilvie's results are reproduced by the first two columns. Our results for the same special case (backward difference, $v_x = v_y = 0$) that Aglietti considers [12] are listed in Table II and agree with Aglietti where they overlap. The three columns of the forward differ-

TABLE I. The coefficients c_{mnl} used in the velocity renormalization when expanded in powers of the velocity to $O(\tilde{v}^6)$ according to Eq. (41). s is the order of the velocity term, found by summing the indices; $s=m+n+l$. The first set is for the forward-difference action; the second set is for the backward-difference action. If you consider the velocity in only one direction, then only the top row is relevant.

c_{mnl}		Forward difference				Backward difference			
n	l	$s=0$	$s=2$	$s=4$	$s=6$	$s=0$	$s=2$	$s=4$	$s=6$
0	0	−28.07(3)	−4.977(6)	−1.093(3)	−0.458(2)	11.78(1)	0.33(2)	−0.88(3)	−2.03(3)
2	0		−4.292(6)	−2.100(6)	−1.380(6)		10.26(2)	9.49(6)	7.0(2)
4	0			−1.010(3)	−1.346(6)			7.62(3)	28.1(2)
2	2			−1.005(6)	−1.36(1)			9.53(6)	43.4(3)
6	0				−0.469(2)				39.98(6)
4	2				−4.54(2)				109.6(6)

ence are: $c(\tilde{v})$ according to Eq. (40), its expansion through sixth order in small velocity according to Eq. (41), and its expansion through second order of the velocity expansion using only the first three terms of Eq. (41). The latter confirms Mandula's and Ogilvie's result; however, the sixth order result (using the coefficients of Table I) is in much better agreement with the exact result (as one would expect). Although Table II only considers motion along a single axis, our more general results indicate that for the forward-difference action it is sufficient to use the velocity expansion to sixth order.

For the more general case of all the spatial velocities not equal to zero, we present the results for the forward-difference action at small velocities in Table III. This is the factor, $c_z(\tilde{v})$, which renormalizes the $\hat{z}$ component of the velocity according to Eq. (38). The renormalizations for the v_x and v_y components can be deduced from the table by symmetry. Notice that the v_z renormalization is affected by each component of $\vec{v}$, not merely by v_z. The numerical size of the perturbative factors in Tables II and III are both large.

TABLE II. This table lists the velocity renormalization for both forward-difference (our choice) and backward-difference actions for the special case $v_x=v_y=0$. The last two columns solve the expanded equation through the superscripted order. The $c(\tilde{v})$ entries are exact, that is, not expanded in the velocity. Note that the $\tilde{v}\rightarrow 0.0$ limit is considered even though there is no need to calculate the renormalization coefficient when $\tilde{v}=0$. (c has no interpretation in the static limit.)

	(Backward) $c(\tilde{v})$	$c(\tilde{v})$	(Forward) $c^{(6)}$	$c^{(2)}$
$c(\tilde{v}\rightarrow 0.0)$	11.779(4)	−28.06(1)	−28.06(1)	—
$c(\tilde{v}=0.1)$	11.899(5)	−28.40(1)	−28.40(1)	−28.38(1)
$c(\tilde{v}=0.2)$	12.275(5)	−29.44(1)	−29.44(1)	−29.42(1)
$c(\tilde{v}=0.3)$	12.966(5)	−31.35(1)	−31.35(1)	−31.29(1)
$c(\tilde{v}=0.4)$	14.036(7)	−34.39(1)	−34.39(1)	−34.28(1)
$c(\tilde{v}=0.5)$	15.67(1)	−39.17(1)	−39.17(1)	−38.95(2)
$c(\tilde{v}=0.6)$	18.05(1)	−46.90(1)	−46.90(1)	−46.44(2)
$c(\tilde{v}=0.7)$	20.82(3)	−60.44(2)	−60.45(2)	−59.47(3)

Tadpole improving the perturbative factors, by adding $+\pi^2$ to them as in Eq. (38), does not substantially reduce the size of the perturbative contribution.

To give an idea about the magnitude of the velocity renormalization, we consider $\beta=6.0$ with $|\vec{v}|=0.5$, and use the bare lattice coupling. The non-tadpole improved multiplicative factor is $Z_{v_j}=0.67$; the corresponding tadpole improved

TABLE III. The velocity renormalization, $c_z(\tilde{v})$, for the forward difference action for several general (small) velocities. The uncertainty is at most 2 in the last digit.

$\tilde{v}_z=0.00$	$\tilde{v}_y$ \ $\tilde{v}_x$	0.00	0.05	0.10	0.25
	0.00	−28.06	−28.15	−28.40	−30.14
	0.05	−28.16	−28.23	−28.47	−30.19
	0.10	−28.39	−28.47	−28.72	−30.46
	0.25	−30.12	−30.20	−30.45	−32.17
$\tilde{v}_z=0.05$	$\tilde{v}_y$ \ $\tilde{v}_x$	0.00	0.05	0.10	0.25
	0.00	−28.16	−28.23	−28.48	−30.19
	0.05	−28.26	−28.34	−28.55	−30.27
	0.10	−28.49	−28.56	−28.82	−30.53
	0.25	−30.20	−30.30	−30.54	−32.26
$\tilde{v}_z=0.10$	$\tilde{v}_y$ \ $\tilde{v}_x$	0.00	0.05	0.10	0.25
	0.00	−28.40	−28.48	−28.72	−30.45
	0.05	−28.49	−28.55	−28.81	−30.53
	0.10	−28.72	−28.80	−29.06	−30.78
	0.25	−30.45	−30.52	−30.79	−32.50
$\tilde{v}_z=0.25$	$\tilde{v}_y$ \ $\tilde{v}_x$	0.00	0.05	0.10	0.25
	0.00	−30.15	−30.21	−30.46	−32.20
	0.05	−30.22	−30.31	−30.56	−32.27
	0.10	−30.48	−30.54	−30.80	−32.54
	0.25	−32.21	−32.27	−32.51	−34.26

TABLE IV. Mass $[x(\tilde{v})]$ and wave function $[e(\tilde{v})]$ renormalization functions for the backward-difference (BD) and forward-difference (FD) actions. The BD numbers reproduce Aglietti's table [12]. The BD and FD numbers for $x(\tilde{v})$ and $e'(\tilde{v})$ should agree *only* in the static limit ($\tilde{v}=0$). $e'(\tilde{v})=e(\tilde{v})-\sigma x(\tilde{v})/v_0$ are the reduced wave function. Notice that we define $x(\tilde{v})$ as the negative of that of Aglietti. In addition, $v_x=v_y=0$.

	Backward difference			Forward difference		
$\tilde{v}$	$x(\tilde{v})$	$e(\tilde{v})$	$e'(\tilde{v})$	$x(\tilde{v})$	$e(\tilde{v})$	$e'(\tilde{v})$
0.0	−19.92(3)	24.43(4)	4.53(1)	−19.93(1)	−15.40(1)	4.530(4)
0.1	−19.87(3)	24.64(4)	4.875(4)	−19.99(1)	−15.75(1)	4.141(2)
0.2	−19.69(3)	25.24(4)	5.97(1)	−20.17(1)	−16.82(1)	2.935(2)
0.3	−19.34(3)	26.36(4)	7.91(1)	−20.47(1)	−18.78(1)	0.759(4)
0.4	−18.75(3)	28.14(4)	10.96(1)	−20.97(1)	−21.91(1)	−2.694(6)
0.5	−17.72(3)	30.94(5)	15.60(2)	−21.72(2)	−26.83(2)	−8.015(8)
0.6	−15.79(3)	35.44(5)	22.82(4)	−22.89(1)	−34.74(2)	−16.44(2)
0.7	−11.15(3)	44.2(1)	36.27(10)	−24.79(2)	−48.56(4)	−30.85(2)

number is $Z_{v_j}^{\mathrm{tad}}=0.75$. If the boosted coupling, g^2/u_0^4 [21], is used then $Z_{v_j}^{\mathrm{tad}}=0.59$. As the slope of the Isgur-Wise function essentially depends quadratically on the velocity renormalization, this makes perturbation theory unreliable to analyze the simulation data and thus numerical renormalization techniques must be used [4,5,13,14].

Aside: Slow HQET. In Aglietti's [12] initial calculations, the velocity renormalization was presented as a function of the velocity. However, Mandula and Ogilvie [13] expanded the velocity renormalization in a power series in the velocity, which allowed them to compare their perturbative results with the numbers from their numerical renormalization technique. The expansion of the renormalization factor in velocity seems to be similar to Aglietti's [29] idea of slow HQET, where the $v\cdot D$ term is a perturbation on the static theory. Slow HQET was studied in perturbation theory by Aglietti and Giménez [30], where they demonstrated that slow HQET agreed with HQET in the infrared and ultraviolet limits. It would be interesting to understand the connection between slow HQET and the HQET formalism of Mandula and Ogilvie.

We have found expressions for the velocity renormalization in terms of the coefficients for the backward-difference action (Table I) and note that the c_{042} coefficient of the backward difference is rather large, at 109.6(6)—much larger than the equivalent coefficient for the forward-difference action. This could indicate a problem with the expansion for the backward difference; the forward-difference coefficients, which we checked through $O(\tilde{v}^6)$, are all reasonably close to unity. Aglietti and Giménez do not calculate the sixth-order coefficient for the velocity renormalization (although they take the other renormalizations to this order); however, the renormalization factors are only required to quadratic order in the velocity in order to compute the slope of the Isgur-Wise function from simulations of slow HQET. (The slow HQET formalism is used to directly calculate the derivatives of the Isgur-Wise function, using the ''moments'' technique [29]. Aglietti and Giménez [30] found that the expressions for the higher order derivatives of the Isgur-Wise function, beyond the slope, contained operators that diverged with an inverse power of the lattice spacing and that must be subtracted off in the simulation.)

B. Mass renormalization

As with the velocity renormalization, we define $x(\tilde{v})$ as the mass renormalization without the g^2 prefactor. For comparison, Aglietti [12] also does this; however, we prefer (for comparison to the static limit of the forward-difference NRQCD theory) to have our $x(\tilde{v})$ proportional to $+\Sigma(0,\tilde{v})$. So, our x is the negative of Aglietti's. (See Appendix C for a comparison between groups.) We also include the effect of tadpole improvement.

$$\delta M=-\Sigma(0,\tilde{v})-\sigma v_0\ln u_0=-\frac{g^2C_F}{16\pi^2}\frac{x(\tilde{v})}{a}-\sigma v_0\ln u_0. \tag{42}$$

Recall that $\ln u_0$ is $O(g^2)$. From Eq. (33),

$$x(\tilde{v})=\frac{2v_0}{\pi}\int\frac{d^3k}{\sqrt{\Delta_3^2+4\Delta_3}}\frac{-z_\sigma(k)+\sum_j\frac{\tilde{v}_j^2}{u_0^2}\cos^2\left(\frac{k_j}{2}\right)}{\left[\sigma\left(\frac{1}{u_0}z_\sigma(k)-1\right)+\sum_j\frac{\tilde{v}_j}{u_0}\sin(k_j)\right]}+\sigma 8\pi^2v_0(0.154933), \tag{43}$$

where the $8\pi^2(0.154933)$ is from the tadpole contribution ($\Sigma^{\rm tad}$) which is partially canceled by the second term in Eq. (42) as it should be. The u_0 are again perturbatively expanded such that, at this order, they can be replaced with unity. They are included here as a reminder that in the next order they will have an effect. The values of this integral are listed in Table IV. As they are relevant to the reduced wave function, we will discuss these there.

C. Wave function renormalization

The results of the wave function and reduced wave function renormalization can be compared not only to Aglietti[12] (backward difference, HQET) and Mandula and Ogilvie [13] (forward difference, HQET), but also to Eichten and Hill [25,31] (backward difference, static theory) and the static limit of Davies and Thacker [22] (forward difference, NRQCD). (Appendix C compares the notations between groups.)

Recall that the wave function renormalization can be found as Eq. (34). During this calculation, as with the velocity renormalization, the k_0 integration is done analytically with the same comments as were made earlier. This introduces the residue from the negative-energy gluon pole, $z_\sigma(k)$. Again using the $\sigma=\pm 1$ to distinguish between the actions, the result for δZ is

$$\delta Z=\frac{g^2C_F}{16\pi^2}\frac{2v_0^2}{\pi}\int\frac{d^3k}{\sqrt{\Delta_3^2+4\Delta_3}}\left\{\frac{-2\sigma z_\sigma(k)+\sum_j(\tilde{v}_j^3/u_0)\sin(k_j)}{\left[\sigma[(1/u_0)z_\sigma(k)-1]+\sum_j(\tilde{v}_j/u_0)\sin(k_j)\right]}+\frac{\left[z_\sigma(k)-\sum_j(\tilde{v}_j^2/u_0^2)\cos^2\left(\frac{k_j}{2}\right)\right]\left[z_\sigma(k)-\sum_j\tilde{v}_j^2\cos(k_j)\right]}{\left\{\sigma[(1/u_0)z_\sigma(k)-1]+\sum_j(\tilde{v}_j/u_0)\sin(k_j)\right\}^2}\right\}+\frac{g^2C_F}{16\pi^2}8\pi^2(0.154933),$$

where again the $8\pi^2(0.154933)$ is from the tadpole contribution and the u_0 are again perturbatively expanded such that at this order, they can be replaced with unity. This three-dimensional integral has a logarithmic divergence. The way with which this is typically dealt is to add and subtract an integral with the same logarithmic divergence which is solvable analytically. We call this integral δZ^c and use the small k limit because we are interested in the infrared (low energy) divergence. The difference $\delta Z-\delta Z^c$ is finite and calculated numerically. δZ^c (found analytically) will have a finite piece, which is added back to the numerical calculation, as well as a divergent piece. The divergent piece contributes to the coefficient of the $\ln(\lambda^2a^2)$ term in the renormalization of the lattice Isgur-Wise function.

Although the ''continuum-like'' limit of δZ is actually

$$\frac{g^2C_F}{16\pi^2}\frac{2v_0^2}{\pi}\int\frac{d^3k}{2\sqrt{k^2+\lambda^2a^2}}\left\{\frac{-2\sigma+\sum_j(\tilde{v}_j^3/u_0)k_j}{\left[(\sqrt{k^2+\lambda^2a^2})+\sum_j(\tilde{v}_j/u_0)k_j\right]}+\frac{\left[1-\sum_j(\tilde{v}_j^2/u_0^2)\right]\left[1-\sum_j\tilde{v}_j^2\right]}{\left[(\sqrt{k^2+\lambda^2a^2})+\sum_j(\tilde{v}_j/u_0)k_j\right]^2}\right\}+\frac{g^2C_F}{16\pi^2}8\pi^2(0.154933),$$

the first and third terms are finite. Since we are interested in the infrared divergent piece, we will define δZ^c as the second term. By taking advantage of $u_0|_{\rm pert}=1+O(g^2)$ as well as by using the velocity normalization $[(1-\Sigma_j\tilde{v}_j^2)=1/v_0^2]$ we get

$$\delta Z^c=\frac{g^2C_F}{16\pi^2}\frac{1}{\pi v_0^2}\int_0^R\frac{d^3k}{\sqrt{k^2+\lambda^2a^2}}\frac{1}{[\sqrt{k^2+\lambda^2a^2}+\Sigma_j\tilde{v}_jk_j]^2}.$$

The upper limit R is arbitrary because this term is added and subtracted. Interestingly, this is the same integral for both actions. The result of this integral is (with $\tilde{v}=\sqrt{\Sigma_j\tilde{v}_j^2}$)

$$\frac{g^2C_F}{16\pi^2}\frac{1}{v_0^2}\left[\frac{2}{1-\tilde{v}^2}\ln\left(\frac{\sqrt{R^2+\lambda^2a^2}+R}{\sqrt{R^2+\lambda^2a^2}-R}\right)-\frac{2}{\tilde{v}(1-\tilde{v}^2)}\ln\left(\frac{\sqrt{R^2+\lambda^2a^2}+\tilde{v}R}{\sqrt{R^2+\lambda^2a^2}-\tilde{v}R}\right)\right]$$

$$\underset{\lambda\ll R}{\rightarrow}\frac{g^2C_F}{16\pi^2}\left[2\ln\left(\frac{4R^2}{\lambda^2a^2}\right)-\frac{2}{\tilde{v}}\ln\left(\frac{1+\tilde{v}}{1-\tilde{v}}\right)\right]. \qquad (44)$$

The divergent piece is $-2\ln(\lambda^2a^2)$. This is the wave function contribution to the divergence in the renormalization of the Isgur-Wise function. We can, for convenience, set $R=\frac{1}{2}$.

As with the mass renormalization, the results for the wavefunction renormalization are listed in Table IV, but discussed further in the next subsection. Note that, as with $x(\tilde{v})$ for the mass renormalization, the wave function renormalization is referred to by $e(\tilde{v})$ and defined by

$$\delta Z = \frac{g^2 C_F}{16\pi^2}[-2\ln(\lambda^2 a^2) + e(\tilde{v})]. \tag{45}$$

D. Renormalization of the reduced wave function

The perturbative factors for various heavy quark effective field theories depend subtly [12,22,25,32] on whether the forward or backward time derivative is used in the action. It is expected that in the Euclidean formulation, the propagator as a function of time and the residual three-momentum (i.e., Fourier transforming k_4 into t) will have the dependence $e^{-\varepsilon t} = e^{-mt}$. However, it turns out (Appendix B) to have the dependence $Ae^{-m(t-\sigma)}$ where $\sigma = \pm 1$ distinguishes the actions. Eichten and Hill [25] noticed this relation and found that if one fits, instead, to $A'e^{-mt}$ (where $A' = Ae^{m\sigma}$) this changes the wave function renormalization by subtracting (or adding) the mass renormalization. It also ''reduces'' the wave function renormalization to a common answer for both the forward- and backward-difference actions. Since it is convenient to fit to e^{-mt} and the reduced value is the same for both actions, this is a popular choice. Unfortunately, in lattice HQET away from the static limit the reduced values (for the forward- and the backward-difference cases) are not equal, as we will show.

Equation (42) defines $x(\tilde{v})$ in terms of the mass renormalization. Equation (45) defines $e(\tilde{v})$ in terms of the wave function renormalization. Appendix B derives Eq. (B9),

$$e'(\tilde{v}) \equiv e(\tilde{v}) - \sigma x(\tilde{v})/v_0^{\text{ren}}, \tag{46}$$

which is the relation for the reduced value of the wave function renormalization. The tadpole term is in $x(\tilde{v})$ [it gets canceled in the mass renormalization of Eq. (42)] and, as noticed for the static case in [27,28], the wave function and reduced wavefunction renormalizations remain unaffected by tadpole improvement. Table IV lists our values for these functions for a large velocity range. Notice that in the static limit, the reduced value for the two actions is the same. This is the expected result. Also notice that our forward-difference value for $e(0.0)$ agrees with Davies and Thacker [22] (in their notation $C = Z + aA = -15.4$). Our backward-difference table agrees with Aglietti [12]; and the static limit of the backward difference action, $e(0.0) = 24.44$, is also in agreement with Eichten and Hill [25]. While it is still convenient to use the reduced result and fit to e^{-mt} in the static limit, the forward and backward difference will have different reduced wave function renormalizations away from the static limit.

E. Vertex correction

The vertex correction also has differences between the actions and a divergence which must be subtracted as was done for the wave function renormalization. However, this has the further complication that it depends on the velocities of both the incoming and the outgoing quarks. So whereas the wave function renormalization is a function of $\tilde{v}$, the vertex correction, $\delta V(\tilde{v},\tilde{v}')$, is a function of the initial and final velocities.

After analytically doing the contour integration over the k_4 variable and dealing with the poles as discussed previously, we find

$$\delta V(\tilde{v},\tilde{v}') = \frac{g^2 C_F}{16\pi^2}\frac{2}{\pi}\int \frac{d^3k}{\sqrt{\Delta_3^2 + 4\Delta_3}}\left\{ -\frac{v_0 v_0'}{u_0} z_\sigma(k) + \sum_j \frac{v_j v_j'}{u_0^2}\cos^2\left(\frac{k}{2}\right)\right\} \Big/ \left\{ \left[v_0 \sigma\left(\frac{1}{u_0} z_\sigma(k) - 1\right) + \sum_j \frac{v_j}{u_0}\sin(k_j)\right]\left[v_0' \sigma\left(\frac{1}{u_0} z_\sigma(k) - 1\right) + \sum_j \frac{v_j'}{u_0}\sin(k_j)\right]\right\}. \tag{47}$$

The u_0 are once again perturbatively expanded such that at this order, they can be replaced with unity. (They are included here as a reminder that in the next order they will have an effect.) For the rest of this section, we will explicitly set $u_0 = 1$.

As claimed by Aglietti [12], this must have the form

$$\frac{g^2 C_F}{16\pi^2}\frac{2}{\pi}[2(v\cdot v')r(v\cdot v')\ln(\lambda^2 a^2) + d(\tilde{v},\tilde{v}')]. \tag{48}$$

The lattice coefficient must have this form if it is to cancel the continuum divergence $\ln(\mu/\lambda)$ (which was computed by Falk *et al.* [19]). Of primary interest is that it be a function of $v\cdot v'$, the only nontrivial invariant constructible from the heavy quark velocities v and v'. We find that numerically the lattice divergence coefficient agrees with the continuum divergence coefficient—these are listed in Table V. Aglietti only gives results for the $\vec{v} = v_z\hat{z}$ with $\tilde{v}' = 0$ case. We have found a problem with an equation which he uses and have introduced a better expression for finding the divergence. We have also extended the calculation to the forward-difference action and to nonzero $\tilde{v}'$.

1. Vertex correction with $\tilde{v}'\to 0$

In the $\tilde{v}'\to 0$ limit, we can reproduce Aglietti's results; the numerical values are listed in the next subsection. Originally, we could not satisfactorily reproduce Aglietti's numbers, especially for the $\tilde{v}=0.0$ result. As we investigated this, we found a problem with the $\lambda\to 0$ limit, specifically there was a subtlety with the interchange of limits ($\lambda\to 0$ versus $\tilde{v}\to 0$). We believe that Aglietti's choice of integral subtrac-tion can be improved. This subsection discusses this subtlety. Tables VI, VII, and VIII were produced with our choice.

Since the δV integral is divergent, a technique similar to that used for δZ can be used; however, it needs to be modified because the ''continuum-like'' limit of this integral, δV^c, is not analytically manageable. However, a second integral, δV^{cc}, can be taken such that $\delta V-\delta V^c$ and $\delta V^c-\delta V^{cc}$ are each finite. These numeric integrals are then done separately and added together along with the finite piece of δV^{cc}.

Aglietti refers to δV^c as I, and our δV^{cc} is analogous to his L. Aglietti uses the notation $\delta(\tilde{v})=L-I$; we will use the analogous definition $\delta'(\tilde{v})=\delta V^{cc}-\delta V^c$. To be explicit, in the small-k limit Eq. (47) reduces to

$$\delta V^c(\tilde{v},\tilde{v}')=\frac{g^2C_F}{16\pi^2}\frac{2}{\pi}\int\frac{d^3k}{2\sqrt{k^2+\lambda^2a^2}}\frac{-1+\sum_j\tilde{v}_j\tilde{v}'_j}{\left[\sqrt{k^2+\lambda^2a^2}+\sum_j\tilde{v}_jk_j\right]\left[\sqrt{k^2+\lambda^2a^2}+\sum_j\tilde{v}'_jk_j\right]},\tag{49}$$

$$\delta V^c(\tilde{v},0)=\frac{g^2C_F}{16\pi^2}\frac{2}{\pi}\int\frac{d^3k}{2(k^2+\lambda^2a^2)}\frac{-1}{\left[\sqrt{k^2+\lambda^2a^2}+\sum_j\tilde{v}_jk_j\right]}.\tag{50}$$

Aglietti approximates $\delta V^c(\tilde{v},\tilde{v}'=0)$ with

$$L=\frac{g^2C_F}{16\pi^2}\frac{2}{\pi}\int\frac{d^3k}{2(k^2+\lambda^2a^2)}\frac{-1}{[k+\Sigma_j\tilde{v}_jk_j]}=\frac{g^2C_F}{16\pi^2}\frac{-2}{\tilde{v}}\ln\left(\frac{1+\tilde{v}}{1-\tilde{v}}\right)\ln\left(\frac{4R^2}{\lambda^2a^2}\right).\tag{51}$$

However, this gives a λ-dependent $\delta(\tilde{v})$. In spherical coordinates, $\delta(\tilde{v})$ has the form

$$\delta(\tilde{v})=\frac{g^2C_F}{16\pi^2}\frac{2}{\tilde{v}}\int\frac{k\,dk}{k^2+\lambda^2a^2}\left[\ln\left(\frac{\sqrt{k^2+\lambda^2a^2}+\tilde{v}k}{\sqrt{k^2+\lambda^2a^2}-\tilde{v}k}\right)-\ln\left(\frac{1+\tilde{v}}{1-\tilde{v}}\right)\right].\tag{52}$$

So long as $\tilde{v}$ is finite, we can take the limit as $\lambda\to 0$. However, if we want both $\tilde{v}\to 0$ and $\lambda\to 0$, a problem arises: the result in the limit $\lambda\to 0$ is not the result at $\lambda=0$. This is a case in which the limits cannot be interchanged. To be rigorous, we break up the integration into a region for $k<\lambda$ and for $k>\lambda$:

$$\delta(\tilde{v}\to 0,\lambda\to 0)=\frac{g^2C_F}{16\pi^2}4\left[\lim_{\epsilon\to 0}\int_\epsilon^\lambda dk\left(\frac{k^2}{(k^2+\lambda^2a^2)^{3/2}}-\frac{k}{(k^2+\lambda^2a^2)}\right)+\int_\lambda^R dk\left(\frac{k^2}{(k^2+\lambda^2a^2)^{3/2}}-\frac{k}{(k^2+\lambda^2a^2)}\right)\right]\tag{53}$$

$$=\frac{g^2C_F}{16\pi^2}4\lim_{\lambda\to 0}\left[-1+\frac{1}{2}\ln\left(4\frac{R^2}{\lambda^2a^2}\right)-\frac{1}{4}\ln\left(\frac{R^2}{\lambda^2a^2}\right)\right]+\lim_{\lambda\to 0}O(\lambda^2a^2/R^2)+\lim_{\lambda\gg\epsilon\to 0}O(\epsilon^3/\lambda^3)\tag{54}$$

$$=\infty.\tag{55}$$

While a λ divergence was expected for δV, the difference $\delta(\tilde{v})$ must be finite. $\delta(\tilde{v}\to 0,\lambda\to 0)$ is infinite because the logarithms do not cancel exactly. It happens that $\delta(\lambda)$ has a minimum around $\lambda\approx 10^{-5}$; at this value, if $\tilde{v}$ is taken to zero, then Aglietti's $d(\tilde{v}=0,\tilde{v}'=0)=-4.53$ can be calculated from $\lim_{\tilde{v}\to 0}[\delta V(\tilde{v},0)-\delta V^c(\tilde{v},0)+\text{finite part of}V^{cc}(\tilde{v},0)]=-5.75$ and $\delta(\tilde{v}\to 0,\ \lambda\approx 10^{-5})=-1.22$. However, Eq. (54) clearlyshows that $\delta(\lambda\to 0)$ blows up. This can be seen for $\lambda<10^{-5}$.

To avoid this problem, we write Eq. (50) as

TABLE V. The coefficient of the lattice divergent $\ln(\lambda a)$ piece must and does reproduce the continuum divergent coefficient $[4(v\cdot v')r(v\cdot v')]$ in order to correctly cancel the $\ln\lambda$. Errors are at most three in the last digit shown. In addition, $v_x=v_y=v'_x=v'_y=0$.

$\tilde v$ \ $\tilde v'$	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70
0.00	4.001	4.014	4.050	4.127	4.237	4.396	4.618	4.953
0.10	4.012	4.004	4.015	4.059	4.139	4.269	4.462	4.753
0.20	4.053	4.011	4.002	4.018	4.064	4.160	4.319	4.573
0.30	4.126	4.058	4.014	4.000	4.016	4.079	4.194	4.412
0.40	4.238	4.137	4.066	4.016	4.000	4.021	4.095	4.257
0.50	4.398	4.267	4.159	4.077	4.022	4.000	4.026	4.133
0.60	4.622	4.460	4.316	4.197	4.095	4.028	3.998	4.040
0.70	4.956	4.753	4.571	4.412	4.257	4.135	4.041	4.000

$$\delta V^{c}(\tilde v,\tilde v'=0)=\frac{g^2C_F}{16\pi^2}\frac{2}{\pi}\int\frac{d^3k}{2(k^2+\lambda^2a^2)^{3/2}}\times\frac{-1}{[1+\vec{\tilde v}\cdot\vec k/\sqrt{k^2+\lambda^2a^2}]},\tag{56}$$

which expands as follows and allows a better definition of $\delta V^{cc}(\tilde v,\tilde v'=0)$:

$$\delta V^{c}(\tilde v,\tilde v'=0)\approx\frac{g^2C_F}{16\pi^2}\frac{2}{\pi}\int\frac{d^3k}{2(k^2+\lambda^2a^2)^{3/2}}\times\frac{-1}{\{1+\vec{\tilde v}\cdot\hat k[1-\frac{1}{2}(\lambda^2a^2/k^2)]\}}\tag{57}$$

$$\delta V^{cc}(\tilde v,\tilde v'=0)=\frac{g^2C_F}{16\pi^2}\frac{2}{\pi}\int\frac{d^3k}{2(k^2+\lambda^2a^2)^{3/2}}\frac{-1}{[1+\vec{\tilde v}\cdot\hat k]}.\tag{58}$$

We find that this makes $\delta'(\tilde v)$ stable to small λ and that it generalizes to give useful results when $\tilde v'\neq 0$. In the equation analogous to Eq. (54), the logarithms cancel and the result is finite in the $\lambda\to 0$ limit.

2. Vertex correction with $\tilde v'\neq 0$

This case requires the continuum-like expression for Eq. (47). Recall Eq. (49). Again there are problems if we use Aglietti's trick of setting λ to zero in the factors with $\tilde v$ and $\tilde v'$. The problems are (1) the λ dependence is incorrect (which implies that the difference is λ-dependent), (2) the limit as $\tilde v'\to 0$ does not reproduce the results of the previous section, and (3) the integral is too difficult. So, once again, we will try to retain the λ dependence as follows: we approximate

$$\delta V^{c}(\tilde v,\tilde v')=\frac{g^2C_F}{16\pi^2}\frac{2}{\pi}\int\frac{d^3k}{2(k^2+\lambda^2a^2)^{3/2}}\times\frac{-1+\sum_j\tilde v_j\tilde v'_j}{[1+\vec{\tilde v}\cdot\vec k/\sqrt{k^2+\lambda^2a^2}][1+\vec{\tilde v}'\cdot\vec k/\sqrt{k^2+\lambda^2a^2}]}$$

by

$$\delta V^{cc}(\tilde v,\tilde v')=\frac{g^2C_F}{16\pi^2}\frac{2}{\pi}\int\frac{d^3k}{2(k^2+\lambda^2a^2)^{3/2}}\times\frac{-1+\sum_j\tilde v_j\tilde v'_j}{[1+\vec{\tilde v}\cdot\hat k][1+\vec{\tilde v}'\cdot\hat k]}.$$

While this does solve both the λ-dependence problem and the $\tilde v'\to 0$ problem, it only barely solves the difficulty of the integral. However, in spherical coordinates, it allows the $|k|$ integral to be separated from the angular integration. We can solve this integral by doing the $|k|$ integration analytically. (Since this is where the λ divergence exists, it is the only piece that needs to be done analytically anyway.) Having thus removed the $\ln(\lambda)$ term, we can numerically calculate and add back the angular integration along with the finite piece of the $|k|$ integration.

TABLE VI. The finite piece of the backward-difference vertex correction $d(\tilde v,\tilde v')$ for $v_x=v_y=v'_x=v'_y=0$.

$\tilde v$ \ $\tilde v'$	0.0	0.1	0.2	0.3	0.4	0.5	0.6
0.0	−4.528(2)	−4.579(2)	−4.756(2)	−5.088(3)	−5.642(3)	−6.592(4)	−8.454(5)
0.1	−4.583(2)	−4.459(2)	−4.446(2)	−4.567(3)	−4.892(3)	−5.562(3)	−7.018(4)
0.2	−4.755(2)	−4.447(2)	−4.228(3)	−4.131(3)	−4.186(3)	−4.523(3)	−5.525(4)
0.3	−5.088(3)	−4.570(3)	−4.126(3)	−3.768(3)	−3.514(3)	−3.457(4)	−3.890(4)
0.4	−5.640(3)	−4.890(3)	−4.183(3)	−3.517(3)	−2.882(4)	−2.320(4)	−2.007(5)
0.5	−6.598(4)	−5.556(3)	−4.523(4)	−3.460(4)	−2.320(4)	−1.071(5)	0.283(6)
0.6	−8.452(4)	−7.022(4)	−5.529(5)	−3.884(4)	−2.003(5)	0.280(6)	3.322(8)

TABLE VII. The finite piece of the forward difference vertex correction $d(\tilde{v},\tilde{v}')$ for $v_x=v_y=v'_x=v'_y=0$.

$\tilde{v}$ \ $\tilde{v}'$	0.0	0.1	0.2	0.3	0.4	0.5	0.6
0.0	−4.528(2)	−4.513(2)	−4.471(2)	−4.402(3)	−4.280(3)	−4.102(3)	−3.822(4)
0.1	−4.516(2)	−4.527(2)	−4.509(2)	−4.462(3)	−4.370(3)	−4.221(3)	−3.971(4)
0.2	−4.471(2)	−4.508(2)	−4.513(3)	−4.487(3)	−4.437(3)	−4.315(3)	−4.108(4)
0.3	−4.402(3)	−4.457(3)	−4.494(3)	−4.505(3)	−4.472(3)	−4.396(3)	−4.224(4)
0.4	−4.283(3)	−4.371(3)	−4.431(3)	−4.474(3)	−4.481(3)	−4.446(4)	−4.325(4)
0.5	−4.108(4)	−4.219(3)	−4.316(3)	−4.398(4)	−4.443(4)	−4.459(4)	−4.402(4)
0.6	−3.822(4)	−3.973(4)	−4.112(4)	−4.226(4)	−4.320(4)	−4.398(4)	−4.424(4)

Since this is symmetric in $\tilde{v}$ and $\tilde{v}'$, the results for the vertex correction should be also. Our extension to nonzero $\tilde{v}'$ shows that this is the case: Tables VI and VII show our results for backward- and forward-difference vertex correction at general velocities. Notice that the results are symmetric about the diagonal, $\tilde{v}=\tilde{v}'$. Notice also that the first row and the first column of Table VI both reproduce the backward-difference results of $\tilde{v}'=0$ in Table VIII. Table VII shows the results for the forward-difference vertex correction at general velocities and the first row and column reproduce the forward-difference results of Table VIII.

IV. LATTICE TO CONTINUUM MATCHING

To renormalize the Isgur-Wise function, which is proportional to the current in Eq. (2), we need the current renormalization, which can be assembled from the wave function and vertex renormalizations calculated in the previous section. This involves

$$Z_Q^{1/2}(v)Z_V(v,v')Z_Q^{1/2}(v')=\left[1+\frac{1}{2}\delta Z_Q(v)\right][1+\delta V(v,v')]\times\left[1+\frac{1}{2}\delta Z_Q(v')\right]=\left\{1+\frac{1}{2}[\delta Z_Q(v)+\delta Z_Q(v')]+\delta V(v,v')\right\}. \tag{59}$$

So, following Aglietti's lead [12], we define

$$f(\tilde{v},\tilde{v}')=\tfrac{1}{2}[e(\tilde{v})+e(\tilde{v}')]+d(\tilde{v},\tilde{v}'),$$

$$f'(\tilde{v},\tilde{v}')=\tfrac{1}{2}[e'(\tilde{v})+e'(\tilde{v}')]+d(\tilde{v},\tilde{v}')$$

where a reduced Isgur-Wise correction, f', is defined using the reduced wavefunction, e', which was used with a fit model of the form e^{-mt}. Since the wavefunction reduction-does not affect the vertex correction, d, the perturbative factor for the lattice Isgur-Wise function is:

TABLE VIII. The finite piece of the vertex correction, $d(\tilde{v},\tilde{v}')$, of the forward- and backward-difference actions for $v_x=v_y=0$ and $\vec{v}'=\vec{0}$. The backward-difference action results should and do reproduce Aglietti's table up to the correction mentioned in the text of this paper. Also listed is the current correction $f(\tilde{v},\tilde{v}')=\frac{1}{2}e(\tilde{v})+\frac{1}{2}e(\tilde{v}')+d(\tilde{v},\tilde{v}')$ and the reduced current correction $f'(\tilde{v},\tilde{v}')=\frac{1}{2}e'(\tilde{v})+\frac{1}{2}e'(\tilde{v}')+d(\tilde{v},\tilde{v}')$ which form the correction for the lattice Isgur-Wise function.

	Backward difference			Forward difference		
$\tilde{v}$	$d(\tilde{v},\tilde{v}'=0)$	$f(\tilde{v},\tilde{v}'=0)$	$f'(\tilde{v},\tilde{v}'=0)$	$d(\tilde{v},\tilde{v}'=0)$	$f(\tilde{v},\tilde{v}'=0)$	$f'(\tilde{v},\tilde{v}'=0)$
0.0	−4.526(2)	19.92(1)	0.000(2)	−4.527(2)	−19.94(1)	0.000(2)
0.1	−4.578(2)	19.96(1)	0.122(2)	−4.511(2)	−20.09(1)	−0.174(2)
0.2	−4.757(2)	20.08(1)	0.489(2)	−4.474(2)	−20.58(1)	−0.740(2)
0.3	−5.089(2)	20.33(2)	1.129(2)	−4.401(2)	−21.50(1)	−1.755(2)
0.4	−5.639(4)	20.63(2)	2.100(4)	−4.282(2)	−22.93(1)	−3.364(2)
0.5	−6.597(4)	21.09(2)	3.459(4)	−4.104(4)	−25.21(1)	−5.844(2)
0.6	−8.432(6)	21.50(2)	5.23(1)	−3.800(4)	−28.89(1)	−9.755(4)
0.7	−14.80(1)	19.53(4)	5.57(2)	−3.354(6)	−35.32(2)	−16.529(8)

$$Z_\xi^{\mathrm{l}}=1+\frac{g^2}{12\pi^2}\{-2[1-(v\cdot v')r(v\cdot v')]\ln(\lambda a)^2+f'(\tilde{v},\tilde{v}')\} \tag{60}$$

where the divergences have been isolated to calculate the finite pieces and $r(v\cdot v')$ has been defined by Eq. (8). If we did not wish to use the reduced value, the divergence would stay the same and we would merely replace the f' with f. We have already shown (Table V) that the divergent piece of the lattice vertex correction cancels exactly with that of the continuum; thus the lattice logarithm coefficient is written with the same form as for the continuum correction, Eq. (7). Now the continuum correction, Z_ξ^{c}, can be divided by the lattice correction, Z_ξ^{l}, to find the lattice to continuum matching factor

$$Z_\xi^{\mathrm{cl}}(v,v')=1+\frac{g^2}{12\pi^2}\{2[1-(v\cdot v')r(v\cdot v')]\times\ln(\mu a)^2-f'(\tilde{v},\tilde{v}')\}. \tag{61}$$

The expression in Eq. (61) is suitable for renormalizing the Isgur-Wise function extracted by taking ratios of two- and three-point functions [33]. However, to improve statistics, HQET simulations extract the Isgur-Wise function using ratios of three-point functions only [4,14,34]. We discuss this additional complication below.

Our results for d, f, and f' are listed in the following tables. Recall Tables VI and VII show our results for backward- and forward-difference vertex correction at general velocities. Table VIII lists our results for the vertex correction, the current correction, and the reduced current correction in the backward- and forward-difference actions for $\tilde{v}'=0$. The backward difference reproduces Aglietti's results. Tables IX and X show our results for backward-difference current and reduced current corrections at general velocities. Again, the results are symmetric about the diagonal. Tables XI and XII for the forward-difference current and reduced current corrections at general velocities are also symmetric about the diagonal. Notice that the first rows and columns of Tables VI, VII, IX–XII reproduce Table VIII. Notice also that although the different actions give the same result in the static limit ($v\to 0$, $v'\to 0$), this is not the case at any other velocity.

For continuum HQET in the $\overline{\mathrm{MS}}$ renormalization scheme at zero recoil ($v\cdot v'=1$), $Z_\xi^{\mathrm{c}}=1$ and the finite piece is zero [15]. This corresponds to the diagonal ($v=v'$) of the tables which contain our results. On the lattice, however, if the conserved current is not used, $f'(v,v)$ is not constrained to be zero. We account for this next.

To deal with the finite piece of the renormalization, we note that the numeric extraction of the Isgur-Wise function on the lattice does not calculate the Isgur-Wise function directly. The numerical extraction is more manageable using the technique of Mandula and Ogilvie [34] where the ratio of the three-point quark propagator, G (defined explicitly in our concurrent numerical paper [5]), gives a ratio of lattice Isgur-Wise functions

$$\frac{4v_0v_0'}{(v_0+v_0')^2}\frac{G^{v,v'}(\tau)G^{v',v}(\tau)}{G^{v,v}(\tau)G^{v',v'}(\tau)}\underset{\tau\gg 1}{\longrightarrow}\frac{|\xi(v,v')||\xi(v',v)|}{|\xi(v,v)||\xi(v',v')|}. \tag{62}$$

This technique exploits the continuum normalization of the Isgur-Wise function at zero recoil

$$\xi(v\cdot v)=\xi(1)=1. \tag{63}$$

Since v_μ^2 is normalized to 1, the denominator of Eq. (62) can be set to unity in the continuum. This ratio also allows the normalizations and smearing-function dependence to cancel, so we expect that

$$\frac{|Z_\xi^{\mathrm{cl}}(v,v')\xi^{\mathrm{latt}}(v,v')||Z_\xi^{\mathrm{cl}}(v',v)\xi^{\mathrm{latt}}(v',v)|}{|Z_\xi^{\mathrm{cl}}(v,v)\xi^{\mathrm{latt}}(v,v)||Z_\xi^{\mathrm{cl}}(v',v')\xi^{\mathrm{latt}}(v',v')|}\underset{a\to 0}{\longrightarrow}\frac{|\xi^{\mathrm{cont}}(v\cdot v')|^2}{|\xi^{\mathrm{cont}}(1)|^2}=|\xi^{\mathrm{cont}}(v\cdot v')|^2.$$

Thus, our unrenormalized calculation of

$$\xi_{\mathrm{ratio}}(v,v')\equiv\left(\frac{\xi^{\mathrm{latt}}(v,v')\xi^{\mathrm{latt}}(v',v)}{\xi^{\mathrm{latt}}(v,v)\xi^{\mathrm{latt}}(v',v')}\right)^{1/2}$$

must be renormalized by

$$Z_{\mathrm{ratio}}^{\mathrm{cl}}(v,v')=\left(\frac{Z_\xi^{\mathrm{cl}}(v,v')Z_\xi^{\mathrm{cl}}(v',v)}{Z_\xi^{\mathrm{cl}}(v,v)Z_\xi^{\mathrm{cl}}(v',v')}\right)^{1/2} \tag{64}$$

written as

$$Z_{\mathrm{ratio}}^{\mathrm{cl}}(v,v')\xi_{\mathrm{ratio}}(v,v')\underset{a\to 0}{\longrightarrow}\xi^{\mathrm{cont}}(v\cdot v'). \tag{65}$$

On the lattice, $\xi^{\mathrm{latt}}(v,v)$ does not obey Eq. (63) unless a conserved current is used; nevertheless, $\xi_{\mathrm{ratio}}(v,v')$ (by definition) acts like the continuum Isgur-Wise function even if the conserved current is not used. Without the conserved current, $\xi^{\mathrm{latt}}(v,v)\neq 1$, but the normalization cancels in the ratio so that $\xi_{\mathrm{ratio}}(v,v)=1$. Thus, $Z_{\mathrm{ratio}}^{\mathrm{cl}}$ will be symmetric in v and v' and will have the property $Z_{\mathrm{ratio}}^{\mathrm{cl}}(v,v)=1$.

Expanding Eq. (64), we find

$$\begin{aligned}Z_{\mathrm{ratio}}^{\mathrm{cl}}=1+\frac{1}{2}\frac{g^2C_F}{16\pi^2}\{2[(1-(v\cdot v')r(v\cdot v'))\\+(1-(v'\cdot v)r(v'\cdot v))-(1-(v\cdot v)r(v\cdot v))\\-(1-(v'\cdot v')r(v'\cdot v'))]\ln(\mu a)^2-f'(\tilde{v},\tilde{v}')\\-f'(\tilde{v}',\tilde{v})+f'(\tilde{v},\tilde{v})+f'(\tilde{v}',\tilde{v}')\}.\end{aligned} \tag{66}$$

Using $v\cdot v=v'\cdot v'=r(1)=1$ and $f'(\tilde{v},\tilde{v}')=f'(\tilde{v}',\tilde{v})$, this reduces to

TABLE IX. The finite piece of the backward-difference current correction, $f(\tilde{v},\tilde{v}')$, for $v_x=v_y=0$ and $v'_x=v'_y=0$.

$\tilde{v}$ \ $\tilde{v}'$	0.0	0.1	0.2	0.3	0.4	0.5	0.6
0.0	19.903(8)	19.958(8)	20.100(9)	20.324(10)	20.654(9)	21.093(9)	21.51(1)
0.1	19.961(8)	20.195(10)	20.485(8)	20.921(10)	21.516(10)	22.245(10)	23.03(1)
0.2	20.087(8)	20.498(9)	21.022(9)	21.662(9)	22.522(9)	23.565(10)	24.84(1)
0.3	20.320(9)	20.947(9)	21.681(9)	22.57(1)	23.73(1)	25.20(1)	27.02(1)
0.4	20.646(10)	21.494(9)	22.495(9)	23.74(1)	25.25(1)	27.24(1)	29.78(1)
0.5	21.090(9)	22.228(9)	23.579(10)	25.175(10)	27.20(1)	29.86(1)	33.48(1)
0.6	21.499(10)	23.017(10)	24.84(1)	27.02(1)	29.75(1)	33.47(1)	38.75(1)

TABLE X. The finite piece of the backward-difference reduced current correction, $f'(\tilde{v},\tilde{v}')$, for $v_x=v_y=0$ and $v'_x=v'_y=0$.

$\tilde{v}$ \ $\tilde{v}'$	0.0	0.1	0.2	0.3	0.4	0.5	0.6
0.0	0.004(3)	0.118(2)	0.491(2)	1.134(3)	2.098(3)	3.458(4)	5.228(6)
0.1	0.119(2)	0.422(3)	0.975(2)	1.821(3)	3.021(3)	4.675(4)	6.822(6)
0.2	0.488(2)	0.974(2)	1.733(3)	2.810(3)	4.273(4)	6.258(5)	8.858(6)
0.3	1.128(3)	1.822(3)	2.812(3)	4.142(4)	5.922(4)	8.292(5)	11.484(7)
0.4	2.096(3)	3.026(4)	4.274(4)	5.922(4)	8.073(5)	10.954(6)	14.875(7)
0.5	3.459(4)	4.672(4)	6.250(4)	8.291(5)	10.957(6)	14.504(7)	19.477(9)
0.6	5.219(6)	6.839(6)	8.856(6)	11.477(7)	14.893(7)	19.470(9)	26.10(1)

TABLE XI. The negative of the finite piece of the forward-difference current correction, $-f(\tilde{v},\tilde{v}')$, for $v_x=v_y=0$ and $v'_x=v'_y=0$.

$\tilde{v}$ \ $\tilde{v}'$	0.0	0.1	0.2	0.3	0.4	0.5	0.6
0.0	19.923(6)	20.086(6)	20.591(7)	21.500(7)	22.953(7)	25.223(8)	28.897(9)
0.1	20.095(7)	20.282(7)	20.786(7)	21.714(7)	23.206(7)	25.513(8)	29.217(8)
0.2	20.591(7)	20.784(7)	21.342(7)	22.297(7)	23.810(7)	26.136(8)	29.882(10)
0.3	21.491(7)	21.742(7)	22.298(7)	23.268(8)	24.819(8)	27.188(9)	30.987(10)
0.4	22.941(8)	23.192(8)	23.794(7)	24.824(8)	26.394(9)	28.829(9)	32.66(1)
0.5	25.220(8)	25.498(9)	26.138(8)	27.177(8)	28.804(9)	31.283(10)	35.15(1)
0.6	28.885(9)	29.22(1)	29.897(9)	30.993(10)	32.64(1)	35.16(1)	39.16(1)
0.7	35.32(1)	35.68(1)	36.41(1)	37.59(1)	39.30(1)	41.90(1)	46.01(2)

TABLE XII. The negative of the finite piece of the forward-difference reduced current correction, $-f'(\tilde{v},\tilde{v}')$, for $v_x=v_y=0$ and $v'_x=v'_y=0$.

$\tilde{v}$ \ $\tilde{v}'$	0.0	0.1	0.2	0.3	0.4	0.5	0.6
0.0	−0.004(3)	0.178(2)	0.737(2)	1.755(2)	3.365(3)	5.843(3)	9.773(4)
0.1	0.181(2)	0.380(2)	0.968(2)	2.008(2)	3.645(3)	6.154(3)	10.113(4)
0.2	0.740(2)	0.967(2)	1.583(2)	2.649(3)	4.311(3)	6.849(3)	10.851(4)
0.3	1.759(2)	2.009(3)	2.652(3)	3.742(3)	5.441(3)	8.020(4)	12.061(5)
0.4	3.366(3)	3.642(3)	4.312(3)	5.442(3)	7.173(3)	9.796(4)	13.881(5)
0.5	5.841(3)	6.154(3)	6.858(3)	8.022(4)	9.796(4)	12.450(5)	16.618(6)
0.6	9.772(4)	10.118(4)	10.855(4)	12.059(5)	13.886(5)	16.610(6)	20.845(8)
0.7	16.525(6)	16.911(6)	17.687(6)	18.964(7)	20.839(8)	23.675(9)	27.996(10)

$$Z^{\mathrm{cl}}_{\mathrm{ratio}}(v,v')=1+\frac{g^2}{12\pi^2}\bigg[2(1-(v\cdot v')r(v\cdot v'))\ln(\mu a)^2 - f'(\tilde{v},\tilde{v}')+\frac{f'(\tilde{v},\tilde{v})+f'(\tilde{v}',\tilde{v}')}{2}\bigg] \quad (67)$$

which not only has the correct divergent coefficient but we also see a new finite piece which is manifestly zero on the diagonal. The wave function renormalization cancels explicitly in Eq. (67), so f' can be replaced by the vertex correction d.

V. CONCLUSIONS

We have calculated the renormalization of the lattice $b\to c$ current by considering the lattice Isgur-Wise function. This calculation extends previous work by including tadpole improvement, by extending to nonzero initial and final velocities, and by considering forward as well as backward-difference actions.

By considering the forward-difference action and the backward-difference action side-by-side, we find nontrivial differences between the two. The practical difference in a lattice calculation is that the backward difference requires a matrix inversion at each step of the calculation. The differences in the renormalization are that the gluon poles over which one integrates are interchanged; away from the static limit, the reduced values are no longer equal; and the velocity renormalization, when expanded as powers of the velocity, stays small for the forward difference, but grows large for the backward difference.

Of greater concern is that the velocity renormalization is not terribly small. We have shown that the velocity renormalization can be expanded in small velocity and that the coefficients remain on the order of unity at higher orders (at least for the forward-difference action). These coefficients are given here to $O(v^6)$. The nonperturbative calculations are giving smaller renormalizations [13,14] and these should be, in principle, more reliable. This should be considered in more detail, especially the slow HQET for the forward-difference action.

Although our results confirm other groups' calculations where they overlap, the integrals and divergences are subtle and must be managed with care. When we combine our renormalizations into a current correction with the ratio introduced by Mandula and Ogilvie [34], such that the finite piece of the current correction is $-f'(\tilde{v},\tilde{v}')+\frac{1}{2}[f'(\tilde{v},\tilde{v})+f'(\tilde{v}',\tilde{v}')]$, we find that all of our results have the appropriate limits and cancelations. These expressions are used in our concurrent numerical paper [5] to compute the slope of the Isgur-Wise function using lattice HQET.

APPENDIX A: TADPOLE IMPROVEMENT

Tadpole improvement is a mean field improvement [21] which (at lowest order) cancels the effects of the large ''tadpole'' Feynman diagrams. In the HQET, there is no coefficient (analogous to κ in the Wilson action) which is common to both U_t and U_j and which allows one to *a posteriori* tadpole improve any previous calculation which was not tadpole improved. Fortunately, as noticed by Mandula and Ogilvie [13], the evolution equation can be written such that the u_0 is grouped with $\tilde{v}_j=v_j/v_0$. Thus, tadpole-improved (tad) Monte-Carlo data can be *constructed* from the non-tadpole-improved (not) data by replacing $v^{\mathrm{not}}\to v^{\mathrm{tad}}$ and by including two overall multiplicative factors ($v_0^{\mathrm{not}}/v_0^{\mathrm{tad}}$ was not included by Mandula and Ogilvie):

$$G^{\mathrm{tad}}(t;\tilde{v}^{\mathrm{tad}},v_0^{\mathrm{tad}})=u_0^{-t}\frac{v_0^{\mathrm{not}}}{v_0^{\mathrm{tad}}}G^{\mathrm{not}}(t;\tilde{v}^{\mathrm{not}},v_0^{\mathrm{not}}). \quad (A1)$$

In addition, the tadpole-improvement of a simulation requires adjusting the velocity (analogous to adjusting κ) according to $\tilde{v}^{\mathrm{tad}}=u_0\tilde{v}^{\mathrm{not}}$, subject to the normalization $(v^{\mathrm{tad}})^2=1$ and $(v^{\mathrm{not}})^2=1$. The adjustment on the velocity is then

$$\begin{aligned} v_0^{\mathrm{tad}}&=v_0^{\mathrm{not}}[1+(1-u_0^2)(v_j^{\mathrm{not}})^2]^{-1/2}\\ v_j^{\mathrm{tad}}&=u_0v_j^{\mathrm{not}}[1+(1-u_0^2)(v_j^{\mathrm{not}})^2]^{-1/2}. \end{aligned} \quad (A2)$$

The tadpole improved data is at a velocity which is shifted from the original tadpole unimproved data. Previous HQET calculations have either not included tadpole improvement [12] or have had difficulties with it [13]. Although one should start with a tadpole-improved action, we find it convenient to be able to tadpole improve a calculation *a posteriori* because there are choices for how one can determine the mean-field value u_0 [21].

APPENDIX B: REDUCED RENORMALIZATIONS

One can define a ''reduced'' wave function renormalization and relate it to the fit-model exponential. We begin by considering the propagator as a function of time t and the residual momentum $\vec{k}$,[1]

$$\begin{aligned} iH(t,\vec{k}) &= \int\frac{dk_4}{2\pi}\frac{e^{ik_4}}{\left\{v_0\sigma[(1/u_0)e^{i\sigma k_4}-1]+\sum_j (v_j/u_0)\sin(k_j)\right\}}\\ &=\Theta\left(t+\frac{1-\sigma}{2}\right)\frac{u_0^{\sigma t}}{v_0}e^{-(t-\sigma)\ln[1-\sigma\Sigma_j(\tilde{v}_j/u_0)\sin(k_j)]^{-\sigma}}. \end{aligned} \quad (B1)$$

Since $iH\sim e^{-\varepsilon t}$, the energy-momentum relation can be found:

$$\varepsilon=-\sigma\ln\left[1-\sigma\sum_j\frac{\tilde{v}_j}{u_0}\sin(k_j)\right]\approx\sum_j\frac{\tilde{v}_j}{u_0}\sin(k_j). \quad (B2)$$

[1]Recall that the residual momentum, rather than the full momentum, is conjugate to the position.

Aglietti [12] notes that the energy goes to zero for both $\vec{k}=\vec{0}$ and $\vec{k}=\vec{\pi}$, but provides a physical argument for why this doubling problem has a negligible effect in the HQET.

In Eq. (B1), it may be noticed that the Θ-function has a different argument for the different actions. Though it was phrased differently, this was also noticed by Davies and Thacker [22] who give recursive expressions for the Green function evolution equation for the two cases of a forward or a backward difference in their NRQCD action.

In order to consider the renormalization effects of the fitting form, consider the next loop-order of the propagator as a function of the time and the residual three-momentum,

$$iH^{(2)}(t,\vec{k})=\int\frac{dk_4}{2\pi}e^{ik_4}\{iH(k_4,\vec{k})+iH(k_4,\vec{k})\Sigma(k)iH(k_4,\vec{k})\}. \tag{B3}$$

Following Aglietti [12], we will make use of

$$\begin{aligned}\Sigma(k)&=\Sigma(0)+k_4X_4+\sum_j k_jX_j+O(k^2)\\&=-\delta M^{\rm tad}+[-\sigma v_0(1-\delta Z)+\delta v_0]\ln(u_0)\\&\quad+\delta Z\left[v_0\sigma\left(\frac{1}{u_0}e^{i\sigma k_4}-1\right)+\sum_j\frac{v_j}{u_0}\sin(k_j)\right]\\&\quad-\delta v_0\sigma\left(\frac{1}{u_0}e^{i\sigma k_4}-1\right)-\sum_j\delta\left(\frac{v_j}{u_0}\right)\sin(k_j),\end{aligned} \tag{B4}$$

where $\delta M^{\rm tad}$ is the tadpole improved mass renormalization (versus $\delta M^{\rm not}$ the not-tadpole improved mass renormalization) defined by

$$\delta M^{\rm tad}=-\Sigma(0,\tilde{v})-\sigma v_0\ln u_0=\delta M^{\rm NT}-\sigma v_0\ln u_0. \tag{B5}$$

It may also be noticed that since $\ln u_0\sim O(g^2)$, the $[(v_0\sigma\delta Z+\delta v_0)\ln u_0]$ can be neglected as $O(g^4)$. We further note that terms of the residual momentum, $O(\vec{k})$, can be neglected.[2] (The residual momentum can be adjusted by introducing a ''residual mass.'') Finally, we note that the δv_0 and $\delta(v_j/u_0)$ can be collected with the bare velocity in precisely the proportion necessary to renormalize each velocity. To solve these integrals, one needs to put Eq. (B3) into a form which allows the use of

$$\int_{-\infty}^{\infty}\frac{dx}{2\pi}\frac{e^{iax}}{(e^{ix}-1)}=\Theta(a), \tag{B6}$$

$$\int_{-\infty}^{\infty}\frac{dx}{2\pi}\frac{e^{iax}}{(e^{ix}-1)^2}=(a-1)\Theta(a). \tag{B7}$$

[2]The calculation including these terms is available from author J.C.

With these relationships, we find (eventually[3])

$$\begin{aligned}iH^{(2)}(t,\vec{k})&=\Theta\left(t+\frac{1-\sigma}{2}\right)\frac{u_0(1+\delta Z)}{v_0(1+\delta v_0/v_0)}\\&\quad\times\exp\{-(t-\sigma)[M_\sigma]\}\\&\quad\times[1+O(g^2)+O(\tilde{v}^2)]\end{aligned}$$

where M_σ is an action-dependent function of the renormalizations, of the velocity, and of the momentum; and $v_0(1+\delta v_0/v_0)=v_0^{\rm ren}$. The relevant point is that, as was said previously, for the forward-difference action one should fit to a form of $\exp(-M_f[t-1])$; whereas for the backward-difference action one should fit to a form of $\exp(-M_b[t+1])$ [i.e., fit to $\exp\{-M_\sigma(t-\sigma)\}$]. However, if one chooses to fit to the form $\exp(-Mt)$, then the coefficient $Z=(1+\delta Z)$ gets changed to $Ze^{\sigma M_\sigma}\approx Z(1+\sigma M_\sigma)\approx(1+\delta Z+\sigma M_\sigma)$. To $O(g^2)$, neglecting $O(k)$ terms, $M_\sigma=(\delta M^{\rm tad}+\sigma v_0^{\rm ren}\ln u_0)/v_0^{\rm ren}=-\Sigma(0)/v_0^{\rm ren}$ [recall Eq. (B5)]. So, to this order, the ''reduced'' wave function renormalization is

$$\begin{aligned}Z'&=Z-\sigma\Sigma(0)/v_0^{\rm ren}\\&=[1+\delta Z-\sigma\Sigma(0)/v_0^{\rm ren}]\\&=\left(1+\frac{g^2C_F}{16\pi^2}[-2\ln(\lambda^2a^2)+e(\tilde{v})-\sigma x(\tilde{v})/v_0^{\rm ren}]\right).\end{aligned} \tag{B8}$$

This is also written in terms of the finite pieces

$$e'(\tilde{v})\equiv e(\tilde{v})-\sigma x(\tilde{v})/v_0^{\rm ren}. \tag{B9}$$

The tadpole term is in $x(\tilde{v})$ [it gets canceled in the mass renormalization of Eq. (42)] and, as noticed for the static case in [27,28], the wave function and reduced wave function renormalizations remain unaffected by tadpole improvement.

APPENDIX C: NOTATION

When comparing between the results of HQET, NRQCD and the static theory, the difference in notation starts to become a factor. Where Davies and Thacker (NRQCD) used A for $\Sigma(0)$, Aglietti (HQET) uses $A(p)$ for the nontadpole portion of the self-energy as well as using A for a particular grouping of terms for convenience in the calculation. We are going to maintain Davies' and Thacker's use of A and give new names to Aglietti's A's. However, since Aglietti considers the velocity-dependence of various quantities, we will use Aglietti's notation for a variety of velocity-dependent functions. The velocity will be relevant for the HQET, but

[3]We found that there are two Θ terms. One goes as $\Theta[t+(1-\sigma/2)]\equiv\theta_1$, the other as $\Theta[t+(3-\sigma/2)]\equiv\theta_3$. We resolved this assuming we were interested in late enough times $(t>-1)$ that $\theta_3=\theta_1=1$.

TABLE XIII. Comparison of notation between Aglietti [12] and Mandula and Ogilvie [13]. Note also that Aglietti only considers motion in the z direction. Finally note that in the last row, Mandula and Ogilvie consider $\delta\tilde{v}_i$, but Aglietti considers δv_z ($u_i=\tilde{v}_i={}^{v_i}/_{v_0}$). To convert between the two, one must include a factor of v_0^2.

Mandula and Ogilvie	Aglietti	Comparison
$X_0=-iX_4$	X	$X\|_{Ag}=X_4\|_{MO}=iX_0\|_{MO}$
X_i	Y	$Y\|_{Ag}=X_3\|_{MO}$
$\tilde{v}_i=\frac{v_i}{v_0}$	$u_z=\frac{v_z}{v_0}$	$u_z\|_{Ag}=\tilde{v}_z\|_{MO}$
$\delta\tilde{v}_i=-\frac{1}{v_0}(X_i-\tilde{v}_iX_0)$	$\delta v_z=-iv_0v_zX-v_0^2Y$	$\frac{\delta v}{v}=v_0^2\frac{\delta\tilde{v}}{\tilde{v}}$
$=-\frac{1}{v_0}(X_i+i\tilde{v}_iX_4)$	$=-v_0^2(Y+iu_zX)$	

not for the *static* theory nor for the NRQCD. In the HQET, the functional dependence is on $\tilde{v}$ defined by

$$\tilde{v}=\sum_j \tilde{v}_j^2=\sum_j \frac{v_j^2}{v_0^2}. \tag{C1}$$

Note that Aglietti calls this u.

Aglietti calls the mass renormalization δM; he also puts in a negative sign, which we leave out. Aglietti notes that for the HQET, this is velocity dependent, and defines a function $x(u)$ which is proportional to his δM

$$\delta M|_{\rm Ag}=-g^2A|_{\rm DT}=\frac{g^2C_F}{16\pi^2}\frac{x(u)|_{\rm Ag}}{a}. \tag{C2}$$

v_0 does not appear in NRQCD and is 1 in the static limit.

In calculating the wave function renormalization, $\partial\Sigma(p)/\partial p_\mu|_{p=0}$ is needed. Mandula and Ogilvie use the notation X_μ. This is a useful notation and does not conflict with either Davies and Thacker or Eichten and Hill. Aglietti names these as $X|_{Ag}=X_0|_{MO}$ and $Y|_{Ag}=X_3|_{MO}$. See Table XIII for an explicit comparison. We choose to use Mandula's and Ogilvie's notation.

In the definition of the velocity renormalization, there is a further subtlety. Mandula and Ogilvie consider $\delta\tilde{v}_i\equiv\tilde{v}_i^{(\rm ren)}-\tilde{v}_i$, but Aglietti considers $\delta v_z\equiv v_z^{(\rm ren)}-v_z$ (with the definition $u_i=\tilde{v}_i=v_i/v_0$). As shown in Table XIII, a factor of v_0^2 must be included to translate between $\delta v/v$ and $\delta\tilde{v}/\tilde{v}$. In addition, Mandula and Ogilvie include the prefactor $g^2C_F/16\pi^2$ in their definition of $c(\tilde{v})$ in Eq. (39).

Mandula and Ogilvie do not calculate the wavefunction renormalization, therefore we will compare Aglietti's wavefunction renormalization to Davies and Thacker (while using Mandula's and Ogilvie's notation for X_μ). Aglietti uses $\delta Z=Z-1$ for the wave function renormalization. To relate this to Davies and Thacker, we note that

$$\begin{aligned} Z|_{Ag}&=1+\left(v_0X_0-\sum_j v_jX_j\right)\\ &=1+\frac{g^2C_F}{16\pi^2}[-2\ln(a\lambda)^2+e(\tilde{v})], \end{aligned} \tag{C3}$$

where the ln($a\lambda$) term comes from doing the self-energy integral. It is $\delta Z=Z-1$ which is Davies' and Thacker's C:

$$\frac{C_F}{16\pi^2}[-2\ln(a\lambda)^2+e(\tilde{v})]=C|_{DT}=Z|_{DT}+aA|_{DT}. \tag{C4}$$

In addition, because of some discrepancies discussed in Sec. III D, it will be convenient to define a ''reduced value of e,'' $[e^{(R)}(\tilde{v})\equiv e'(\tilde{v})]$:

$$\frac{C_F}{16\pi^2}[-2\ln(a\lambda)^2+e'(\tilde{v})]=Z|_{DT}. \tag{C5}$$

This reduced value can be found from $e(\tilde{v})$ and $x(\tilde{v})$ as expressed in Eq. (B9).

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