Nonequilibrium evolution and symmetry structure of the large-N Φ^4 model at finite temperature

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We consider the large-N Φ^4 theory with spontaneously broken symmetry at finite temperature. We study, in the large-N limit, quantum states which are characterized by a time-dependent, spatially homogeneous expectation value of one of the field components, $\phi_N(t)$, and by quantum fluctuations of the other $N-1$ components, which evolve in the background of the classical field. Investigating such systems out of equilibrium has recently been shown to display several interesting features. We extend here this type of investigation to finite-temperature systems. Essentially, the novel features observed at $T=0$ carry over to finite temperature. This is not unexpected, as the main mechanisms that determine the late-time behavior remain the same. We extend two empirical, presumably exact, relations for the late-time behavior to finite temperature and use them to define the boundaries between the regions of different asymptotic regimes. This results in a phase diagram with the temperature and initial value of the classical field as parameters, the phases being characterized by spontaneous symmetry breaking and symmetry restoration, respectively. The time evolution is computed numerically and agrees very well with expectations.

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I. INTRODUCTION

The investigation of the $O(N)$ vector model at large N has a long-standing history in quantum field theory [1–3]. One of the main aspects was the question of symmetry restoration at high temperature, which for some time was controversial. The dynamical exploration of a special class of nonequilibrium properties has been developed only recently [4–7].

The out-of equilibrium configuration that has been studied mainly is characterized by an initial state in which one of the components has a spatially homogeneous classical expectation value $\phi(t)$. This implies that the other $N-1$ components $\psi_i(\mathbf{x},t)$, $i=1,\ldots,N-1$, have a mass that is different from the mass in the ground state. This means that their initial state is related to the Fock-space vacuum state by a Bogoliubov transformation. The evolution of the system is governed by the classical equation of motion for the field $\phi(t)$ and by the mode equations for the quantum fields $\psi(\mathbf{x},t)$. The expectation value $\langle\psi(\mathbf{x},t)\psi(\mathbf{x},t)\rangle$ appears in both equations of motion: this constitutes the quantum back reaction. In the one-loop approximation, in contrast to the large-N approximation, this quantum back reaction only appears in the classical equation of motion. This leads to decisive differences in the late-time behavior.

We have previously [8] carried out such dynamical computations for the $O(N)$ vector model in the limit of large N at finite temperature for the case of unbroken symmetry, i.e., with a positive mass term. Here we will consider the case of spontaneously broken symmetry. In this case, at low temperatures the fields $\psi_i(\mathbf{x},t)$ will be the Goldstone modes. This is the case for the ground state at $T=0$ and at finite temperature; for nonequilibrium initial states, these modes become massless when the system settles to a stationary state at late times. Symmetry restoration happens at high temperature *and* at large values of the initial field $\phi(0)$; then, at late times these modes stay massive while the classical field vanishes, and thereby the spontaneous symmetry breaking disappears.

Our investigation, as well as the analogous ones at $T=0$, is limited to fields, masses (as solutions of the gap equation), and temperatures much smaller than the scale of the Landau ghost, $m_x=m_1\exp(8\pi^2/\lambda)$, where m_1 is a renormalization scale, taken of order $\sqrt{\lambda}v$. So the question of symmetry nonrestoration at ''really'' high temperatures [3] will not be addressed here.

The plan of the paper is as follows. In Sec. II we introduce the model and set up the equations governing the nonequilibrium evolution. In Sec. III we discuss the renormalization of the equations of motion and of the energy-momentum tensor: some details are referred to the Appendix. In Sec. IV we discuss the phase structure of the system as a function of temperature and initial conditions. In Sec. V we present the results of the numerical computations. Some conclusions are drawn in Sec. VI.

II. FORMULATION OF THE MODEL

We consider the $O(N)$ vector model with the Lagrangian

$$\mathcal{L}=\frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i-\frac{\lambda}{4N}(\phi^i\phi^i-Nv^2)^2, \tag{2.1}$$

where ϕ^i, $i=1,\ldots,N$, are N real scalar fields. The nonequilibrium state of the system is characterized by a classical expectation value which we take in the direction of ϕ_N. We split the field into its expectation value ϕ and the quantum fluctuations ψ via

$$\phi^i(\mathbf{x},t)=\delta^i_N\sqrt{N}\phi(t)+\psi^i(\mathbf{x},t). \tag{2.2}$$

In the large-N limit one neglects, in the Lagrangian, all terms which are not of order N. In particular, terms containing the

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fluctuation ψ_N of the component ϕ_N are at most of order $\sqrt{N}$ and are dropped therefore. The fluctuations of the other components are identical: their summation produces factors $N-1=N[1+O(1/N)]$. In the broken symmetry case these are the Goldstone modes. Identifying all the fields $\psi_1,\ldots,\psi_{N-1}$ as ψ, the leading order term in the Lagrangian then takes the form

$$\mathcal{L}=N(\mathcal{L}_\phi+\mathcal{L}_\psi+\mathcal{L}_I), \tag{2.3}$$

with

$$\mathcal{L}_\phi=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{\lambda}{4}(\phi^2-v^2)^2, \tag{2.4}$$

$$\mathcal{L}_\psi=\frac{1}{2}\partial_\mu\psi\partial^\mu\psi+\frac{\lambda}{2}v^2\psi^2+\frac{\lambda}{4}(\psi^2)^2, \tag{2.5}$$

$$\mathcal{L}_I=-\frac{\lambda}{2}\psi^2\phi^2, \tag{2.6}$$

where ψ^2 is to be identified with $\Sigma\psi^i\psi^i/N$.

We decompose the fluctuating field into momentum eigenfunctions via

$$\psi(\mathbf{x},t)=\int\frac{d^3k}{(2\pi)^3 2\omega_{k0}}[a_{\mathbf{k}}U_k(t)e^{i\mathbf{k}\cdot\mathbf{x}}+a_{\mathbf{k}}^\dagger U_k^*(t)e^{-i\mathbf{k}\cdot\mathbf{x}}], \tag{2.7}$$

with $\omega_{k0}=\sqrt{m_0^2+k^2}$. The mass m_0 will be specified below. This field decomposition defines a vacuum state as being annihilated by the operators $a_{\mathbf{k}}$.

The equations of motion for the field $\phi(t)$ and of the fluctuations $U_k(t)$ have been derived in this formalism by various authors [9–11].

We include in the following the counterterms that we will need later in order to write the renormalized equations. The equation of motion for the field ϕ becomes

$$\ddot{\phi}(t)+\delta m^2\phi(t)-\lambda v^2\phi(t)+(\lambda+\delta\lambda)\phi(t)[\phi^2(t)+\mathcal{F}(t,T)]=0. \tag{2.8}$$

Here $\mathcal{F}(t,T)$ is the divergent fluctuation integral; it is given by the average of the fluctuation fields defined by the initial density matrix. For a thermal initial state of quanta with energy $\omega_{k0}=\sqrt{k^2+m_0^2}$, it is given by

$$\mathcal{F}(t,T)=\langle\psi^2(\mathbf{x},t)\rangle=\int\frac{d^3k}{(2\pi)^3 2\omega_{k0}}\coth\frac{\beta\omega_{k0}}{2}|U_k(t)|^2. \tag{2.9}$$

The mode functions satisfy the equation

$$\left[\frac{d^2}{dt^2}+\omega_k^2(t)\right]U_k(t)=0 \tag{2.10}$$

and the initial conditions

$$U_k(0)=1,\quad \dot{U}_k(0)=-i\omega_{k0}. \tag{2.11}$$

The time-dependent frequency $\omega_k(t)$ is given by

$$\omega_k^2(t)=k^2+\mathcal{M}^2(t), \tag{2.12}$$

with the time-dependent mass

$$\mathcal{M}^2(t)=-\lambda v^2+\delta m^2+(\lambda+\delta\lambda)[\phi^2(t)+\mathcal{F}(t)]. \tag{2.13}$$

Using this definition, the classical equation of motion can be rewritten as

$$\ddot{\phi}(t)+\mathcal{M}^2(t)\phi(t)=0, \tag{2.14}$$

which is the same equation as the one for $U_k(t)$ with $k=0$ (zero mode). Of course, the initial conditions are different and $\phi(t)$ is real.

As in our previous work, we rewrite the mode equation in the form

$$\left[\frac{d^2}{dt^2}+\omega_{k0}^2\right]U_k(t)=-\mathcal{V}(t)U_k(t), \tag{2.15}$$

whereby we have defined the time-dependent potential $\mathcal{V}(t)=\mathcal{M}^2(t)-\mathcal{M}^2(0)$; we further identify $m_0=\mathcal{M}(0)$ as the ''initial mass.''

The average of energy with respect to the initial density matrix is given by[1]

$$\mathcal{E}=\frac{1}{2}\dot{\phi}^2(t)+\frac{1}{2}(-\lambda v^2+\delta m^2)\phi^2(t)+\frac{\lambda+\delta\lambda}{4}\phi^4(t)+\delta\Lambda+\int\frac{d^3k}{(2\pi)^3 2\omega_{k0}}\coth\frac{\beta\omega_{k0}}{2}\left\{\frac{1}{2}|\dot{U}_k(t)|^2+\frac{1}{2}\omega_k^2(t)|U_k(t)|^2\right\}-\frac{\lambda+\delta\lambda}{4}\mathcal{F}^2(t,T). \tag{2.16}$$

It is easy to check, using the equations of motion (2.14) and (2.10), that the energy is conserved. The energy density is the 00 component of the energy-momentum tensor. The average of the energy momentum tensor for our system is diagonal: its space-space components define the pressure, which is given by

$$p=\phi^2(t)-\mathcal{E}+\delta\xi\frac{d^2}{dt^2}[\phi^2(t)+\mathcal{F}(t,T)]+\int\frac{d^3k}{(2\pi)^3 2\omega_{k0}}\coth\frac{\beta\omega_{k0}}{2}\left(\omega_{k0}^2+\frac{k^2}{3}\right)|U_k(t)|^2. \tag{2.17}$$

Here $\delta\xi$ is the renormalization of the conformal coupling term $\xi(g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)\phi^2$, which has been used for the improved energy momentum tensor [12].

[1]Note that twice the last term, with positive sign, is included in the fluctuation energy, since $\omega_k^2(t)$ contains $\mathcal{F}(t,T)$.

III. RENORMALIZED EQUATION OF MOTION

The expressions for the time-dependent mass $\mathcal{M}^2(t)$, the energy density $\mathcal{E}(t)$, and the pressure are still undefined as they involve divergent integrals over the fluctuations. Our approach to regularization and renormalization has been presented previously [13,8]. It is based on expanding the fluctuations $U_k(t)$ and subsequently the various integrals involving these fluctuations with respect to the time-dependent potential $\mathcal{V}(t)$. As this procedure has been presented elsewhere in detail, we just give the outline here.

The expansion of the fluctuations with respect to $\mathcal{V}(t)$ is given in the Appendix. We use this perturbative expansion in order to single out the divergent contributions in the fluctuation integral. One finds

$$\mathcal{F}(t)=I_{-1}(m_0,T)-I_{-3}(m_0,T)[\mathcal{M}^2(t)-\mathcal{M}^2(0)] + \mathcal{F}_{\rm fin}(t,T), \tag{3.1}$$

where the finite part of $\mathcal{F}(t,T)$ can be written as

$$\mathcal{F}_{\rm fin}(t,T)=\int \frac{d^3k}{(2\pi)^3}\frac{1}{4\omega_{k0}^3}\int_0^t dt' \times\cos[2\omega_{k0}(t-t')]\dot{\mathcal{V}}(t')\coth\frac{\beta\omega_{k0}}{2} + \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{k0}}[2\,\mathrm{Re}\,\overline{f_k^{(2)}}(t) + |\overline{f_k^{(1)}}(t)|^2]\coth\frac{\beta\omega_{k0}}{2}, \tag{3.2}$$

and where the divergent integrals are defined as

$$I_{-1}(m_0,T)=\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{k0}}\left(1+\frac{2}{e^{\beta\omega_0}-1}\right) =I_{-1}(m_0)+\Sigma_{-1}(m_0,T), \tag{3.3}$$

$$I_{-3}(m_0,T)=\int\frac{d^3k}{(2\pi)^3}\frac{1}{4\omega_{k0}^3}\left(1+\frac{2}{e^{\beta\omega_0}-1}\right) =I_{-3}(m_0)+\Sigma_{-3}(m_0,T). \tag{3.4}$$

The integrals $I_{-k}(m_0)$ are the genuine divergences which appear in the renormalization at $T=0$. Their dimensionally regularized form is given by

$$I_{-3}(m_0)=\left\{\int\frac{d^3k}{(2\pi)^3}\frac{1}{4\omega_{k0}^3}\right\}_{\rm reg}=\frac{1}{16\pi^2}\left\{\frac{2}{\epsilon}+\ln\frac{4\pi\mu^2}{m_0^2}-\gamma\right\}, \tag{3.5}$$

$$I_{-1}(m_0)=\left\{\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{k0}}\right\}_{\rm reg} =-\frac{m_0^2}{16\pi^2}\left\{\frac{2}{\epsilon}+\ln\frac{4\pi\mu^2}{m_0^2}-\gamma+1\right\} =-m_0^2 I_{-3}(m_0)-\frac{m_0^2}{16\pi^2}. \tag{3.6}$$

The additional temperature-dependent terms $\Sigma_{-k}(m_0,T)$ are finite. They are defined as

$$\Sigma_{-1}(m_0,T)=\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_{k0}(e^{\beta\omega_{k0}}-1)}, \tag{3.7}$$

$$\Sigma_{-3}(m_0,T)=\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{k0}^3(e^{\beta\omega_{k0}}-1)}. \tag{3.8}$$

It is convenient to include these finite terms into the definition of $\mathcal{F}_{\rm fin}(t,T)$. Then the time-dependent mass takes the form

$$\mathcal{M}^2(t)=\lambda(\phi^2-v^2)+\delta\lambda\,\phi^2+\delta m^2+(\lambda+\delta\lambda)[I_{-1}(m_0) - I_{-3}(m_0)\mathcal{V}(t)+\tilde{\mathcal{F}}_{\rm fin}(t,T)], \tag{3.9}$$

with

$$\tilde{\mathcal{F}}_{\rm fin}(t,T)=\Sigma_{-1}(m_0,T)-\mathcal{V}(t)\Sigma_{-3}(m_0,T)+\mathcal{F}_{\rm fin}(t,T). \tag{3.10}$$

The time-dependent mass (3.9) contains both renormalization constants δm and $\delta\lambda$. Furthermore, its definition by this equation is implicit: $\mathcal{M}^2(t)$ also appears on the right-hand side of Eq. (3.9) in $\mathcal{V}(t)$.

We now have to fix the renormalization counterterms in such a way that the relation between the time-dependent mass and $\phi(t)$ becomes finite. An additional constraint derives from the requirement that the renormalization counterterms should not depend on the initial condition, but only on the parameters appearing in the Lagrangian, i.e., λ and v and renormalization conventions.

We first determine $\delta\lambda$ by considering the difference

$$\mathcal{V}(t)=\mathcal{M}^2(t)-\mathcal{M}^2(0)=(\lambda+\delta\lambda)[\phi^2(t)-\phi^2(0) - I_{-3}(m_0)\mathcal{V}(t)+\tilde{\mathcal{F}}_{\rm fin}(t,T)-\tilde{\mathcal{F}}_{\rm fin}(0,T)]. \tag{3.11}$$

The divergent parts depend on the initial mass m_0. We have to replace this by a renormalization scale independent of the initial conditions. In Ref. [8] we had chosen the scale m, where m was the mass parameter appearing in the Lagrangian. Here the analogous mass squared would be $m^2=-\lambda v^2$ and so m would be imaginary. We therefore choose another scale m_1 which we do not specify here. In the numerical computations we have used the physical Higgs boson mass $m_1^2=m_H^2=2\lambda v^2$.

We rewrite the implicit equation for $\mathcal{V}(t)$ as

$$\mathcal{V}(t)[1+(\lambda+\delta\lambda)I_{-3}(m_1)] = (\lambda+\delta\lambda)\{\phi^2(t)-\phi^2(0)-[I_{-3}(m_0) - I_{-3}(m_1)]\mathcal{V}(t)+\tilde{\mathcal{F}}_{\rm fin}(t,T)-\tilde{\mathcal{F}}_{\rm fin}(0,T)\} \tag{3.12}$$

and require

$$\frac{\lambda+\delta\lambda}{1+(\lambda+\delta\lambda)I_{-3}(m_1)}=\lambda. \tag{3.13}$$

Solving with respect to $\delta\lambda$, we find

$$\delta\lambda=\frac{\lambda^2 I_{-3}(m_1)}{1-\lambda I_{-3}(m_1)}. \tag{3.14}$$

Inserting this relation into Eq. (3.12) we find

$$\mathcal{V}(t)=\lambda\mathcal{C}[\phi^2(t)-\phi^2(0)+\tilde{\mathcal{F}}_{\rm fin}(t,T)-\tilde{\mathcal{F}}_{\rm fin}(0,T)], \tag{3.15}$$

with

$$\mathcal{C}=\frac{1}{1+\lambda[I_{-3}(m_0)-I_{-3}(m_1)]}=\frac{1}{1+\frac{\lambda}{16\pi^2}\ln\left(\frac{m_1^2}{m_0^2}\right)}. \tag{3.16}$$

Equation (3.15) is a finite relation for the potential $\mathcal{V}(t)$ since the difference $[I_{-3}(m_0)-I_{-3}(m_1)]$ is finite. Going back to Eq. (3.10), we realize that $\tilde{\mathcal{F}}_{\rm fin}$ on the right-hand side of Eq. (3.15) contains a term proportional to $\mathcal{V}(t)$. Taking account of this term, we rewrite $\mathcal{V}(t)$ in terms of $\mathcal{F}_{\rm fin}$ as

$$\mathcal{V}(t)=\lambda\mathcal{C}_T[\phi^2(t)-\phi^2(0)+\mathcal{F}_{\rm fin}(t,T)], \tag{3.17}$$

with

$$\mathcal{C}_T=\frac{1}{1+\frac{\lambda}{16\pi^2}\ln\left(\frac{m_1^2}{m_0^2}\right)+\lambda\Sigma_{-3}(m_0,T)}. \tag{3.18}$$

Recall that $\mathcal{F}_{\rm fin}(t)$ is the mode integral of second order in $\mathcal{V}(t)$ and vanishes at $t=0$.

We now go back to Eq. (3.9), which we take at the initial time $t=0$:

$$m_0^2\equiv\mathcal{M}^2(0)=\lambda[\phi^2(0)-v^2]+\delta\lambda\,\phi^2(0)+\delta m^2+(\lambda+\delta\lambda) \times[I_{-1}(m_0)+\tilde{\mathcal{F}}_{\rm fin}(0,T)]. \tag{3.19}$$

This is an implicit relation between m_0 and $\phi(0)$, which, however, contains still the infinite quantities $\delta\lambda$, δm, and $I_{-1}(m_0)$. Using Eq. (3.6), we can rewrite Eq. (3.19) as

$$m_0^2=(-\lambda v^2+\delta m^2)+(\lambda+\delta\lambda)\left[\phi^2(0)-m_0^2 I_{-3}(m_0) - \frac{m_0^2}{16\pi^2}+\tilde{\mathcal{F}}_{\rm fin}(0,T)\right]. \tag{3.20}$$

As the renormalization condition, we require m_0 to vanish for temperature $T=0$ at the minimum of the potential $\phi=v$, as is the case at the tree level. We note that $m_0^2=0$ is *not* the curvature of the tree-level potential at $\phi=v$, which is $m_H^2=2\lambda v^2$. It is the mass of the fluctuations at $\phi=v$ in the large-N approximation. For $T=0$ we have $\tilde{\mathcal{F}}_{\rm fin}(t=0,T=0)=\Sigma_{-1}(m_0,T=0)=0$. Setting $m_0=0$, $\phi(0)=v$ in the gap equation (3.20), we get immediately

$$\delta m^2=-\delta\lambda v^2=-\frac{\lambda^2 v^2 I_{-3}(m_1)}{1-\lambda I_{-3}(m_1)}. \tag{3.21}$$

Inserting this into Eq. (3.20), we obtain the renormalized gap equation

$$m_0^2=\lambda\mathcal{C}\left[\phi^2(0)-v^2-\frac{m_0^2}{16\pi^2}+\Sigma_{-1}(m_0,T)\right]. \tag{3.22}$$

For the numerical computation it is easier to choose some $m_0^2\geqslant 0$ and to use the gap equation solved for $\phi^2(0)$:

$$\phi^2(0)=\frac{m_0^2}{\lambda}+v^2+\frac{m_0^2}{16\pi^2}\left(1+\ln\frac{m_1^2}{m_0^2}\right)-\Sigma_{-1}(m_0,T). \tag{3.23}$$

For $t>0$, the renormalized relation for the mass squared $\mathcal{M}^2(t)$ we find, using Eqs. (3.15) and (3.22), is

$$\mathcal{M}^2(t)=m_0^2+\mathcal{V}(t)=\lambda\mathcal{C}\left[\phi^2(t)-v^2-\frac{m_0^2}{16\pi^2}+\tilde{\mathcal{F}}_{\rm fin}(t,T)\right]. \tag{3.24}$$

Having thus obtained a finite relation between $\phi(t)$ and $\mathcal{M}(t)$, the equations of motion for the classical field $\phi(t)$ and for the modes $U_k(t)$ are well defined and finite.

The way in which we have renormalized has made the cutoff disappear. This was possible only to the extent that we could safely neglect corrections of order ϵ in the evaluation of the divergent integrals. One way of achieving this is to take the limit $\epsilon\to 0$. This implies, for the bare coupling λ_0,

$$\lambda_0=\lim_{\epsilon\to 0}\frac{\lambda}{1-\frac{\lambda}{16\pi^2}\frac{2}{\epsilon}}=0^-, \tag{3.25}$$

so this is the case of ''negative bare coupling'' as discussed in [3]. One can leave the cutoff finite, however, as long as the masses and momenta are much smaller than the scale of the Landau ghost, $m_x=m_1^2\exp(8\pi^2/\lambda)$. This will be the case here. This is not related to a pragmatic momentum cutoff that we apply to the convergent integrals of the finite part.

While we have found here the gap equation as a self-consistency condition, it can also be derived [5,7] from a potential (free energy), which here takes the form

$$V(m_0^2,\Phi^2,T)=\frac{m_0^2}{2}\left\{\phi^2-v^2-\frac{m_0^2}{2\lambda}+\frac{m_0^2}{32\pi^2}\left[\ln\left(\frac{m_0^2}{m_1^2}\right)-\frac{3}{2}\right]\right\}+\int\frac{d^3k}{(2\pi)^3}\frac{1}{\beta}\ln[1-\exp(-\beta\omega_0)]. \quad (3.26)$$

The gap equation then follows from the condition

$$\frac{\partial V(m_0^2,\phi^2,T)}{\partial m_0^2}=0. \quad (3.27)$$

It should be mentioned here that the gap equation has two solutions, one of which lies above the scale of the Landau ghost, $m_x=m_1\exp(8\pi^2/\lambda^2)$. In the sense that we consider here the model as giving rise to a low-energy effective theory, we discard this high-mass solution and its discussion. The solution we consider is the low-energy one, which is of order $\sqrt{\lambda}v$ or m_1.

The energy density is given by

$$\mathcal{E}=\frac{1}{2}\dot{\phi}^2(t)+\frac{1}{4}(\lambda+\delta\lambda)(\phi^2-v^2)^2+\delta\Lambda+\mathcal{E}_{\rm fl}(t,T)-\frac{\lambda+\delta\lambda}{4}\mathcal{F}^2(t,T). \quad (3.28)$$

Here we have used already that $\delta m^2=-\delta\lambda v^2$, and part of the ''cosmological constant'' counterterm $\delta\Lambda$ is included in $\delta\lambda v^4/4$. The fluctuation energy is given by

$$\mathcal{E}_{\rm fl}(t,T)=\int\frac{d^3k}{(2\pi)^3 2\omega_{k0}}\coth\frac{\beta\omega_{k0}}{2}\left\{\frac{1}{2}|\dot{U}_k(t)|^2+\frac{1}{2}\omega_k^2(t)|U_k(t)|^2\right\}. \quad (3.29)$$

We again split off the temperature-dependent contribution via

$$\mathcal{E}_{\rm fl}(t,T)=\mathcal{E}_{\rm fl}(t,0)+\Delta\mathcal{E}_{\rm fl}(t,T), \quad (3.30)$$

where the second term on the right-hand side,

$$\Delta\mathcal{E}_{\rm fl}(t,T)=\int\frac{d^3k}{(2\pi)^3 2\omega_{k0}}\frac{2}{e^{\beta\omega_{k0}}-1}\left\{\frac{1}{2}|\dot{U}_k(t)|^2+\frac{1}{2}\omega_k^2(t)|U_k(t)|^2\right\}, \quad (3.31)$$

is finite. The divergences of the first term are given [13] by the decomposition

$$\mathcal{E}_{\rm fl}(t,0)=I_1(m_0)+\frac{1}{2}\mathcal{V}(t)I_{-1}(m_0)-\frac{1}{4}\mathcal{V}^2(t)I_{-3}(m_0)+\mathcal{E}_{\rm fl,fin}(t,0), \quad (3.32)$$

with

$$\mathcal{E}_{\rm fl,fin}(t,0)=\frac{1}{2}\int\frac{d^3k}{(2\pi)^3 2\omega_{k0}}\left\{\frac{1}{2}\overline{|\dot{f}_k^{(1)}|^2}+\frac{\mathcal{V}(t)}{2}[2\,\mathrm{Re}\,\overline{f_k^{(1)}}+\overline{|f_k^{(1)}|^2}]+\frac{\mathcal{V}^2(t)}{8\omega_{k0}^2}\right\}. \quad (3.33)$$

We denote the sum of $\mathcal{E}_{\rm fl,fin}(t,0)$ and $\Delta\mathcal{E}_{\rm fl}(t,T)$ finite contributions as $\mathcal{E}_{\rm fl,fin}(t,T)$. The expression for the energy then takes the form

$$\mathcal{E}=\frac{1}{2}\dot{\phi}^2+\frac{\lambda+\delta\lambda}{4}(\phi^2-v^2)^2+\mathcal{E}_{\rm fl,fin}(t,T)+I_1(m_0)+\frac{1}{2}\mathcal{V}(t)I_{-1}(m_0)-\frac{1}{4}\mathcal{V}^2(t)I_{-3}(m_0)-\frac{\lambda+\delta\lambda}{4}\mathcal{F}^2(t,T)+\delta\Lambda. \quad (3.34)$$

In addition to the divergences arising from $\mathcal{E}_{\rm fl}(t,T)$, we have to take into consideration those of $\mathcal{F}^2(t,T)$, which we have analyzed above. If all divergences and the renormalization constant $\delta\lambda$ are inserted, the expression turns out to be finite: i.e., the remaining counterterm $\delta\Lambda$ is needed only for a finite renormalization. We require the energy to vanish at $T=0$ for $\phi(t)\equiv v$, which implies $m_0=0$. Then $\delta\Lambda=0$. There remains a finite constant dependent on the initial condition

$$\Delta\Lambda=\frac{m_0^4}{128\pi^2}\left(1+\frac{2\lambda\mathcal{C}}{16\pi^2}\right), \quad (3.35)$$

and the energy is given by

$$\mathcal{E}=\frac{1}{2}\dot{\phi}^2+\frac{\lambda}{4}\mathcal{C}(\phi^2-v^2)^2+\frac{1}{2}\Delta m^2(\phi^2-v^2)+\mathcal{E}_{\rm fl,fin}(t,T)-\frac{\lambda}{4}\mathcal{C}\tilde{\mathcal{F}}_{\rm fin}^2(t,T)+\Delta\Lambda. \quad (3.36)$$

Here Δm^2 is given by

$$\Delta m^2=-\lambda\mathcal{C}\frac{m_0^2}{16\pi^2}. \quad (3.37)$$

We write the pressure in the form

$$p=\dot{\phi}^2(t)-\mathcal{E}+p_{\rm fl}(t,T)+\delta\xi\frac{d^2}{dt^2}[\phi^2(t)+\mathcal{F}(t,T)]. \quad (3.38)$$

The renormalization does not differ form the case of unbroken symmetry discussed in Ref. [8] and is not presented again. We find

$$\delta\xi=\frac{\lambda x}{6(1-\lambda x)}=\frac{\lambda I_{-3}(m)}{6[1-\lambda I_{-3}(m)]}. \quad (3.39)$$

The final result for the renormalized pressure reads

$$p=\dot{\phi}^2(t)-\mathcal{E}+p_{\mathrm{fl,fin}}(t,T)-\frac{m_0^4}{96\pi^2}-\frac{m_0^2}{48\pi^2}\mathcal{V}(t)$$
$$-\frac{1}{96\pi^2}\left[\ln\left(\frac{m_1^2}{m_0^2}\right)+2\right]\ddot{\mathcal{V}}(t), \tag{3.40}$$

with

$$p_{\mathrm{fl,fin}}(t,0)=\int\frac{d^3k}{(2\pi)^3 2\omega_{k0}}\left\{\left(\omega_{k0}^2+\frac{\vec{k}^2}{3}\right)\right.$$
$$\times[2\,\mathrm{Re}\,\overline{f_k^{(2)}}(t)+|\overline{f_k^{(1)}}(t)|^2]$$
$$+\left(\frac{1}{6\omega_{k0}^2}-\frac{m_0^2}{24\omega_{k0}^2}\right)\int_0^t dt'\cos 2\omega_{k0}(t-t')\dddot{\mathcal{V}}(t')$$
$$+\left(\frac{1}{12\omega_{k0}^2}+\frac{m_0^2}{24\omega_{k0}^4}\right)\cos(2\omega_{k0}t)\ddot{\mathcal{V}}(0)$$
$$+|\dot{\overline{f_k^{(1)}}}(t)|^2-2\,\mathrm{Re}[i\omega_{k0}\dot{\overline{f_k^{(1)}}}(t)$$
$$\left.+i\omega_{k0}\overline{f_k^{(1)}}(t)\overline{f_k^{(1)*}}(t)]\right\}. \tag{3.41}$$

IV. ANALYSIS OF THE GAP EQUATION AND OF THE PHASE STRUCTURE

The dynamical evolution of the nonequilibrium system depends on two parameters, the temperature T and the initial amplitude of the classical field $\phi(0)=\phi_0$, which in analogy with thermal equilibrium systems can be considered as an external parameter. There are two regions from which we can start the system, which we will call regions II and III. There is, in addition, one region into which the system can evolve when one considers $\phi(t=\infty)=\phi_\infty$ and not ϕ_0 as the external parameter. We call it region I. In this section we will characterize these regions and describe the dynamical evolution to be expected from the analysis at $T=0$. This analysis is based on certain empirical results [6,7] that, though unproven, seem to be at least almost exact. We will generalize these results to finite temperature in a plausible heuristic way, to be confirmed by the numerical computations. We think that the way in which we generalize these results will give a further clue to understanding them.

A. Region I: $m_0^2<0$

The gap equation requires m_0^2 to be positive. The point where $m_0^2=0$ marks an initial condition that leads to a solution $\phi=\mathrm{const}$, if $\dot{\phi}_0=0$, as we will assume in the following. For $T=0$ this stationary amplitude is $\phi=v$. For $T>0$ we can easily find this amplitude as well. Indeed, for $m_0=0$ the integral $\Sigma_{-1}(m_0,T)$ is given by its value for massless quanta, i.e.,

$$\Sigma_{-1}(0,T)=\frac{T^2}{12}, \tag{4.1}$$

and therefore

$$\phi_1^2(T)=\phi_0^2|_{m_0=0}=v^2-\frac{T^2}{12}. \tag{4.2}$$

For $\phi_0<\phi_1(T)$ the gap equation has no real solution m_0. The region below the boundary (4.2) is region I.

If nevertheless one wants to start the system with ϕ_0 in region I, one faces the problem that in this region the gap equation requires m_0^2 to be negative. Then the low-momentum modes with $k^2<-m_0^2$ have imaginary frequencies. So from an orthodox point of view (to which we adhere here) the system cannot be quantized properly. One may avoid this problem by redefining the dispersion relation for the initial frequencies via $\omega_{k0}^2=k^2+|m_0^2|$ in this region. Of course, at $t>0$, $\mathcal{M}^2(t)$ will be negative, so the ''mass squared'' changes sign at $T=0$, a situation called ''quench'' in analogy to a similar transition from a stable to an unstable state by a sudden drop of temperature or inversion of a magnetic field. On the other hand, the amplitude $\phi(t)$ can reach this region at late times, but then it is in a quantum state different from the ones we use as initial states.

B. Region II: $m_0^2>0$, $m_\infty^2=0$

We now assume ϕ_0 is started above the boundary value (4.2). If ϕ_0 is not too large, the system may, at $t>0$, enter a region where $\mathcal{M}^2(t)<0$, i.e., region I. Then the quantum fluctuations with momenta $k^2<-\mathcal{M}^2(t)$ will increase exponentially, signaling instability. This causes $\mathcal{M}^2(t)$ to increase so that it is driven back to a value $\mathcal{M}^2(t)>0$. If the initial amplitude ϕ_0 is sufficiently small, this forth-and-back reaction will lead $\mathcal{M}^2(t)$ to stabilize at $\mathcal{M}_\infty^2=0$. So at late times $\tilde{\mathcal{F}}_{\mathrm{fin}}(t,T)$ is determined by quantum modes $U_k(t)$ that oscillate with time-independent frequencies $\omega_\infty=k$: it becomes stationary as well and will be positive. Therefore $\phi(t)$ stabilizes at some value

$$\phi_\infty^2=v^2-\tilde{\mathcal{F}}_{\mathrm{fin}}(\infty,T)<v^2-\tilde{\mathcal{F}}_{\mathrm{fin}}(0,T). \tag{4.3}$$

This is entirely analogous to the behavior found at $T=0$ [7]. We call the region of initial values ϕ_0 leading to this late-time behavior region II.

The stabilization by back reaction onto the fluctuations obtained in the large-N approximation is not present in the *one-loop approximation*. In this approximation, once $\phi(t)$ dips into the unstable region $\phi(t)<v/\sqrt{3}$, the mass squared of the fluctuations becomes negative and the low-momentum modes evolve exponentially. The effective mass of the classical field increases exponentially as well and continues to do so, but the mass squared of the fluctuations stays negative. The amplitude $\phi(t)$ is driven towards zero. Nevertheless, the classical energy continues to increase, as the field oscillates faster and faster, this energy being extracted from the energy of the quantum fluctuations. Obviously, this signals the instability of the quantum vacuum, as is already apparent from

the fact that the effective potential is complex in this region. We will illustrate this by a numerical example, to be presented in the next section.

At $T=0$ the final value ϕ_∞ was found to be related to the initial value ϕ_0 by an empirical relation

$$\phi_\infty^2=\sqrt{\phi_0^2(2v^2-\phi_0^2)},\quad T=0. \tag{4.4}$$

It is not obvious how to generalize this relation to finite temperature. It was remarked in Ref. [7] that the relation only depends on the initial, purely classical, energy, which is given by $E=\lambda(\phi^2-v^2)^2/4$. Obviously, it satisfies the constraints that $\phi_\infty^2=v^2$ if $\phi_0=v^2$ and that $\phi_\infty=0$ if classically the system can reach the maximum of the potential; this happens at $\phi_0^2=\phi_2^2(T=0)=2v^2$. So Eq. (4.4) seems to be related to energy considerations. We further observe that the classical turning point is at $\overline{\phi_0^2}=2v^2-\phi_0^2$ so that one may write Eq. (4.4) as the geometric mean

$$\phi_\infty^2=\sqrt{\phi_0^2\overline{\phi_0^2}}. \tag{4.5}$$

This form turns indeed out to lead to the correct generalization for finite temperature.

Obviously, the relation is characterized by the motion at early times when the quantum fluctuations have not yet evolved. When discussing renormalization we have made an expansion with respect to the ''potential'' $\mathcal{V}(t)$, which vanishes at $t=0$. So the same expansion can be used to study the early-time behavior. In the energy the coefficients of the terms of first and second order in $\mathcal{V}$ have been absorbed into renormalization constants. However, the *thermal* fluctuations are not absorbed in this way and will add to the classical terms in an early-time expansion. These appear in the energy [see Eq. (3.34)] via

$$\Delta\mathcal{E}_{\rm fl}(t,T)=\Sigma_1(m_0)+\frac{1}{2}\mathcal{V}(t)\Sigma_{-1}(m_0)-\frac{1}{4}\mathcal{V}^2(t)\Sigma_{-3}(m_0)+O(\mathcal{V}^3) \tag{4.6}$$

as a part of $\mathcal{E}_{\rm fl,fin}(t,T)$ and via Eq. (4.10) in $\tilde{\mathcal{F}}_{\rm fin}(t,T)$. Taking these expansions into account, the energy can be written in the form

$$E\simeq\frac{\lambda}{4}\mathcal{C}[a\phi^4+\tilde{a}\phi_0^4+b\phi^2+\tilde{b}\phi_0^2+c\phi^2\phi_0^2]+\text{const} \tag{4.7}$$

up to terms of order $\mathcal{V}^3$. We need the coefficients

$$a=1-\lambda\mathcal{C}_T\Sigma_{-3}(m_0,T), \tag{4.8}$$

$$b=-2[v^2-\Sigma_{-1}(m_0,T)], \tag{4.9}$$

$$c=\lambda\mathcal{C}\mathcal{C}_T\Sigma_{-3}(m_0,T). \tag{4.10}$$

The classical turning point is given by

$$\overline{\phi_0^2}=-\frac{b+(a+c)\phi_0^2}{a}=\frac{1}{1-\lambda\mathcal{C}_T\Sigma_{-3}}[2v^2-\Sigma_{-1}-(1+\lambda\mathcal{C}_T\Sigma_{-3})\phi_0^2], \tag{4.11}$$

so that we are led to suppose

$$\phi_\infty^2(T)=\sqrt{\frac{1}{1-\lambda\mathcal{C}_T\Sigma_{-3}}}\times\sqrt{\phi_0^2[2v^2-2\Sigma_{-1}-(1+\lambda\mathcal{C}_T\Sigma_{-3})\phi_0^2]}. \tag{4.12}$$

We find indeed (see below) that this relation is very well fulfilled numerically. According to this formula, region II is limited by the requirement that the expression in the square root be positive, so the boundary between region II and the new region III is given by

$$\phi_2^2=2\frac{v^2-\Sigma_{-1}(m_0,T)}{1+\lambda\mathcal{C}_T\Sigma_{-3}(m_0,T)}. \tag{4.13}$$

We note that the relation is implicit: the value of m_0 that appears on the right-hand side is related to ϕ_2^2 on the left-hand side by the gap equation.

C. Region III: $\phi_\infty=0$, $\mathcal{M}_\infty^2>0$

If the value ϕ_0 becomes larger than ϕ_2, the stationary state with constant ϕ and vanishing mass $\mathcal{M}^2(t)$ is no longer attained, and the system reaches another asymptotic regime where $\mathcal{M}^2(\infty)\neq 0$, whereas $\phi(t)\to 0$. This regime is similar to the one that describes the late-time behavior for the unbroken symmetry case. We call the region of initial values ϕ_0 that leads to such a behavior region III.

There are two phenomena that characterize the transition to this region. On the one hand, the stabilization of the system is taken over by the phenomenon of parametric resonance. On the other hand, the system has enough energy so that $\phi(t)$ can move over the maximum of the potential at $\phi=0$ and indeed will oscillate around $\phi=0$. Accordingly, the threshold value of ϕ_0 at which these two phenomena set in can be characterized by two, *a priori* unrelated, criteria. Both rely on plausible assumptions, which at $T=0$ lead to the same prediction for the critical value of ϕ_0.

The criterion based on the energy consideration has been presented in the previous subsection: we now describe the criterion supplied by the phenomenon of parametric resonance. For the case of unbroken symmetry, it was found at zero [6] and finite temperature [8] that the late-time behavior is described by an empirical sum rule which relates $\mathcal{M}_\infty^2$ to the initial amplitude. For $T=0$ an analogous sum rule was found to hold for the case of spontaneously broken symmetry as well [7]. It is given by

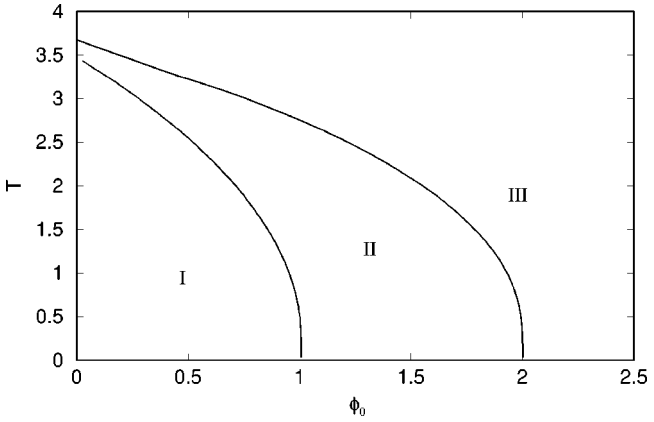

FIG. 1. Phase diagram in the ϕ_0^2-T plane.

$$\mu_\infty^2 = -1 + \frac{\eta_0^2}{2}. \tag{4.14}$$

Here μ and η are normalized in such a way that the classical equation of motion at early times, i.e., in the parametric resonance regime without back reaction, reads

$$\eta'' - \eta + \eta^3 = 0, \tag{4.15}$$

where the primes denote a derivative with respect to $\tau = \alpha t$ and where $\eta = \beta\phi$, also $\mu = \mathcal{M}/\alpha$. With $\eta(\tau)$ a solution of Eq. (4.15), the mode equation becomes a Lamé equation. The sum rule implies [6] that the frequencies $\omega^2(t) = \mathcal{M}^2(t) + k^2$ are shifted outside the parametric resonance band of the Lamé equation. Though there is no rigorous derivation for the sum rule, it accordingly seems related to the parametric resonance phenomenon.

As the shift of the frequencies outside the parametric resonance region must have happened at the end of the phase where the evolution of the system is described by parametric resonance, we will again consider the initial classical evolution. Again, in addition to the classical terms, we have to take into account the terms due to the thermal fluctuations. In terms of the parameters introduced in the previous section, the equation of motion is given by

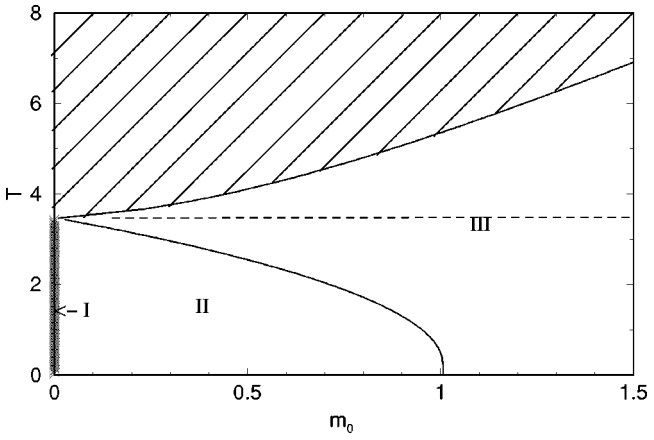

FIG. 2. Phase diagram in the m_0^2-T plane.

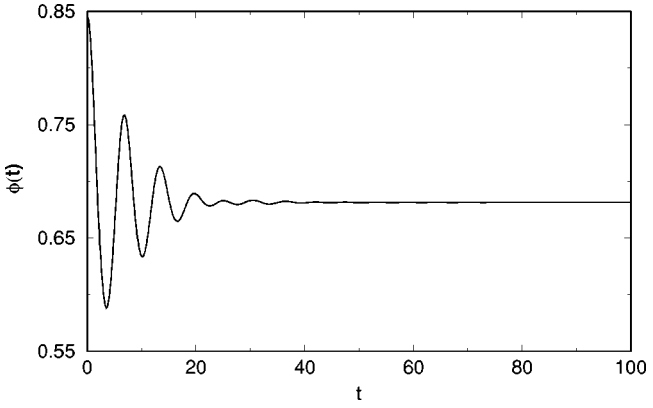

FIG. 3. Evolution of classical field in region II.

$$\ddot{\phi} + \lambda \mathcal{C} a \phi^3 + \frac{\lambda}{3}\mathcal{C}(b + c\phi_0^2)\phi = 0. \tag{4.16}$$

Comparing to the normalized equation (4.15), we determine the factors α and β to be

$$\alpha = \sqrt{\frac{\lambda \mathcal{C}}{2}}\sqrt{b + c\phi_0^2}, \tag{4.17}$$

$$\beta = \sqrt{-\frac{2a}{b + c\phi_0^2}}, \tag{4.18}$$

so that the asymptotic mass is given by

$$\begin{aligned}\mathcal{M}_\infty^2 &= \alpha^2\left(-1 + \frac{1}{2}\beta^2\phi_0^2\right) \\ &= \lambda\mathcal{C}\Big\{-v^2 + \Sigma_{-1}(m_0, T) \\ &\quad + \frac{1}{2}[1 + \lambda \mathcal{C}_T \Sigma_{-3}(m_0, T)]\phi_0^2\Big\}.\end{aligned} \tag{4.19}$$

Again, ϕ_0 and m_0 are related by the gap equation. At the transition from region II to region III, the asymptotic mass vanishes. It is easily seen that this criterion leads to an identical equation for the boundary, i.e., Eq. (4.13).

The field amplitude decreases to zero at late times in this regime. So the symmetry is restored dynamically at high excitation characterized by a high value of ϕ_0.

At the critical temperature $T = \sqrt{12}v$, both boundaries $\phi_1(T)$ and $\phi_2(T)$ become zero. Above T_C the behavior of the system is the same as for region III for all initial values of ϕ_0. While at the border between region I and II there was a lowest value for ϕ_0 for obtaining real solutions of the gap equation, now there is a lowest value of m_0, the one for which $\phi_0 = 0$. It is obtained by solving the gap equation for $\phi_0 = 0$ and agrees with the thermodynamical equilibrium value m_β at that temperature, as defined, e.g., in Eq. (3.38) of Ref. [2]. Of course, with $\phi_0 = 0$ the system remains static.

Having defined the three regions by the two boundaries (4.2) and (4.13), we present, in Fig. 1, a phase diagram in the ϕ_0^2-T plane. Figure 2 shows the phase diagram in the m_0^2-T

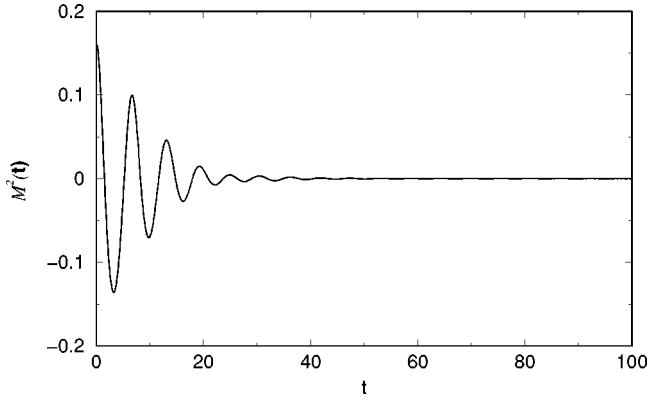

FIG. 4. Evolution of $\mathcal{M}^2(t)$ in region II.

plane, displaying, above T_C, the region $m_0 < m_\beta$, which is excluded as an initial condition. We have to stress that the boundary between regions II and III relies on an empirical relation.

The symmetry restoration above a critical temperature is expected naively. However, if the temperature becomes nonperturbatively large, $T \simeq \sqrt{12} m_1 \exp(8\pi^2/\lambda)$, the gap equation does not have solutions any longer. Then the free energy attains its maximum at the boundary $m_0 = 0$ and the $O(N)$ symmetry is again broken [3]. This phenomenon of ''symmetry nonrestoration,'' as well as the existence of the second solution of the gap equation above $m_x = m_1 \exp(8\pi^2/\lambda)$, will not be discussed here, as it is not part of the low-energy effective theory.

V. NUMERICAL RESULTS

We have discussed already in the previous section the type of nonequilibrium behavior to be expected in the different regions of phase space. The numerical results follow these expectations. We have chosen generally the parameters $v = 1$ and $\lambda = 1$. We present results for the various regions in the T-ϕ_0 plane. The critical temperature is $2\sqrt{3} = 3.464$. We choose temperatures between $T = 1$ and 4, the latter one being above the phase transition. The numerical method has been described in [8]. We just recall that all the integrals

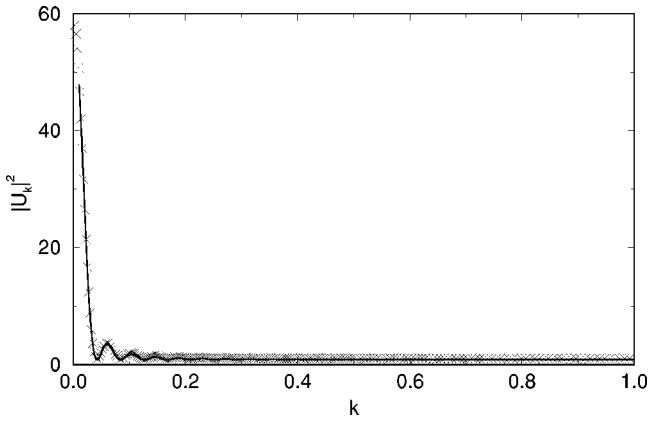

FIG. 5. The momentum spectrum for $T = 1$ at $t = 75$ displaying ''dynamical Bose-Einstein condensation,'' with a fit $\sin^2(kt)/k^2$.

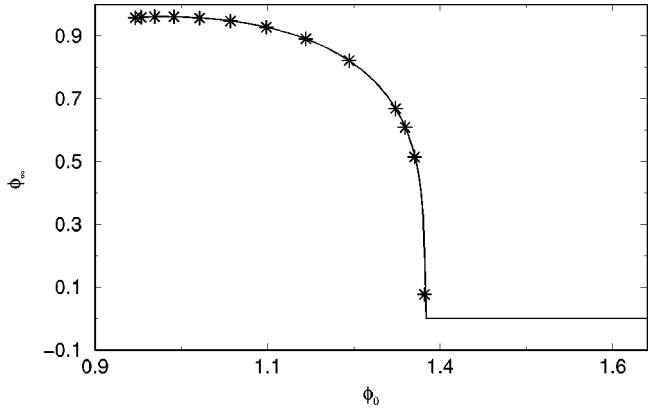

FIG. 6. Late-time amplitude $\phi(\infty)$ vs. initial amplitude ϕ_0 for $T = 1$ (asterisks), compared with Eq. (4.12) (solid line).

computed numerically are finite, so cutting off the momentum integration at some reasonable value is unrelated to cutoffs used for renormalization.

We first consider initial conditions in region II. The expectation value of ϕ, shown in Fig. 3, becomes constant and different from zero as $t \to \infty$. This signals spontaneous breakdown of the $O(N)$ symmetry. As displayed in Fig. 4, the mass $\mathcal{M}^2(t)$ vanishes as $t \to \infty$, as expected form the Goldstone theorem. The momentum distribution of the quantum fluctuations peaks at $k = 0$ as $|U_k(t)|^2 \propto k^{-2}$, leading to long-range correlations, a phenomenon called ''dynamical Bose-Einstein condensation'' in Ref. [7] and investigated further, for finite volume, in Ref. [18]. We show an example of the momentum distribution in Fig. 5, but we have not studied the phenomenon in detail.

The relation between the asymptotic value as $t \to \infty$ for $\phi(t)$ and the initial amplitude ϕ_0 is displayed in Figs. 6–8 for $T = 1$, 2.5, and 3. We compare the data with our generalization (4.12) of the empirical formula (4.4) given in Ref. [7]. The data are obtained by averaging over the second half of the time interval. The agreement is excellent, except at the phase boundary where the averaging converges slowly.

As an illustration of the behavior of the system in the unstable region in the one-loop approximation, we show, in Fig. 9, the evolution of the field amplitude and, in Fig. 10, the exponential behavior of the fluctuation integral and of the

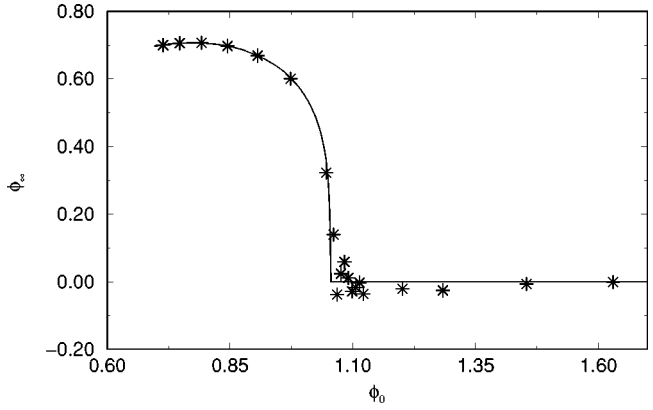

FIG. 7. The same as Fig. 3 for $T = 2$.

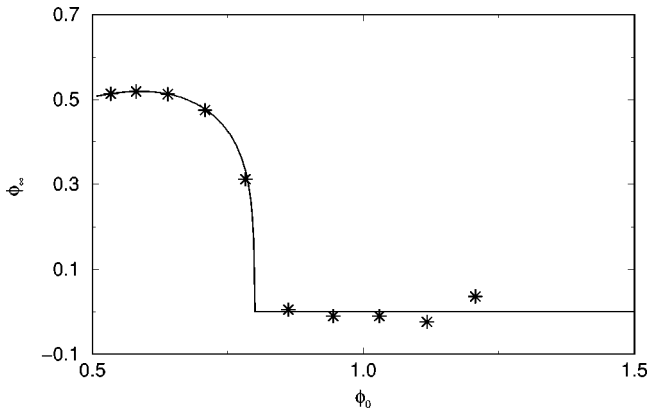

FIG. 8. The same as Fig. 3 for $T=3$.

effective mass squared $\mathcal{M}^2(t)$ of the classical field.

The behavior of the system in region III is displayed in Figs. 11 and 12. The amplitude $\phi(t)$ is seen to decrease to zero. The decrease is power like, not exponential, a phenomenon called anomalous relaxation in Ref. [6]. Figure 12 shows the squared mass $\mathcal{M}^2(t)$, which is seen to converge to an asymptotic value $\mathcal{M}^2_\infty$. The sum rule for this asymptotic value, Eq. (4.12), is compared to the data in Fig. 13 for $T=1.5$, 2.5, and 4. The agreement is again excellent.

We have not presented the results for the pressure and the ratio of pressure and energy, which varies between 0 for a nonrelativistic and 1/3 for an ultrarelativistic ensemble. Here these are dominated, already at $T=1$, by the purely thermal contributions, so that the fluctuations generated by the motion of the field $\phi(t)$ are relatively unimportant.

VI. CONCLUSIONS AND OUTLOOK

The dynamical exploration of the quantum states of the $O(N)$ $\lambda\Phi^4$ theory in the limit $N\to\infty$ has been extended here to finite temperature. We have performed numerical simulations with various initial fields $\phi_0=\phi(0)$ and initial masses m_0 related by the gap equation and for various temperatures T. Depending on the initial conditions we find, in analogy to computations at zero temperature [6,7], final states with restored $O(N)$ symmetry and final states for which the symmetry is spontaneously broken. The resulting phase diagrams resemble typical phase diagrams of thermodynamical systems, with the temperature and an external variable as parameters. Instead of, e.g., the magnetic field or the pressure, we have here the initial value ϕ_0 as external parameter. While the initial states are thermal states, the final states are not.

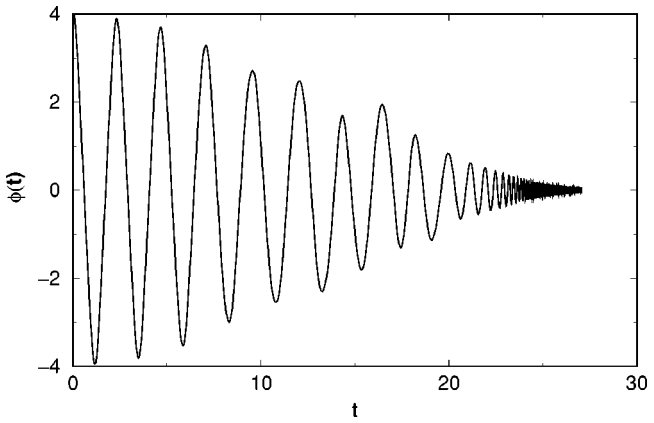

FIG. 9. Evolution of the classical field in the one-loop approximation.

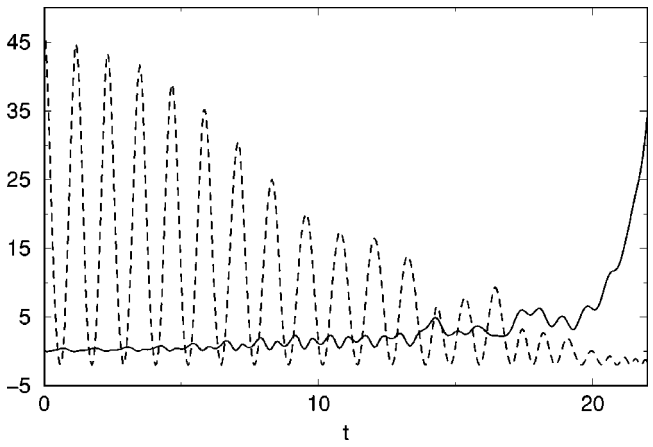

FIG. 10. The fluctuation integral (solid line) and $\mathcal{M}^2(t)$ (dashed line) in the one-loop approximation.

We have generalized two empirical formulas, the relation between the initial and asymptotic field amplitudes in region II and the formula for the asymptotic value of $\mathcal{M}^2(t)$ in region III to finite temperature, extending the plausibility arguments given in [7]. While we have not been able, either, to derive these formulas, the way of generalizing them may give some clue for such a derivation. Both relations are linked as they give the same formula for the boundary between regions II and III, though the arguments for their heuristic derivation are seemingly different. Furthermore, it is clear that both of them are based on the early-time behavior. Obviously, the fluctuations have to be included up to order $\mathcal{V}^2(t)$ in a perturbative expansion. At $T=0$ these terms are essentially absorbed into renormalization constants, so that the purely classical behavior prevails. One may also formulate the modifications at finite temperature in terms of temperature-dependent masses and couplings. It is the role of the large-N quantum back reaction to transmit the early-time

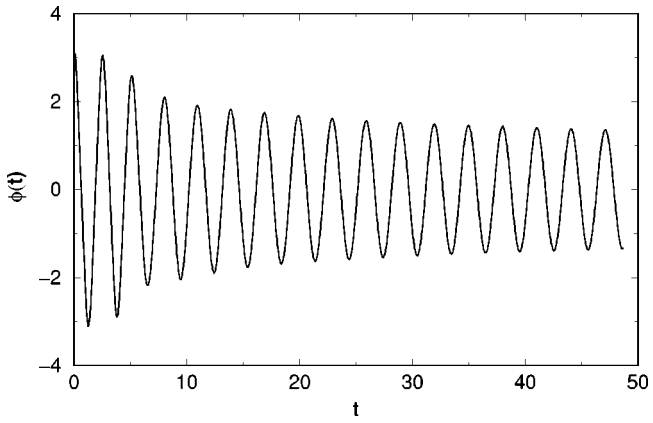

FIG. 11. Evolution of the classical field in region III.

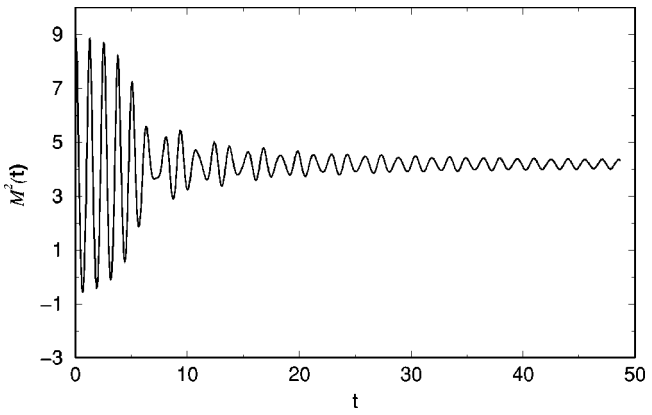

FIG. 12. Evolution of $\mathcal{M}^2(t)$ in region III.

behavior into the late-time one.

Unfortunately, there are many interesting models for which the large-N approach is not possible or not adequate. The one-loop approximation, on the other hand, can be applied in general. However, it shows features that seem to make it obsolete for describing nonequilibrium phenomena, especially for theories with spontaneous symmetry breaking. As an illustration, we have shown the typical behavior of a spontaneously broken $\lambda\Phi^4$ model in the one-loop approximation. The system does not reach a stationary state at late times: the effective mass of the classical field diverges exponentially, while the effective mass of the quantum fluctuations is and stays negative. This is due to the lack of the quantum back reaction onto the fluctuations. The fact that one finds such a pathological behavior may, however, indicate the correct physics and is not necessarily a consequence of an inadequate approximation. It is known that the system is indeed unstable for spatially constant *static* fields: it is an instability with respect to formation of domains [14]. For *space-dependent* fields like minimal bubble configurations, the one-loop approximation to the effective *action* does not display any unplausible features [15–17], though the effective *potential* is complex in the unstable region. So it is not clear whether the ''taming'' of the instability introduced by the large-N approximation necessarily improves the understanding of the physics.

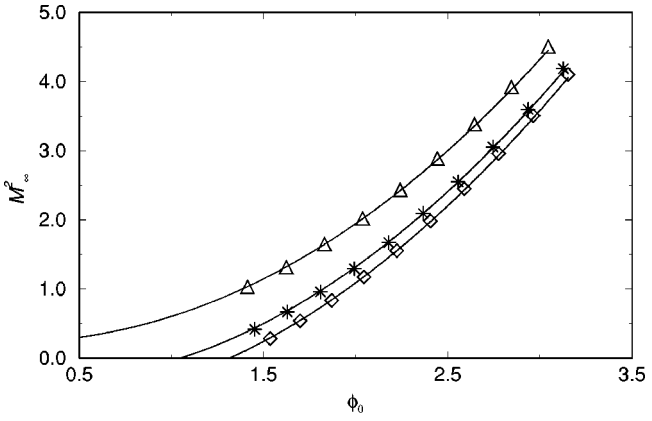

FIG. 13. The asymptotic sum rule for $\mathcal{M}^2(t)$. The data for $T=1.5$ (diamonds), $T=2.5$ (asterisks), and $T=4$ (triangles) are compared to Eq. (4.19) (solid lines).

In this situation it is certainly very important to develop new approaches to the evolution of quantum systems for theories with spontaneously broken symmetry [19,20]. There are indications in a large-N quantum mechanical system [20] that the large-N limit may be misleading, as the next-to-leading corrections become large especially at late times. It is not clear, however, what the impact of these results on quantum field theory will be. One of the problems is that, in contrast to the large-N and one-loop approximations, alternative wave functionals pose problems with renormalization [19]. This is not only a technical problem. It is connected (trivially) to the fact that the higher the dimension of space, the more the ultraviolet behavior of the system will be important.

A surprising feature of the nonequilibrium evolution in the large-N approximation is its lack of thermalization. Our evolution equations are exact in the large-N limit, so thermalization is expected to be related to $1/N$ corrections. Such corrections would include the rescattering of the quantum modes; implementing them into numerical computations in $3+1$ dimensions seems out of scope at present. Rescattering of *classical* fluctuations is taken into account in the 3D lattice simulations of Ref. [21]. It is observed that rescattering smoothes out the parametric resonance structures, but no thermal distributions have been presented. Various lattice simulations in $1+1$ dimensions find thermalization. Aarts and Smit [22] find thermal distributions of fermions in a lattice simulation of the $(1+1)$-dimensional Abelian Higgs model. There the bosonic fluctuations are treated classically. For the bosonic modes an approximate thermalization is found in an Abelian Higgs model coupled to an inflaton [23]. An effective temperature, defined by matching the observed distributions to thermal ones, is found to be moderately momentum dependent.

It should be noted generally that a classical ensemble at fixed energy cannot thermalize in the strict sense because of the Rayleigh-Jeans divergence. A more general difficulty arises from the fact that one cannot expect the distributions functions to be those of a free theory while one is considering the dynamics of an interacting one. In a recent report [24], Berges and Cox have considered the nonequilibrium quantum evolution of a ϕ^4 theory in $1+1$ dimensions, including two-loop corrections. They find the system to thermalize with a distribution derived in finite-temperature quantum field theory in the same approximation.

Thermalization is also a question of time scales. Recent lattice simulations of a ϕ^4 model [25] show that the early-time behavior is well described by the Hartree approximation (as appropriate for $N=1$), while the system thermalizes later on. Likewise, the large-N approximation was found [20] to describe the early-time behavior of a quantum mechanical system with N degrees of freedom. The time scale one which the approximation breaks down is found to depend on N: below a ''critical'' value of $N\approx 20$, the approximation describes only the first oscillations of the system. How these results, and the one of Ref. [24], translate to three-dimensional systems remains to be seen. Tentatively one

may ask the question how our phase diagrams will change if the system finally comes to thermal equilibrium. The temperature T_∞ of such a final state will depend only on the initial energy, which is a function of m_0 and T_0. This dependence can easily be computed. The transition between the symmetric and the broken symmetry phases occurs at $T_\infty = \sqrt{12}v$. This phase boundary is plotted as a dashed line in Fig. 2. One finds that region III in which the nonequilibrium system tends to a symmetric final state with $\phi_\infty = 0$, $\mathcal{M}_\infty \neq 0$ is divided into two parts. In the region below the new phase boundary, the thermalized system is in the broken symmetry phase. We recall that the temperature-dependent potential is flat for $\phi^2 < v^2(T) = v^2 - T^2/12$, so that ϕ_∞ is not fixed uniquely.

It has to be seen how the results found in lower-dimensional and/or classical systems translate to nonequilibrium quantum field theory in $3+1$ dimensions. One may expect that the approximation will be good at lower values of N and for larger times than in the lower-dimensional models. One of the essential findings of the previous analysis of Refs. [6], [7] and of this work is the close relationship between the late-time behavior and the initial conditions. This implies that the essential features of the late-time behavior are fixed already by the back reaction at early times. However, the system will evolve, and possibly thermalize, at later times: this initial period will be important for the further evolution of the nonequilibrium system.

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APPENDIX: PERTURBATIVE EXPANSION

The mode functions $U_k(t)$ with the initial conditions introduced in Sec. II satisfy the integral equation

$$U_k(t) = e^{-i\omega_{k0}t} + \int_0^\infty dt' \Delta_{k,\mathrm{ret}}(t-t')\mathcal{V}(t')U_k(t'), \tag{A1}$$

with

$$\Delta_{k,\mathrm{ret}}(t-t') = -\frac{1}{\omega_{k0}}\Theta(t-t')\sin[\omega_{k0}(t-t')]. \tag{A2}$$

We separate $U_k(t)$ into the trivial part corresponding to the case $\mathcal{V}(t)=0$ and a function $f_k(t)$ which represents the reaction to the potential by making the ansatz

$$U_k(t) = e^{-i\omega_{k0}t}[1+f_k(t)]. \tag{A3}$$

Here $f_k(t)$ satisfies then the integral equation

$$f_k(t) = \int_0^t dt' \Delta_{k,\mathrm{ret}}(t-t')\mathcal{V}(t')[1+f_k(t')]e^{i\omega_{k0}(t-t')} \tag{A4}$$

and an equivalent differential equation

$$\ddot{f}_k(t) - 2i\omega_{k0}\dot{f}_k(t) = -\mathcal{V}(t)[1+f_k(t)], \tag{A5}$$

with the initial conditions $f_k(0) = \dot{f}_k(0) = 0$. We expand now $f_k(t)$ with respect to orders in $\mathcal{V}(t)$ by writing

$$f_k(t) = f_k^{(1)}(t) + f_k^{(2)}(t) + f_k^{(3)}(t) + \cdots \tag{A6}$$

$$= f_k^{(1)}(t) + \overline{f_k^{(2)}}(t), \tag{A7}$$

where $f_k^{(n)}(t)$ is of nth order in $\mathcal{V}(t)$ and $\overline{f_k^{(n)}}(t)$ is the sum over all orders beginning with the nth one:

$$\overline{f_k^{(n)}}(t) = \sum_{l=n}^{\infty} f_k^{(l)}(t). \tag{A8}$$

The $f_k^{(n)}$ are obtained by iterating the integral equation (A4) or the differential equation (A5). The function $\overline{f_k^{(1)}}(t)$ is identical to the function $f_k(t)$ itself, which is obtained by solving Eq. (A5). The function $\overline{f_k^{(2)}}(t)$ can again be obtained by iteration via

$$\overline{f_k^{(2)}}(t) = \int_0^t dt' \Delta_{k,\mathrm{ret}}(t-t')\mathcal{V}(t')\overline{f_k^{(1)}}(t')e^{i\omega_{k0}(t-t')}. \tag{A9}$$

The integral equations can be used in order to derive the asymptotic behavior as $\omega_{k0}\to\infty$ and to separate divergent and finite contributions. This has been described previously in extenso [13]. We illustrate the procedure by calculating the relevant leading terms for $f_k^{(1)}(t)$. We have

$$f_k^{(1)}(t) = \frac{i}{2\omega_{k0}}\int_0^t dt' \{\exp[2i\omega_{k0}(t-t')]-1\}\mathcal{V}(t'). \tag{A10}$$

Integrating by parts, we obtain

$$f_k^{(1)}(\tau) = -\frac{i}{2\omega_{k0}}\int_0^t dt' \mathcal{V}(t') - \frac{1}{4gq_{k0}^2}\mathcal{V}(t) + \frac{1}{4\omega_{k0}^2}\int_0^t dt' \exp[2i\omega_{k0}(t-t')]\dot{\mathcal{V}}(t'). \tag{A11}$$

For the expansion of the fluctuation integral $\mathcal{F}(t)$, we need the real part of $f_k^{(1)}$, for which we find

$$\mathrm{Re}\, h_k^{(1)}(t) = -\frac{1}{4\omega_{k0}^2}\mathcal{V}(t) + \frac{1}{4\omega_{k0}^2} \times \int_0^t dt' \cos[2\omega_{k0}(t-t')]\dot{\mathcal{V}}(t'). \tag{A12}$$

The second term decreases at least as ω_{k0}^{-3}. In terms of the perturbative expansion for the functions f_k, we can the mode functions appearing in the fluctuation integral as

$$|U_k|^2 = 1 + 2\,\mathrm{Re}\, f_k^{\overline{(1)}} + |f_k^{\overline{(1)}}|^2. \tag{A13}$$

Using Eq. (A12), the leading behavior of this expression is

$$1 + 2\,\mathrm{Re}\, f_k^{\overline{(1)}} + |f_k^{\overline{(1)}}|^2 \simeq 1 - \frac{1}{2\omega_{k0}^2}\mathcal{V}(t). \tag{A14}$$

Similarly, the integrand of the energy density and pressure can be expanded [13]. As these are more divergent, the calculations require more integrations by parts in order to single out the leading powers in ω_{k0} and they become more involved.

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