Relating different approaches to nonlinear QCD evolution at finite gluon density

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We analyze the relation between evolution equations at low x that have been derived in different approaches in the last several years. We show that the equation derived by Balitsky and Kovchegov is obtained from the Jalilian-Marian–Kovner–Leonidov–Weigert (JKLW) equation in the limit of small induced charge density. We argue that the higher nonlinearities resummed by the JKLW equation correspond, in physical terms, to the breakdown of the eikonal approximation when the gluon fields in the target are large.

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I. INTRODUCTION

In recent years, there has been renewed interest in the understanding of the physics of systems with a large number of partons. These studies have been essentially motivated by two large experimental programs—low-x deep inelastic scattering (DIS) at the DESY ep collider HERA and heavy ion collision experiments at the BNL Relativistic Heavy Ion Collider (RHIC) and CERN Large Hadron Collider (LHC). Both physical situations involve a large number of participating gluons. In low-x DIS these gluons are generated in the proton light cone wave function by the evolution to low x, whereas in the nuclear collision this evolution is enhanced since the nuclear wave function contains many gluons already at moderate values of energy.

The growth of gluon density leads to interesting physical consequences, the physical understanding of which has been steadily improving. One universal feature now believed to be true is the saturation of gluon densities. Apparently, the number of gluons per unit phase space volume practically saturates and at large densities grows only very slowly (logarithmically) as a function of the parameter that triggers the growth. The relevant parameter could be $1/x$ in the low-x regime or the atomic number of the nucleus A in heavy ion collisions. This saturation takes place at values of transverse momentum below a certain saturation momentum k_s, which itself depends on $1/x$ and A. The nature of this dependence is less well understood. In the analysis based on the Balitskiǐ-Fadin-Kuraev-Lipatov (BFKL) evolution [1] and on the double logarithmic approximation (DLA) [2] the dependence is powerlike $k_s \propto (1/x)^{\alpha_s \delta}$, while other approaches [3] suggest a much slower dependence. In the case of power dependence, the saturation momentum at HERA is estimated to be in the range 1–2 GeV with similar, slightly higher, values at LHC. Optimistically, one can hope that the saturation region is itself semiperturbative; that is, the value of the coupling constant is reasonably small and, therefore, weak coupling methods can be applied to the quantitative analysis of the phenomenon.

The physics of saturation must have experimental manifestations. The simplest and the most direct, in a way, is the unitarization of the total DIS cross section. This is, however, also the least interesting one. First, since the effect of unitarization is almost kinematical, one does not need high partonic density; it is enough to have a large number of partons, not necessarily in the same bin of the phase volume [4,5]. Second, because the experimental status of unitarization is unclear. So far, all DIS data on the total cross section can be reasonably well described by linear Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution without the need to include nonlinear effects [6]. Although physically it is hard to believe that the leading twist perturbative approximation can be applied at Q^2 as low as 1 GeV^2 and although some aspects of the gluon distribution that emerge from these fits [6] are intuitively not satisfactory, present inclusive DIS data cannot be considered as an unambiguous confirmation of nonlinear effects.

The realm of nonlinear effects is, however, much richer than the total cross section. In particular, one expects qualitative changes in the structure of the final states as one moves into the saturation region. The study of these effects has, however, not started in earnest yet and we have a long way to go before being able to make verifiable quantitative predictions.

In particular, one needs a well-defined formal framework to perform calculations. Several approaches to the problem have been developed in recent years by different groups. They all rely on the smallness of the coupling constant while resuming the effects of a large number of partons/partonic density. The aim of all these approaches is essentially to derive the evolution of the hadronic scattering cross section with $1/x$. They, however, utilize different techniques and conceptual frameworks and the resulting evolution equations look rather different. It is the aim of this paper to explore the relation between some of these different approaches in an attempt to understand where they diverge from each other in terms of physics input.

In particular, we will concern ourselves with three recent works, Refs. [7,8,9–13]. In Ref. [7] the evolution equation for the scattering amplitude is derived using the effective action and the eikonal approximation in the target rest frame. Reference [8] uses the dipole model method of Refs. [14,15]. And, finally, Refs. [9–12] use the effective action in the projectile rest frame to derive the evolution of the hadron

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light cone wave function with $1/x$. We will refer to the resulting evolution equation as the Jalilian-Marian–Leonidov–Weigert (JKLW) equation.

The outline of this paper is the following. In Sec. II we rederive the evolution equation of Ref. [7] in a simple and intuitive way. This derivation makes it obvious that this approach is equivalent to the approach of Ref. [8] up to subleading corrections in $1/N_c$. This is not new and was noted already in Ref. [8]. In the following, we will refer to this evolution equation as the Balistskiĭ-Kovchegov (BK) equation.[1] We discuss the physical picture of this evolution and resulting unitarization of the total cross section in both target and projectile rest frames and point out the effects due to which the approximations involved should break down at extremely small x. The breakdown of the approximation should have very little effect on the unitarization of the total cross section, since, especially for large targets like nuclei, the black disk limit should be reached while the approximation is still valid. However, one does expect the structure of the final states to be strongly affected. Our discussion here is, in large measure, parallel to that of Refs. [4,5].

In Sec. III we relate explicitly the calculation of Ref. [7] to that of Refs. [9–12]. In particular, we calculate the basic physical quantities appearing in the evolution equation of Refs. [9–12] in the approximation of Ref. [7]. We show that the results of Ref. [7] are recovered from Refs. [9–12] in the limit of small induced fields. We also show that the double logarithmic limit of the evolution of Ref. [7] is trivial. That is, in the double logarithmic limit, the evolution equation for the gluon distribution function (defined operatorially as the number of gluons in the light cone gauge in the infinite momentum frame) becomes linear and does not contain any Gribov-Levin-Ryskin (GLR-) type corrections.[2] This is in contrast with the result of Ref. [13], where it was shown that the double logarithmic limit of the evolution Refs. [9–12] results in a nonlinear equation. We point out that this is indeed a very natural result from the point of view of the dipole model approach.

In Sec. IV we transform the full calculation of Refs. [9–12] into the framework of Ref. [7]. We show that in the approach of Ref. [7] it corresponds to abandoning the eikonal approximation or, equivalently, to the inability to fully describe the target by a classical A^+ field. We also point out technical reasons which lead us to believe that, in fact, in the framework of the effective action of Ref. [7] such a failure is expected when the evolution is continued to very low values of x. Finally, we conclude with a brief discussion in Sec. V.

[1]The $O(1/N_c)$ differences between the equations derived in Refs. [8] and [7] do not carry essential new physics. They therefore do not affect our understanding of the relationship between the generic frameworks of the BK and JKLW equations.

[2]This is not to say that the evolution of the DIS cross section for which the equation of Refs. [7] and [8] has been derived is linear in the double logarithmic limit. The GLR-type nonlinearity does indeed appear in the evolution equation for the virtual photon cross section due to the nonlinear relation between the cross section and the gluon distribution function.

II. A SIMPLE DERIVATION OF THE BK EQUATION

In this section we will give a simple derivation of the evolution equation first derived in Ref. [7] and discuss the physical picture behind it. Consider the deep inelastic scattering at low x. We will work in the frame in which the photon fluctuates into an energetic quark-antiquark pair long before it reaches the target, but where most of the energy resides in the target hadron which moves very fast. The scattering of the quark-antiquark pair is dominated by its interaction with the gluons in the target. Since the target hadron moves fast, the time evolution of the gluon fields is slowed by Lorentz time dilation. Also, due to Lorentz contraction, the gluon fields are well localized in the plane perpendicular to the direction of motion, which we take to be the positive x_3 axis. The target can, therefore, be modeled by a distribution of static gluon fields localized at $x^-=0$. As the scattering energy increases (x decreases) the gluon fields of the target change due to contributions of quantum fluctuations. It is this evolution in x of the hadronic ensemble that we intend to describe in terms of the evolution equation.

A. The BK equation

In this section we will use the light cone gauge $A^-=0$. In this gauge, following Ref. [7], we take the vector potentials representing the relevant gluon field configurations to be of the form

$$b^i=0, \quad b^+=b(x_\perp)\,\delta(z^-). \tag{1}$$

Here and in the rest of this section, unless otherwise specified, we use the matrix notation for the gauge field $b^+=b_a^+t_a$, etc., where t_a are the generators of the SU(N) group in the fundamental representation. One can reasonably ask whether the vector potential of this form is the only relevant one. This turns out to be a nontrivial question. In fact, we will argue later in Sec. IV that this is not quite the case if we want to be able to describe the evolution up to arbitrarily small values of x. At this point, however, we follow Ref. [7]. We will return to this question in Sec. IV.

The DIS structure function can be written in the following general form:

$$F_2(x,Q^2)=\frac{Q^2}{4\pi^2\alpha_{em}}\int\frac{dz\,dx_\perp dy_\perp}{4\pi}\times\Phi(x_\perp-y_\perp,z)N(x_\perp,y_\perp,y). \tag{2}$$

Here, $x_\perp$ and $y_\perp$ are the transverse coordinates of the quark and the antiquark in the pair, z is the fraction of the pairs longitudinal momentum carried by the quark, and y is the rapidity of the slowest particle in the pair. Also, $\Phi(x_\perp-y_\perp,z)$ is the square of the ‘‘wave function’’ of the photon—the probability that the virtual photon fluctuates into the pair with given coordinates and momenta—and $N(x_\perp,y_\perp,y)$ is the cross section for the scattering of the pair.

The wave function Φ is well known. It is given, for example, in Ref. [8], but its explicit form will not be of interest to us. We concentrate our discussion on the scattering cross section N. If the quark-antiquark pair is energetic enough the scattering cross section is eikonal,

$$N(x_\perp,y_\perp)=\mathrm{tr}\langle V(x_\perp)V^\dagger(y_\perp)-1\rangle_A, \tag{3}$$

where $V(V^\dagger)$ is the eikonal phase for the scattering of the energetic quark (antiquark),

$$V(x^+=0,x_\perp)=\mathcal{P}\exp\left[-ig\int_{-\infty}^{+\infty}dx^-A^+(x^+=0,x_\perp,x^-)\right], \tag{4}$$

with the vector potential in the fundamental representation. We, therefore, have to calculate the average of $\langle V(x^+=0,x_\perp)V^\dagger(y^+=0,y_\perp)\rangle_A$ over the hadronic wave function as indicated by $\langle\cdots\rangle_A$. In our frame, the quark and the antiquark move with the velocity of light in the negative x_3 direction. All the fields in Eq. (4), therefore, have a vanishing x^+ coordinate. This will also be the case for all the fields in the rest of this section. For simplicity, we suppress the x^+ coordinate in the following. In the leading approximation the vector potential is given by Eq. (1) and the scattering amplitude is

$$N(x_\perp,y_\perp)=\langle\mathrm{tr}[U(x_\perp)U^\dagger(y_\perp)-1]\rangle_b, \tag{5}$$

with

$$U(x_\perp)=\mathcal{P}\exp\left[-i\int_{-\infty}^{+\infty}dx^-b^+(x_\perp,x^-)\right], \tag{6}$$

To calculate the order α_s correction to this expression we write the vector potential as

$$A^+=\frac{1}{g}b^++a^+, \tag{7}$$

with a^+ being a small fluctuation and expand the eikonal factors to second order in a^+.

Recalling that the classical background vector potential is a δ function in x^-, we have

$$V(x_\perp)=\mathcal{P}\exp\left[-ig\int_{-\infty}^{0}dx^-a^+(x_\perp,x^-)\right]U(x_\perp)\mathcal{P}\exp\left[-ig\int_{0}^{+\infty}dx^-a^+(x_\perp,x^-)\right] \tag{8a}$$

$$\begin{aligned}&=U(x_\perp)-ig\left\{\int_{-\infty}^{0}dx^-a^+(x_\perp,x^-)U(x_\perp)+U(x_\perp)\int_0^{+\infty}dx^-a^+(x_\perp,x^-)\right\}\\&-g^2\left\{\int_{-\infty}^{0}dx^-dy^-\,\theta(y^--x^-)a^+(x_\perp,x^-)a^+(x_\perp,y^-)U(x_\perp)\right.\\&+\int_{-\infty}^{0}dx^-a^+(x_\perp,x^-)U(x_\perp)\int_0^{+\infty}dx^-a^+(x_\perp,x^-)\\&\left.+U(x_\perp)\int_0^{+\infty}dx^-dy^-\,\theta(y^--x^-)a^+(x_\perp,x^-)a^+(x_\perp,y^-)\right\}.\end{aligned} \tag{8b}$$

All contributions break down into x^- ordered pieces because of the x^- structure in Eq. (1). Now, together with the analogous expansion for $V^\dagger$, we insert this into Eq. (3) and obtain

$$\begin{aligned}&\mathrm{tr}\langle V(x_\perp)V^\dagger(y_\perp)\rangle_A-\mathrm{tr}\langle U(x_\perp)U^\dagger(y_\perp)\rangle_b \qquad (9)\\&=g^2\mathrm{tr}\left\langle\int_{-\infty}^{0}dw^-a^+(x_\perp,w^-)U(x_\perp)U^\dagger(y_\perp)\int_{-\infty}^{0}dz^-a^+(y_\perp,z^-)\right.\\&+\int_{-\infty}^{0}dw^-a^+(x_\perp,w^-)U(x_\perp)\int_0^{+\infty}dz^-a^+(y_\perp,z^-)U^\dagger(y_\perp)\\&+U(x_\perp)\int_0^{+\infty}dw^-a^+(x_\perp,w^-)U^\dagger(y_\perp)\int_{-\infty}^{0}dz^-a^+(y_\perp,z^-)\\&+U(x_\perp)\int_0^{+\infty}dw^-a^+(x_\perp,w^-)\int_0^{+\infty}dz^-a^+(y_\perp,z^-)U^\dagger(y_\perp)\end{aligned}$$

$$-\int_{-\infty}^{0} dw^- dz^- \,\theta(z^- - w^-) a^+(x_\perp, w^-) a^+(x_\perp, z^-) U(x_\perp) U^\dagger(y_\perp)$$

$$-\int_{-\infty}^{0} dw^- a^+(x_\perp, w^-) U(x_\perp) \int_0^{+\infty} dz^- a^+(x_\perp, z^-) U^\dagger(y_\perp)$$

$$-U(x_\perp) \int_0^{+\infty} dw^- dz^- \,\theta(z^- - w^-) a^+(x_\perp, w^-) a^+(x_\perp, z^-) U^\dagger(y_\perp)$$

$$-U(x_\perp) U^\dagger(y_\perp) \int_{-\infty}^{0} dw^- dz^- \,\theta(w^- - z^-) a^+(y_\perp, w^-) a^+(y_\perp, z^-)$$

$$-U(x_\perp) \int_0^{+\infty} dw^- a^+(y_\perp, w^-) U^\dagger(y_\perp) \int_{-\infty}^{0} dz^- a^+(y_\perp, z^-)$$

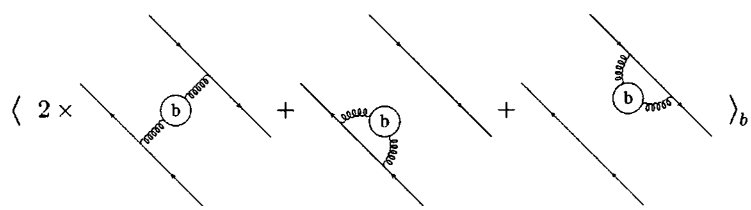
$$-U(x_\perp) \int_0^{+\infty} dw^- dz^- \,\theta(w^- - z^-) a^+(y_\perp, w^-) a^+(y_\perp, z^-) U^\dagger(y_\perp) \Bigg\rangle_{b,a}.$$

In writing Eq. (9) we have anticipated that $\langle a_u^+ \rangle_a = 0$ as in the free case. This can easily be shown using the explicit expression for the fluctuation propagator given below.

Although this expression is a little cumbersome, the physical meaning of the various terms is very clear. Put more compactly the structure of the above is determined by

$$\mathrm{tr}\langle V(x_\perp) V^\dagger(y_\perp) \rangle_A - \mathrm{tr}\langle U(x_\perp) U^\dagger(y_\perp) \rangle_b = g^2 \Bigg\langle \langle a_u^+ a_v^+ \rangle_a \frac{1}{2} \Bigg[2 \frac{\delta}{\delta b_u^+} U(x_\perp) \frac{\delta}{\delta b_v^+} U^\dagger(y_\perp) + \left(\frac{\delta}{\delta b_u^+} \frac{\delta}{\delta b_v^+} U(x_\perp) \right) U^\dagger(y_\perp) + U(x_\perp) \left(\frac{\delta}{\delta b_u^+} \frac{\delta}{\delta b_v^+} U^\dagger(y_\perp) \right) \Bigg] \Bigg\rangle_b . \tag{10}$$

Diagramatically the right-hand side (rhs) can be represented as follows:



(11)

The straight lines represent the eikonal factors U, while the curly lines denote the gluon fluctuation propagator $\langle a_u^+ a_v^+ \rangle_a$, evaluated in the fixed background $b^+(x)$. The terms in Eq. (10) with first order derivatives correspond to processes where the gluon is emitted by the quark and absorbed by the antiquark (or vice versa). Those will be hereafter referred to as ''exchange contributions.''

The terms in Eq. (10) with second derivatives acting on U (or $U^\dagger$) correspond to the diagrams where the quark (or antiquark) emits the gluon and then reabsorbs it at a later time: typical self-energy corrections. To contrast them against the exchange contributions we will also refer to them as ''nonexchange contributions.''

In Eq. (9) we are looking at a x^- ordered breakdown of the diagrams in Eq. (11) with the first four terms summing up to the exchange contribution and the rest to the nonexchange contributions. With the vertices known, we only lack an explicit expression for $\langle a_u^+ a_v^+ \rangle_a$. The QCD action expanded to second order in the fluctuation field a^+, in the presence of the classical background b^+, in our light cone gauge is

$$S=\tfrac{1}{2}\{a_a^+[-(\partial^-)^2]a_a^+-2(\partial^i a_a^+)(\partial^- a_a^i)\\ -a_a^i[(2D_{ab}^+[b]\partial^- -(\partial_\perp)^2\delta_{ab})\delta^{ij}+\partial^i\partial^j\delta_{ab}]a_b^j\}. \tag{12}$$

Recall that we are interested only in the propagator of the fields at equal x^+. Consequently, it is only the on-shell part of the propagator that is relevant for our purposes. We can, therefore, use the classical equation of motion for a^+

$$a^+=\frac{\partial^i}{\partial^-}a^i. \tag{13}$$

Substituting this in Eq. (12) we get

$$S=-\tfrac{1}{2}a_a^i(D^2)_{ab}\delta^{ij}a_b^j, \tag{14}$$

where

$$(D^2)_{ab}=2D_{ab}^+[b]\partial^- -(\partial_\perp)^2\delta_{ab}. \tag{15}$$

The propagator of the spatial components of the vector potential is, therefore, simply given by $-i/D^2$. The explicit form is very simple and can be found, for example, in Ref. [16].

$$-i\left[\frac{1}{(D^2)}\right]_{ab}=\int\frac{dp^-}{2p^-(2\pi)^3}[\theta(x^- -y_-)\theta(p^-)\\ -\theta(y^- -x^-)\theta(-p^-)]\\ \times\int d^2p_\perp d^2q_\perp e^{-ip\cdot x+iq\cdot y}\\ \times\int\frac{d^2z_\perp}{(2\pi)^2}e^{-i(p_\perp-q_\perp)z_\perp}\tilde{U}_{ab}^{-1}(x^-,y^-,z_\perp), \tag{16}$$

with $p^+=p_\perp^2/2p^-$, $q^+=q_\perp^2/2p^-$ and $q^-=p^-$. The adjoint color matrix[3]

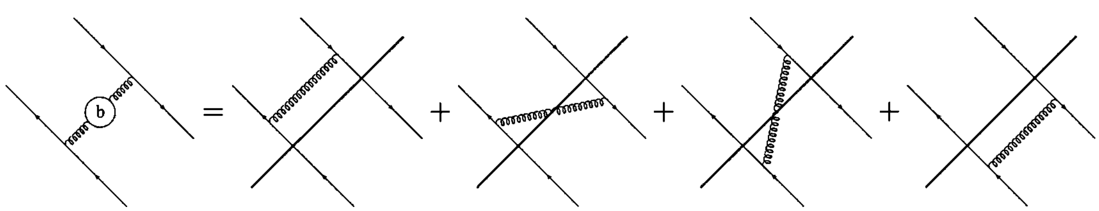
$$\tilde{U}_{ab}^{-1}(x^-,y^-,z_\perp)=[\theta(x^-)\theta(y^-)+\theta(-x^-)\theta(-y^-)]\delta_{ab}\\ +\theta(-x^-)\theta(y^-)\tilde{U}_{ab}(z_\perp)\\ +\theta(x^-)\theta(-y^-)\tilde{U}_{ab}^\dagger(z_\perp) \tag{17}$$

represents a phase factor one picks up when crossing the $x^-=0$ plane due to interaction with a field of type Eq. (1). Here, $\tilde{U}_{ab}(z_\perp)$ is the adjoint version of the fundamental U in Eq. (6). If the $x^-=0$ plane is not crossed the propagation remains free.

We can now write the on-shell correlator of the ''+'' component of the vector potential as

$$\langle a_a^+(x^+=0,x_\perp,x^-)a_b^+(y^+=0,y_\perp,y^-)\rangle_a\\ =\left\langle\frac{\partial_x^i}{\partial_x^-}a_a^i(x^+=0,x_\perp,x^-)a_b^i(y^+=0,y_\perp,y^-)\frac{\partial_y^i}{\partial_y^-}\right\rangle\\ =-\partial_x^i\partial_y^i\int\frac{dp^-}{(p^-)^3}\frac{1}{4\pi}[\theta(x^- -y^-)\theta(p^-)\\ -\theta(y^- -x^-)\theta(-p^-)]\\ \times\int d^2z_\perp\frac{d^2p_\perp}{(2\pi)^2}\frac{d^2q_\perp}{(2\pi)^2}e^{ip_\perp(x_\perp-z_\perp)+iq_\perp(z_\perp-y_\perp)}\\ \times e^{-ip_\perp^2/2p^-x^-+iq_\perp^2/2p^-y^-}\\ \times\{\theta(x^-)\theta(y^-)+\theta(-x^-)\theta(-y^-)\\ +\theta(-x^-)\theta(y^-)\tilde{U}(z_\perp)+\theta(x^-)\theta(-y^-)\tilde{U}^\dagger(z_\perp)\}. \tag{18}$$

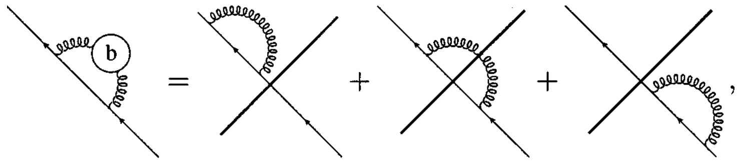
This expression displays a separation into x^- ordered contributions that seamlessly matches up with what we have already seen for the vertices in Eq. (9). Diagrammatically, the x^- ordered exchange contributions are given by

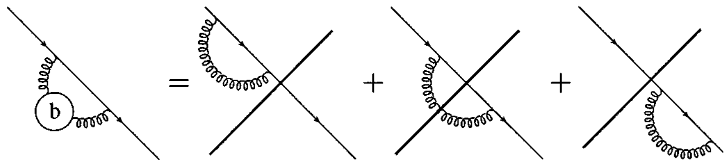


(19)

while the (connected parts of the) non-exchange ones are represented by

[3]More rigorously, the structure of $\tilde{U}^{-1}$ is given by $\tilde{U}_{ab}^{-1}(z_\perp)=e^{i[\theta(x^-)-\theta(y^-)]b(z_\perp)}$. However, the difference between this expression and that given in Eq. (17) only shows up if it is multiplied by ∂^+ derivatives or $\delta(x^-)$ factors. Since we encounter no such factors in our calculation we will be using Eq. (17) throughout.


(20a)


(20b)

Whenever the $x^- = 0$ plane cuts the fluctuation propagator, the diagram contains a factor $U^{(\dagger)}_{ab}$; otherwise the fluctuation propagator is free.

Algebraically, the corrections to the scattering cross section involve integrals of this propagator with respect to x^- and y^- from zero to either $+$ or $-$ infinity. This is straightforward to do. For example, for the second term in Eq. (9) we need

$$\int_{-\infty}^{0} dw^- \int_0^{+\infty} dz^- \langle a_a^+(x_\perp, w^-) a_b^+(y_\perp, z^-)\rangle$$
$$= \partial_x^i \partial_y^i \int_0^{+\infty} \frac{dp^-}{p^-} \frac{1}{\pi} \int d^2 z_\perp \int \frac{d^2 p_\perp}{(2\pi)^2} \frac{d^2 q_\perp}{(2\pi)^2}$$
$$\times \frac{e^{ip_\perp(x_\perp - z_\perp) + iq_\perp(z_\perp - y_\perp)}}{p_\perp^2 q_\perp^2} \tilde{U}_{ab}(z_\perp)$$
$$= \frac{1}{\pi} \int_0^{+\infty} \frac{dp^-}{p^-} \left\langle x_\perp \left| \frac{\partial^i}{\partial_\perp^2} \tilde{U}_{ab} \frac{\partial^i}{\partial_\perp^2} \right| y_\perp \right\rangle. \tag{21}$$

Here, $\langle x_\perp | O | y_\perp \rangle$ means the matrix element of the operator O in the coordinate basis in the usual sense. We treat $\tilde{U}$ as an operator in the coordinate space with matrix elements $\langle x_\perp | \tilde{U} | y_\perp \rangle = \tilde{U}(x_\perp)\,\delta(x_\perp - y_\perp)$ and the products in the last line are understood in the operatorial sense. Explicitly,

$$\int_{-\infty}^{0} dw^- \int_0^{+\infty} dz^- \langle a_a^+(x_\perp, w^-) a_b^+(y_\perp, z^-)\rangle$$
$$= \frac{1}{4\pi^3} \int_0^{+\infty} \frac{dp^-}{p^-} \int d^2 z_\perp \frac{(x-z)_\perp \cdot (y-z)_\perp}{(x-z)_\perp^2 (y-z)_\perp^2} \cdot \tilde{U}_{ab}(z_\perp). \tag{22}$$

In the same way we obtain for the other contributions with interaction with the background

$$\int_0^{+\infty} dw^- \int_{-\infty}^{0} dz^- \langle a_a^+(x_\perp, w^-) a_b^+(y_\perp, z^-)\rangle$$
$$= \frac{1}{\pi} \int_0^{+\infty} \frac{dp^-}{p^-} \left\langle x_\perp \left| \frac{\partial^i}{\partial_\perp^2} \tilde{U}^\dagger_{ab} \frac{\partial^i}{\partial_\perp^2} \right| y_\perp \right\rangle$$
$$= \frac{1}{4\pi^3} \int_0^{+\infty} \frac{dp^-}{p^-} \int d^2 z_\perp \frac{(x-z)_\perp \cdot (y-z)_\perp}{(x-z)_\perp^2 (y-z)_\perp^2} \cdot \tilde{U}^\dagger_{ab}(z_\perp) \tag{23}$$

$$-\int_{-\infty}^{0} dw^- \int_0^{+\infty} dz^- \langle a_a^+(x_\perp, w^-) a_b^+(x_\perp, z^-)\rangle$$
$$= -\frac{1}{\pi} \int_0^{+\infty} \frac{dp^-}{p^-} \left\langle x_\perp \left| \frac{\partial^i}{\partial_\perp^2} \tilde{U}_{ab} \frac{\partial^i}{\partial_\perp^2} \right| x_\perp \right\rangle$$
$$= -\frac{1}{4\pi^3} \int_0^{+\infty} \frac{dp^-}{p^-} \int d^2 z_\perp \frac{1}{(x-z)_\perp^2} \cdot \tilde{U}_{ab}(z_\perp) \tag{24}$$

$$-\int_0^{+\infty} dw^- \int_{-\infty}^{0} dz^- \langle a_a^+(y_\perp, w^-) a_b^+(y_\perp, z^-)\rangle$$
$$= -\frac{1}{\pi} \int_0^{+\infty} \frac{dp^-}{p^-} \left\langle y_\perp \left| \frac{\partial^i}{\partial_\perp^2} \tilde{U}^\dagger_{ab} \frac{\partial^i}{\partial_\perp^2} \right| y_\perp \right\rangle$$
$$= -\frac{1}{4\pi^3} \int_0^{+\infty} \frac{dp^-}{p^-} \int d^2 z_\perp \frac{1}{(y-z)_\perp^2} \cdot \tilde{U}^\dagger_{ab}(z_\perp). \tag{25}$$

To simplify the color structure of these expressions we use the identity

$$\tilde{U}_{ab}(z_\perp)(t_a U(x_\perp))^{\alpha\beta}(t_b U^\dagger(y_\perp))^{\gamma\delta}$$
$$= 2\,\mathrm{tr}[t_a U(z_\perp) t_b U^\dagger(z_\perp)](t_\alpha U(x_\perp))^{\alpha\beta}(t_b U^\dagger(y_\perp))^{\gamma\delta}$$
$$= \frac{1}{2N_c}[N_c(U(z_\perp)\cdot U^\dagger(y_\perp))^{\alpha\delta}(U^\dagger(z_\perp)\cdot U(x_\perp))^{\gamma\beta}$$
$$- U(x_\perp)^{\alpha\beta} U^\dagger(y_\perp)^{\gamma\delta}]. \tag{26}$$

Note that the integral over the frequency p^- logarithmically diverges. In fact, we have to integrate only over a finite interval of frequencies. The gluon field modes of very low frequency have been already included in the background field b^+ and, therefore, the fluctuation fields at these low frequencies should not be considered. The lower cutoff on the frequency of the modes that are being integrated is inversely proportional to the initial value of x_0 at which we start the evolution. The upper limit on the high frequency side is furnished by the maximal rapidity of the quark (or antiquark) in the virtual photon which is of the order of $1/x$. The ratio of these two cutoffs is of order x_0/x. Thus, in the leading logarithmic approximation we identify

$$\int \frac{dp^-}{p^-} = \ln \frac{x_0}{x}. \tag{27}$$

The calculation of the remaining contributions (the ones with no interaction with the background) proceeds along similar lines and is given in the Appendix.

Collecting all contributions together we obtain

$$\begin{aligned}
&\mathrm{tr}\langle V(x_\perp)V^\dagger(y_\perp)\rangle_A - \mathrm{tr}\langle U(x_\perp)U^\dagger(y_\perp)\rangle_b \\
&= \frac{g^2}{8\pi^3}\ln\left(\frac{x_0}{x}\right)\int d^2z_\perp \Bigg\langle \{\mathrm{tr}[U(x_\perp)U^\dagger(z_\perp)]\mathrm{tr}[U^\dagger(y_\perp)U(z_\perp)] - 2N_c\mathrm{tr}[U(x_\perp)U^\dagger(y_\perp)]\}\frac{(x-z)_\perp\cdot(y-z)_\perp}{(x-z)_\perp^2(y-z)_\perp^2} \\
&\quad - \{\mathrm{tr}[U(x_\perp)U^\dagger(z_\perp)]\mathrm{tr}[U(z_\perp)U^\dagger(y_\perp)] - N_c\mathrm{tr}[U(x_\perp)U^\dagger(y_\perp)]\}\frac{1}{(x-z)_\perp^2} \\
&\quad - \{\mathrm{tr}[U^\dagger(y_\perp)U(z_\perp)]\mathrm{tr}[U^\dagger(z_\perp)U(x_\perp)] - N_c\mathrm{tr}[U^\dagger(y_\perp)U(x_\perp)]\}\frac{1}{(y-z)_\perp^2}\Bigg\rangle_b .
\end{aligned} \tag{28}$$

The eikonal factors V themselves should be considered as functions of x, so that $U = V(x_0)$. Differentiating this equation with respect to $\ln 1/x$ we recover the evolution step for $\mathrm{tr}\langle V(x_\perp)V^\dagger(y_\perp)\rangle_A$. This is precisely what was found in Ref. [7].[4]

At large N_c the products of traces in Eq. (28) factorize:

$$\langle \mathrm{tr}[U(x_\perp)\cdot U^\dagger(z_\perp)]\mathrm{tr}[U^\dagger(y_\perp)\cdot U(z_\perp)]\rangle \xrightarrow{N_c\to\infty} \langle \mathrm{tr}[U(x_\perp)\cdot U^\dagger(z_\perp)]\rangle\langle \mathrm{tr}[U^\dagger(y_\perp)\cdot U(z_\perp)]\rangle. \tag{29}$$

Equation (28) then becomes a closed equation for the evolution of $N(x_\perp, y_\perp) = \langle \mathrm{tr}[U(x)U^\dagger(y) - 1]\rangle$. It is identical to the nonlinear evolution equation of Ref. [8].

B. The physical interpretation

Now, let us discuss the physical picture of this evolution. As always with DIS, the physical picture depends on the frame in which one chooses to view the process. We have specified the frame to some extent by declaring that the photon fluctuates into a $q\bar{q}$ pair long before the target. However, we are still free to put the subsequent evolution in x either into the evolution of the photon wave function or into the evolution of the gluon field distribution in the target. We will refer to the former picture as the ‘‘projectile evolution picture’’ and to the latter as the ‘‘target evolution picture.’’ The calculation is, fortunately, noncommittal on this point and we will consider both pictures in turn.

In the projectile evolution picture, the higher energy of the scattering is achieved by boosting the $q\bar{q}$ pair. In this picture, the quark and antiquark have very high energy and consequently their wave function develops extra gluon components. The growth of the cross section with $1/x$ then is interpreted as due to the scattering of extra gluons in the projectiles wave function. This is precisely how the low-x evolution is viewed in the dipole model of Mueller [14,15]. Our calculation of this section has a simple interpretation from this point of view. The quark and the antiquark is the pair of pointlike color charges moving with the velocity of light and located at $x^+ = 0$ and transverse coordinates $x_\perp$ and $y_\perp$. These color charges carry with them a fluctuating gluon field. When the pair is boosted to higher rapidity the gluon fields ‘‘freeze’’ due to the time dilation and become static. In the approximation when the gluon fields are frozen, they are given by the Weizsäcker-Williams static $(p^+ = 0)$ fields created by the $q\bar{q}$ pair. In the leading order in α_s the Weizsäcker-Williams (WW) fields are small and are emitted independently by the quark and the antiquark. The total WW field is

$$A^i = g\frac{1}{\partial^-}\frac{\partial^i}{\partial_\perp^2}[j_q^- + j_{\bar{q}}^-], \tag{30}$$

[4]As it stands, Eq. (28) does not provide a closed equation—it has to be supplemented by evolution equations for arbitrary products $\langle V_1^{(\dagger)}\otimes\cdots\otimes V_n^{(\dagger)}\rangle_A$. The evolution of these higher correlators is derived following the same procedure as described above and leads to the full set of operator equations derived in Ref. [7]. We will give a compact representation of the whole set of the evolution equations in Eq. (75), Sec. IV.

where j_q^- $(j_{\bar{q}}^-)$ is the color current due to the quark (antiquark) which in our frame has only a ''$-$'' component. For pointlike quark and antiquark the charge densities are δ functions in the transverse coordinates and in x^+. The WW field is therefore

$$A^i(p^-,z_\perp)=g\frac{1}{p^-}\left[\tau\frac{x^i-z^i}{(x-z)^2}+\tau'\frac{y^i-z^i}{(y-z)^2}\right]. \quad (31)$$

Here, τ and τ' are fundamental color matrices corresponding to the orientation of the quark and antiquark wave functions in the color space. Their exact form does not matter for our purposes. The WW field, if written in the particle basis, can be thought of as representing equivalent gluons. The number of gluons at a given transverse position is given by the familiar expression

$$n_{\mathrm{WW}}(z_\perp)\propto\int_0^\infty dp^-p^-\mathrm{tr}F^{-i}(p^-,z_\perp)F^{-i}(p^-,z_\perp)$$
$$=\alpha_s\int_0^\infty\frac{dp^-}{p^-}\mathrm{tr}\left[\tau\frac{x^i-z^i}{(x-z)^2}+r'\frac{y^i-z^i}{(y-z)^2}\right]$$
$$\times\left[\tau\frac{x^i-z^i}{(x-z)^2}+\tau'\frac{y^i-z^i}{(y-z)^2}\right]. \quad (32)$$

If we do not take the trace over the color indices, this expression gives the probability to have one extra WW gluon in the wave function of the $q\bar{q}$ pair (at the transverse position $z_\perp$ with a particular color orientation). These WW gluons scatter on the gluon field of the target eikonally just like the quark and the antiquark, apart from the fact that they carry adjoint charge, and so their eikonal amplitude is given by $\tilde{U}$ rather than U. The terms in this expression are in one-to-one correspondence with the real contributions in Eq. (9), that is, the terms in which the gluons interact with the target (background). The rest of the terms in Eq. (9)—the virtual terms—as usual serve to restore the correct normalization of the wave function.

To summarize, in the projectile evolution picture our calculation describes emission of the WW gluons into the wave function of $q\bar{q}$ long before the scattering. The transverse coordinates of these gluons are frozen due to the Lorentz time dilation. Subsequently, both the q and $\bar{q}$, and also the gluons, scatter eikonally and independently of each other on the target gluon field. Clearly, this picture is identical to the dipole evolution picture of Mueller which was used in Ref. [8] to derive a nonlinear evolution equation. The only difference is that the dipole model uses the simplifications in the color algebra which arise in the large N_c limit.

The calculation presented above also has a simple interpretation in the target evolution picture. In this picture, it is the target rather than the projectile that is boosted when going to lower x. As already made explicit above by writing $\langle\cdots\rangle_b$, one should think about the target as being represented by an ensemble of the configurations of b^+. The corresponding statistical weight $Z[b]$ is determined, of course, by the structure of the target at the relevant resolution scale. We will have more to say about it in Sec. IV. The boost of the target freezes the gluon field fluctuations around the target background b^+ and, consequently, some field modes which were not important at higher x are now capable of inducing scattering. Thus, the ensemble of the relevant field configurations which characterizes the target changes. In fact, every b^+ now forks into a ''subensemble'' $b^{+\prime}=b^++a^+$. In the weak coupling regime a^+ have Gaussian distribution with the width determined by the inverse of their correlation function Eq. (18). One can work back from here and calculate the modification of the distribution of the background fields. We will do this in the following sections. The fluctuations of a^+ are, therefore, considered in the target evolution picture as modifying the ensemble of the target background fields very much like in the approach of Refs. [9–12], [17].

C. Unitarization in different approximations

From what has been said so far, it is clear that although the calculation presented in this section includes in the evolution some nonlinear effects, it is not the end of the story. At very low x, this approximation should break down. There are clear reasons why this should happen in both pictures. In the projectile evolution picture, it is not true indefinitely that the WW fields are emitted independently from the partons in the projectile. Due to the evolution, more and more gluons are emitted into the wave function of the projectile and so the density of partons grows. At some point, the approximation of independent emissions as well as of independent scattering of the partons on the target must break down. This is the point at which, in the parlance of Ref. [8], the Pomeron loop diagrams must come into play.

In the target evolution picture, the problematic point is the eikonal approximation for the scattering of the $q\bar{q}$ pair. One starts the evolution at some initial value of $x=x_0$ with all the available energy in the $q\bar{q}$ pair and the target fields not too strong. Since the $q\bar{q}$ pair is very energetic, the eikonal approximation is perfectly valid at this initial point. However, with the evolution the strength of the target fields grows, whereas the energy of the $q\bar{q}$ pair, on the other hand, stays fixed. Corrections to the eikonal approximation are of order $\alpha G(x)/s$, where $s\propto 1/x_0$ is the energy squared of the pair, and $G(x)$ is the density of the target fields. The field density grows due to evolution (at least initially in the linear regime) as

$$G(x)\propto\left(\frac{1}{x}\right)^{\alpha_P-1}, \quad (33)$$

with α_P the BFKL Pomeron intercept. At very small x the fields are strong enough so that the quark and antiquark start losing a finite fraction of their energy and, therefore, the no recoil eikonal approximation cannot stay valid indefinitely. Parametrically these corrections are of the same order as the Pomeron loops in the projectile frame, which suggests that they have the same physical origin.

The effect of the nonlinear evolution Eq. (28) on the behavior of the total cross section was studied in Refs. [18,19]. It was concluded that the nonlinearities slow down the BFKL-type rise of the cross section and lead to its unitarization so that the cross section approaches the black disk limit. We want to conclude this section with a comment on the nature of the unitarization in this approximation. Essentially, the unitarization is brought about by purely kinematical effects. This is especially clear in the projectile evolution picture. At the initial value of $x=x_0$ one starts with the $q\bar{q}$ pair as the only relevant component of the photon wave function, which has a certain probability $P_{q\bar{q}}$ to scatter on the target. So, initially, the total scattering probability P_{x_0} is

$$P_{x_0}=P_{q\bar{q}}. \tag{34}$$

At lower $x=x_0-\delta x$ the wave function also contains a component with an extra gluon. Let the probability to have an extra gluon in the wave function be Δ and the probability for this gluon to scatter on the target P_g. In the linear approximation (the BFKL limit) the total probability of scattering is additive

$$P_x=(1-\Delta)P_{q\bar{q}}+\Delta(P_{q\bar{q}}+P_g)=P_{q\bar{q}}+\Delta P_g. \tag{35}$$

However, this is, in fact, overcounting, since there are events where both the gluon and the $q\bar{q}$ pair undergo scattering and those events are counted twice in the linear approximation. One should, therefore, subtract the probability of these double scattering events from the total probability. This deficiency is corrected by writing.

$$\begin{aligned}P_x&=(1-\Delta)P_{q\bar{q}}+\Delta(P_{q\bar{q}}+P_g-P_{q\bar{q}}P_g)\\&=P_{q\bar{q}}+\Delta(P_g-P_{q\bar{q}}P_g)\\&=P_{x+\delta x}+\Delta(1-P_{x+\delta x})P_g.\end{aligned} \tag{36}$$

At arbitrary low x, the same argument leads to a similar expression where $P_{x+\delta x}$ denotes the total scattering probability of the projectile (which itself contains the $q\bar{q}$ pair and some number of gluons) at a slightly higher value of x. This is precisely the nonlinear term in the evolution equation (28), with the only difference that the extra gluon in the wave function can have arbitrary transverse coordinate and one should, of course, integrate over this extra degree of freedom. It is clear that this negative nonlinear correction leads to the unitarization of the cross section since as P_x tends to unity the emission of the extra gluon does not increase the total scattering probability. This effect is somewhat similar to the Glauber mechanism, not in the sense that each parton undergoes multiple scattering, but that the unitarization is of a purely geometrical nature. A similar discussion, in the framework of the dipole model, is given in Refs. [4,5]. In the next section we will show how to relate the approach just discussed with that of Refs. [9–12,17].

III. THE JKLW EQUATION AND THE SMALL INDUCED FIELD LIMIT

We start this section by recalling the framework and results of Refs. [9–13,17].

A. The JKLW equation

In this approach, following Refs. [20,21], the averages of gluonic observables in a hadron are calculated via the following path integral:[5]

$$\begin{aligned}\langle O(A)\rangle=\int D\alpha^i DA^\mu O(A)Z[\alpha]\\ \times\exp\Big\{-i\int d^4x\frac{1}{4}\mathrm{tr}F^{\mu\nu}F_{\mu\nu}\\ -\frac{1}{N_c}\int d^2x_\perp dx^-\delta(x^-)\partial^i\alpha_a^i(x_\perp)\\ \times\mathrm{tr}T_a\mathcal{W}_{-\infty,\infty}[A^-](x^-,x_\perp)\Big\},\end{aligned} \tag{37}$$

where the gluon field strength tensor is given by

$$F_a^{\mu\nu}=\partial^\mu A_a^\nu-\partial^\nu A_a^\mu-gf_{abc}A_b^\mu A_c^\nu \tag{38}$$

and $\mathcal{W}$ is the Wilson line in the adjoint representation along the x^+ axis

$$\begin{aligned}&\mathcal{W}_{-\infty,+\infty}[A^-](x^-,x_\perp)\\&=\mathcal{P}\exp\Big[+ig\int dx^+A_a^-(x^+,x^-,x_\perp)T_a\Big].\end{aligned} \tag{39}$$

The hadron is represented by an ensemble of chromoelectric fields, localized in the plane $x^-=0$, of the form

$$f^{+i}=\frac{1}{g}\delta(x^-)\alpha^i(x_\perp), \tag{40}$$

where the two-dimensional vector potential $\alpha^i(x_\perp)$ is ‘‘pure gauge’’

$$\partial^i\alpha_a^j-\partial^j\alpha_a^i-f_{abc}\alpha_b^i\alpha_c^j=0. \tag{41}$$

In Eq. (37), $Z[\alpha]$ is the statistical weight of a configuration $\alpha_i(x_\perp)$ in the hadronic ensemble.

The evolution in Refs. [9–12] is derived in the target evolution picture where decreasing x corresponds to boosting the hadronic target. This leads to freezing of part of the gluonic degrees of freedom. Integrating out these slow modes of the vector potential generates the renormalization group equation, which has the form of the evolution equation for the statistical weight Z [9–12,17]

[5] An alternative form of the effective action was suggested in Ref. [22], where it was also shown that it leads to the same evolution equation.

$$\frac{d}{d\ln\frac{1}{x}}Z=\alpha_s\left\{\frac{1}{2}\,\frac{\delta^2}{\delta\alpha(u)\,\delta\alpha(v)}[Z\chi(u,v)]\right.$$
$$\left.-\frac{\delta}{\delta\alpha(u)}[Z\sigma(u)]\right\}. \qquad (42)$$

In the compact notation used in Eq. (42), both u and v stand for color and rotational index and transverse coordinates, with summation and integration over repeated occurrences implied. This evolution equation for the statistical weight can be rewritten as the set of the evolution equations for the correlation functions of the chromoelectric field

$$\frac{d}{d\ln\frac{1}{x}}\langle\alpha^{i_1}_{a_1}(x_1)\cdots\alpha^{i_n}_{a_n}(x_n)\rangle$$
$$=\alpha_s\Bigg[\sum_{0<l<n+1}\langle\alpha^{i_1}_{a_1}(x_1)\cdots\alpha^{i_{l-1}}_{a_{l-1}}(x_{l-1})$$
$$\times\alpha^{i_{l+1}}_{a_{l+1}}(x_{l+1})\cdots\alpha^{i_n}_{a_n}(x_n)\sigma^{i_l}_{a_l}(x_l)\rangle$$
$$+\sum_{0<m<k<n+1}\langle\alpha^{i_1}_{a_1}(x_1)\cdots\alpha^{i_{m-1}}_{a_{m-1}}(x_{m-1})$$
$$\times\alpha^{i_{m+1}}_{a_{m+1}}(x_{m+1})\cdots\alpha^{i_{k-1}}_{a_{k-1}}(x_{k-1})\,\alpha^{i_{k+1}}_{a_{k+1}}(x_{k+1})$$
$$\times\cdots\alpha^{i_n}_{a_n}(x_n)\chi^{i_m i_k}_{a_m a_k}(x_m,x_k)\rangle\Bigg]. \qquad (43)$$

The quantities $\chi[\alpha]$ and $\sigma[\alpha]$ have the meaning of the mean fluctuation and the average value of the induced vector potential which arises from the field modes which become frozen due to extra boost of the hadronic target. In the leading logarithmic approximation of Refs. [9–12] the two quantities χ and σ completely specify the low-x evolution. We give here an explicit expression for the mean fluctuation χ which will be the focus of our interest throughout this section,

$$\chi^{ij}_{ab}(x_\perp,y_\perp)=2\left\langle x_\perp\left|\left\{\frac{D^i}{D^2_\perp}[D^2_\perp-S^{-1}]\frac{D^j}{D^2_\perp}\right\}_{ab}\right|y_\perp\right\rangle. \qquad (44)$$

For convenience, we have defined

$$\alpha^i_{ab}=f_{abc}\alpha^i_c,$$
$$D^i_{ab}=\partial^i\delta_{ab}+\alpha^i_{ab}. \qquad (45)$$

The operator S in Eq. (44) is given by

$$S=\frac{1}{D^2_\perp}+2\left[\frac{\partial^i}{\partial^2_\perp}-\frac{D^i}{D^2_\perp}\right]\left[\frac{\partial^i}{\partial^2_\perp}-\frac{D^i}{D^2_\perp}\right]$$
$$=\frac{1}{D^2_\perp}-2\,\frac{1}{\partial^2_\perp}\,\partial_\perp\alpha\,\frac{1}{D^2_\perp}+2\,\frac{1}{D^2_\perp}\,D_\perp\alpha\,\frac{1}{\partial^2_\perp}. \qquad (46)$$

B. Where does it come from?

Technically, these results are derived as follows. One considers the quantum corrections in the classical background field Eq. (40). The calculation is performed in the light cone gauge $A^+=0$ with the residual gauge fixing $\partial^i A^i(x^-\to-\infty)=0$ which fixes the gauge completely. In this gauge the chromoelectric field Eq. (40) corresponds to the background vector potential

$$b^i=\theta(x^-)\,\alpha^i(x_\perp). \qquad (47)$$

Note that, as opposed to the previous section, here we are using a different light cone gauge: $A^+=0$. As a consequence, the background vector potential has a different form.

The complete set of on-shell small fluctuation solutions of the classical equations is

$$a^i_{p^-,r}=e^{ip^-x^+}\int d^2p_\perp\Bigg[\theta(-x^-)$$
$$\times\exp\left(i\frac{p^2_\perp}{2p^-}x^--ip_\perp x_\perp\right)v^i_{-,r}(p_\perp)$$
$$+\theta(x^-)U(x_\perp)\exp\left(i\frac{p^2_\perp}{2p^-}x^--ip_\perp x_\perp\right)$$
$$\times[U^\dagger v^i_{+,r}](p_\perp)+\theta(x^-)\,\gamma^i_{+,r}\Bigg]. \qquad (48)$$

Here, r is the degeneracy label, which labels independent solutions with the frequency p^-. In the free case it is conventionally chosen as the transverse momentum, $\{r\}=\{p_\perp\}$. The matrix $U(x_\perp)$ is the SU(N) matrix that parameterizes the two-dimensional ''pure gauge'' vector potential $\alpha^i(x_\perp)$,

$$\alpha^i(x_\perp)=iU(x_\perp)\,\partial^i U^\dagger(x_\perp).$$

The auxiliary functions $\gamma^i_+,v^i_\pm$ are all determined in terms of one vector function. Choosing this independent function as v^i_- we have

$$v^i_{+,r}=[T^{ij}-L^{ij}][t^{jk}-l^{jk}]v^k_{-,r}, \qquad (49)$$

$$\gamma^i_{+,r}=2D^i\left[\frac{D^j}{D^2_\perp}-\frac{\partial^j}{\partial^2_\perp}\right][t^{jk}-l^{jk}]v^k_{-,r}, \qquad (50)$$

where we have defined the projection operators

$$T^{ij}\equiv\delta^{ij}-\frac{D^iD_j}{D^2_\perp},\quad L^{ij}\equiv\frac{D^iD^j}{D^2_\perp},$$
$$t^{ij}\equiv\delta^{ij}-\frac{\partial^i\partial^j}{\partial^2_\perp},\quad l^{ij}\equiv\frac{\partial^i\partial^j}{\partial^2_\perp}. \qquad (51)$$

The γ_+ piece of the eigenfunction Eq. (48) is responsible for the induced vector potential since this is the only contribution that does not vanish at $x^-\to\infty$, so that

$$\chi^{ij}_{ab}(x_\perp,y_\perp)=4\pi\int dp^-\langle\gamma^i_{+,a}(x_\perp,p^-)\gamma^j_{+,b}(y_\perp,-p^-)\rangle. \quad (52)$$

Note that the essential nonlinearity of the expression Eq. (44) is due to the denominator in the operator S^{-1} Eq. (46). The reason this arises is due to the nontrivial normalization of the small fluctuation eigenfunctions. As discussed in detail in Refs. [9–12] the proper normalization of the eigenfunctions requires v^i_- to be chosen as a complete set of eigenfunctions of the two-dimensional Hermitian operator O^{-1}

$$[(t-l)O^{-1}(t-l)]^{ij}_{ab}(x_\perp,y_\perp)=\langle x_\perp|\delta^{ij}_{ab}-2\left\{\left[\partial^i\frac{1}{\partial^2_\perp}-D^i\frac{1}{D^2_\perp}\right]\times S^{-1}\left[\frac{1}{\partial^2_\perp}\partial^j-\frac{1}{D^2_\perp}D^j\right]\right\}_{ab}|y_\perp\rangle, \quad (53)$$

such that

$$\int d^2r_\perp v^i_{-,r,a}(x_\perp)v^{*j}_{-,r,b}(y_\perp)=\frac{1}{4\pi|p^-|}[O^{-1}]^{ij}_{ab}(x_\perp,y_\perp). \quad (54)$$

This nontrivial normalization is the consequence of the presence of the γ_+ piece in the solution Eq. (48). Equation (52), supplemented by Eq. (50) and the normalization Eq. (54), leads to the final expression Eq. (44).

If the contribution of γ_+ could be neglected in the normalization condition, the normalization of the eigenfunctions would be trivial and we would have $O=1$ in Eq. (54). One can consider the limit in which γ^i_+, or equivalently χ, is small. In the leading order in the expansion in γ_+ we have a very simple expression for χ,[6]

$$\tilde{\chi}^{ij}_{ab}(x_\perp,y_\perp)=4\left\langle x_\perp\left|\left[D^i\left\{\frac{1}{\partial^2_\perp}+\frac{1}{D^2_\perp}-\frac{1}{\partial^2_\perp}\partial_\perp D_\perp\frac{1}{D^2_\perp}-\frac{1}{D^2_\perp}D_\perp\partial_\perp\frac{1}{\partial^2_\perp}\right\}D^j\right]_{ab}\right|y_\perp\right\rangle. \quad (55)$$

Note that this is a different limit than the one in which the JKLW evolution reduces to the BFKL equation [9–12]. The BFKL limit corresponds to the expansion in powers of the background field α^i. Now, we are not assuming that α^i is small, but rather that the correction induced by the evolution γ^i is small.

We will now see that Eq. (55) is reproduced precisely by translating the calculation of the previous section into the language of the JKLW evolution. Thus the BK equation is recovered from the JKLW equation in the limit when the induced field γ^i is small.

[6]If so desired this expression can be written in a simple form in terms of the unitary matrix U, since operatorially $D^i=U\partial^iU^\dagger$. In Fourier space this gives convolutions of $U(p)$ and powers of transverse momentum.

C. BK to JKLW: transforming between the gauges

In the previous section, following Ref. [7], we used the gauge $A^-=0$. This is a very convenient gauge from the point of view of the projectile evolution since the eikonal amplitudes in this gauge are given by simple Wilson line factors. We will refer to this gauge as the ‘‘projectile light cone gauge,’’ or the ‘‘projectile gauge’’ for short. The JKLW approach, on the other hand, uses the $A^+=0$ gauge, which is convenient for the target evolution picture since it simplifies the relation between the distribution functions and the correlators of the gluon fields. We will call this gauge the ‘‘target light cone gauge’’ or, simply, the ‘‘target gauge.’’ Our immediate aim is, therefore, to calculate χ using the results of the calculation in the projectile gauge.

To do this, note that the relation between the fields in the target and projectile gauges is given by

$$\frac{1}{g}B^\mu+A^\mu=V\left(\frac{1}{g}b^\mu+a^\mu\right)V^\dagger+\frac{i}{g}V\partial^\mu V^\dagger. \quad (56)$$

To simplify the notation, from now on we will denote the fields in the target $(A^+=0)$ gauge by capital letters and fields in the projectile $(a^-=0)$ gauge by lower case letters. This we do for both the background part of the field and for the small fluctuation part. The field-dependent matrix V is given by

$$V=P\exp\left[-i\int_{-\infty}^{x^-}dx^-(b^++ga^+)\right]. \quad (57)$$

The condition $A^+=0$ does not by itself specify the lower limit of the integration over x^- in the exponential. However, choosing this limit to be at minus infinity ensures that $V(x^-\to-\infty)=1$ and, as a consequence, $A^i(x^-\to-\infty)=a^i(x^-\to-\infty)$. The projectile gauge fields satisfy the standard vanishing boundary conditions at infinity. This choice of the lower limit of the integration, therefore, guarantees that the target gauge fields also vanish at $x^-\to-\infty$ and, further, satisfy the residual gauge condition $\partial^iA^i(x^-\to-\infty)=0$ that was imposed in Refs. [9–12]. To calculate χ we only need to consider the linearized relation between the small fluctuations of the fields in the two gauges. To do this we need to expand V to first order in a^+. This has been done in the previous section. Taking only linear terms in a^+ in Eq. (8a) and substituting them into Eq. (56) we find for the transverse components of the field

$$A^i_a(x)=\theta(-x^-)\left[a^i_a(x)-\int_{-\infty}^{x^-}dx^-\partial^ia^+\right]+\theta(x^-)\left[\tilde{U}^{ab}a^i_b(x)-D^i_{ab}\times\left(\int_{-\infty}^{0}dx^-a^+_b+\tilde{U}_{bc}\int_0^{x^-}dx^-a^+_c\right)\right]=\left[\delta^{ij}-W\partial^iW^\dagger\frac{1}{\partial^+\partial^-}W\partial^jW^\dagger\right]_{ab}(Wa^j)_b. \quad (58)$$

Here, the matrix $\tilde{U}$ is the same as in the previous section and is related to the classical background by

$$\tilde{U}(x_\perp)=\mathcal{P}\exp\left\{-i\int_{-\infty}^{+\infty}dx^-b^+\right\},$$

$$B^i=\theta(x^-)i\tilde{U}\partial^i\tilde{U}^\dagger \tag{59}$$

as per Eq. (47). We have also defined

$$W(x^-)=\mathcal{P}\exp\left\{-i\int_{-\infty}^{x^-}dx^-b^+(x^-)\right\}$$
$$=\theta(-x^-)1+\theta(x^-)\tilde{U}, \tag{60}$$

which is essentially the classical part of V. The operator $1/\partial^+$ in the last line of Eq. (58) is defined as the integral from $-\infty$.[7] We will further simplify this expression by using the on-shellness conditions

$$a^+=\frac{\partial^i}{\partial^-}a^i,$$

$$(2\partial^-D^+[b^+]-\partial_\perp^2)a^i=0. \tag{61}$$

The resulting relation between the on-shell transverse fields in the two gauges is

$$A_a^i=\theta(-x^-)(t-l)^{ij}a_a^j(x^-)+\theta(x^-)\left\{\tilde{U}_{ab}(t-l)^{ij}a_b^j(x^-)-2D_{ab}^i\left[\frac{\partial^j}{\partial_\perp^2}a_b^j(x^-\to0^-)-\tilde{U}_{bc}\frac{\partial^j}{\partial_\perp^2}a^j(x^-\to0^+)\right]\right\}. \tag{62}$$

Note that we have to specify on which side of $x^-=0$ the fields are taken since the solutions of the small fluctuation equations in the projectile gauge are discontinuous at zero. Now, recall that a^i satisfies, everywhere except at $x^-=0$, the free equations of motion. With this in mind we can compare this equation with Eq. (48). We see that Eq. (62) is indeed precisely of the form Eq. (48) with

$$\gamma_{+,a}^i=-2D_{ab}^i\left[\frac{\partial^j}{\partial_\perp^2}a_b^j(x^-\to0^-)-\tilde{U}_{bc}\frac{\partial^j}{\partial_\perp^2}a_c^j(x^-\to0^+)\right],$$

$$v_{-,a}^i=(t-l)^{ij}a_a^j(x^-\to0^-),$$

$$v_{+,a}^i=\tilde{U}_{ab}(t-l)^{ij}a_b^j(x^-\to0^+). \tag{63}$$

Remembering that (see, for example, Ref. [16])

$$a^i(x^-\to0^+)=\tilde{U}^\dagger a^i(x^-\to0^-), \tag{64}$$

we see that the functions v_+^i, v_-^i, and γ_+^i are related precisely by the same relations as in Eq. (50). We have established, therefore, that if a^i satisfies the equation of motion in the projectile gauge, then the transformed field A^i of Eq. (56) satisfies the equations of motion in the target gauge. The only remaining question is that of the normalization of the eigenfunctions. Recall that the functions a^i in the calculation of Ref. [7], which was reproduced in the previous section, were normalized in the same way as the eigenfunctions of the free theory. That is to say, the full set of on-shell eigenfunctions is obtained by choosing $a_{p_\perp}^i(x^-\to0^-)$ as a complete set of normalized eigenfunctions of the unit operator in the transverse space

$$\int\frac{d^2p_\perp}{4\pi^2}a_{p_\perp}^i(x_\perp,x^-\to0^-)a_{p_\perp}^{j*}(y_\perp,x^-\to0^-)$$
$$=\delta^{ij}\delta(x_\perp-y_\perp). \tag{65}$$

Since $t-l$ is a unitary operator, Eq. (63) tells us that v_-^i is also normalized to unity rather than to a nontrivial operator O as in the JKLW calculation Eq. (54). Using this, as well as the relations Eq. (63) and Eq. (52), we find that when translated into the language of JKLW, the results of Ref. [7] give Eq. (55) as the mean fluctuation of the induced chromoelectric field. The essential nonlinearity of Eq. (44) is, therefore, absent in this calculation.

So far, we have only considered the real part of the JKLW kernel, χ. Of course, the same method can be applied to find what is the form of the virtual part σ, Eq. (42), that arises from the calculation of Ref. [7]. To reproduce the virtual part it is clearly necessary to keep the quadratic terms in the relation between A^i and a^i. Thus the quadratic terms in Eq. (8a) will be important in this calculation. Other than that, the calculation is straightforward. Again the gauge invariance ensures that all the ‘‘kinematical’’ factors of σ of Refs. [9–12] are reproduced in the projectile gauge calculation and the only difference comes from the difference in the normalization of the eigenfunctions. It is clear, therefore, that the result of such a calculation is again the lowest order expansion of σ in powers of γ_+,

$$\tilde{\sigma}_a^i=\left[\frac{D^i}{D^2}\right]_{ab}\left(\frac{N_c}{2}(\partial^j\alpha_b^j)\left\langle x_\perp\left|\frac{1}{\partial^2}\right|x_\perp\right\rangle-f_{bcd}\langle x_\perp|\left[4D^j\frac{1}{\partial^2}\partial D\frac{D^j}{D^2}+2\frac{1}{\partial^2}\partial\alpha-2\alpha\partial\frac{1}{\partial^2}+4\alpha^j\frac{1}{\partial^2}\alpha^j\right]_{cd}|x_\perp\rangle\right)$$
$$-2\epsilon^{ij}\left[\frac{D^i}{D^2}\right]_{ab}(x,y)f_{bcd}\epsilon^{kl}\langle y_\perp|\left[D^k\left(\frac{1}{\partial^2}+\frac{1}{D^2}-\frac{1}{\partial^2}\partial D\frac{1}{D^2}-\frac{1}{D^2}D\partial\frac{1}{\partial^2}\right)D^l\right]_{cd}|y_\perp\rangle. \tag{66}$$

[7]Equation (58) has been derived also in Refs. [23,24]. The only difference in our derivation is that the meaning of the $1/p^\pm$ pole is entirely unambiguous and, as discussed above, is dictated by the residual gauge condition.

D. The doubly logarithmic limit

Before exploring the relationship between the two approaches further in the next section, we want to make a comment about the form Eq. (55). Although this equation certainly gives $\overline{\chi}$ in general as a nonlinear function of the background field α^i, this nonlinearity disappears in the double logarithmic limit. Following Ref. [13] we take the double logarithmic limit as the limit when the background field α^i does not depend on $x_\perp$. In this limit, the covariant and the simple derivatives commute and it is easy to see that Eq. (55) reduces to

$$\tilde{\chi}^{ij}=4\,\frac{\alpha^2}{\partial^2}\,\frac{D^iD^j}{D^2} \tag{67}$$

or

$$\mathrm{tr}\tilde{\chi}=4\,\mathrm{tr}\frac{\alpha^2}{\partial^2}. \tag{68}$$

When substituted into the evolution equation (43) this gives the simple linear double logarithmic DGLAP evolution for the gluon distribution function $G\propto \mathrm{tr}\alpha^2$ (see Ref. [13] for a detailed derivation). This is in contrast with the situation discussed in Ref. [13] where the double logarithmic limit of Eq. (44) was studied. It was shown there that the nonlinearities in Eq. (44) survive in the doubly logarithmic limit and, in fact, lead even in this limit to the ''almost saturation'' of the gluon distribution.

The absence of the nonlinearities is in contradiction with the explicit calculation of Mueller and Qiu [25] who showed that the QCD evolution of the gluon distribution in the doubly logarithmic approximation does indeed contain contributions from higher twist operators. This again underscores our observation that the nonlinearities included in the evolution of Ref. [7] are not the whole story. Those are the ''kinematical'' nonlinearities in the sense discussed in the previous section and do not include interesting dynamical effects which come into play when the parton density becomes large.

In fact, the triviality of the doubly logarithmic limit of the calculation of the previous section is easy to understand using the intuition based on the dipole model approach. In the projectile evolution picture, in every step in the evolution one extra gluon is emitted into the virtual photon wave function.[8] The doubly logarithmic limit is achieved by assuming that this extra gluon has the smallest transverse momentum or, in the coordinate space, has the largest transverse coordinate [8]. In the large N_c limit the extra gluon is equivalent to an extra $q\tilde{q}$ pair. Thus, if initially one starts (as in Ref. [7] or [8]) from a fundamental dipole, after one step in the evolution the wave function has a component with two fundamental dipoles. The scattering events involving simultaneous scattering of both these dipoles lead to a nonlinear GLR-type quadratic term [26] in the evolution equation for the scattering cross section. This result indeed has been derived in Ref. [8].

The same process can be viewed from the point of view of adjoint dipoles. Since in the doubly logarithmic approximation the newly emitted gluon is assumed to be very far in the transverse plain from the partons previously present in the wave function, the part of the wave function that contains it essentially looks like one adjoint dipole of large transverse size. One leg of this dipole is the newly emitted gluon, while the other leg is the remainder partons which are closely bunched together in the coordinate space.[9] The cross section for the scattering of the adjoint dipole in the large N_c limit is simply related to the cross section for the fundamental dipole $\sigma_{\mathrm{adj}}=2\sigma_{\mathrm{fund}}-\sigma^2_{\mathrm{fund}}$. Thus starting from the fundamental dipole in the initial state, the nonlinear GLR evolution in this picture follows due to the nonlinear relation between the scattering cross section of the adjoint and the fundamental dipoles.

However, if we want to consider the evolution of the gluon distribution itself, the initial state should contain an adjoint rather than a fundamental dipole. This can be achieved by considering ''DIS'' of a virtual particle that couples to $\mathrm{tr}F^2$ [27]. In this case, in any step in the doubly logarithmic evolution, the state contains only one adjoint dipole. Thus for this initial configuration the nonlinear relation between the adjoint and fundamental cross sections is entirely irrelevant. The probability for the appearance of a larger dipole in the approximation of independent emissions is itself proportional to the number of gluons. The evolution of the gluon distribution in this approximation is, therefore, naturally linear and is merely the simple DGLAP DLA.

IV. MORE ON THE TARGET VERSUS PROJECTILE GAUGE

The discussion of the previous section may seem a little paradoxical on the purely technical level. Indeed, we have been calculating the same physical quantity in two different ways. The quantity in question is the equal (light cone) time propagator of the transverse components of the vector potential A^i in the target light cone gauge. The first way of performing the calculation is to work entirely in the target gauge as was done in Refs. [9–12]. This gives the result Eq. (44). The second way to calculate the same quantity is to first calculate the propagator of a^i in the projectile gauge and then gauge transform the result into the target gauge using Eqs. (58,62). This results in an inequivalent expression Eq. (55).

[8]This extra gluon can be emitted either from a quark or from an antiquark present in the original wave function, or from any other parton emitted in the previous steps in the evolution. So the total amplitude for emission is given by the sum of the amplitudes for these processes. Note, however, that the amplitude for the processes where two gluons are emitted simultaneously (say from q and $\tilde{q}$) is higher order in α_s and such processes are not present in the dipole model.

[9]These partons are in the adjoint representation of the color group, since together with the extra emitted gluon the state must be an overall singlet.

A. The $i\epsilon$ complication

Our first aim in this section is to resolve this technical paradox. To do this let us consider in more detail the calculation of Refs. [9–12] and its transformation into the projectile gauge. The equal time propagator of the transverse components of vector potential is calculated in the following way [9–12]. One starts with the quadratic part of the action for the small fluctuations of A^μ. Integrating A^- it is reduced to a quadratic action for the small fluctuations of the transverse components of the vector potential

$$S=\int d^4x d^4y A^i(x) G^{-1ij}(x,y) A^j(y). \qquad (69)$$

For the purpose of this discussion we use somewhat simplified notations and omit the color indices on the fields. The explicit form of G^{-1} is given in Refs. [9–12]. One then finds properly normalized eigenfunctions of G^{-1}

$$G^{-1ij}(x,y)A^j_{\lambda,p^-,r}(y)=\lambda A^i_{\lambda,p^-,r}(x),$$

$$\int d^4x A^i_{\lambda,p^-,r}(x) A^{i*}_{\lambda',p^{-\prime},r'}(x)=\delta(\lambda-\lambda')\,\delta(p^- - p^{-\prime}) \times \delta^2(r-r'). \qquad (70)$$

Using the complete set of eigenfunctions one constructs the propagator with the standard $i\epsilon$ prescription as

$$G^{ij}(x,y)=\int \frac{d\lambda}{\lambda+i\epsilon}\int dp^- d^2r A^i_{\lambda,p^-,r}(x) A^{j*}_{\lambda,p^-,r}(y). \qquad (71)$$

The limit $x^+=y^+$, and $x^-,y^-\to\infty$ is then taken to calculate χ^{ij}. Clearly, the equal time limit selects the on-shell eigenfunctions $\lambda=0$ and, therefore, when transforming into the projectile gauge it is important to keep track of the $i\epsilon$ prescription. The simplest way to do this is to include the $i\epsilon$ term directly in the action

$$S_t=\int d^4x d^4y A^i(x)[G^{-1ij}(x,y)+i\epsilon\delta^{ij}\delta(x-y)]A^j(y). \qquad (72)$$

The propagator Eq. (71) is then just the inverse of the quadratic form in Eq. (72) without any additional regulators.

To transform this expression into the projectile gauge one has to use Eqs. (58),(62). The gauge invariance of the QCD action ensures that the first term in Eq. (72) under this transformation transforms into

$$a^i(x)D^2(x,y)a^i(y), \qquad (73)$$

which is exactly the action used in Sec. II to calculate the projectile gauge propagator. However, the $i\epsilon$ term is not so simple. If the transformation Eq. (62) was unitary, the norm of the field A^i would be preserved and the $i\epsilon$ term in Eq. (72) would transform into the standard $i\epsilon\int d^4x a_i(x)a_i(x)$ term in the projectile gauge. The problem is that, as we saw in the previous section, the transformation Eq. (62) is not unitary. A normalized function a^i is transformed into a function A^i normalized not to unity but rather to an eigenvalue of the operator O in Eq. (53).[10] This was precisely the root of the discrepancy between χ and $\overline{\chi}$. The resulting projectile gauge action can be written as[11]

$$S_p=\int d^4x a^i(x)D^2a^i(x)+i\epsilon\int d^4x\Big\{a^i(x)a^i(x)+2a^i(0^-) \times\left[\partial^i\frac{1}{\partial_\perp^2}-D^i\frac{1}{D_\perp^2}\right]D_\perp^2\left[\partial^i\frac{1}{\partial_\perp^2}-D^i\frac{1}{D_\perp^2}\right]a^i(0^-)\Big\}. \qquad (74)$$

Thus, the standard $i\epsilon$ prescription in the target gauge is equivalent to a fairly complicated momentum-dependent prescription in the projectile gauge. Since the calculation of Sec. II, following Ref. [7], was performed using the standard $i\epsilon$ prescription in the projectile gauge the result is, indeed, expected to differ from that of Refs. [9–12].

While the technical reason for the difference between the results of Refs. [9–12] and Ref. [7] is clear, the physics behind it is not so obvious. In the rest of this section, we will make an attempt to understand the physical reason for this difference.

As we have just explained, the calculation of Refs. [9–12] is equivalent to a calculation in the projectile gauge with a nonstandard momentum dependent $i\epsilon$ prescription. It is well known that such a change of prescription is equivalent to a calculation not in the vacuum state but rather in a state which contains gluons [28,29]. We, therefore, ask ourselves why the projectile gauge calculation should be performed in a state which, on top of the background field b^+, also contains additional gluons.

B. Evolution as renormalization group in the projectile gauge

To answer this question let us first try to reformulate the projectile gauge calculation of Sec. II in terms of the Wilson renormalization group akin to the approach of Refs. [9–12]. The hadron is represented as a statistical ensemble of the static b^+ fields of the form Eq. (1) with a statistical weight $Z[b]$. Evolution in x generates induced vector potential which changes the statistical weight. Strictly speaking the induced vector potential is not static. It has components in the frequency range $p^-<\Lambda\propto 1/x$. However, as long as the frequency of the components of the projectile wave function are large enough, one can treat the induced potential as static during the interaction with the $q\overline{q}$ pair. Also, as long as the wavelength of the projectile in the x^- direction is large

[10]The fact that the transformation between the two gauges is nonunitary is not unusual. Even though it is a gauge transformation and therefore formally unitary, the gauge parameter itself depends on the dynamical field. Such transformations are generically nonunitary and do not preserve the scalar product.

[11]In writing this expression we have made use of the fact that the $i\epsilon$ term is important only for functions a^i that satisfy $D^2a^i=0$.

enough (p^+ is small) the induced vector potential can be approximated by a $\delta(x^-)$ shaped function. One can, equivalently, describe the hadron by a statistical ensemble of V and $V^\dagger$ with some statistical weight $Z[V,V^\dagger]$. It is clear that both descriptions encode exactly the same information. For this purpose one has to define analogs of χ and σ, that is, the (connected) fluctuation correlation functions of order α_s. The resulting evolution equation

$$\frac{d}{d\ln\frac{1}{x}}Z[U,U^\dagger]=\alpha_s\Bigg[\frac{1}{2}\Bigg(\frac{\delta^2}{\delta U(u)\,\delta U^\dagger(v)}[Z\chi_{q\bar{q}}(u,v)]+\frac{\delta^2}{\delta U^\dagger(u)\,\delta U(v)}[Z\chi_{\bar{q}q}(u,v)]+\frac{\delta^2}{\delta U(u)\,\delta U(v)}[Z\chi_{qq}(u,v)]+\frac{\delta^2}{\delta U^\dagger(u)\,\delta U^\dagger(v)}[Z\chi_{\overline{qq}}(u,v)]\Bigg)-\frac{\delta}{\delta U(u)}[Z\sigma_q(u)]-\frac{\delta}{\delta U^\dagger(u)}[Z\sigma_{\bar{q}}(u)]\Bigg] \tag{75}$$

is the analog of Eq. (42).

Using the formulas of Sec. II and the Appendix we find

$$\sigma_q^{\alpha\beta}(x_\perp)=-\frac{1}{2\pi^2}\int d^2z_\perp\frac{1}{(x-z)_\perp^2}\{\mathrm{tr}[U(x_\perp)\cdot U^\dagger(z_\perp)][U(z_\perp)]^{\alpha\beta}-N_c[U(x_\perp)]^{\alpha\beta}\}, \tag{76}$$

$$\chi_{q\bar{q}}^{\alpha\beta,\gamma\delta}(x_\perp,y_\perp)=\frac{1}{2\pi^2}\int d^2z_\perp\frac{(x-z)_\perp\cdot(y-z)_\perp}{(x-z)_\perp^2(y-z)_\perp^2}\{[U(z_\perp)\cdot U^\dagger(y_\perp)]^{\alpha\delta}[U^\dagger(z_\perp)\cdot U(x_\perp)]^{\gamma\beta}+[U(x_\perp)\cdot U^\dagger(z_\perp)]^{\alpha\delta}[U^\dagger(y_\perp)\cdot U(z_\perp)]^{\gamma\beta}-\delta^{\alpha\delta}[U^\dagger(y_\perp)\cdot U(x_\perp)]^{\gamma\beta}-[U(x_\perp)\cdot U^\dagger(y_\perp)]^{\alpha\delta}\delta^{\gamma\beta}\}, \tag{77}$$

$$\chi_{qq}^{\alpha\beta,\gamma\delta}(x_\perp,y_\perp)=-\frac{1}{2\pi^2}\int d^2z_\perp\frac{(x-z)_\perp\cdot(y-z)_\perp}{(x-z)_\perp^2(y-z)_\perp^2}\{[U(z_\perp)]^{\alpha\delta}[U(y_\perp)\cdot U^\dagger(z_\perp)\cdot U(x_\perp)]^{\gamma\beta}+[U(x_\perp)\cdot U^\dagger(z_\perp)\cdot U(y_\perp)]^{\alpha\delta}[U(z_\perp)]^{\gamma\beta}-[U(x_\perp)]^{\alpha\delta}[U(y_\perp)]^{\gamma\beta}-[U(y_\perp)]^{\alpha\delta}[U(x_\perp)]^{\alpha\beta}\}. \tag{78}$$

$\sigma_{\bar{q}}$ is obtained from σ_q by replacing Us by $U^\dagger s$. The same is true for $\chi_{\overline{qq}}$ and χ_{qq}. $\chi_{\bar{q}q}$ is obtained from $\chi_{q\bar{q}}$ by swapping $(x,\alpha\beta)$ with $(y,\gamma\delta)$ as this exchanges the q and $\bar{q}$ lines there. We stress that Eqs. (75)–(78) contain all the information that is contained in the BK equation as well as in the equations for higher correlation functions of U that appear in Ref. [7].

C. The snag

There is one implicit assumption in this procedure: namely, that the a^+ component of the vector potential is the only relevant one. One assumes that if, for example, an a^i component is generated in the evolution it does not affect the subsequent evolution of the physical cross section. This is, however, not quite right. What is important for the interaction with the projectile is not merely b^+ but rather the F^{+i} component of the color electric field. The interaction between the projectile and the target is due to the term $F^{-i}F^{+i}$ in the QCD Lagrangian. The F^{-i} component is the Weiszäker-Williams field of the $q\bar{q}$ pair, while the F^{+i} component is generated by the color charges in the target. In the eikonal approximation it is true that $F^{+i}=\partial^i b^+$ and, therefore, the coupling can be written as b^+J^-, where $J^-=\partial^i F^{-i}$. However, if there is a contribution to F^{+i} coming from the transverse component of the vector potential, it should be taken into account.

It is easy to see that such a contribution is indeed generated by the low-x evolution. Suppose one starts the evolution initially with the background field configuration as in Eq. (1). In the first step of the evolution one generates both the increment in a^+ and the increment in a^i. The two are related by the condition Eq. (61)

$$a^+=\frac{\partial^i}{\partial^-}a^i. \tag{79}$$

Naively, one would expect that since all the fluctuation fields in this step have small frequencies, it should be true that $a^+\gg a^i$ and, therefore, it should be safe to forget about a^i. The reason this is incorrect is that the on-shell solutions for a^i are discontinuous at $x^-=0$. Therefore, even though the field a^i is indeed small, it has a large derivative with respect to x^- which contributes to the field strength. In fact, the induced chromoelectric field is

$$\delta F^{+i}=\partial^i a^+-D^+a^i. \tag{80}$$

Recalling that on-shell a^i satisfy the second of the equations Eq. (61), we see that the second term in this expression is

$(\partial_1^2/2\partial^- a^i)$ and is of the same order as the first term $(\partial^i\partial^j/\partial^-)a^j$. Clearly, even if one starts initially from a background which only contains b^+, after long enough evolution a large transverse component of the vector potential is generated. When the contribution of the transverse component to the field strength is comparable to the contribution of the "+" component, the eikonal approximation breaks down and the evolution discussed in Sec. II ceases to be valid. It looks indeed very natural that in order to take into account the presence of the (potentially large) transverse field, the calculation in the projectile gauge should be performed around a state that contains transverse gluons apart from the b^+ background.

One could try to argue that the transverse part of the vector potential can be somehow gauged away and the calculation could still be performed consistently around a pure b^+ background. Even if this is possible the evolution of the background defined by such a procedure will be different from the evolution of Sec. II. In any case, we do not see how such "regauging" is possible.

It is instructive to see in more detail how the gauge fixing works in both the projectile and the target gauges and why the two seem to have different status as far as the renormalization-group structure is concerned. As we mentioned above, the chromoelectric field is created by the color charges in the target. In fact, the whole renormalization group procedure can be formulated in terms of the color charge density j^+ rather than the vector potentials themselves, which was in fact originally done in Refs. [9–11]. The background vector potentials are found as static solutions of classical equations of motion in the presence of the color charge density $j^+=\rho\delta(x^-)$,

$$F^{ij}=0,$$

$$D^iF^{+i}=D^i[D^ib^+-\partial^+b^i]=j^+. \tag{81}$$

An important property of these equations is that for a given ρ they have infinitely many solutions. By considering an arbitrary unitary matrix $V(x_\perp,x^-)$ it is straightforward to see that all the following are solutions:

$$b^i=iV^\dagger\partial^iV,$$

$$b^+=\frac{1}{D_\perp^2}[j^++D^i\partial^+b^i]. \tag{82}$$

The difference between the target and the projectile gauges at this point becomes important. In the target gauge $B^+=0$, the equations reduce to

$$B^i=iV^\dagger\partial^iV,$$

$$D^i\partial^+B^i=-j^+. \tag{83}$$

We, therefore, get rid of almost all the solutions, the only residual degeneracy being the value of the matrix V at $x^-\to-\infty$. The imposition of the residual gauge condition $\partial^iA^i(x^-\to-\infty)$ then removes all solutions except one. In the projectile gauge the situation is very different. The condition $a^-=0$ does not eliminate any of the infinite number of solutions Eq. (82). The choice of the residual gauge fixing is thus crucial to eliminate the redundant solutions. If those are not eliminated, the perturbative calculation will be plagued with zero mode problems. The calculation in Sec. II was in fact performed with the residual gauge fixing $\partial^ia^i(p^-=0)=0$. This gauge fixing does indeed eliminate all the solutions except the one which has vanishing a^i and has therefore precisely the form of Eq. (1).

Now, consider the renormalization-group (RG) calculation. Here, we have to integrate out modes which have higher frequency p^-. In the target gauge this is straightforward: the residual gauge condition does not care about frequency. It therefore eliminates nonzero frequency fluctuation modes which do not vanish at $x^-\to-\infty$ in the same way as it eliminated the static background solutions with this behavior. As a result, the fluctuation modes have a very similar structure to the background field and the induced field is similar to the background. It is, therefore, straightforward to formulate a self-similar renormalization-group transformation in this gauge. The situation is quite different in the projectile gauge. The residual gauge condition, although it fixes unambiguously the background, has nothing to say about the fluctuations—it only fixes the static modes. It is impossible therefore to ensure that the fluctuations will have the same form as the static background. In fact, as we have seen above, it will not be the case. The first equation of Eq. (61) in the projectile gauge is just one of the equations of motion (with or without the external source). This means that it is possible to have nonvanishing a^+ with vanishing a^i only at exactly zero frequency. At any finite frequency nonvanishing, a^i is required. As we have seen, this a^i contributes to the induced chromoelectric field, or equivalently to the induced color charge density. The calculation thus explicitly lacks a self-similar structure and proper renormalization-group setup does not seem possible[12] unless extra eikonal approximation is invoked.

The discussion of this section leads us to conclude that the projectile gauge calculation, as formulated in Ref. [7] and Sec. II is only valid as long as the eikonal approximation is applicable. When the evolution is continued for a large span of $1/x$, the eikonal approximation breaks down and the higher nonlinear corrections of Refs. [9–12] should become important. This is not to say that one cannot learn much from this simplified evolution. Quite to the contrary—clearly there is a range of x values where this evolution captures the relevant physics. This is particularly true when the target is large— is the case of a large nucleus discussed in Ref. [8]. In this case, the eikonal cross section is significantly different from the simple perturbative one which assumes single scattering. The nonlinearity of the evolution becomes important much faster than for a small hadron. One, therefore, expects

[12] It may be possible to reformulate RG so that it would include also transverse background fields or equivalently finite number of gluons in addition to b^+. This seems, however, to be quite a complicated problem and is far beyond the scope of our discussion here.

unitarization to appear already within the eikonal regime. Subsequent appearance of other nonlinear corrections will not change the fact that the total cross section has unitarized. It must, however, affect other more exclusive properties of the process such as the structure of final states. The spectrum of the target gauge fields is presumably directly related to the spectrum of the emitted gluons [27,30]. Thus, when the evolution of these fields changes even locally one expects this change to be visible in the spectrum of final state gluons. Assuming the local parton-hadron duality this then has to be mirrored in the spectrum of final state hadrons.

V. CONCLUSIONS

In recent years several approaches to the evolution of dense gluonic systems in the saturation regime were developed. The approaches differ from one another in many technical respects and the relationship between the physics is also not always clear. In this paper, our aim was to relate two of these approaches and thereby to try and reduce the entropy in the field. We have shown that the nonlinear JKLW equation of Refs. [9–12] coincides with the BK evolution equation derived in Refs. [7] and [8] as long as the gluon field induced by the evolution is small. We have argued that the approach of Ref. [7] should break down when the field is large enough so that the eikonal approximation intrinsic in the derivation of Ref. [7] ceases to be valid. We have also argued that the evolution of Refs. [9–12] when translated into the language of Ref. [7] corresponds to taking into account some noneikonal contributions.

We should note that our discussion puts into perspective the discrepancy between the double logarithmic limit (DLL) of the evolution of Refs. [9–12] and the evolution suggested in Refs. [31–33]. The AGL equation [31–33] has been shown to arise from the BK equation [8] in the regime where the evolution on the projectile side is dominated by production of small size dipoles, or equivalently, large transverse momentum gluons [8,19]. This is a natural regime when the target is a small object rather than a large object of the typical hadronic size. It was suggested in Ref. [19] that these configurations also dominate in usual DIS in the saturation regime. It seems to us that this point warrants further study. In any case this is not the standard DLL, where the evolution on the projectile side is dominated by large dipoles. There is, therefore, no reason to expect that the DLL of Refs. [9–13] has much to do with the AGL equation. In fact, as we have shown, the DLL of the BK evolution is itself extremely simple when considered as the evolution of the gluon distribution operator rather than the physical DIS cross section. It turns out to be entirely devoid of nonlinear corrections and coincides with the standard DGLAP double logarithmic equation. The DIS cross section still evolves nonlinearly and in fact saturates in this limit due to the nonlinear Glauber-type relation between the cross section and the gluon distribution [18]. On the other hand, the DLL of Refs. [9–13] is also nonlinear for the gluon distribution and as a result the evolution is slowed down already on the level of the gluon distribution [34].

We hope that this paper clarifies to some extent the relationship between the different approaches to the nonlinear low-x evolution. There are still many questions to be answered. In particular, it is desirable to find a more explicit relation between the nonlinearities of the JKLW equation and the breakdown of the eikonal approximation and to better understand the physics of these nonlinearities. It would be extremely useful to understand on the level of Feynman diagrams the differences between the BK and the JKLW evolution. At this point unfortunately we are unable to do that. Perhaps the most interesting question concerns the effect of these nonlinearities on the structure of the final states. Some work on the analytic understanding of quantities less inclusive than the total cross section has appeared recently [35–37]. There is also an ongoing numerical effort in connection with heavy-ion physics [38,39] in the framework of the McLerran-Venugopalan model [20,21]. Further progress in this direction is extremely important both for our understanding of nonlinear physics and for disentangling linear and nonlinear effects in the existing data.

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APPENDIX

In this appendix, we give a more detailed derivation of the evolution equation of Sec. II including the calculation of the contributions where the glue does not interact with the background (target). We start with the quadratic action for small fluctuations in the projectile gauge

$$S=\tfrac{1}{2}(a_a^+[-(\partial^-)^2]a_a^+-2(\partial^i a_a^+)(\partial^- a_a^i) -a_a^i\{[2D_{ab}^+[b]\partial^- -(\partial_\perp)^2\delta_{ab}]\delta^{ij}+\partial^i\partial^j\delta_{ab}\}a_b^j). \tag{A1}$$

The equation of motion for a^+ is

$$a^+=\frac{\partial^i}{\partial^-}a^i. \tag{A2}$$

Substituting this in Eq. (A1) we get

$$S=-\tfrac{1}{2}a_a^i(D^2)_{ab}\delta^{ij}a_b^j, \tag{A3}$$

where

$$(D^2)_{ab}=2D_{ab}^+[b]\partial^- -(\partial_\perp)^2\delta_{ab} \tag{A4}$$

is the same as the inverse propagator for a charged scalar field in the presence of a background field b^+. It has been calculated, for example, in Ref. [16]. The most useful form for us is

$$-i\left[\frac{1}{(D^2)}\right]_{ab}=\int\frac{dp^-}{2p^-(2\pi)^3}[\theta(x^--y^-)\theta(p^-)\theta(y^--x^-)\theta(-p^-)]\times\int d^2p_\perp d^2q_\perp e^{-ip\cdot x+iq\cdot y}\int\frac{d^2z_\perp}{(2\pi)^2}e^{-i(p_\perp-q_\perp)z_\perp}\tilde{U}^{-1}_{ab}(x^-,y^-,z_\perp), \tag{A5}$$

with $p^+=p_\perp^2/2p^-$, $q^+=q_\perp^2/2p^-$, and $q^-=p^-$. The color matrix $\tilde{U}^{-1}_{ab}(x^-,y^-,z_\perp)$ is[13]

$$\tilde{U}^{-1}_{ab}(x^-,y^-,z_\perp)=[\theta(-x^-)\theta(-y^-)+\theta(x^-)\theta(y^-)]\delta_{ab}+\theta(-x^-)\theta(y^-)\tilde{U}_{ab}(z_\perp)+\theta(x^-)\theta(-y^-)\tilde{U}^\dagger_{ab}(z_\perp). \tag{A6}$$

The on-shell two-point correlator of a^+ can be written as

$$\begin{aligned}&\langle a^+_a(x^+=0,x_\perp,x^-)a^+_b(y^+=0,y_\perp,y^-)\rangle\\&=\left\langle\frac{\partial^i_x}{\partial^-_x}a^i_a(x^+=0,x_\perp,x^-)a^i_b(y^+=0,y_\perp,y^-)\frac{\partial^i_y}{\partial^-_y}\right\rangle\\&=-\partial^i_x\partial^i_y\int\frac{dp^-}{p^-}\frac{1}{2(2\pi)}\frac{1}{(p^-)^2}[\theta(x^--y^-)\theta(p^-)-\theta(y^--x^-)\theta(-p^-)]\\&\quad\times\int d^2z_\perp\int\frac{d^2p_\perp}{(2\pi)^2}e^{+ip_\perp(x_\perp-z_\perp)}\int\frac{d^2q_\perp}{(2\pi)^2}e^{+iq_\perp(z_\perp-y_\perp)}\\&\quad\times e^{-ip_\perp^2/2p^-x^-}e^{+iq_\perp^2/2p^-y^-}\tilde{U}^{-1}_{ab}(x^-,y^-,z_\perp).\end{aligned} \tag{A7}$$

We now need to expand the eikonal factors

$$V(x_\perp)=\mathcal{P}\exp\left[-\int_{-\infty}^{+\infty}dx^-(b^++ga^+)(x_\perp,x^-)\right] \tag{A8}$$

to second order in the fields. Recalling that the background part of the field $b^+\propto\delta(x^-)$ and that the fluctuation field a^+ is nonsingular at $x^-=0$, this becomes

$$V(x^+=0,x_\perp)=\mathcal{P}\exp\left[-ig\int_{-\infty}^{0}dx^-a^+(x^+=0,x_\perp,x^-)\right]U(x_\perp)\mathcal{P}\exp\left[-ig\int_0^{+\infty}dx^-a^+(x^+=0,x_\perp,x^-)\right], \tag{A9}$$

where

$$U(x_\perp)=\mathcal{P}\exp\left[-i\int_{-\infty}^{+\infty}dx^-b^+(x^+=0,x_\perp,x^-)\right] \tag{A10}$$

is the classical part of the eikonal factor.

To second order in a^+ we have

[13]More rigorously, the structure of $\tilde{U}^{-1}$ is given by $\tilde{U}^{-1}_{ab}(z_\perp)=e^{i[\theta(x^-)-\theta(y^-)]b(z_\perp)}$. However, the difference between this expression and that given in Eq. (17) only shows up if it is multiplied by ∂^+ derivatives or $\delta(x^-)$ factors. Since we encounter no such factors in our calculation we will be using Eq. (17) throughout.

$$V(x^+=0,x_\perp)=\left\{1-ig\int_{-\infty}^{0}dx^-a^+(x^+=0,x_\perp,x^-)-g^2\int_{-\infty}^{0}dx^-dy^-\theta(y^--x^-)a^+(x^+=0,x_\perp,x^-)a^+(x^+=0,x_\perp,y^-)\right\}$$
$$\times U(x_\perp)\left\{1-ig\int_{0}^{+\infty}dx^-a^+(x^+=0,x_\perp,x^-)-g^2\int_{0}^{+\infty}dx^-dy^-\theta(y^--x^-)\right.$$
$$\left.\times a^+(x^+=0,x_\perp,x^-)a^+(x^+=0,x_\perp,y^-)\right\} \tag{A11}$$

or

$$V(x_\perp)=U(x_\perp)-ig\left\{\int_{-\infty}^{0}dw^-a^+(x^+=0,x_\perp,w^-)U(x_\perp)+U(x_\perp)\int_{0}^{+\infty}dw^-a^+(x^+=0,x_\perp,w^-)\right\}$$
$$-g^2\left\{\int_{-\infty}^{0}dw^-dz^-\theta(z^--w^-)a^+(x^+=0,x_\perp,w^-)a^+(x^+=0,x_\perp,z^-)U(x_\perp)\right.$$
$$+U(x_\perp)\int_{0}^{+\infty}dw^-dz^-\theta(z^--w^-)a^+(x^+=0,x_\perp,w^-)a^+(x^+=0,x_\perp,z^-)$$
$$\left.+\int_{-\infty}^{0}dw^-a^+(x^+=0,x_\perp,w^-)U(x_\perp)\int_{0}^{+\infty}dz^-a^+(x^+=0,x_\perp,z^-)\right\}. \tag{A12}$$

Similarly,

$$V^\dagger(y_\perp)=U^\dagger(y_\perp)+ig\left\{U^\dagger(y_\perp)\int_{-\infty}^{0}dz^-a^+(y^+=0,y_\perp,z^-)+\int_{0}^{+\infty}dz^-a^+(y^+=0,y_\perp,z^-)U^\dagger(y_\perp)\right\}$$
$$-g^2\left\{U^\dagger(y_\perp)\int_{-\infty}^{0}dw^-dz^-\theta(w^--z^-)a^+(y^+=0,y_\perp,w^-)a^+(y^+=0,y_\perp,z^-)\right.$$
$$+\int_{0}^{+\infty}dw^-dz^-\theta(w^--z^-)a^+(y^+=0,y_\perp,w^-)a^+(y^+=0,y_\perp,z^-)U^\dagger(y_\perp)$$
$$\left.+\int_{0}^{+\infty}dw^-a^+(y^+=0,y_\perp,w^-)U^\dagger(y_\perp)\int_{-\infty}^{0}dz^-a^+(y^+=0,y_\perp,z^-)\right\}. \tag{A13}$$

Rather than directly calculating the eikonal cross section, we will first calculate the tensor product of two eikonal factors and later take the trace over the color indices. We use the notation $A\otimes B=A^{\alpha\beta}B^{\gamma\delta}$. To second order in the fluctuation field we have

$$\langle V(x_\perp)\otimes V^\dagger(y_\perp)\rangle-\langle U(x_\perp)\otimes U^\dagger(y_\perp)\rangle$$
$$=g^2\left\langle\int_{-\infty}^{0}dw^-a^+(x^+=0,x_\perp,w^-)U(x_\perp)\otimes U^\dagger(y_\perp)\int_{-\infty}^{0}dz^-a^+(y^+=0,y_\perp,z^-)\right.$$
$$+\int_{-\infty}^{0}dw^-a^+(x^+=0,x_\perp,w^-)U(x_\perp)\otimes\int_{0}^{+\infty}dz^-a^+(y^+=0,y_\perp,z^-)U^\dagger(y_\perp)$$
$$+U(x_\perp)\int_{0}^{+\infty}dw^-a^+(x^+=0,x_\perp,w^-)\otimes U^\dagger(y_\perp)\int_{-\infty}^{0}dz^-a^+(y^+=0,y_\perp,z^-)+U(x_\perp)\int_{0}^{+\infty}dw^-a^+(x^+=0,y_\perp,w^-)$$
$$\otimes\int_{0}^{+\infty}dz^-a^+(y^+=0,y_\perp,z^-)U^\dagger(y_\perp)-\int_{-\infty}^{0}dw^-dz^-\theta(z^--w^-)a^+(x^+=0,x_\perp,w^-)a^+(x^+=0,x_\perp,z^-)U(x_\perp)\otimes U^\dagger(y_\perp)$$
$$-\int_{-\infty}^{0}dw^-a^+(x^+=0,x_\perp,w^-)U(x_\perp)\int_{0}^{+\infty}dz^-a^+(x^+=0,x_\perp,z^-)\otimes U^\dagger(y_\perp)-U(x_\perp)$$

$$\times\int_0^{+\infty}dw^-dz^-\,\theta(z^--w^-)a^+(x^+=0,x_\perp,w^-)a^+(x^+=0,x_\perp,z^-)\otimes U^\dagger(y_\perp)$$

$$-U(x_\perp)\otimes U^\dagger(y_\perp)\int_{-\infty}^0 dw^-dz^-\,\theta(w^--z^-)a^+(y^+=0,y_\perp,w^-)a^+(y^+=0,y_\perp,z^-)$$

$$-U(x_\perp)\otimes\int_0^{+\infty}dw^-a^+(y^+=0,y_\perp,w^-)U^\dagger(y_\perp)\int_{-\infty}^0 dz^-a^+(y^+=0,y_\perp,z^-)$$

$$-U(x_\perp)\otimes\int_0^{+\infty}dw^-dz^-\,\theta(w^--z^-)a^+(y^+=0,y_\perp,w^-)a^+(y^+=0,y_\perp,z^-)U^\dagger(y_\perp)\Big\rangle. \tag{A14}$$

For calculational purposes it is fruitful to separate the different contributions into the set terms where the glue interacts with the background field [the first four terms in Eq. (A14)] and the set of contributions where there is no such interaction [the other terms in Eq. (A14)].

Let us first consider the contribution where the glue is exchanged between the quark in the negative half plane ($x^-<0$) in the amplitude and an antiquark in the positive half plane in the complex conjugate amplitude:

$$\int_{-\infty}^0 dw^-\int_0^{+\infty}dz^-\langle a_a^+(x^+=0,x_\perp,w^-)a_b^+(y^+=0,y_\perp,z^-)\rangle\cdot[(t_aU(x_\perp))\otimes(t_bU^\dagger(y_\perp))]. \tag{A15}$$

One notices that only the $\theta(-p^-)$ term in Eq. (A7) survives and that, Eq. (A6), the color matrix $\tilde{U}^{-1}_{ab}(w^-,z^-,z_\perp)$ reduces to $\tilde{U}_{ab}(z_\perp)$.

$$\int_{-\infty}^0 dw^-\int_0^{+\infty}dz^-\langle a_a^+(x^+=0,x_\perp,w^-)a_b^+(y^+=0,y_\perp,z^-)\rangle$$
$$=-\partial_x^i\partial_y^i\int\frac{dp^-}{p^-}[-\theta(-p^-)]\frac{1}{2(2\pi)}\frac{1}{(p^-)^2}\int d^2z_\perp\int\frac{d^2p_\perp}{(2\pi)^2}e^{+ip_\perp(x_\perp-z_\perp)}\int\frac{d^2q_\perp}{(2\pi)^2}e^{+iq_\perp(z_\perp-y_\perp)}$$
$$\times\left(\int_{-\infty}^0 dw^-e^{-ip_\perp^2/2p^-w^-}\int_0^{+\infty}dz^-e^{+iq_\perp^2/2p^-z^-}\right)\cdot\tilde{U}_{ab}(z_\perp). \tag{A16}$$

The w^- and z^- integrations are easily performed

$$\int_{-\infty}^0 dw^-e^{-ip_\perp^2/2p^-w^-}\int_0^{+\infty}dz^-e^{+iq_\perp^2/2p^-z^-}=-\frac{(2p^-)^2}{p_\perp^2q_\perp^2}. \tag{A17}$$

Noting that

$$\int\frac{dp^-}{p^-}\theta(-p^-)=\int_{-\infty}^0\frac{dp^-}{p^-}=-\int_0^{+\infty}\frac{dp^-}{p^-}=-\int\frac{dp^-}{p^-}\theta(p^-), \tag{A18}$$

we have

$$\frac{1}{\pi}\int_0^{+\infty}\frac{dp^-}{p^-}\int d^2z_\perp\partial_x^i\partial_y^i\int\frac{d^2p_\perp}{(2\pi)^2}\frac{1}{p_\perp^2}e^{+ip_\perp(x_\perp-z_\perp)}\int\frac{d^2q_\perp}{(2\pi)^2}\frac{1}{q_\perp^2}e^{+iq_\perp(z_\perp-y_\perp)}\cdot\tilde{U}_{ab}(z_\perp). \tag{A19}$$

The transverse momenta integrals yield

$$\int\frac{d^2p_\perp}{(2\pi)^2}\frac{1}{p_\perp^2}e^{+ip_\perp(x_\perp-z_\perp)}=-\frac{1}{4\pi}\log(x_\perp-z_\perp)^2. \tag{A20}$$

Taking the derivatives

$$\frac{1}{4\pi^3}\int_0^{+\infty}\frac{dp^-}{p^-}\int d^2z_\perp\frac{(x-z)_\perp\cdot(y-z)_\perp}{(x-z)_\perp^2(y-z)_\perp^2}\cdot\tilde{U}_{ab}(z_\perp), \tag{A21}$$

we use the following identity valid for SU(N) group:

$$\tilde{U}_{ab}(z_\perp)(t_a U(x_\perp))\otimes(t_b U^\dagger(y_\perp))=2\,\mathrm{tr}[t_a U(z_\perp)t_b U^\dagger(z_\perp)](t_a U(x_\perp))^{\alpha\beta}(t_b U^\dagger(y_\perp))^{\gamma\delta}$$
$$=\frac{1}{2N_c}[N_c(U(z_\perp)\cdot U^\dagger(y_\perp))^{\alpha\delta}(U^\dagger(z_\perp)\cdot U(x_\perp))^{\gamma\beta}-U(x_\perp)^{\alpha\beta}U^\dagger(y_\perp)^{\gamma\delta}]. \tag{A22}$$

This contribution, the second term in Eq. (A14), therefore is

$$\frac{1}{4\pi^3}\int_0^{+\infty}\frac{dp^-}{p^-}\int d^2z_\perp\frac{(x-z)_\perp\cdot(y-z)_\perp}{(x-z)_\perp^2(y-z)_\perp^2}\,\frac{1}{2N_c}[N_c(U(z_\perp)\cdot U^\dagger(y_\perp))^{\alpha\delta}(U^\dagger(z_\perp)\cdot U(x_\perp))^{\gamma\beta}-U(x_\perp)^{\alpha\beta}U^\dagger(y_\perp)^{\gamma\delta}]. \tag{A23}$$

The third contribution, due to the exchange of the gluon between the quark in the positive half plane and the antiquark in the negative half plane, is calculated similarly with the only difference that now $\tilde{U}^{-1}_{ab}(w^-,z^-,z_\perp)=\tilde{U}^\dagger_{ab}(z_\perp)$ and we pick up the $\theta(p^-)$ term in the propagator.

$$\int_0^{+\infty}dw^-\int_{-\infty}^0 dz^-\langle a_a^+(x^+=0,x_\perp,w^-)a_b^+(y^+=0,y_\perp,z^-)\rangle\cdot[(U(x_\perp)t_a)\otimes(U^\dagger(y_\perp)t_b)]$$
$$=\frac{1}{4\pi^3}\int_0^{+\infty}\frac{dp^-}{p^-}\int d^2z_\perp\frac{(x-z)_\perp\cdot(y-z)_\perp}{(x-z)_\perp^2(y-z)_\perp^2}\times\frac{1}{2N_c}[N_c(U(x_\perp)\cdot U^\dagger(z_\perp))^{\alpha\delta}(U^\dagger(y_\perp)\cdot U(z_\perp))^{\gamma\beta}$$
$$-U(x_\perp)^{\alpha\beta}U^\dagger(y_\perp)^{\gamma\delta}]. \tag{A24}$$

For the quark self-energy correction everything is the same as in Eq. (A23) except that the transverse coordinates of the two fields coincide ($y_\perp\to\tilde{x}_\perp=x_\perp$),

$$-\int_{-\infty}^0 dw^-\int_0^{+\infty}dz^-\langle a_a^+(x^+=0,x_\perp,w^-)a_b^+(y^+=0,\tilde{x}_\perp=x_\perp,z^-)\rangle\cdot[(t_aU(x_\perp)t_b)\otimes U^\dagger(y_\perp)]$$
$$=-\frac{1}{4\pi^3}\int_0^{+\infty}\frac{dp^-}{p^-}\int d^2z_\perp\frac{1}{(x-z)_\perp^2}\times\frac{1}{2N_c}[N_c\mathrm{tr}((U(x_\perp)\cdot U^\dagger(z_\perp))(U(z_\perp))^{\alpha\beta}-(U(x_\perp))^{\alpha\beta}]U^\dagger(y_\perp)^{\gamma\delta}. \tag{A25}$$

Finally, the antiquark self-energy correction:

$$-\int_0^{+\infty}dw^-\int_{-\infty}^0 dz^-\langle a_a^+(x^+=0,y_\perp,w^-)a_b^+(y^+=0,\tilde{y}_\perp=y_\perp,z^-)\rangle\cdot[U(x_\perp)\otimes(t_at_bU^\dagger(y_\perp))]$$
$$=-\frac{1}{4\pi^3}\int_{-\infty}^0\frac{dp^-}{p^-}\int d^2z_\perp\frac{1}{(y-z)_\perp^2}\times\frac{1}{2N_c}[N_c\mathrm{tr}(U^\dagger(y_\perp)\cdot U(z_\perp))(U^\dagger(z_\perp))^{\gamma\delta}-(U^\dagger(y_\perp))^{\gamma\delta}]U(x_\perp)^{\alpha\beta}. \tag{A26}$$

Now let us look at the other set of contributions. Since for all of these there is no interaction with the background we have, Eq. (A6), $\tilde{U}^{-1}_{ab}(w^-,z^-,z_\perp)=\delta_{ab}$. The $z_\perp$ integral is trivial for all terms in this set (it yields a δ function for the transverse momenta). However, as we want to combine both sets of contributions in the final result, this integral will not be performed.

Consider the first term in Eq. (A14)—the exchange between the quark and the antiquark in the negative half-plane—

$$\int_{-\infty}^0 dw^-dz^-\langle a_a^+(x^+=0,y_\perp,w^-)a_b^+(y^+=0,\tilde{y}_\perp=y_\perp,z^-)\rangle\cdot[(t_aU(x_\perp))\otimes(U^\dagger(y_\perp)t_b)]. \tag{A27}$$

In the correlator part now both θ functions survive, corresponding to the two possible orderings of w^- and z^-

$$\int_{-\infty}^{0} dw^- dz^- \langle a_a^+(x^+=0,x_\perp,w^-) a_b^+(y^+=0,y_\perp,z^-)\rangle$$
$$= -\partial_x^i \partial_y^i \int \frac{dp^-}{p^-}\frac{1}{2(2\pi)}\frac{1}{(p^-)^2}\int d^2z_\perp \int \frac{d^2p_\perp}{(2\pi)^2} e^{+ip_\perp(x_\perp - z_\perp)} \int \frac{d^2q_\perp}{(2\pi)^2} e^{+iq_\perp(z_\perp - y_\perp)}$$
$$\times\left(\int_{-\infty}^{0} dw^- dz^- [\theta(w^- - z^-)\theta(p^-) - \theta(z^- - w^-)] e^{-i(p_\perp^2/2p^-)w^-} e^{+i(q_\perp^2/2p^-)z^-}\right)\cdot \delta_{ab}. \tag{A28}$$

The w^- and z^- integrals now have to be performed with more care. Doing this we get

$$-\frac{1}{\pi}\partial_x^i \partial_y^i \int \frac{dp^-}{p^-}\int d^2z_\perp \int \frac{d^2p_\perp}{(2\pi)^2} e^{+ip_\perp(x_\perp - z_\perp)} \int \frac{d^2q_\perp}{(2\pi)^2} e^{+q_\perp(z_\perp - y_\perp)}$$
$$\times\left[\theta(p^-)\left(\frac{1}{p_\perp^2 q_\perp^2} - \frac{1}{p_\perp^2(q_\perp^2 - p_\perp^2)}\right) + \theta(-p^-)\left(-\frac{1}{p_\perp^2 q_\perp^2} - \frac{1}{q_\perp^2(q_\perp^2 - p_\perp^2)}\right)\right]\cdot \delta_{ab}. \tag{A29}$$

Recalling Eq. (A18) we obtain for the momentum denominators

$$\int \frac{dp^-}{p^-}\left[\theta(p^-)\left(\frac{1}{p_\perp^2 q_\perp^2} - \frac{1}{p_\perp^2(q_\perp^2 - p_\perp^2)}\right) + \theta(-p^-)\left(-\frac{1}{p_\perp^2 q_\perp^2} - \frac{1}{q_\perp^2(q_\perp^2 - p_\perp^2)}\right)\right]$$
$$= \int_0^{+\infty} \frac{dp^-}{p^-}\left[2\frac{1}{p_\perp^2 q_\perp^2} + \left(\frac{1}{q_\perp^2(q_\perp^2 - p_\perp^2)} - \frac{1}{p_\perp^2(q_\perp^2 - p_\perp^2)}\right)\right]$$
$$= \int_0^{+\infty} \frac{dp^-}{p^-}\frac{1}{p_\perp^2 q_\perp^2}. \tag{A30}$$

The correlator part is then

$$\int_{-\infty}^{0} dw^- dz^- \langle a_a^+(x^+=0,x_\perp,w^-) a_b^+(y^+=0,y_\perp,z^-)\rangle = -\frac{1}{4\pi^3}\int_0^{+\infty}\frac{dp^-}{p^-}\int d^2z_\perp \frac{(x-z)_\perp\cdot(y-z)_\perp}{(x-z)_\perp^2 (y-z)_\perp^2}\cdot \delta_{ab}. \tag{A31}$$

The color algebra raises no problems,

$$\delta_{ab}(t_a U(x_\perp)) \otimes (U^\dagger(y_\perp) t_b) = \delta_{ab}(t_a U(x_\perp))^{\alpha\beta}(U^\dagger(y_\perp) t_b)^{\gamma\delta} = \frac{1}{2N_c}[N_c \delta^{\alpha\delta}(U^\dagger(y_\perp)\cdot U(x_\perp))^{\gamma\beta} - U(x_\perp)^{\alpha\beta} U^\dagger(y_\perp)^{\gamma\delta}] \tag{A32}$$

and so the first term in Eq. (A14) is

$$\frac{1}{4\pi^3}\int_0^{+\infty}\frac{dp^-}{p^-}\int d^2z_\perp \frac{(x-z)_\perp\cdot(y-z)_\perp}{(x-z)_\perp^2 (y-z)_\perp^2}\times\frac{1}{2N_c}[N_c \delta^{\alpha\delta}(U^\dagger(y_\perp)\cdot U(x_\perp))^{\gamma\beta} - U(x_\perp)^{\alpha\beta} U^\dagger(y_\perp)^{\gamma\delta}]. \tag{A33}$$

The quark to antiquark exchange in the positive half plane gives

$$\int_0^{+\infty} dw^- dz^- \langle a_a^+(x^+=0,y_\perp,w^-) a_b^+(y^+=0,\tilde{y}_\perp y_\perp,z^-)\rangle\cdot[(U(x_\perp)t_a)\otimes(t_b U^\dagger(y_\perp))]$$
$$= -\frac{1}{4\pi^3}\int_0^{+\infty}\frac{dp^-}{p^-}\int d^2z_\perp \frac{(x-z)_\perp\cdot(y-z)_\perp}{(x-z)_\perp^2 (y-z)_\perp^2}\times\frac{1}{2N_c}\{N_c \delta^{jk}[U(x_\perp)\cdot U^\dagger(y_\perp)]^{\alpha\delta} - U(x_\perp)^{\alpha\beta} U^\dagger(y_\perp)^{\gamma\delta}\}. \tag{A34}$$

Now we combine the two terms that give corrections to the quark line—the fifth and the seventh terms in Eq. (A14). It is easy to see that they have the same color structure and will also yield the same transverse structure.

The color algebra is trivial

$$\delta_{ab}(t_a t_b U(x_\perp)) \otimes U^\dagger(y_\perp) = \delta_{ab}(t_a t_b U(x_\perp))^{\alpha\beta}(U^\dagger(y_\perp))^{\gamma\delta} = \frac{(N_c^2 - 1)}{2N_c} U(x_\perp)^{\alpha\beta} U^\dagger(y_\perp)^{\gamma\delta}. \tag{A35}$$

The fifth term is the same as Eq. (A33), but with $y_\perp \to \tilde{x}_\perp = x_\perp$ and only the $\theta(-p^-)$ term surviving,

$$-\int_{-\infty}^{0} dw^- dz^- \theta(z^- - w^-)\langle a_a^+(x^+=0,x_\perp,w^-)a_b^+(x^+=0,x_\perp,z^-)\rangle \cdot \delta_{ab}(t_a t_b U(x_\perp))\otimes U^\dagger(y_\perp)$$

$$= -\frac{1}{\pi}\partial_x^i \partial_y^i \int_0^{+\infty}\frac{dp^-}{p^-}\int d^2z_\perp \int \frac{d^2p_\perp}{(2\pi)^2}e^{+ip_\perp(x_\perp - z_\perp)}\int \frac{d^2q_\perp}{(2\pi)^2}e^{+iq_\perp(z_\perp - \tilde{x}_\perp)}$$

$$\times\left[\frac{1}{p_\perp^2 q_\perp^2}+\frac{1}{q_\perp^2(q_\perp^2-p_\perp^2)}\right]\frac{(N_c^2-1)}{2N_c}U(x_\perp)^{\alpha\beta}U^\dagger(y_\perp)^{\gamma\delta}. \tag{A36}$$

And the seventh term is the same as Eq. (A34),

$$-\int_{0}^{+\infty} dw^- dz^- \theta(z^- - w^-)\langle a_a^+(x^+=0,x_\perp,w^-)a_b^+(x^+=0,x_\perp,z^-)\rangle \cdot \delta_{ab}(U(x_\perp)t_a t_b)\otimes U^\dagger(y_\perp)$$

$$= -\frac{1}{\pi}\partial_x^i \partial_y^i \int_0^{+\infty}\frac{dp^-}{p^-}\int d^2z_\perp \int \frac{d^2p_\perp}{(2\pi)^2}e^{+ip_\perp(x_\perp - z_\perp)}\int \frac{d^2q_\perp}{(2\pi)^2}e^{+iq_\perp(z_\perp - \tilde{x}_\perp)}$$

$$\times\left[\frac{1}{p_\perp^2 q_\perp^2}-\frac{1}{p_\perp^2(q_\perp^2-p_\perp^2)}\right]\frac{(N_c^2-1)}{2N_c}U(x_\perp)^{\alpha\beta}U^\dagger(y_\perp)^{\gamma\delta}. \tag{A37}$$

Adding the two terms Eqs. (A36),(A37) and then performing the transverse integrations we get

$$\frac{1}{4\pi^3}\int_0^{+\infty}\frac{dp^-}{p^-}\int d^2z_\perp \frac{1}{(x-z)_\perp^2}\frac{(N_c^2-1)}{2N_c}U(x_\perp)^{\alpha\beta}U^\dagger(y_\perp)^{\gamma\delta}. \tag{A38}$$

The correction to the antiquark line—the eighth and the tenth terms in Eq. (A18)—give similarly

$$\frac{1}{4\pi^3}\int_0^{-\infty}\frac{dp^-}{p^-}\int d^2z_\perp \frac{1}{(y-z)_\perp^2}\frac{(N_c^2-1)}{2N_c}U(x_\perp)^{\alpha\beta}U^\dagger(y_\perp)^{\gamma\delta}. \tag{A39}$$

Finally, combining all the terms together we get

$$\begin{aligned}
&\langle V(x_\perp)^{\alpha\beta}V^\dagger(y_\perp)^{\gamma\delta}\rangle - \langle U(x_\perp)^{\alpha\beta}U^\dagger(y_\perp)^{\gamma\delta}\rangle \\
&= \frac{1}{8\pi^3}\log\left(\frac{x_0}{x}\right)\int d^2z_\perp \cdot \Bigg\{[(U(z_\perp)\cdot U^\dagger(y_\perp))^{\alpha\delta}(U^\dagger(z_\perp)\cdot U(x_\perp))^{\gamma\beta} + (U(x_\perp)\cdot U^\dagger(z_\perp))^{\alpha\delta}(U^\dagger(y_\perp)\cdot U(z_\perp))^{\gamma\beta} \\
&\quad - \delta^{\alpha\delta}(U^\dagger(y_\perp)\cdot U(x_\perp))^{\gamma\beta} - (U(x_\perp)\cdot U^\dagger(y_\perp))^{\alpha\delta}\delta^{\beta\gamma}]\cdot \frac{(x-z)_\perp\cdot(y-z)_\perp}{(x-z)_\perp^2(y-z)_\perp^2} \\
&\quad - [\mathrm{tr}(U(x_\perp)\cdot U^\dagger(z_\perp))(U(z_\perp))^{\alpha\beta} - N_c U(x_\perp)^{\alpha\beta}]U^\dagger(y_\perp)^{\gamma\delta}\cdot \frac{1}{(x-z)_\perp^2} \\
&\quad - U(x_\perp)^{\alpha\beta}[\mathrm{tr}(U^\dagger(y_\perp)\cdot U(z_\perp))(U^\dagger(z_\perp))^{\gamma\delta} - N_c U^\dagger(y_\perp)^{\gamma\delta}]\cdot \frac{1}{(z-y)_\perp^2}\Bigg\}.
\end{aligned} \tag{A40}$$

This coincides with the result of Ref. [7]. When comparing this evolution equation with the results of Ref. [7], one should keep in mind that there the evolution is considered with respect to the variable ζ. The relation between the two evolution equations is given by $d/d\ln\frac{1}{x}\langle\cdots\rangle = -2d/d\ln\zeta\langle\cdots\rangle$. Now taking trace over the color indices we obtain the evolution equation for the scattering cross section given in Sec. II.

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