Noncommutative geometry on a discrete periodic lattice and gauge theory

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(Received 1 November 1999; published 23 October 2000)

We discuss the quantum mechanics of a particle in a magnetic field when its position x^μ is restricted to a periodic lattice, while its momentum p^μ is restricted to a periodic dual lattice. Through these considerations we define non-commutative geometry on the lattice. This leads to a deformation of the algebra of functions on the lattice, such that their product involves a ''diamond'' product, which becomes the star product in the continuum limit. We apply these results to construct non-commutative U(1) and U(M) gauge theories, and show that they are equivalent to a pure U(NM) matrix theory, where N^2 is the number of lattice points.

PACS number(s): 11.15.Ha, 11.25.−w

I. INTRODUCTION AND RESULTS

Recently non-commutative geometry has found applications in string and M theory in the B-field background [1,2]. The non-commutative geometry in question is described by a deformation of the ordinary algebra of functions $f(x)g(x)$ on R^d into a non-commutative albeit associative algebra, with a star product [2]

$$f(x)*g(x)=\exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right)f(x)g(y)\Bigg|_{y=x}. \quad (1)$$

$\theta^{\mu\nu}$ is given in terms of a constant background B field that has even rank d:

$$\theta^{\mu\nu}=-(2\pi\alpha')^2\left(\frac{1}{g+2\pi\alpha' B}B\frac{1}{g-2\pi\alpha' B}\right)^{\mu\nu}. \quad (2)$$

In the limit $\alpha'\to 0$ and $g_{\mu\nu}\sim(\alpha')^2$, string theory is correctly represented by non-commutative gauge theory, with $\theta^{\mu\nu}=f(B^{-1})^{\mu\nu}$. Effectively this is the large B limit. The indices μ label a Euclidean space[1] $\mu=1,\ldots,d$. The star product is related to the Moyal brackets [3–6]. When this product is used instead of the ordinary product of functions in a gauge theory, the resulting non-commutative gauge theory represents string theory in a large $B_{\mu\nu}$ limit, including the non-perturbative effects of the background B field [1,2].

In order to further analyze non-commutative gauge theory, a cutoff version would be useful. With this in mind we define non-commutative gauge theory on a discrete periodic lattice that has two parameters: the periodicity characterized by a length L and the lattice spacing a. The ratio of these is the number of steps $n=L/a$ in each direction labeled by μ. In effect, this lattice is the d-dimensional discretized torus T^d in d-dimensions, with n steps in every direction, which we will denote by $(T_n)^d$. There are altogether n^d lattice points on the discrete torus. A less uniform lattice would have different number of steps in the various directions μ, such that the total number of lattice points would be Πn_μ, instead of n^d. In most of the paper we will concentrate on the uniform lattice for simplicity, but we will also discuss some interesting aspects of a non-uniform lattice in which the number of lattice points is not the same in every direction, but are taken equal in pairs, such that n_1 for both $\mu=1,2$, and n_2 for both $\mu=3,4$, etc. By identifying

$$N=n_1 n_2\cdots n_{d/2}, \quad (3)$$

(d is even) we see that the positions x^μ live on the N^2 points of the periodic lattice $(T_n)^d$. The gauge fields $A_\mu(x)$ or other functions on the lattice are defined only on these N^2 space-time points.

We then construct a ''diamond product'' which is a lattice version of the star product. We will be guided by a previous construction that introduced the discrete Moyal bracket [7] as a cutoff version of the Moyal bracket with a different application in mind [8–17]. The first step is to provide an explicit map $\hat{\Delta}_I^J(x)$ from the N^2 lattice points x^μ to a $N\times N$ matrix that has N^2 entries. Then any function $f(x^\mu)$ defined on the N^2 lattice points can be rewritten in terms of a matrix $\hat{f}$ with x-independent matrix elements $\hat{f}_I^J$, $I,J=1,2,\ldots,N$, as follows (matrices are denoted by the hat symbol)

$$f(x)=\frac{1}{N}\mathrm{Tr}(\hat{\Delta}(x)\hat{f}),\quad \hat{f}_I^J=\sum_{x\in(T_n)^d}\hat{\Delta}_I^J(x)f(x). \quad (4)$$

The properties of the map $\hat{\Delta}_I^J(x)$ are obtained by studying the quantum mechanics of particles in a constant magnetic field $B_{\mu\nu}$, such that the particle positions x^μ are at the n^d lattice points on $(T_n)^d$, while their momenta p_μ ($\partial/\partial x^\mu$ in

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[1]In the string theory derivation, these dimensions correspond to the Euclidean dimensions of a D-brane along which the string background field $B_{\mu\nu}$ does not vanish. The field $B_{\mu\nu}$ can also be thought of as a constant magnetic field with a potential $A_\mu=\frac{1}{2}x^\nu B_{\nu\mu}$ interacting with charged points $x^\mu(\tau)$ at the end of a string. However, taken as an independent starting point, there does not seem to be any problem in allowing one of the dimensions to be timelike. Our discussion does not change if the space is purely Euclidean or Minkowski.

continuum) are on n^d points on the dual lattice. The dual lattice $(\tilde{T}_n)^d$ is similar to $(T_n)^d$ but its lattice spacing is measured in terms of momentum units. Then the map is given by

$$\hat{\Delta}_I^J(x) = \frac{1}{N} \sum_{p^\mu \in (\tilde{T}_n)^d} e^{-ip\cdot x}[\exp(ip\cdot X)]_I^J \tag{5}$$

where $[\exp(ip\cdot X)]_I^J$ is a matrix that will be given explicitly. Roughly, this map is the matrix elements of a delta function $\delta^{(d)}(X-x)$ with X^μ non-commutative operators and x^μ defined only on the periodic lattice. The map contains all the information about non-commutative geometry on the periodic lattice. Using this map and the definitions in Eq. (4), the diamond product is constructed as follows [7]:

$$\begin{aligned} f(x)\diamond g(x) &= \frac{1}{N}\mathrm{Tr}(\hat{\Delta}(x)\hat{f}\hat{g}),\\ &= \frac{1}{N}\sum_{y,z\in(T_n)^d} f(y)g(z)\exp\big(2iB^{\mu\nu}(x_\mu - y_\mu)\\ &\quad \times(x_\nu - z_\nu)\big). \end{aligned} \tag{6}$$

It is physically interesting to note that the sums in the diamond product are weighted by exponentials of the flux that passes through the area defined by the three lattice points x,y,z. We will show that the diamond product reduces to the star product (1) in the continuum limit. In this way the diamond (or star) product is explicitly related to ordinary matrix product $(\hat{f}\hat{g})_I^J$.

Using this formulation we show that the non-commutative U(1) gauge theory on the periodic lattice can be rewritten as a U(N) pure matrix theory where all spacetime positions x^μ have been converted to matrix elements by using the map. The non-Abelian U(M) non-commutative gauge theory on the lattice can also be discussed in the same non-commutative formalism by generalizing to a U(NM) matrix theory.

The $U(M)$ non-commutative gauge theory action on the periodic lattice is constructed by using the diamond product $A_\mu(x)\diamond A_\nu(x)$ whenever gauge fields need to be multiplied with each other, and by substituting the derivative $\partial_\mu A_\nu(x)$ by a suitable lattice version, but otherwise keeping the same general form of the Yang-Mills action. By using the map $\hat{\Delta}_I^J(x)$ the U(M) lattice action is rewritten in the following pure matrix version:

$$\begin{aligned} S &= \frac{1}{4N^2}\sum_{x\in(T_n)^d} (F_{\mu\nu}(x))_a^{a'}\diamond(F_{\mu\nu}(x))_{a'}^{a},\\ &= -\frac{1}{4}\mathrm{Tr}([a_\mu,a_\nu]^2), \end{aligned} \tag{7}$$

where the x^μ-independent a_μ is an $NM\times NM$ matrix related to the $M\times M$ gauge field $(A_\mu(x))_a^{a'}$ in a way that will be indicated.

It is possible to interpret the non-Abelian U(M) theory in d-dimensions as an Abelian U(1) theory in $d+2$ dimensions. This comes about by considering a non-uniform lattice as described above. Then the U(1) Abelian theory in d dimensions is described by a U(N) pure matrix theory (7) with $N = n_1 n_2 \cdots n_{d/2}$, whereas the non-Abelian U(M) theory in d dimensions can be regarded as a U(1) theory with two more non-commutative discretized dimensions, with lattice steps $n_{(d+2)/2}\equiv M$, so that $NM = n_1 n_2\cdots n_{d/2} n_{(d+2)/2}$. Thus, in the U$(NM)$ matrix theory (7), $N = n_1 n_2\cdots, n_{d/2}$ relates to space and $M\equiv n_{(d+2)/2}$ relates to two more non-commutative discrete dimensions that replace the internal space.

The form of the action (7) could be related to the reduced models of gauge theories [18], or more precisely, to the fully reduced matrix theory version written in the form $\mathrm{Tr}([X_\mu,X_\nu]^2)$ [11]. However, in the present version, the physical meaning of the matrix is quite different. Namely, the space-time interpretation is obtained via the map $\hat{\Delta}_I^J(x)$ related to the factor $N = n_1 n_2\cdots, n_{d/2}$, and the internal symmetry information is in the factor M, which are different than the spacetime–internal symmetry interpretation of the reduced models. Thus the existing computational technology of reduced models and matrix models could be adapted to the current problem provided one takes care of the physical interpretation via the map (4) and the meaning of N,M. Some recent computations in [19,20] also seem to be related to our observations, but with a somewhat different spacetime interpretation.

The organization of this paper is as follows: First we discuss the quantum mechanics of particles in a lattice in a magnetic field and show how to derive non-commutative geometry on the lattice from such considerations. This leads directly to an explicit expression for the map $\hat{\Delta}_I^J(x)$. We apply these results to the non-commutative U(1) and U(M) gauge theories on the lattice, and show that they are equivalent to a pure U(NM) matrix theory, as in Eq. (7).

The larger project of studying non-commutative gauge theories in this cutoff version should be worthwhile, but it is not pursued in the current paper.

II. NON-COMMUTATIVE GEOMETRY ON THE LATTICE

It is well known that the quantum mechanics of a particle in a constant magnetic field $B_{\mu\nu}$ produces non-commutative momenta [21] $[K_\mu,K_\nu] = iB_{\mu\nu}$. In order to map this problem to the string theory setting we define "coordinates" $X_\mu = (B^{-1})_{\mu\nu}K^\nu$ which satisfy the commutation rules of non-commutative geometry [1,2]

$$[X^\mu,X^\nu] = i(B^{-1})^{\mu\nu}\equiv i\theta^{\mu\nu}. \tag{8}$$

For simplicity, we begin the discussion of the lattice version of this setup for the special form of $B^{\mu\nu}$ that is block diagonal, with 2×2 blocks along the diagonal, each of them proportional to the Pauli matrix $i\sigma_2$ with various proportionality constants, and zero entries otherwise (it is always possible to rotate $B^{\mu\nu}$ into such a basis). At the end we generalize to an arbitrary form of $B^{\mu\nu}$. For the special form of $\theta^{\mu\nu}$ non-commutativity occurs in pairs of coordinates

$$[X^1,X^2]=\frac{i}{B_{12}},\quad [X^3,X^4]=\frac{i}{B_{34}},\quad \cdots. \tag{9}$$

There does not seem to be anything special about a distinction between timelike or spacelike coordinates since all signs may be absorbed into a redefinition of $B_{\mu\nu}$. We will first discuss the pair (X^1,X^2) and later include all the X^μ. We will then follow the construction of the 2D diamond product in [7] whose discussion we generalize to higher dimensions.

A. Two-torus

Since X^1,X^2 do not commute, they cannot be diagonalized simultaneously. Consider diagonalizing X^1. In the continuum the eigenvalues are on the real line. Consider a periodic lattice, with period L and lattice spacing a in the X^1 direction. The eigenstates of X^1 are labeled as $|j_1\rangle$, $j_1=0,1,\ldots,n-1$, with $n=L/a$, and the eigenvalues of X^1 are restricted to the discrete set $x^1=aj_1$. Furthermore there is a periodicity condition

$$X^1|j_1\rangle=aj_1|j_1\rangle,\quad |j_1+n\rangle=|j_1\rangle; \tag{10}$$

therefore the eigenvalues x^1 take discrete values on the circle of perimeter L:

$$x^1=a(j_1 \bmod n). \tag{11}$$

According to Eq. (9) the operator $B_{12}X_2$ acts like infinitesimal translations on the eigenspace of X^1. On the lattice only finite translations make sense. Taking the commutation rules (9) into account, the translation operator by one lattice unit is $\exp(iaB_{12}X^2)$:

$$\langle j_1|\exp(iaX^2B_{12})=\langle j_1+1|. \tag{12}$$

Its matrix elements take the form

$$\hat{g}_{j_1}^{j_1'}=\langle j_1|\exp(iaX^2B_{12})|j_1'\rangle=\delta_{(1+j_1)\bmod n}^{j_1'\bmod n}. \tag{13}$$

Including the periodicity condition, $\hat{g}_{j_1}^{j_1'}$ becomes the well known circular matrix that has also a non-trivial entry in location $\hat{g}_{n-1}^{0}=1$

$$\hat{g}=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \ddots & \vdots\\ \vdots & 0 & 0 & \ddots & 0\\ 0 & \vdots & \ddots & \ddots & 1\\ 1 & 0 & \cdots & 0 & 0\end{pmatrix}. \tag{14}$$

k_1 units of translation along X^1 is obtained by taking k_1 powers of $\hat{g}$

$$\exp(ik_1aX^2B_{12})\rightarrow(\hat{g}^{k_1})_{j_1}^{j_1'}. \tag{15}$$

Due to periodicity, n units of translation must give the same state. Indeed this is reflected in the property of the circular matrix $\hat{g}$

$$\hat{g}^n=1. \tag{16}$$

Similarly we consider diagonalizing X^2 on a periodic lattice with periodicity L and lattice spacing a in the X^2 direction such that $an=L$. The eigenstates $|j_2\rangle$ are associated with the eigenvalues $a(j_2 \bmod n)$. In the eigenspace of X^2 the translation operator by one unit is $\exp(-iaX^1B_{12})$ and it has similar properties to $\hat{g}$. However, acting on the eigenspace of X^1 defined in Eq. (10), this operator is a diagonal matrix

$$\hat{h}_{j_1}^{j_1'}=\langle j_1|\exp(-iaX^1B_{12})|j_1'\rangle=\delta_{j_1}^{j_1'}e^{-i(j_1 \bmod n)a^2B_{12}}. \tag{17}$$

Taking into account the periodicity of the lattice in the X^2 direction, n powers of $\hat{h}$ should be the identity operator for any state. This requires

$$a^2B_{12}=\frac{2\pi b_{12}}{n} \tag{18}$$

where b_{12} is an integer. Therefore the magnetic flux a^2B_{12} passing through a lattice unit surface a^2 in the 1-2 plane is quantized as b_{12} units of $2\pi/n$.

It is convenient to define ω as the nth root of the identity

$$\omega=\exp(-ia^2B_{12})=e^{-i(2\pi b_{12}/n)},\quad \omega^n=1. \tag{19}$$

The matrix elements of $\hat{h}$ can then be written in the form

$$\hat{h}=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0\\ 0 & \omega & 0 & \cdots & 0\\ 0 & 0 & \omega^2 & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & \omega^{n-1}\end{pmatrix}. \tag{20}$$

If one diagonalizes the matrix $\hat{g}$, the result must be the matrix $\hat{h}$ since the roles of X^1,X^2 can be reversed. Indeed, one can find the explicit unitary transformation

$$\hat{g}=\hat{U}\hat{h}\hat{U}^\dagger,\quad \hat{U}_j^{j'}=\frac{1}{\sqrt{n}}\omega^{jj'}. \tag{21}$$

The unitary matrix $\hat{U}$ also satisfies the periodicity property under $j\rightarrow j+n$ thanks to the fact that ω is the nth root of unity. The commutation property of these matrices is well known:

$$\hat{g}\hat{h}=\hat{h}\hat{g}\omega. \tag{22}$$

They follow from the non-commutative properties of the coordinates $[X^1,X^2]=i/B_{12}$ by using $\exp(\beta X^2)\exp(\alpha X^1)$

$=\exp(\alpha X^1)\exp(\beta X^2)\exp[\beta X^2,\alpha X^1]$. Thus the matrices $\hat{g},\hat{h}$ capture the essence of the non-commutative geometry on the lattice.

On the entire quantum space, whether X^1 or X^2 is diagonal, the only operators that are meaningful are all the possible translations given by $\exp(ia(k^1X^2-k^2X^1)B_{12})$ with k_1,k_2 integers modulo n. Their matrix elements are given by

$$\langle j_1|\exp(ia(k^1X^2-k^2X^1)B_{12})|j_1'\rangle=\omega^{k_1k_2/2}(\hat{h}^{k_2}\hat{g}^{k_1})_{j_1}^{j_1'} \equiv(\hat{v}_{k_1k_2})_{j_1}^{j_1'} \tag{23}$$

where we have used the formula $\exp(A+B)=\exp A\exp B\exp(-[A,B]/2)$ on the left hand side and then evaluated the matrix elements.

It is useful to define a momentum lattice given by $p_\mu=ak^\nu B_{\nu\mu}$ where the lattice distance is measured by aB_{12} and the integers k_1,k_2 are defined modulo n:

$$p_1=-aB_{12}(k_2\bmod n),\quad p_2=aB_{12}(k_1\bmod n). \tag{24}$$

This lattice is the *dual lattice* to the position lattice, its steps are measured in units of momentum. Then the full set of n^2 translation operators take a more suggestive form of a plane wave operator, or ''vertex operator,'' whose matrix elements are $(\hat{v}_{p^\mu})_{j_1}^{j_1'}$

$$\exp(ip_\mu X^\mu)\to\hat{v}_p=\hat{h}^{k_2}\hat{g}^{k_1}\omega^{k_1k_2/2}. \tag{25}$$

These translations are the only meaningful operators that need to be considered for the quantum mechanics of the particle on the non-commutative discrete torus. They have the well known property that under matrix multiplication they form a group algebra

$$\hat{v}_p\hat{v}_{p'}=\hat{v}_{p+p'}\omega^{((1/2)\varepsilon^{\mu\nu}k_\mu k_\nu')} \tag{26}$$

$$=\hat{v}_{p+p'}\exp\left(-i\frac{2\pi b_{12}}{2n}(k_1k_2'-k_2k_1')\right) \tag{27}$$

which is derived by using $\hat{g}^a\hat{h}^b=\hat{h}^b\hat{g}^a\omega^{ab}$.

One final remark is in order: To avoid confusion, one should not think of h, g as being obtained by exponentiating matrices X^1, X^2 that are $n\times n$ matrices. This is *not* how we presented them. Rather, we have evaluated the matrix elements of exponentials of the *operators* X^1, X^2 and obtained finite $n\times n$ matrices, because we used only a discrete set of states that represent the n lattice points. We argued that only exponentials of operators X^1, X^2 that correspond to finite lattice translations are needed to discuss the lattice. These exponentials clearly are finite $n\times n$ matrices on the lattice, as we have seen. For discussing the lattice, only discrete powers of the same exponentials are used, while other functions of the operators X^1, X^2 are never needed. Thus, while X^1, X^2 are opearators acting on an infinite Hilbert space, only a finite set of that space comes into play thanks to the fact that only the exponentials of X^1, X^2 enter in the lattice theory. The infinite space becomes relevant when the lattice spacing goes to zero or n goes to infinity. If one wishes, one may *define* $n\times n$ matrices $\tilde{X}^1$, $\tilde{X}^2$ as the logarithms of the matrices h, g but these should not be identified with the operators X^1, X^2. Obviously the commutation rules in Eq. (9) are true for the operators X^1, X^2 but not true for the matrices $\tilde{X}^1$, $\tilde{X}^2$. As n goes to infinity $\tilde{X}^1$, $\tilde{X}^2$ would approach the matrix elements of the operators X^1, X^2.

B. *d* dimensions

Now we generalize the previous section to d-dimensions. Consider any other pair in the set of non-commuting operators, such as X^3,X^4. The story is the same as in the previous section. The eigenspace of the operator X^3 is labeled by $|j_3\bmod n\rangle$, and the eigenvalues are $x^3=a(j_3\bmod n)$. The set of all operators that need to be considered are $\exp(ip_3X^3+ip_4X^4)$ with

$$p_3=-aB_{34}(k^3\bmod n),\quad p_4=aB_{34}(k^4\bmod n), \tag{28}$$

and with a quantization rule for the flux

$$a^2B_{34}=\frac{2\pi b_{34}}{n},\quad b_{34}=\text{integer}, \tag{29}$$

that leads to a phase ω_{34}

$$\omega_{34}=\exp(-ia^2B_{34})=e^{-i(2\pi b_{34}/n)},\quad(\omega_{34})^n=1. \tag{30}$$

The corresponding translation matrices $\hat{h}_{34},\hat{g}_{34}$ satisfy $\hat{g}_{34}\hat{h}_{34}=\hat{h}_{34}\hat{g}_{34}\omega_{34}$ and they lead to the group algebra (26) with ω_{34},k_3,k_4 inserted instead of ω,k_1,k_2.

The combined non-commutative geometry for all the operators can be treated by taking a direct product of the eigenspaces of $X^1,X^3,X^5,\ldots,X^{d-1}$

$$|j_1,j_3,\ldots,j_{d-1}\rangle,\quad x_{2i-1}=a(j_{2i-1}\bmod n). \tag{31}$$

The remaining operators $X_2,X_4,\ldots,X_d$ cannot be simultaneously diagonalized with the above. But in the space in which they are diagonal [obtained by applying the transformation U in Eq. (21)] these X_{2i} have eigenvalues that are similar to those above $x_{2i}=a(j_{2i}\bmod n)$.

The flux is quantized because of the periodicity of the lattice

$$a^2B_{\mu\nu}=\frac{2\pi b_{\mu\nu}}{n},\quad b_{\mu\nu}=\text{integer} \tag{32}$$

$$\omega_{\mu\nu}=\exp(-ia^2B_{\mu\nu}),\quad(\omega_{\mu\nu})^n=1, \tag{33}$$

and the momentum lattice is defined by

$$p_\mu=ak^\nu B_{\nu\mu}=k^\nu\frac{2\pi b_{\nu\mu}}{an},\quad p_\mu\in(\tilde{T}_n)^d. \tag{34}$$

The set of all possible lattice translations $\exp(i\Sigma_{\mu=1}^d p_\mu X^\mu)$, which is similar to a ''vertex operator'' in string theory, has the matrix elements

$$\langle j_1,j_3,\ldots,j_{d-1}|\exp\left(i\sum_{\mu=1}^{d} p_\mu X^\mu\right)|j_1',j_3',\ldots,j_{d-1}'\rangle \tag{35}$$

$$\equiv(\hat{V}_p)_{j_1j_3\cdots j_{d-1}}^{j_1'j_3'\cdots j_{d-1}'}=(\hat{v}_{p_1p_2})_{j_1}^{j_1'}\cdots(\hat{v}_{p_{d-1}p_d})_{j_{d-1}}^{j_{d-1}'}. \tag{36}$$

In matrix notation, the set of all translation operators takes the direct product form

$$\hat{V}_p=\hat{v}_{p_1p_2}\otimes\hat{v}_{p_3p_4}\otimes\cdots\otimes\hat{v}_{p_{d-1}p_d}. \tag{37}$$

The matrices $\hat{V}_p$ satisfy the group algebra

$$\hat{V}_p\hat{V}_{p'}=\hat{V}_{p+p'}\exp\left(-i\frac{\pi b_{\mu\nu}}{n}k_\mu k_\nu'\right) \tag{38}$$

$$=\hat{V}_{p+p'}\exp\left(-\frac{i}{2}\theta^{\mu\nu}p_\mu p_\nu'\right) \tag{39}$$

which follows from Eq. (26). Under tracing one gets a Kronecker delta function

$$\mathrm{Tr}\,\hat{V}_p=N\,\delta_{p_\mu,0},\quad \mathrm{Tr}(\hat{V}_p\hat{V}_{p'})=N\,\delta_{p_\mu,-p_\mu'}. \tag{40}$$

The relation (38) looks formally the same as the continuum, but in the present case it takes into account the momentum lattice $(\tilde{T}_n)^d$ by having discrete momenta p_μ, and the position lattice $(T_n)^d$ by taking discrete eigenvalues x_μ. Only half of the x_μ label the matrix elements of the matrices $(\hat{V}_p)_J^{J'}$ where J is a label for the direct product space $J=(j_1,j_3,j_5,\ldots,j_{d-1})$. For the more general lattice the rank of these matrices is $N=n_1n_2\ldots n_{d/2}$. Both the position and momentum lattices are periodic and this is manifest in the expression (37) for $\hat{V}_p$.

Although this result was derived by taking a block diagonal $\theta_{\mu\nu}$, it is easy to generalize. The final result (38) is valid for the general quantized antisymmetric matrix $b_{\mu\nu}$, or general quantized $\theta_{\mu\nu}$

$$\theta_{\mu\nu}=\frac{na^2}{4\pi}(b_{\mu\nu})^{-1}. \tag{41}$$

C. Map $(\hat{\Delta}(x))_J^{J'}$ from position lattice to matrix

Consider the Fourier transform of the matrix $(\hat{V}_p)_J^{J'}$ that represents all possible translations on the lattice

$$(\hat{\Delta}(x))_J^{J'}\equiv\sum_{p\in(\tilde{T}_n)^d}(\hat{V}_p)_J^{J'}\frac{e^{ip^\mu x_\mu}}{N},\quad x_\mu\in(T_n)^d \tag{42}$$

$$=\frac{1}{N}\sum_{p\in(\tilde{T}_n)^d}(e^{ip^\mu(x_\mu-X_\mu)})_J^{J'}. \tag{43}$$

The inverse transform is (recall $N=n^{d/2}$ or $n_1n_2\cdots n_{d/2}$)

$$(\hat{V}_p)_J^{J'}=\sum_{x_\mu\in(T_n)^d}(\hat{\Delta}(x))_J^{J'}\frac{e^{-ip^\mu x_\mu}}{N},\quad p\in(\tilde{T}_n)^d. \tag{44}$$

These finite Fourier transforms are defined with both positions and momenta taken on lattices, and follow from the completeness and orthogonality properties of the periodic lattice functions $f_p(x)$

$$f_p(x)=\frac{\exp(ip\cdot x)}{N},\quad p^\mu\in(\tilde{T}_n)^d,\quad x^\mu\in(T_n)^d, \tag{45}$$

which are given by

$$\sum_{p\in(\tilde{T}_n)^d}\frac{e^{ip\cdot x}}{N}\frac{e^{-ip\cdot x'}}{N}=\delta_{x,x'},\qquad \sum_{x\in(T_n)^d}\frac{e^{ip\cdot x}}{N}\frac{e^{-ip'\cdot x}}{N}=\delta_{p,p'}. \tag{46}$$

These are verified by performing finite sums, e.g. the sum over $p^1=-k_2(2\pi b_{12}/na)$ gives

$$\sum_{k_2=0}^{n-1}\frac{1}{n}e^{-ik_2(2\pi b_{12}/n)j_1}e^{ik_2(2\pi b_{12}/n)j_1'} \tag{47}$$

$$=\sum_{k_2=0}^{n-1}\frac{1}{n}(\omega^{j_1-j_1'})^{k_2}=\frac{1}{n}\frac{1-(\omega^{j_1-j_1'})^n}{1-\omega^{j_1-j_1'}}=\delta_{j,j'}. \tag{48}$$

The numerator is always zero since $\omega^n=1$, but the denominator also vanishes provided $j_1-j_1'=0$, thus $\delta_{j,j'}$ is the correct answer. Likewise, in the definition of $(\hat{\Delta}(x))_J^{J'}$, by concentrating on any one of the sums over p^μ, e.g. $p^1=-k_2(2\pi b_{12}/n)$, using Eqs. (37), (23), one finds

$$\sum_{k_2=1}^{n-1}\hat{h}^{k_2}\omega_{12}^{k_1k_2/2}\exp\left(i\frac{2\pi b_{12}}{n}j_1k_2\right) \tag{49}$$

$$=\sum_{k_2=1}^{n-1}(\hat{h}(\omega_{12})^{j_1+k_1/2})^{k_2}=\frac{1-\hat{h}^n((\omega_{12})^n)^{j_1+k_1/2}}{1-\hat{h}(\omega_{12})^{j_1+k_1/2}}. \tag{50}$$

The numerator is proportional to 1 since $\hat{h}^n=1$, and it vanishes by using $\omega_{12}^n=1$ when k_1 is even (there is a further sum over k_1). So, the result of the sum would be zero (for fixed even k_1) except for the fact that the matrix in the denominator also has one eigenvalue that vanishes. In fact, formally $(\hat{\Delta}(x))_J^{J'}$ are the matrix elements of the delta function $\delta(x_\mu-X_\mu)$, with non-commutative operators X_μ, and lattice points $x_\mu\in(T_n)^d$.

Under matrix multiplication $\hat{\Delta}(x)\hat{\Delta}(y)$ satisfies a closed algebra, and yields a Kronecker delta function upon tracing

$$(\hat{\Delta}(x)\hat{\Delta}(y))_J^{J'}=\frac{1}{N}\sum_{z_\mu\in(T_n)^d}(\hat{\Delta}(z))_J^{J'}e^{2iB^{\mu\nu}(x_\mu-z_\mu)(y_\nu-z_\nu)} \tag{51}$$

$$\mathrm{Tr}(\hat{\Delta}(x)\hat{\Delta}(y))=N\,\delta_{x,y} \tag{52}$$

$$\mathrm{Tr}(\hat{\Delta}(x)) = N^2\,\delta_{x,0}. \tag{53}$$

These are derived by using the group algebra (38), thus

$$\hat{\Delta}(x)\hat{\Delta}(y) = \sum_{p,p'} \hat{V}_p \hat{V}_{p'} \frac{e^{ip^\mu x_\mu}}{N}\frac{e^{ip'^\mu y_\mu}}{N}, \tag{54}$$

$$= \sum_{p,p'} \hat{V}_{p+p'} \exp\left(-\frac{i}{2}\theta^{\mu\nu}p_\mu p'_\nu\right)\frac{e^{ip^\mu x_\mu}}{N}\frac{e^{ip'^\mu y_\mu}}{N}, \tag{55}$$

$$= \sum_{p,p'}\sum_z \hat{\Delta}(z)\frac{e^{-i(p^\mu + p'^\mu)z_\mu}}{N}\exp\left(-\frac{i}{2}\theta^{\mu\nu}p_\mu p'_\nu\right) \times \frac{e^{ip^\mu x_\mu}}{N}\frac{e^{ip'^\mu y_\mu}}{N},$$

using the orthogonality-completeness relations (46) to perform the sum over p_μ one finds Eq. (51). Also, using Eq. (40) or (46) in Eq. (54) one derives Eq. (52).

D. Diamond product

We may now define functions on the lattice, such as gauge fields $A(x)$. Since there are only N^2 points on the lattice, these functions really consist of only N^2 numbers. Therefore, it makes sense to set up a map to a $N\times N$ matrix $\hat{A}_J^{J'}$ by using the map $(\hat{\Delta}(x))_J^{J'}$

$$\hat{A}_J^{J'} = \sum_{x_\mu \in (T_n)^d} A(x)(\hat{\Delta}(x))_J^{J'}, \quad A(x) = N^{-1}\mathrm{Tr}(\hat{\Delta}(x)\hat{A}). \tag{56}$$

All the information in $A(x)$ on the N^2 lattice points is contained in the N^2 entries of $\hat{A}_J^{J'}$. The matrix $\hat{A}$ can be viewed as an operator acting on the quantum Hilbert space of noncommutative geometry. To define products among the $\hat{A}_\mu$ it is natural to adopt the usual product of operators in quantum mechanics, which in this case, corresponds to ordinary matrix product $(\hat{A}_\mu \hat{A}_\nu)_J^{J'}$. Having defined the product, we introduce the diamond product in x^μ space as the one that is equivalent to the matrix product via the map (56)

$$(A_\mu \diamond A_\nu)(x) \equiv N^{-1}\mathrm{Tr}(\hat{\Delta}(x)\hat{A}_\mu \hat{A}_\nu), \tag{57}$$

$$(\hat{A}_\mu \hat{A}_\nu)_J^{J'} = \sum_{x_\mu \in (T_n)^d} A_\mu(x)\diamond A_\nu(x)(\hat{\Delta}(x))_J^{J'}. \tag{58}$$

This expression can be rewritten purely in terms of $A_\mu(x)$ by using the correspondence (56) and then using the formulas in Eqs. (51), (52)

$$A_\mu(x)\diamond A_\nu(x) = N^{-1}\sum_{y,z\in(T_n)^d}\mathrm{Tr}(\hat{\Delta}(x)\hat{\Delta}(y) \times \hat{\Delta}(z))A_\mu(y)A_\nu(z)$$
$$= N^{-1}\sum_{y,z\in(T_n)^d} e^{2iB^{\mu\nu}(x_\mu - y_\mu)(x_\nu - z_\nu)} \times A_\mu(y)A_\nu(z). \tag{59}$$

It is physically interesting to note that the sums in the diamond product are weighted by exponentials of the flux that passes through the area defined by the three lattice points x,y,z.

For the complete set of periodic functions $f_p(x)$ given in Eq. (45) it is interesting to note their matrix map according to Eq. (56)

$$(\hat{f}_p)_J^{J'} = \sum_{x_\mu\in(T_n)^d}\frac{\exp(ip\cdot x)}{N}(\hat{\Delta}(x))_J^{J'} = (\hat{V}_{-p})_J^{J'}.$$

The result of applying the diamond product on them is

$$\frac{\exp(ip\cdot x)}{N}\diamond\frac{\exp(ip'\cdot x)}{N} = \exp\left(-\frac{i}{2}\theta^{\mu\nu}p_\mu p'_\nu\right)\frac{\exp(i(p+p')\cdot x)}{N} \tag{60}$$

$$f_p(x)\diamond f_{p'}(x) = \exp\left(-\frac{i}{2}\theta^{\mu\nu}p_\mu p'_\nu\right)f_{p+p'}(x). \tag{61}$$

It is important to emphasize that all positions x^μ and all momenta p^μ are on their respective lattices with only N^2 allowed values for each. The form of this result has a complete parallel in the continuum limit when the positions and momenta are continuous and the star product (1) is used instead of the diamond product (59)

$$e^{ip\cdot x} * e^{ip'\cdot x} = \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial^x_\mu \partial^y_\nu\right)e^{ip\cdot x}e^{ip'\cdot y}\Big|_{y=x} \tag{62}$$

$$= \exp\left(-\frac{i}{2}\theta^{\mu\nu}p_\mu p'_\nu\right)e^{i(p+p')\cdot x}. \tag{63}$$

Since the plane waves $e^{ip\cdot x}$ form a complete set of functions in the continuum theory, this shows that the continuum limit of the diamond product is the star product given in Eq. (1).

We have shown through Eq. (57) that the diamond product $(A_\mu \diamond A_\nu)(x)$ is equivalent to the finite $N\times N$ matrix product $(\hat{A}_\mu \hat{A}_\nu)_J^{J'}$ thanks to the map $(\hat{\Delta}(x))_J^{J'}$. Going over to the continuum corresponds to a particular large N limit (there are many possible large N limits since $N = n_1 n_2 \cdots n_{d/2}$, and any of the factors could be large in independent ways). When all $n_i = n \to \infty$ are large, the star product can be associated with the large N limit of the diamond product. Note that in taking the large n limit to reach the continuum, one must keep $B_{\mu\nu}=2\pi b_{\mu\nu}n/(na)^2$ and the products an, and $nb_{\mu\nu}$ finite (the number of flux lines $b_{\mu\nu}$ per lattice plaquette goes to zero as the lattice distance vanishes).

III. DISCRETE NON-COMMUTATIVE GAUGE THEORY

To construct a gauge theory we also need to define a lattice version of the derivative of fields, such as $\partial_\mu A_\nu(x)$. When x^μ is on the lattice we will write symbolically $\hat{\partial}_\mu A_\nu(x)$ where $\hat{\partial}_\mu$ is a discrete operation that we define. The simplest way is to define it in matrix space as a commutator $-i[\hat{P}_\mu,\hat{A}_\nu]_J^{J'}$ with a fixed set of $N\times N$ matrices $\hat{P}_\mu$, and then map it to x-space using the map (56)

$$\hat{\partial}_\mu A_\nu(x)=N^{-1}\mathrm{Tr}(-i[\hat{P}_\mu,\hat{A}_\nu]\hat{\Delta}(x)) \tag{64}$$

$$-i[\hat{P}_\mu,\hat{A}_\nu]_J^{J'}=\sum_{x_\mu\in(T_n)^d}\hat{\partial}_\mu A_\nu(x)(\hat{\Delta}(x))_J^{J'}. \tag{65}$$

The important property of the definition is that this lattice derivative is distributive when the diamond product is used,

$$\hat{\partial}_\mu(A_\nu(x)\diamondsuit A_\lambda(x))$$
$$=(\hat{\partial}_\mu A_\nu(x))\diamondsuit A_\lambda(x)+A_\nu(x)\diamondsuit(\hat{\partial}_\mu A_\lambda(x)).$$

Furthermore, $\hat{\partial}_\mu$ is commutative $\hat{\partial}_\mu\hat{\partial}_\nu=\hat{\partial}_\nu\hat{\partial}_\mu$ if the matrices $\hat{P}_\mu$ commute with each other $\hat{P}_\mu\hat{P}_\nu=\hat{P}_\nu\hat{P}_\mu$. That is, one can immediately show

$$\hat{\partial}_\mu(\hat{\partial}_\nu A_\lambda(x))=\hat{\partial}_\nu(\hat{\partial}_\mu A_\lambda(x)), \tag{66}$$

by using the definition (64) and matrix Jacobi identities.

A. Non-commutative U(1) gauge theory

With these definitions we give the covariant derivative applied on any function $\psi(x)$ defined on the periodic lattice

$$\hat{D}_\mu\psi(x)\equiv\hat{\partial}_\mu\psi(x)-iA_\mu(x)\diamondsuit\psi(x)+i\psi(x)\diamondsuit A_\mu(x). \tag{67}$$

Both the function and the covariant derivative transform covariantly under gauge transformations provided $A_\mu(x)$ also transforms as follows:

$$\delta A_\mu(x)=\hat{D}_\mu\Lambda(x), \tag{68}$$

$$\delta\psi(x)=i\Lambda(x)\diamondsuit\psi(x)-i\psi(x)\diamondsuit\Lambda(x), \tag{69}$$

$$\delta(\hat{D}_\mu\psi(x))=i\Lambda(x)\diamondsuit(\hat{D}_\mu\psi(x))-i(\hat{D}_\mu\psi(x))\diamondsuit\Lambda(x). \tag{70}$$

Using the map $(\hat{\Delta}(x))_J^{J'}$ each one of these equations can be written in the equivalent matrix space for $\hat{\psi},\hat{A}_\mu$. The covariant derivative becomes the matrix commutator

$$\hat{D}_\mu\psi(x)\rightarrow[-i(\hat{P}_\mu+\hat{A}_\mu),\hat{\psi}]. \tag{71}$$

The transformation laws are

$$\delta\hat{\psi}=i[\hat{\Lambda},\hat{\psi}],\quad \delta\hat{A}_\mu=[-i(\hat{P}_\mu+\hat{A}_\mu),\hat{\Lambda}] \tag{72}$$

and the covariance can be checked explicitly by using matrix Jacobi identities

$$\delta[-i(\hat{P}_\mu+\hat{A}_\mu),\hat{\psi}]=i[\hat{\Lambda},[-i(\hat{P}_\mu+\hat{A}_\mu),\hat{\psi}]].$$

Next we can define the covariant field strength in position space and in matrix space as the commutator of the covariant derivatives, with the usual map relating the two

$$F_{\mu\nu}(x)=N^{-1}\mathrm{Tr}(\hat{F}_{\mu\nu}\hat{\Delta}(x)) \tag{73}$$

$$(\hat{F}_{\mu\nu})_J^{J'}=\sum_{x_\mu\in(T_n)^d}F_{\mu\nu}(x)(\hat{\Delta}(x))_J^{J'} \tag{74}$$

and

$$F_{\mu\nu}(x)=-i\hat{D}_{[\mu}\diamondsuit\hat{D}_{\nu]}=\hat{\partial}_{[\mu}A_{\nu]}(x)+A_{[\mu}\diamondsuit A_{\nu]}(x) \tag{75}$$

$$(\hat{F}_{\mu\nu})_J^{J'}=-i[(\hat{P}_\mu+\hat{A}_\mu),(\hat{P}_\nu+\hat{A}_\nu)]_J^{J'}=-i[\hat{a}_\mu,\hat{a}_\nu]_J^{J'} \tag{76}$$

where in the last line we have used the definition for the matrix $\hat{a}_\mu$

$$(\hat{a}_\mu)_J^{J'}=(\hat{P}_\mu+\hat{A}_\mu)_J^{J'} \tag{77}$$

that appears everywhere. The matrices $\hat{P}_\nu,\hat{A}_\nu$ appear everywhere only in the combination $\hat{a}_\mu$, therefore the theory is expressed only in terms of the matrix $\hat{a}_\mu$.

The action for the pure U(1) non-commutative gauge theory is then written in either discrete position space or in matrix space

$$S=\frac{1}{4N^2}\sum_{x\in(T_n)^d}F_{\mu\nu}(x)\diamondsuit F_{\mu\nu}(x) \tag{78}$$

$$=\frac{1}{4}\sum_{x\in(T_n)^d}\mathrm{Tr}(\hat{F}_{\mu\nu}\hat{F}_{\mu\nu})=-\frac{1}{4}\mathrm{Tr}[\hat{a}_\mu,\hat{a}_\nu]^2. \tag{79}$$

To derive the last line from the first line one can use the map (57) for the product $F_{\mu\nu}\diamondsuit F_{\mu\nu}$ and then use $N^{-3}\Sigma_{x\in(T_n)^d}\hat{\Delta}(x)=\hat{1}$.

Matter, including fermions, can be added naturally both in the lattice and the matrix formulation. The supersymmetric version is also straightforward.

B. Non-commutative U(M) gauge theory

The U(M) gauge theory is naturally constructed by attaching indices on the gauge fields $(A_\mu(x))_a^{a'}$ with $a,a'=1,2,\ldots,M$. Then the diamond product is combined with matrix product $(A_\mu(x)\diamondsuit A_\mu(x))_a^{a'}$. In the matrix version the matrix has the following indices $\hat{A}_{Ja}^{J'a'}$. This is equivalent to enlarging the direct product space $J=(j_1j_3\cdots j_{d-1})$ to $(j_1j_3\cdots j_{d-1}a)$. From the point of view of our discussion we can interpret the additional index a as arising from two extra non-commuting dimensions $[X_{d+1},X_{d+2}]=i(B^{-1})_{d+1,d+2}$, with their eigenvalues on the lattice $j_{d+1}\equiv a=1,2,\ldots,n_{(d+2)/2}$, where $n_{(d+2)/2}\equiv M$. Then we can regard the U(M) non-commutative gauge theory in d dimensions, as a U(1) non-commutative gauge theory in $d+2$ dimensions. In the matrix version its action takes the form

$$S=-\frac{1}{4}\mathrm{Tr}[\hat{a}_\mu,\hat{a}_\nu]^2 \tag{80}$$

where now $\hat{a}_\mu$ is a $NM\times NM$ matrix given by

$$(\hat{a}_\mu)_{Ja}^{J'a'}=(\hat{P}_\mu)_J^{J'}\delta_a^{a'}+(\hat{A}_\mu)_{Ja}^{J'a'}. \tag{81}$$

Since $(\hat{A}_\mu)_{Ja}^{J'a'}$ is the most general matrix, $(\hat{a}_\mu)_{Ja}^{J'a'}$ is also the most general $NM\times NM$ matrix. The form $(\hat{P}_\mu)_J^{J'}\delta_a^{a'}$ that seems to be pulled out artificially serves only to distinguish between the space directions and the internal directions.

If we take this point of view, the U(1) non-commutative gauge field may be labeled by $A_\mu(x^\mu,\vec{\sigma})$ where σ_1,σ_2 are the extra coordinates that take values at the M^2 lattice points in the (σ_1,σ_2) plane. This point of view was explored a long time ago in [7], where it was shown that the U(M) gauge transformations at finite M may also be regarded as discrete diffeomorphism transformations of the discrete torus. As discussed in [7] these discrete area preserving transformations can be embedded in SL($2,Z_M$).

The action above is not yet a full $d+2$ dimensional gauge theory because two additional fields $A_{d+1}(x^\mu,\vec{\sigma})$ and $A_{d+2}(x^\mu,\vec{\sigma})$ (or their matrix counterparts $\hat{a}_{d+1}$ and $\hat{a}_{d+2}$) are missing. However, if the original U(M) non-commutative gauge theory is enlarged by including two additional scalars in the adjoint representation of $U(M)$, then those two scalars could be interpreted as the extra space components of the gauge field in $d+2$ dimensions, to complete it to a full U(1) non-commutative gauge theory in $d+2$ dimensions.

As in the U(1) case, matter fields can be easily added and the theory can be supersymmetrized.

IV. OUTLOOK

In this paper we have discussed a discrete version of non-commutative geometry that arises in string theory in the B field background. We have presented a formalism that introduced the diamond product as a lattice version of the star product, and thus suggested a cutoff version of non-commutative gauge theory.

One may ask what relation could one establish between our results and some other attempt at providing a non-commutative version of Wilson's lattice gauge theory formalism. In the same way that non-commutative gauge theory in the continuum can be recast as a usual gauge theory with an infinite number of high derivative terms [2], we suspect that our results can be rewritten as a complicated Wilsonian type lattice action. It would be interesting to compare the 't Hooft limits of ordinary and non-commutative Yang-Mills on the lattice and verify their equivalence as claimed in [22] for the continuum.

The similarity to reduced models could be further explored. Wilson loop variables for non-commutative Yang-Mills have their counterparts in the reduced Yang-Mills theory, but now the tracing must be done over both internal and external matrix indices Ja. It would be interesting to understand the relevance of this formulation of Wilson loop variables in the extrapolation of the AdS-CFT correspondence in the presence of the background B field, as studied in [23].

In our version one could analyze the theory at finite N which provides a cutoff. For a sufficiently small N the analysis can be done with the help of a computer. Also, since the action is very simple, analytic computations may not be out of reach.

ACKNOWLEDGMENTS

We thank O. Aharony, R. Corrado, E. Gimon, A. Hashimoto, N. Itzhaki, and E. Witten for discussions. This research was partially supported by the U.S. Department of Energy under grant number DE-FG03-84ER40168.

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