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$O(\alpha^2\Gamma, \alpha^3\Gamma)$ binding effects in orthopositronium decay

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We present a new, simplified analysis of the low-energy electron-positron interaction, and use the resulting effective theory to calculate the binding effects that contribute to the decay rate Γ of orthopositronium through $O(\alpha^3 \ln \alpha\,\Gamma)$. We express the total decay rate in terms of the annihilation rate for a free electron and positron at threshold, which has just recently been computed to sufficient precision. Our $O(\alpha^2\Gamma)$ result corrects errors in a previous analysis.

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There has been long standing uncertainty regarding a possible disagreement between theoretical and experimental determinations of the orthopositronium ground-state decay rate [1]. A discrepancy here could have important implications. For example, it might indicate a failure of perturbative expansions in α, the fine structure constant of quantum electrodynamics (QED); or it could signal the presence of new physical phenomena beyond QED. The decay rate is currently of particular interest because important parts of the theoretical calculation have recently been completed: in Ref. [1] the annihilation rate for a free electron and positron at threshold is calculated through $O(m\alpha^5)$, where m is the electron mass. This is two powers of α beyond leading order. Here we convert this decay rate for a free electron and positron into the decay rate for a positronium atom, the electron-positron bound state. We also calculate new contributions at $O(\alpha^3 \ln \alpha\,\Gamma)$.

Traditionally, precision bound-state calculations have been formulated within the context of quantum field theory. A Bethe-Salpeter analysis is an example; more recently, the nonrelativistic QED (NRQED) effective field theory approach has often been used [2,4]. It is, however, much simpler to recast the problem within the familiar framework of nonrelativistic quantum mechanics [3]. Long-range QED interactions correspond to standard long-range potentials in the nonrelativistic Hamiltonian, while short-distance effects are described by a small number of local operators whose coefficients must be determined by comparing with the complete, relativistic formulation of QED. Here we relate the short-distance coefficients in the positronium Hamiltonian to the threshold annihilation rate computed in [1], thereby obtaining the full Hamiltonian needed for computing the decay rate. The decay rate enters our formalism through the non-Hermiticity of our Hamiltonian and the complex energies that result: $\Gamma = -2\,\mathrm{Im}\,E$.

The methods used in this paper are new to QED applications, and have a number of desirable features. A finite ultraviolet cutoff is built into our Hamiltonian. This cutoff excludes high-momentum states that are poorly described by the nonrelativistic dynamics. Unlike traditional approaches we do not take the cutoff to infinity at the end of the calculation; rather it is held fixed at a value of order the electron's mass. Consequently, no divergences occur, and the resulting energy eigenvalue problem can be solved nonperturbatively using simple numerical methods—for example, by evaluating the matrix elements of the Hamiltonian using a finite basis set, and then solving a matrix eigenvalue problem. This eliminates the need for bound-state perturbation theory. Another feature is that only physical inputs—on-shell scattering amplitudes, for example—are required from full QED, and therefore gauge and QED-regulator independence are explicit. Finally, our approach can be adapted in a natural way to multielectron systems such as helium, where the lack of an exact zero-order solution and the complexity of bound state perturbation theory makes a nonperturbative solution particularly convenient.

We now proceed to the effective Hamiltonian. We work in the center-of-momentum frame of the electron and positron, and consider only states of orbital angular momentum 0 (S-states), and spin 1 (triplet- or ortho-states). In order to make use of the threshold results in [1], we give the photon a small mass, λ, which is taken to 0 after the determination of the local operator coefficients; the three final-state photons are not given a mass.

We begin with Hamiltonian

$$H \approx \frac{p^2}{m} - \frac{p^4}{4m^3} + V + iW \tag{1}$$

where V and W are Hermitian. Potential W accounts for the effects of three-photon decay, while V accounts for all other interactions. To determine corrections through $O(\alpha^2\Gamma)$ or $O(\alpha^3\Gamma)$, we need retain only nonannihilation terms that contribute to the positronium S-state binding energy through $O(\alpha^4 m)$ or $O(\alpha^5 m)$, respectively. We write V as the sum of three terms,

$$V(E) = V_0 + V_{\mathrm{rel}} + V_{\mathrm{rad}}(E), \tag{2}$$

where the Coulomb potential V_0 and the leading relativistic corrections V_{rel} are given by

$$\langle l|V_0|k\rangle = \frac{-4\pi\alpha}{|l-k|^2+\lambda^2} \exp\left(-\frac{|l-k|^2+\lambda^2}{2\Lambda^2}\right), \tag{3}$$

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$$\langle l|V_{\rm rel}|k\rangle=\left[\frac{4\pi\alpha}{|l-k|^2+\lambda^2}\left(\frac{2}{3m^2}|l-k|^2-\frac{1}{2m^2}(l^2+k^2)\right)-\frac{4\pi\alpha}{2\Lambda^2}+\frac{4\pi\alpha}{4m^2}\frac{(l^2-k^2)^2}{(|l-k|^2+\lambda^2)^2}\right]\exp\left(-\frac{|l-k|^2+\lambda^2}{2\Lambda^2}\right)$$
$$+\frac{4\pi\alpha}{2m^2}\exp\left(-\frac{|l-k|^2}{2\Lambda^2}\right). \tag{4}$$

We have introduced Gaussian factors to suppress the potentials at high momentum. Typically we take the ultraviolet cutoff $\Lambda\approx m$, although our final decay rate is (and must be) Λ independent. With potentials V_0 and $V_{\rm rel}$ in place, our Hamiltonian correctly reproduces the QED amplitude for one-photon exchange and one-photon virtual annihilation to lowest and first order in $(v/c)^2$ (of the electron). $V_{\rm rad}(E)$ accounts for one-loop radiative corrections:

$$\langle l|V_{\rm rad}(E)|k\rangle=\frac{8\alpha}{3\pi m^2}\langle l|\exp\left(-\frac{p^2}{4\Lambda^2}\right)p^i(p^2/m+V_0-E)\ln\left(\frac{m/4}{p^2/m+V_0-E}\right)p^i\exp\left(-\frac{p^2}{4\Lambda^2}\right)|k\rangle$$
$$+\frac{14\alpha^2}{3m^2}\ln\frac{|l-k|}{m/2}\exp\left(-\frac{|l-k|^2}{2\Lambda^2}\right)+\frac{\alpha^2}{m^2}\{-\tfrac{74}{15}+\tfrac{2}{3}\ln 2+D\}\exp\left(-\frac{|l-k|^2}{2\Lambda^2}\right). \tag{5}$$

$V_{\rm rad}(E)$ gives in positronium the analogue of the nonrecoil Lamb shift and the recoil Salpeter correction in hydrogen. The final term is a local operator accounting for effects at high momentum; parameter D is a counterterm which will be determined shortly.

The annihilation of the electron and positron occurs over distances of order $\Delta x\approx 1/m$, which are much smaller than wavelengths typical of the electron and positron in the atom, $\lambda\approx 1/\alpha m$. Thus the annihilation potential W consists entirely of short-distance interactions, which, to the order of interest, can be parametrized for S states as follows:

$$\langle l|\mathrm{W}|k\rangle=A^{(0)}\left[(1+\alpha A^{(1)}+\alpha^2A^{(2)}+\alpha^3A^{(3)})\right.$$
$$\left.+(B^{(0)}+\alpha B^{(1)})\frac{E}{m}\right]\exp\left(-\frac{|l-k|^2}{2\Lambda^2}\right). \tag{6}$$

We will adjust parameters $A^{(0)}$, $A^{(1)}$, $A^{(2)}$, and $B^{(0)}$ so that our Hamiltonian reproduces QED results for electron-positron annihilation into three photons. Since determination of $A^{(3)}$ and of $B^{(1)}$ requires the as-yet unknown $O(m\alpha^6)$ threshold annihilation rate, and the leading term in the momentum expansion for the $O(m\alpha^4)$ rate, respectively, we simply set $A^{(3)}=B^{(1)}=0$. Doing so introduces an error in the decay rate of $O(\alpha^3\Gamma)$.

The p^4 operator in Eq. (1) is ill-defined at high momenta. To regulate this operator, we replace it by an energy dependent potential [3]:

$$p^4\to m^2[E-(V+iW)]^2. \tag{7}$$

Keeping only the relevant parts of this expression, our final effective Hamiltonian is therefore

$$H(E)=H_0(E)+i\overline{W}(E) \tag{8}$$

where

$$H_0(E)=\frac{p^2}{m}+V(E)-\frac{1}{4m}(E-V_0)^2, \tag{9}$$

$$\overline{W}(E)=W+\frac{1}{2m}(E-V_0)(W_0+W_1). \tag{10}$$

Potentials V [Eqs. (2), (3), (4), (5)], V_0 [Eq. (3)], and W [Eq. (6)] are given above, and

$$\langle l|W_0|k\rangle\equiv\langle l|W_1|k\rangle/A^{(1)}\equiv A^{(0)}\exp\left(-\frac{|l-k|^2}{2\Lambda^2}\right). \tag{11}$$

Parameter D in Eq. (5) is tuned to correctly reproduce the one-loop contribution to $e\overline{e}\to e\overline{e}$ threshold scattering, and is found to be

$$D=-\sqrt{\pi}\left[\frac{-121}{36}\frac{\Lambda}{m}-9\frac{m}{\Lambda}+\frac{5}{3}\left(\frac{m}{\Lambda}\right)^3\right]. \tag{12}$$

The Hermitian Hamiltonian $H_0(E)$ is accurate through $O(\alpha^5m)$.

The parameters in W are determined by considering the imaginary part of the S-wave scattering amplitude for $e\overline{e}\to 3\gamma\to e\overline{e}$, with electron momentum k in the center-of-momentum frame; the optical theorem relates this amplitude to the free-particle annihilation rate. For small k, the imaginary part of this amplitude can be parametrized as

$$\mathcal{T}(k)=\mathcal{T}_0\left\{\left[1+\beta\frac{k^2}{m^2}+O\left(\frac{k^4}{m^4}\right)\right]+\alpha\left[2\frac{m}{\lambda}+a_0+a_1\frac{\lambda}{m}+O\left(\frac{\lambda^2}{m^2}\right)\right]\right.$$
$$\left.+\alpha^2\left[(1+2\ln 2)\frac{m^2}{\lambda^2}+2a_0\frac{m}{\lambda}-\frac{1}{3}\ln\frac{m}{\lambda}+b_0+O\left(\frac{\lambda}{m}\right)\right]+O(\alpha^3)\right\}. \tag{13}$$

We define $\mathcal{T}(k)$ using nonrelativistic normalization for the external particles [7]. Parameters $\mathcal{T}_0$, β, a_0, a_1, b_0 are determined using QED perturbation theory [5].

We now calculate $\mathcal{T}$ in our Hamiltonian theory, and adjust the unknown parameters to reproduce the QED result order by order in α and k^2. Matching at lowest order in α implies that

$$A^{(0)}=\mathcal{T}_0, \quad B^{(0)}=\beta+m^2/\Lambda^2. \tag{14}$$

$A^{(1)}$ and $A^{(2)}$ are determined by matching the $O(\alpha)$ and $O(\alpha^2)$ contributions in $\mathcal{T}(0)$, respectively; we obtain

$$A^{(1)}=a_0+\frac{1}{\sqrt{\pi}}\left[\frac{4}{3}\left(\frac{\Lambda}{m}\right)+3\left(\frac{\Lambda}{m}\right)^{-1}\right]+\left[a_1-\frac{7}{12}+\left(\frac{\Lambda}{m}\right)^{-2}\right]\frac{\lambda}{m}+O\left(\frac{\lambda^2}{m^2}\right), \tag{15}$$

$$A^{(2)}=b_0-2a_1+\frac{1}{\sqrt{\pi}}\left[\frac{4}{3}\left(\frac{\Lambda}{m}\right)+3\left(\frac{\Lambda}{m}\right)^{-1}\right]a_0+\frac{1}{3}\ln\frac{\Lambda}{m}+\frac{1}{\pi\sqrt{\pi}}\left\{\left[-\frac{44\sqrt{6}}{81}\left(\gamma-\ln\frac{2\Lambda^2}{3m^2}-2\right)\right]\left(\frac{\Lambda}{m}\right)^3\right.$$
$$\left.+\left[\frac{7}{3}\ln\frac{\Lambda}{m}+\frac{56\sqrt{6}}{27}\left(\gamma-\ln\frac{2\Lambda^2}{3m^2}-\frac{2}{7}\right)-\frac{37}{15}+\frac{1}{3}\ln 2-\frac{7}{6}\gamma\right]\left(\frac{\Lambda}{m}\right)\right\}+\left(\frac{83}{24\pi}-\frac{11\sqrt{3}}{12\pi}+\frac{11}{48}\right)\left(\frac{\Lambda}{m}\right)^2$$
$$+\left(\frac{25}{2\pi}-\frac{4\sqrt{3}}{3\pi}+\frac{17}{18}-\frac{5}{6}\ln 2-\frac{1}{3}\gamma+\frac{2}{\sqrt{\pi}}\kappa\right)+\left(\frac{49}{6\pi}-\frac{3\sqrt{3}}{2\pi}-\frac{1}{4}\right)\left(\frac{\Lambda}{m}\right)^{-2}+O\left(\frac{\lambda}{m}\right). \tag{16}$$

Here $\gamma=-\psi(1)=0.577216$ is the Euler constant and $\kappa\equiv\int dx\,\ln x^{-1}\exp(-x^2)\mathrm{erf}(x)^2=0.051428$. Having determined all the necessary parameters, we can now safely set $\lambda=0$.

Now that our Hamiltonian is completely specified we finally solve for the decay rate, given by the imaginary part of the ground-state energy eigenvalue. Note that due to the presence of the cutoff, no divergences occur when calculating matrix elements, and no intricate limiting procedures are necessary to solve the eigenvalue problem—renormalization is automatic. To avoid dealing with non-Hermitian matrices we choose to work only to first order in the annihilation potential $\bar{W}$; higher-order terms are suppressed by several powers of α beyond the precision of interest.

We first solve the eigenvalue problem for H_0,

$$H_0(E_0)\,|\psi_0\rangle=E_0\,|\psi_0\rangle, \tag{17}$$

to obtain the ground-state energy and wave function. The energy dependence of H_0 is easily handled by iterating the eigenvalue equation, starting with an approximate energy in H_0; the answer converges to adequate precision after only a few iterations. The eigenfunctions for our energy-dependent Hamiltonian must be normalized so that [8]

$$\langle\psi_0|1-\frac{\partial H_0}{\partial E}|\psi_0\rangle\Big|_{E=E_0}=1. \tag{18}$$

Then the decay rate, to first order in $\bar{W}$, is

$$\Gamma=-2\langle\psi_0|\bar{W}(E_0)|\psi_0\rangle \tag{19}$$
$$=\Gamma_0\left\{\left[1+\alpha A^{(1)}+\alpha^2A^{(2)}+\alpha^3A^{(3)}\right.\right.$$
$$\left.+\left[\tfrac{1}{2}(1+\alpha A^{(1)})+B^{(0)}+\alpha B^{(1)}\right]\frac{E_0}{m}\right]$$
$$\left.\times\alpha^2M_1+\tfrac{1}{2}(1+\alpha A^{(1)})\,\alpha^2M_2\right\} \tag{20}$$

where $\langle O\rangle\equiv\langle\psi_0|O|\psi_0\rangle$, $\langle\delta^3_\Lambda(r)\rangle\equiv(m^3\alpha^5/8\pi)M_1$, $\langle\alpha v_\Lambda(r)\delta^3_\Lambda(r)\rangle\equiv(m^4\alpha^5/8\pi)M_2$ and Γ_0 is the lowest order $1S$ decay rate. The cutoff operators $v_\Lambda(r)$ and $\delta^3_\Lambda(r)=-\nabla^2v_\Lambda(r)/4\pi$ are defined by Fourier transform:

$$\frac{4\pi}{q^2}\exp\left(-\frac{q^2}{2\Lambda^2}\right)\to v_\Lambda(r)\equiv\frac{1}{r}\mathrm{erf}\left(\frac{\Lambda r}{\sqrt{2}}\right). \tag{21}$$

The matrix elements M_1, M_2, in Eq. (19) can be evaluated for any S state of H_0 and the corresponding decay rate computed using this equation. We used bases consisting of 30 to 60 Gaussians, with varying widths, to diagonalize H_0. The numerical eigenvalues accurately reproduce the $^3S_1$ spectrum through $O(\alpha^5m)$ [6]. Our ground-state results, for sev-

TABLE I. Matrix elements and corrections to the ground-state decay rate evaluated at $\alpha^{-1}=137.03599976$ as a function of cutoff Λ. To determine X_Γ, we set $a_0=a_1=b_0=0$.

Λ/m	M_1	M_2	X_Γ
0.25	17839.5823	18.4265	0.5846
0.5	18249.0812	37.6279	0.8518
1.0	18405.3892	75.8182	0.8867
1.5	18410.4321	113.7030	0.8323
2.0	18375.7156	151.2649	0.5852

eral values of Λ, are shown in Table I; there we introduce the dimensionless parameter X_Γ, defined for any S state by

$$\Gamma(nS)\equiv\frac{\Gamma_0}{n^3}\left(1+\alpha a_0+\alpha^2\left[\tfrac{1}{3}(1+\alpha a_0)\ln\alpha+b_0-2a_1-\frac{\beta}{4n^2}-\frac{3}{2\pi}\alpha\ln^2\alpha+X_\Gamma(nS)\right]\right). \tag{22}$$

This definition anticipates the leading $\alpha^2\ln\alpha$ [9] and $\alpha^3\ln^2\alpha$ [10] contributions, which are correctly reproduced by our numerical analysis. The final results for X_Γ are almost independent of Λ, while the changes in the matrix elements from one Λ to the next are two orders of magnitude larger than X_Γ itself. Renormalization theory guarantees that Λ dependence due to the matrix elements cancels, in the final answer, against Λ dependence due to coefficients $A^{(1)}$, $A^{(2)}$, and $B^{(0)}$ in the annihilation potential W. The residual Λ dependence in X_Γ is due to our approximations in potentials V and W, the dominant effect being at $O(\alpha^3\Gamma)$ where we have left out the contributions from $A^{(3)}$ and $B^{(1)}$. These contributions can be large if Λ is set much different from m, as is evident from Table I; but they are of order $1\times\alpha^3\Gamma_0$ when $\Lambda\approx m$. The Λ dependence of our answers would be 100 times smaller ($\approx 1/\alpha$) were we to include $A^{(3)}$ and $B^{(1)}$ in our analysis, or even just the Λ-dependent parts of these couplings.

Our calculation is nonperturbative in potential V and so automatically includes order $\alpha^3\Gamma$ (and higher-order) corrections to the decay rate. To facilitate comparison with other calculations, we suppressed these higher-order effects by extrapolating our analysis to $\alpha=0$. Our results for $\Lambda=m$ and a range of α's are shown in Table II, along with the extrapolated values for $1S$, $2S$, and $3S$ decay rates. The results are accurate to within 1 in the last digit shown. The $1S$ result agrees well with the analytic result in [1].

Since the analysis would be complete through $O(\alpha^3\Gamma)$ with the inclusion of the operators parametrized by $A^{(3)}$ and $B^{(1)}$, and since neither of these operators generates factors of $\ln\alpha$, the complete contribution in the decay rate of $O(\alpha^3\ln\alpha\Gamma)$ is already present, along with that of $O(\alpha^3\ln^2\alpha\Gamma)$. By examining the α dependence of X_Γ, we find

$$X_\Gamma(1S)=0.913-0.665\,\alpha\ln\alpha+O(\alpha). \tag{23}$$

TABLE II. Corrections to decay rate at $\Lambda=m$ as a function of α. Here $A^{(1)}=a_0+2.444821528$, $A^{(2)}=b_0-2a_1+2.444822a_0+6.179923$, and $\alpha_{\rm phys}=1/137.03599976$ is the physical value of α.

α	$X_\Gamma(1S)$	$X_\Gamma(2S)$	$X_\Gamma(3S)$
0.08	0.5362	2.1012	2.4437
0.04	0.7299	2.3607	2.7094
0.02	0.8282	2.4870	2.8373
0.01	0.8749	2.5449	2.8949
$\alpha_{\rm phys}$	0.8867	2.5591	2.9089
0.005	0.8962	2.5703	2.9198
0.0025	0.9057	2.5812	2.9305
$\to 0$	0.913	2.588	2.936

The coefficient of $\alpha\ln\alpha$ is independent of the cutoff Λ. The nonlogarithmic term of $O(\alpha^3\Gamma)$ is cutoff dependent, since we have neglected contributions from the cutoff-dependent $A^{(3)}$ and $B^{(1)}$. Using the values for a_0 and Γ_0 in Ref. [1], the $O(\alpha^3\ln\alpha\Gamma)$ contribution amounts to $2.4\times10^{-5}\ \mu s^{-1}$. Together with a small contribution from five-photon decay of $0.73\times10^{-5}\ \mu s^{-1}$ [11], this brings the current theoretical prediction for the decay rate to $\Gamma=7.039967(10)\ \mu s^{-1}$.

After completing our calculation, we learned of an independent analysis of $O(\alpha^3\ln\alpha\Gamma)$ contributions to positronium decay by Kniehl and Penin [12]. Our result, -0.665 for the coefficient independent of a_0, disagreed with their original result, $-(4/5+8\ln 2/3)/\pi=-0.8430$. To verify our analysis, we compared it with published results on $O(\alpha^3\ln\alpha)$ contributions to muonium hyperfine splitting (HFS). The HFS results involve the same operators, with different coefficients, as those contributing at the same relative order in the decay rate. Terms in the orthopositronium decay rate not involving a_0 arise from second-order perturbations of $V_{\rm rad}$ with the leading decay operator,

$$\frac{\delta\Gamma}{\Gamma_0}=2\frac{\langle V_{\rm rad}\tilde{G}\,\delta^3(r)\rangle}{\langle\delta^3(r)\rangle}+\left\langle\frac{\partial V_{\rm rad}}{\partial E}\right\rangle, \tag{24}$$

where $\tilde{G}$ is the Coulomb Green's function with the ground state pole removed, and expectation values are taken between unperturbed Coulomb eigenfunctions. $V_{\rm rad}$ can be expressed as

$$V_{\rm rad}=\tfrac{2}{3}\alpha^2O_1+\tfrac{7}{6}\alpha^2O_2+(\tfrac{1}{6}\ln 2-\tfrac{37}{30})\alpha^2O_3, \tag{25}$$

where

$$O_1=\frac{1}{\pi\alpha m_r^2}p^i\left(\frac{p^2}{2m_r}-\frac{\alpha}{r}-E\right)\ln\frac{m_r/2}{p^2/2m_r-\alpha/r-E}\,p^i, \tag{26}$$

$$\langle l|O_2|k\rangle=\frac{1}{m_r^2}\ln\frac{|l-k|}{m_r},\quad \langle l|O_3|k\rangle=\frac{1}{m_r^2}.$$

The reduced mass m_r equals $m/2$ in positronium. The logarithmic parts of the matrix elements for the second-order perturbations can be inferred, for the most part, directly from the HFS papers [13,14]:

$$\left\langle 2\frac{\langle O_i\tilde{G}\,\delta^3(r)\rangle}{\langle\delta^3(r)\rangle}+\left\langle\frac{\partial O_i}{\partial E}\right\rangle\right\rangle\rightarrow\frac{\alpha}{\pi}\times\begin{cases}-4\ln^2\alpha-8(-\ln 2+3/4)\ln\alpha, & i=1,\\ \ln^2\alpha+(2\ln 2-1)\ln\alpha, & i=2,\\ 2\ln\alpha, & i=3,\end{cases}\tag{27}$$

The only exception is the coefficient $(2\ln 2-1)$ of $\ln\alpha$ for O_2. A partial analysis in Ref. [14] gives $(2\ln 2+2)$ in place of $(2\ln 2-1)$. We have calculated the full contribution analytically, and find the result shown in Eq. (27) [15]. We have also verified the results in Eq. (27) by direct numerical evaluation, using our Gaussian basis set. Nonlogarithmic terms of $O(\alpha^0,\alpha^1)$ are cutoff dependent, but the logarithmic terms were cutoff independent, as expected. Combining our analytic results with those in Ref. [1], our Eq. (23) becomes

$$X_\Gamma(1S)=\frac{11}{8}-\frac{2}{3}\ln 2+\left(8\ln 2-\frac{229}{30}\right)\frac{\alpha}{\pi}\ln\alpha,\tag{28}$$

in complete agreement with our numerical result.

Soon after performing this additional check, we received word from the authors of Ref. [12] that they had found a contribution missing from their analysis which accounts for the original disagreement. At the same time, we learned that a third group [16] has also calculated the $O(\alpha^3\ln\alpha\Gamma)$ contributions and arrived at the same answer.

We thank the authors of Ref. [1] for sharing their results with us before they were published, and acknowledge discussions with B. Kniehl and A. Penin. We also thank Patrick Labelle for several conversations. This work was supported by a grant from the National Science Foundation.

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[5] The values of these parameters can be found in Ref. [1]. By comparing our Eq. (13) to Eqs. (8), (10), and (12) of Ref. [1], the correspondence between our parameters and theirs must be $a_0+\lambda a_1=A(\lambda)/\pi=[-10.286606(10)+15.39\lambda]/\pi$, $b_0-2a_1=B_2/\pi^2=22.38(26)/\pi^2$, and $\beta=-X/12=-(19\pi^2-132)/12(\pi^2-9)$. Five-photon decay contributes an additional $0.19(1)/\pi^2$ to b_0; see Ref. [11].

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[15] The additional HFS contribution with $(2\ln 2-1)$ in place of $(2\ln 2+2)$ amounts to 0.18 kHz in muonium. Taken alone this would affect the theoretical prediction for the HFS, and consequently the determination of the muon mass, at the 1.5σ level. We are preparing a paper to clarify this situation, giving complete results on $O(\alpha^3\ln\alpha)$ corrections for muoniun and positronium HFS.

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