Dynamical chiral symmetry breaking on the light front. II. The Nambu–Jona-Lasinio model

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An investigation of dynamical chiral symmetry breaking on the light front is made in the Nambu–Jona-Lasinio model with one flavor and N colors. The analysis of the model suffers from extraordinary complexity due to the existence of a ''fermionic constraint,'' i.e., a constraint equation for the bad spinor component. However, to solve this constraint is of special importance. In classical theory, we can exactly solve it and then explicitly check the property of ''light-front chiral transformation.'' In quantum theory, we introduce a bilocal formulation to solve the fermionic constraint by the $1/N$ expansion. Systematic $1/N$ expansion of the fermion bilocal operator is realized by the boson expansion method. The leading (bilocal) fermionic constraint becomes a gap equation for a chiral condensate and thus if we choose a nontrivial solution of the gap equation, we are in the broken phase. As a result of the nonzero chiral condensate, we find an unusual chiral transformation of fields and nonvanishing of the light-front chiral charge. A leading-order eigenvalue equation for a single bosonic state is equivalent to a leading-order fermion-antifermion bound-state equation. We analytically solve it for scalar and pseudoscalar mesons and obtain their light-cone wave functions and masses. All of the results are entirely consistent with those of our previous analysis on the chiral Yukawa model.

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I. INTRODUCTION

Our expectation that light-front (LF) formalism enables us to relate QCD directly to the constituent quark model at a field-theoretic level [1] seriously requires a full understanding of dynamical chiral symmetry breaking (DχSB) on the LF. Central to this issue is, first of all, the well-known problem of how to reconcile a LF ''trivial'' vacuum with a chirally broken vacuum having a nonzero fermion condensate. The secondary problem is to determine the property of ''LF chiral transformation'' which is defined differently from the usual one. The most surprising fact of the LF chiral transformation is that it is an exact symmetry even for a massive free fermion [2].

In the present paper, we discuss this issue within the Nambu–Jona-Lasinio (NJL) model [3] which is a typical example of DχSB. Previously we considered the same problem from a different point of view [4]. Our interest was in describing DχSB of the NJL model, but we actually took a roundabout way in order to apply an idea which works well for spontaneous symmetry breaking of a scalar model, to a fermionic theory. We know that the longitudinal zero modes of scalar fields are responsible for describing spontaneous symmetry breaking on the LF. Indeed, it is achieved by solving the ''zero-mode constraints'' (i.e., constraint equations for the longitudinal zero modes) nonperturbatively [5]. The zero-mode constraint appears in the discretized light-cone quantization (DLCQ) approach [6], where we set periodic boundary conditions for scalars in the longitudinal direction with finite extension. Of course the NJL model has no scalar fields as fundamental degrees of freedom, but we overcame the situation by considering the chiral Yukawa model. This model shows DχSB in the large-N limit (N is the number of fermions) and goes to the NJL model in infinitely heavy mass limit of scalar and pseudoscalar bosons. We showed that the zero-mode constraint of the scalar field correctly produces a gap equation for a chiral condensate and calculated masses of the scalar and pseudoscalar bosons from poles of their propagators. Therefore, in Ref. [4], we succeeded in describing *indirectly* the chiral symmetry breaking of the NJL model on the LF.

Since the very essence of the previous analysis was the existence of scalar fields, one may ask a question: How can one formulate DχSB *without* scalars? In order to answer the question, we treat the NJL model without introducing an auxiliary field. An important key was already shown in Ref. [7]. It was found that the ''fermionic constraint'' plays the same role as that of the zero-mode constraint. Splitting the fermion field as $\Psi=\psi_+ + \psi_-$, $\psi_\pm=\Lambda_\pm\Psi$ by using projectors $\Lambda_\pm=\gamma^\mp\gamma^\pm/2$, we easily find that the ''bad'' component ψ_- is a dependent variable and subject to a constraint equation called the ''fermionic constraint.'' In the LF NJL model, the fermionic constraint is very complicated and it is difficult to solve it as an operator equation. However, we will see that to solve this equation is a crucial step for describing the broken phase and will find a close parallel between the fermionic constraint for DχSB and the zero-mode constraint for spontaneous symmetry breaking in scalar models. Although such special importance of the fermionic constraint might be restricted only to the LF NJL model and therefore most of the analysis might be model dependent, but what we are

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eventually interested in is the physics consequences of the chiral symmetry breaking. And of course we cannot reach the chiral symmetry breaking in QCD unless we understand the simpler and typical example of this phenomenon. Therefore the importance of our analysis is evident.

Let us comment on other attempts for the NJL model on the LF. First of all, Heinzl *et al.* [8] treated the model within the mean-field approximation and insisted on delicacy of the infrared cutoff to obtain a chiral condensate. The meaning and necessity of such a cutoff scheme was clarified in Ref. [9]. As mentioned above, an observation that a gap equation for a chiral condensate emerges from the fermionic constraint was first pointed out by one of the authors [7]. The light-cone (LC) wave function of a pionic state was calculated through the LC projection of the Bethe-Salpeter amplitude which was derived in the equal-time quantization [10]. Bentz *et al.* introduced the auxiliary fields to fermion bilinears and solved the constraint equations for them by the $1/N$ expansion [11]. They obtained an ''effective'' Lagrangian for the broken phase and discussed the structure function of the pionic state. With all these studies, however, there still remains many unknowns concerning basic problems. Especially, we still do not understand well the LF chiral transformation itself. To what extent is it different from the usual chiral symmetry? How is the chiral symmetry breaking realized on the LF? These fundamental problems will be resolved in the present paper.

The paper is organized as follows. The rest of this section is devoted to introduction of the NJL model and our notation. In Sec. II, we discuss the complexity of the fermionic constraint in great detail. We explicitly solve the fermionic constraint in classical treatment and investigate properties of LF chiral transformation. In Sec. III, we solve the fermionic constraint in quantum theory by the $1/N$ expansion. Here we introduce the boson expansion method in order to solve the bilocal fermionic constraint with systematic $1/N$ expansion. We see the emergence of the gap equation for the chiral condensate from the fermionic constraint. We obtain the Hamiltonian with respect to the (bilocal) bosons which is introduced by the boson expansion method. In Sec. IV, some physics consequences of the chiral symmetry breaking are discussed. First of all, we see how the chiral symmetry breaking is realized in the LF formalism. We discuss unusual chiral transformation of fields and nonconservation of the light-front chiral charge. Second, we construct the bound-state equation for mesonic states and solve it for scalar and pseudoscalar mesons. Third, we derive the partially conserving axial vector current (PCAC) relation. Summary and conclusion are given in the last section. Miscellaneous topics with detailed calculation are presented in Appendixes.

Before ending this section, let us fix our model and notation. Since the primary purpose of our paper is to study basic properties of the LF chiral symmetry, we consider only one flavor case for simplicity. Thus the model we discuss is

$$\mathcal{L}=\bar{\Psi}(i\partial\!\!\!/-m_0)\Psi+\frac{g^2}{2}[(\bar{\Psi}\Psi)^2+(\bar{\Psi}i\gamma_5\Psi)^2]. \tag{1.1}$$

Here $\Psi^a(x)$ $(a=1,\ldots,N)$ is a four-component spinor with ''color'' internal symmetry $U(N)$, which has been introduced so that we can use the $1/N$ expansion as a nonperturbative technique. We always work with a nonzero bare mass $m_0\neq 0$. The primary reason is that the Hamiltonian with a massless fermion is plagued with a troublesome situation in 1+1 dimensions: As we will see, if we set $m_0=0$ from the beginning, the canonical LF Hamiltonian P^- of the Gross-Neveu model vanishes altogether. Even in 3+1 dimensions, we will see that absence of the bare mass term causes an inconsistency of the results. The secondary reason is to avoid massless particles which can move in parallel with $x^+=$const surface. The difficulty of describing massless particles is intimately connected with the fact that on the LF, the (massless) Nambu-Goldstone boson becomes physically meaningful only when we first include explicit breaking term and then take the vanishing limit of it [12]. The same situation was observed in the chiral Yukawa model [4].

In practical calculation, it is convenient to introduce the two-component representation for the gamma matrices so that the projectors $\Lambda_\pm$ are expressed as

$$\Lambda_+=\begin{pmatrix}\mathbf{1} & 0\\ 0 & 0\end{pmatrix},\quad \Lambda_-=\begin{pmatrix}0 & 0\\ 0 & \mathbf{1}\end{pmatrix}. \tag{1.2}$$

Then, the projected fermions have only upper or lower components:

$$\psi_+=\Lambda_+\Psi\equiv 2^{-1/4}\begin{pmatrix}\psi\\ 0\end{pmatrix},\quad \psi_-=\Lambda_-\Psi\equiv 2^{-1/4}\begin{pmatrix}0\\ \chi\end{pmatrix}, \tag{1.3}$$

where we defined two-component spinors ψ and χ. Among various representations which satisfy Eq. (1.2), we choose a representation having a similar structure to the chiral representation in 1+1 dimensions, i.e., $\gamma^0=\sigma^1$, $\gamma^1=-i\sigma^2$, $\gamma_5=\gamma^0\gamma^1=\sigma^3$. Explicitly, they are

$$\gamma^0=\begin{pmatrix}0 & \mathbf{1}\\ \mathbf{1} & 0\end{pmatrix},\quad \gamma^3=\begin{pmatrix}0 & -\mathbf{1}\\ \mathbf{1} & 0\end{pmatrix},\quad \gamma^i=\begin{pmatrix}-i\sigma^i & 0\\ 0 & i\sigma^i\end{pmatrix}, \tag{1.4}$$

for $i=1,2$ and

$$\gamma_5=\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}. \tag{1.5}$$

Results of the chiral Gross-Neveu model in 1+1 dimensions can be easily obtained if we make a replacement for the Pauli matrices $\sigma^3\to 1$ and $\sigma^i\to 0$, and regard ψ and χ as one-component spinors. The explicit form of the Lagrangian in the two-component representation is given in Appendix A.

In the previous work [4], we made the longitudinal direction finite in order to carefully treat the longitudinal zero modes of the scalar fields. However, in the present analysis, we work in an infinite longitudinal space. There is no need of introducing finite x^-. When we take the inverse of ∂_-, we need to specify the boundary conditions. We here follow the conventional antiperiodic boundary condition[1] which is standard for free fermions:

$$\psi_a(x^-=-\infty,x_\perp^i)=-\psi_a(x^-=\infty,x_\perp^i), \tag{1.6}$$

$$\chi_a(x^-=-\infty,x_\perp^i)=-\chi_a(x^-=\infty,x_\perp^i). \tag{1.7}$$

So our results must always have a smooth free-field limit.

II. COMPLEXITY OF THE FERMIONIC CONSTRAINT AND THE CHIRAL SYMMETRY

It is highly complicated structure of the fermionic constraint that makes the analysis of the LF NJL model difficult. However, we cannot know anything about LF chiral symmetry unless we confront with this complexity. Therefore in this section, we investigate the fermionic constraint in great detail. First of all, we classically solve the fermionic constraint. Using the explicit solution, we then discuss properties of the LF chiral transformation. Especially we show that LF chiral transformation is no longer an exact symmetry when $m_0 \neq 0$. Finally, we consider the implication of the fermionic constraint in quantum theory.

A. Fermionic constraint and its classical solution

The fermionic constraint is immediately obtained as the Euler-Lagrange equation for χ:

$$i\partial_-\chi_a=\frac{1}{\sqrt{2}}(-\sigma^i\partial_i+m_0)\psi_a-\frac{g^2}{2}\{\psi_a(\psi_b^\dagger\chi_b+\chi_b^\dagger\psi_b)$$
$$+\sigma^3\psi_a(\psi_b^\dagger\sigma^3\chi_b-\chi_b^\dagger\sigma^3\psi_b)\}, \tag{2.1}$$

where summation over color and spinor indices are implied. If we want to solve this equation as an operator equation in quantum theory, we need a commutation relation between χ with ψ which must be given by the Dirac brackets. Since the anticommutator $\{\chi,\psi\}$ is very complicated, it seems almost hopeless to find an exact quantum solution of it. However, in a classical theory where we treat all the variables just as Grassmann numbers, the equation becomes tractable and it is not difficult to solve it. Indeed, the exact solution with antiperiodic boundary condition is given by (see Appendix B for more details)

$$\begin{pmatrix}\chi_{1a}(x)\\ -\chi_{2a}^\dagger(x)\end{pmatrix}=\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}dy^-G_{ab}(x^-,y^-,x_\perp)$$
$$\times\begin{pmatrix}m_0\psi_{1b}(y^-)-\partial_z\psi_{2b}(y^-)\\ -\partial_z\psi_{1b}^\dagger(y^-)+m_0\psi_{2b}^\dagger(y^-)\end{pmatrix}, \tag{2.2}$$

where $\partial_z=\partial_1-i\partial_2$ and the "Green function" $G_{ab}(x^-,y^-,x_\perp)$ is

$$G_{ab}(x^-,y^-,x_\perp)=G^{(0)}(x)\left[\frac{1}{2i}\epsilon(x^--y^-)+C\right]G^{(0)-1}(y), \tag{2.3}$$

$$G^{(0)}(x)=Pe^{ig^2\int_{-\infty}^{x^-}\mathcal{A}(y^-)dy^-}, \tag{2.4}$$

$$\mathcal{A}_{ijab}=\begin{pmatrix}\psi_{1a}\psi_{1b}^\dagger & \psi_{1a}\psi_{2b}\\ \psi_{2a}^\dagger\psi_{1b}^\dagger & \psi_{2a}^\dagger\psi_{2b}\end{pmatrix}. \tag{2.5}$$

The integral constant C is determined so that the solution satisfies the antiperiodic boundary condition. In Eq. (2.4), P stands for the path-ordered product. Note that we easily derive the solution for the chiral Gross-Neveu model by extracting the 1-1 component of $\mathcal{A}$ and neglecting ∂_z. The result is equivalent to the solution of the Thirring model obtained by Domokos [13]. And also, if we take the free fermion limit $g^2\to 0$, we of course recover the free solution due to $G^{(0)}(x)\to 1$ and $G(x^-,y^-)\to\epsilon(x^--y^-)/2i$.

B. Chiral symmetry on the light front

Since the bad component χ is a constrained variable in the LF formalism, we impose the chiral transformation only on the good component $\psi_+\to e^{i\theta\gamma_5}\psi_+$ or in the two-component representation [see Eq. (1.5)]

$$\psi\to e^{i\theta\sigma^3}\psi. \tag{2.6}$$

Now we have completely solved the fermionic constraint for χ, we can explicitly demonstrate its transformation property under the LF chiral transformation. However, before discussing the NJL model, it will be instructive to remind you of the LF chiral symmetry in the free massive fermion.

As we mentioned before, the *massive* free fermion is chiral invariant under the transformation (2.6). Let us see this fact directly in the Lagrangian even though it is a little lengthy. It is convenient to separate the solution of the fermionic constraint $\chi=(\sqrt{2}i\partial_-)^{-1}(-\sigma^i\partial_i+m_0)\psi$ into mass-independent and dependent parts $\chi=\chi^{(0)}+\chi^{(m)}$ as

$$\chi^{(0)}=-\frac{1}{\sqrt{2}}\sigma^i\partial_i\frac{1}{i\partial_-}\psi,\quad \chi^{(m)}=\frac{m_0}{\sqrt{2}}\frac{1}{i\partial_-}\psi.$$

Note that there is a relation between $\chi^{(0)}$ and $\chi^{(m)}$:

$$\sigma^i\partial_i\chi^{(m)}+m_0\chi^{(0)}=0. \tag{2.7}$$

As a result of the LF chiral transformation (2.6), we find

$$\chi^{(0)}\to e^{-i\theta\sigma^3}\chi^{(0)}, \tag{2.8}$$

$$\chi^{(m)}\to e^{i\theta\sigma^3}\chi^{(m)}. \tag{2.9}$$

The free fermion Lagrangian is compactly expressed as $\mathcal{L}_{\rm free}=\psi^\dagger\omega_{\rm EOM}+\chi^\dagger\omega_{\rm FC}$, where $\omega_{\rm EOM}=i\partial_+\psi-1/\sqrt{2}(\sigma^i\partial_i+m_0)\chi=0$ is the equation of motion for ψ and $\omega_{\rm FC}$

[1]For a scalar field, antiperiodic boundary condition in infinite longitudinal space leads to inconsistency [12]. However, fermionic fields are free from such troubles.

$=i\partial_-\chi-1/\sqrt{2}(-\sigma^i\partial_i+m_0)\psi=0$ is the fermionic constraint. The second term is zero and is invariant under the LF chiral transformation. Now substituting $\chi=\chi^{(0)}+\chi^{(m)}$ into the Lagrangian, the first term is decomposed into apparently invariant and (seemingly) noninvariant terms

$$\psi^\dagger\omega_{\rm EOM}=\psi^\dagger\left[i\partial_+\psi-\frac{1}{\sqrt{2}}(\sigma^i\partial_i\chi^{(0)}+m_0\chi^{(m)})\right]+\psi^\dagger\left[-\frac{1}{\sqrt{2}}(\sigma^i\partial_i\chi^{(m)}+m_0\chi^{(0)})\right].$$

The first term consists of the m_0-independent term and quadratically dependent term $\mathcal{O}(m_0^2)$, while the second term linearly depends on m_0. The $\mathcal{O}(m_0)$ term changes under the chiral transformation, but due to the relation (2.7), it eventually vanishes and therefore the Lagrangian is invariant even if there is a mass term. As a result, we have a conserved Noether current [2]

$$j^\mu_{5\rm Free}=\overline{\Psi}\gamma^\mu\gamma_5\Psi-m_0\overline{\Psi}\gamma^\mu\gamma_5\frac{1}{i\partial_-}\gamma^+\psi_+,$$

$$\partial_\mu j^\mu_{5\rm Free}=0,$$

which of course reduces to the usual current in the massless limit.

Now let us consider the NJL model. Decomposition of χ is straightforward:

$$\begin{pmatrix}\chi^{(0)}_{1a}\\ -\chi^{(0)\dagger}_{2a}\end{pmatrix}=-\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}dy^-G_{ab}(x^-,y^-,x_\perp)\begin{pmatrix}\partial_z\psi_{2b}\\ \partial_z\psi^\dagger_{1b}\end{pmatrix},\tag{2.10}$$

$$\begin{pmatrix}\chi^{(m)}_{1a}\\ -\chi^{(m)\dagger}_{2a}\end{pmatrix}=\frac{m_0}{\sqrt{2}}\int_{-\infty}^{\infty}dy^-G_{ab}(x^-,y^-,x_\perp)\begin{pmatrix}\psi_{1b}\\ \psi^\dagger_{2b}\end{pmatrix}.\tag{2.11}$$

Since the matrix $\mathcal{A}$, and thus $G_{ab}(x,y)$ is invariant under the transformation (2.6), it is easy to find that $\chi^{(0)}$ and $\chi^{(m)}$ transform as Eqs. (2.8) and (2.9). Therefore, if $m_0=0$, the LF chiral transformation (2.6) is equivalent to the usual chiral transformation. The chiral current and the chiral charge are given by

$$j_5^\mu=\overline{\Psi}\gamma^\mu\gamma_5\Psi,\tag{2.12}$$

$$Q_5^{\rm LF}=\int_{-\infty}^{\infty}dx^-d^2x_\perp j_5^+(x)=\int_{-\infty}^{\infty}dx^-d^2x_\perp\psi^\dagger\sigma^3\psi.\tag{2.13}$$

How about the massive case? As we explicitly showed above, the mass term does not prevent chiral symmetry in the free fermion case. We must bare in mind such a possibility even in the NJL model. Thus it is worthwhile to check whether the *massive* NJL model is invariant under the LF chiral transformation. To see this, it is convenient to treat the Hermite Lagrangian

$$\mathcal{L}_{\rm Hermite}=\frac{1}{2}i\psi^\dagger\overleftrightarrow{\partial}_+\psi-\frac{1}{2\sqrt{2}}[(\psi^\dagger\sigma^i\partial_i\chi+\partial_i\chi^\dagger\sigma^i\psi)+m_0(\psi^\dagger\chi+\chi^\dagger\psi)].\tag{2.14}$$

Note that this is equivalent to the free Lagrangian except that χ is a solution of Eq. (2.1). Now the apparently noninvariant term is a term linearly depending on m_0:

$$-\frac{1}{2\sqrt{2}}\psi^\dagger(\sigma^i\partial_i\chi^{(m)}+m_0\chi^{(0)})+\text{H.c.}\tag{2.15}$$

In the massive free fermion case, we had the same term but it eventually vanished due to Eq. (2.7). However, in the NJL model, it is evident from Eqs. (2.10) and (2.11) such a relation does not hold because G depends on $x_\perp$. Therefore, we have verified that the massive NJL model is *not* invariant under the LF chiral transformation. If and only if $m_0=0$, the LF chiral transformation is the symmetry of the NJL model and equivalent to the usual chiral transformation. This is of course not a surprising result but must be checked explicitly. Anyway, we do not stick to this problem anymore.

Irrespective of whether we have a mass term or not, we always use the definition for the chiral current Eq. (2.12) which was derived for the massless fermion. In the massless case, it is of course a conserved current $\partial_\mu j_5^\mu=0$, while in the massive case, a usual relation holds

$$\partial_\mu j_5^\mu=2m_0\overline{\Psi}i\gamma_5\Psi,\tag{2.16}$$

which is derived by using the Euler-Lagrange equation for the massive fermion. Equation (2.16) is used when we discuss nonconservation of the chiral charge and the PCAC relation in Sec. IV.

C. Implication of the fermionic constraint

So far we treated the fermionic constraint in classical theory and obtained the exact solution (2.2). However, this solution does not give a nonzero condensate and the resulting Hamiltonian does not describe the broken phase. The situation is very similar to the previous analysis of the chiral Yukawa model [4]. The chiral Yukawa model in the DLCQ approach has three constraint equations. We solved them in classical theory but we could not find any way to describe the broken phase with the classical solutions. What we finally found is that it is very important to treat the constraint equations, especially the zero-mode constraint for a scalar field, nonperturbatively in quantum theory. This fact is true of our present case. To obtain a nonzero condensate, we must treat the fermionic constraint as an operator equation and solve it with some nonperturbative method.

To strengthen this, let us briefly overview the procedure in the previous work [4]. In the chiral Yukawa model, we have three dependent variables: Two of them are the longitudinal zero modes of scalar and pseudoscalar fields

$$\sigma_0(x_\perp)=(1/2L)\int_{-L}^{L}dx^-\sigma(x),$$

$$\pi_0(x_\perp)=(1/2L)\int_{-L}^{L}dx^-\pi(x)$$

where L is an extension of the longitudinal direction $x^-\in[-L,L]$, and the rest is the bad component of a fermion $\psi_-(x)$. So there are three constraints:

$$\left(\frac{\mu^2}{\lambda}-\partial_\perp^2\right)\begin{pmatrix}\sigma_0\\ \pi_0\end{pmatrix}=\frac{\mu^2}{N}\frac{1}{\sqrt{2}}\int_{-L}^{L}\frac{dx^-}{2L}\left[\psi_+^{a\dagger}\begin{pmatrix}-1\\ i\gamma_5\end{pmatrix}\gamma^-\psi_-^a+\psi_-^{a\dagger}\begin{pmatrix}-1\\ i\gamma_5\end{pmatrix}\gamma^+\psi_+^a\right], \tag{2.17}$$

$$2i\partial_-\psi_-^a=(i\gamma_\perp^i\partial_i+m_0+\sigma-i\pi\gamma_5)\gamma^+\psi_+^a, \tag{2.18}$$

where $\lambda=g^2N$ in the present notation and μ is a dimensionless parameter which controls the scalar and pseudoscalar masses. In the infinitely heavy mass limit, $\mu\to\infty$, we recover the NJL model. The procedure of Ref. [4] is as follows: First, we formally solved the fermionic constraint (2.18) and substitute the solution into the zero-mode constraints (2.17). Second, we solved the zero-mode constraints by $1/N$ expansion with a fixed operator ordering and found that the leading order of the scalar zero-mode constraint can be identified with the gap equation. Selecting a nontrivial solution of the gap equation, we again substitute it back to the fermionic constraint. Then we obtain the final expression for the bad component ψ_- in terms of independent variables. Thus we solved three coupled equations step by step. On the other hand, we have only one constraint equation. The procedure in the chiral Yukawa model suggests that we will have to do almost the same procedure *at once* when we solve the fermionic constraint. Note that just the same as in the chiral Yukawa model, a perturbative solution cannot reach the broken phase even in quantum theory. Therefore, we naturally expect that solving the fermionic constraint (2.1) in quantum theory using some nonperturbative method is necessary for describing the chiral symmetry breaking [7].

III. SOLVING THE FERMIONIC CONSTRAINT BY 1/*N* EXPANSION IN QUANTUM THEORY

As we discussed above, it is important to solve the fermionic constraint (2.1) in quantum theory by some nonperturbative method. Here we solve it with a fixed operator ordering by using the $1/N$ expansion. For systematic $1/N$ expansion, we introduce the bilocal formulation. We rewrite the fermionic constraint in terms of bilocal fields and expand it following the Holstein-Primakoff-type expansion of the boson expansion method. We always work with fixed x^+.

A. Quantization and the operator ordering

To solve the constraint in quantum theory, we must first perform the Dirac quantization for constraint systems.[2] After tedious but straightforward calculation of the Dirac brackets, we find a familiar relation for the good component ψ_α ($\alpha=1,2$)

$$\{\psi_\alpha^a(x),\ \psi_\beta^{b\dagger}(y)\}_{x^+=y^+}=\delta_{\alpha\beta}\delta_{ab}\delta(x^--y^-)\delta^{(2)}(x_\perp-y_\perp), \tag{3.1}$$

and so on. We introduce the simplest mode expansion at $x^+=0$ as in Ref. [14]:

$$\psi_\alpha^a(x)=\int_{-\infty}^{\infty}\frac{d^2k_\perp}{2\pi}\int_0^\infty\frac{dk^+}{\sqrt{2\pi}}[b_\alpha^a(\boldsymbol{k})e^{i\boldsymbol{kx}}+d_\alpha^{a\dagger}(\boldsymbol{k})e^{-i\boldsymbol{kx}}], \tag{3.2}$$

where $\boldsymbol{kx}\equiv-k^+x^-+k_\perp^ix_\perp^i$. The vacuum is defined by the annihilation operators as

$$b_\alpha^a(\boldsymbol{k})|0\rangle=d_\alpha^a(\boldsymbol{k})|0\rangle=0. \tag{3.3}$$

When we deal with the quantum fermionic constraint, we have to specify the operator ordering. In many publications discussing the zero-mode constraints, people often choose the Weyl ordering with respect to both constrained and unconstrained variables. However, in a previous paper [4], we discussed that the ideal situation was to find a ''consistent'' operator ordering. For example, let us consider an anticommutator $\{\chi,\psi\}$ in the NJL model. It can be evaluated in two different ways: (i) by using the solution $\chi_{\rm sol}=\chi(\psi)$ of the fermionic constraint and the standard quantization rule (3.1), and (ii) by calculating the Dirac bracket for $\{\chi,\psi\}$. For the case (i), we select a specific operator ordering for the fermionic constraint, and the result depends on the ordering. For the case (ii), we must also determine the ordering in the right-hand side (rhs) of the Dirac bracket $\{\chi,\psi\}_{\rm D}=\ldots$. These two results must be equivalent to each other. We have two ambiguities of the operator ordering: those of the constraint equation in (i) and the rhs of the Dirac bracket in (ii). ''Consistent operator ordering'' should be imposed so that these two quantities be identical. In other words, we determine the operator ordering of the rhs in the Dirac brackets so that it coincides with the direct evaluation. In the chiral Yukawa model, we could not check that the ordering we adopted was consistent or not. Again in the NJL model, this is a very difficult task and we choose a specific operator ordering defined by Eq. (2.1). However, the chiral Gross-Neveu model in 1+1 dimensions allows us to check the consistency of this operator ordering. This is briefly shown in Appendix C.

[2]In Ref. [11], the authors solved the constraint equations for auxiliary fields before canonical quantization was specified and gave a c number to the scalar auxiliary field in leading order of $1/N$. Nevertheless, the condensation in the NJL model is a quantum phenomenon and thus this procedure is not justified.

B. Boson expansion method as 1/N expansion of bilocal operators

How can we solve the ''operator equation'' Eq. (2.1) by the $1/N$ expansion? It is generally difficult to count the order $\mathcal{O}(N^n)$ of an operator instead of its matrix element. What is worse, it is physically hard to justify the $1/N$ expansion of the fermionic field itself. However, as was discussed in Ref. [15], there is a powerful method to this problem. We can perform a systematic $1/N$ expansion of operators if we introduce the bilocal operators and use the boson expansion method. The boson expansion method is one of the traditional techniques in nonrelativistic many-body problems [16]. Originally this was invented for describing bosonic excitations in nonbosonic systems such as collective excitation in nuclei or spin systems.

Let us rewrite the fermionic constraint (2.1) in terms of bilocal operators. We introduce the following ''color'' singlet bilocal operators at equal light-front time

$$\mathcal{M}_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})=\sum_{a=1}^{N}\ \psi_\alpha^{a\dagger}(x^+,\boldsymbol{x})\psi_\beta^a(x^+,\boldsymbol{y}), \tag{3.4}$$

$$\mathcal{T}_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})=\frac{1}{\sqrt{2}}\sum_{a=1}^{N}\ (\psi_\alpha^{a\dagger}(x^+,\boldsymbol{x})\chi_\beta^a(x^+,\boldsymbol{y})+\chi_\beta^{a\dagger}(x^+,\boldsymbol{y})\psi_\alpha^a(x^+,\boldsymbol{x})), \tag{3.5}$$

$$\mathcal{U}_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})=\frac{-i}{\sqrt{2}}\sum_{a=1}^{N}\ (\psi_\alpha^{a\dagger}(x^+,\boldsymbol{x})\chi_\beta^a(x^+,\boldsymbol{y})-\chi_\beta^{a\dagger}(x^+,\boldsymbol{y})\psi_\alpha^a(x^+,\boldsymbol{x})). \tag{3.6}$$

We define the Fourier transformation of them as

$$\mathcal{M}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})=\int_{-\infty}^{\infty}\frac{d^3\boldsymbol{x}}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\frac{d^3\boldsymbol{y}}{(2\pi)^{3/2}}\ \mathcal{M}_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})\ \ e^{-i\boldsymbol{px}-i\boldsymbol{qy}},$$

and so on. Note that this definition allows the longitudinal momenta to take negative values. Using these bilocal operators, the fermionic constraint (2.1) is equivalently rewritten as

$$\begin{aligned}i\frac{\partial}{\partial y^-}\mathcal{T}_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})=&\frac{1}{2}\{-\partial_i^y(\sigma^i_{\beta\gamma}\mathcal{M}_{\alpha\gamma}(\boldsymbol{x},\boldsymbol{y})-\sigma^i_{\gamma\beta}\mathcal{M}_{\gamma\alpha}(\boldsymbol{y},\boldsymbol{x}))\\&+m_0(\mathcal{M}_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})-\mathcal{M}_{\beta\alpha}(\boldsymbol{y},\boldsymbol{x}))\}\\&-\frac{g^2}{2}\{\mathcal{M}_{\alpha\gamma}(\boldsymbol{x},\boldsymbol{y})(\delta_{\gamma\beta}\mathcal{T}(\boldsymbol{y},\boldsymbol{y})+i\sigma^3_{\gamma\beta}\mathcal{U}(\boldsymbol{y},\boldsymbol{y}))\\&-(\delta_{\beta\gamma}\mathcal{T}(\boldsymbol{y},\boldsymbol{y})-i\sigma^3_{\beta\gamma}\mathcal{U}(\boldsymbol{y},\boldsymbol{y}))\mathcal{M}_{\gamma\alpha}(\boldsymbol{y},\boldsymbol{x})\},\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}i^2\frac{\partial}{\partial y^-}\mathcal{U}_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})=&\frac{1}{2}\{-\partial_i^y(\sigma^i_{\beta\gamma}\mathcal{M}_{\alpha\gamma}(\boldsymbol{x},\boldsymbol{y})+\sigma^i_{\gamma\beta}\mathcal{M}_{\gamma\alpha}(\boldsymbol{y},\boldsymbol{x}))\\&+m_0(\mathcal{M}_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})+\mathcal{M}_{\beta\alpha}(\boldsymbol{y},\boldsymbol{x}))\}\\&-\frac{g^2}{2}\{\mathcal{M}_{\alpha\gamma}(\boldsymbol{x},\boldsymbol{y})(\delta_{\gamma\beta}\mathcal{T}(\boldsymbol{y},\boldsymbol{y})\\&+i\sigma^3_{\gamma\beta}\mathcal{U}(\boldsymbol{y},\boldsymbol{y}))+(\delta_{\beta\gamma}\mathcal{T}(\boldsymbol{y},\boldsymbol{y})\\&-i\sigma^3_{\beta\gamma}\mathcal{U}(\boldsymbol{y},\boldsymbol{y}))\mathcal{M}_{\gamma\alpha}(\boldsymbol{y},\boldsymbol{x})\},\end{aligned} \tag{3.8}$$

where we have introduced quantities $\mathcal{T}(\boldsymbol{x},\boldsymbol{y})\equiv\mathcal{T}_{\alpha\alpha}(\boldsymbol{x},\boldsymbol{y})$ and $\mathcal{U}(\boldsymbol{x},\boldsymbol{y})\equiv(\sigma^3)_{\alpha\beta}\mathcal{U}_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})$ so that $\bar{\Psi}\Psi(x)=\mathcal{T}(\boldsymbol{x},\boldsymbol{x})$ and $\bar{\Psi}i\gamma_5\Psi(x)=\mathcal{U}(\boldsymbol{x},\boldsymbol{x})$. In actual calculation, it is more convenient to treat equations for the operators without spinor structure $\mathcal{T}(\boldsymbol{x},\boldsymbol{y})$ and $\mathcal{U}(\boldsymbol{x},\boldsymbol{y})$ because they form closed equations (see Appendix D). Once we solve them, we immediately obtain $\mathcal{T}_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})$ and $\mathcal{U}_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})$ from the above equations.

For systematic $1/N$ expansion of the bilocal fermionic constraints, one must know how to expand $\mathcal{M}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})$. It is the boson expansion method, especially, the Holstein-Primakoff type expansion for large N theories, that enables us to expand $\mathcal{M}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})$ as *operator* quantities:

$$\mathcal{M}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})=N\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{N}}\right)^n\mu^{(n)}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q}). \tag{3.9}$$

According to the Holstein-Primakoff expansion [Eqs. (D5)–(D8)], the first three terms are written in terms of bilocal bosonic variable $B(\boldsymbol{p},\boldsymbol{q})$ as

$$\mu^{(0)}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})=\delta_{\alpha\beta}\delta^{(3)}(\boldsymbol{p}+\boldsymbol{q})\,\theta(p^+)\,\theta(-q^+), \tag{3.10}$$

$$\mu^{(1)}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})=B_{\beta\alpha}(\boldsymbol{q},\boldsymbol{p})\,\theta(p^+)\,\theta(q^+)+B^\dagger_{\alpha\beta}(-\boldsymbol{p},-\boldsymbol{q})\,\theta(-p^+)\,\theta(-q^+), \tag{3.11}$$

$$\begin{aligned}\mu^{(2)}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})=&\int[d\boldsymbol{k}]\sum_\gamma\ B^\dagger_{\alpha\gamma}(-\boldsymbol{p},\boldsymbol{k})B_{\beta\gamma}(\boldsymbol{q},\boldsymbol{k})\\&\times\theta(-p^+)\,\theta(q^+)-\int[d\boldsymbol{k}]\sum_\gamma\\&\times B^\dagger_{\gamma\beta}(\boldsymbol{k},-\boldsymbol{q})B_{\gamma\alpha}(\boldsymbol{k},\boldsymbol{p})\,\theta(p^+)\,\theta(-q^+).\end{aligned} \tag{3.12}$$

where

$$\int\ [d\boldsymbol{q}]=\int_0^\infty dq^+\int_{-\infty}^{\infty}d^2q_\perp\,.$$

Any commutator between $\mathcal{M}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})$'s [such as Eq. (D3)] is correctly reproduced if one uses the following bosonic commutators:

$$[B_{\alpha\beta}(\boldsymbol{p}_1,\boldsymbol{p}_2),\ B^{\dagger}_{\gamma\delta}(\boldsymbol{q}_1,\boldsymbol{q}_2)] = \delta_{\alpha\gamma}\delta_{\beta\delta}\delta^{(3)}(\boldsymbol{p}_1-\boldsymbol{q}_1)\,\delta^{(3)}(\boldsymbol{p}_2-\boldsymbol{q}_2), \quad (3.13)$$

$$[B_{\alpha\beta}(\boldsymbol{p}_1,\boldsymbol{p}_2),\ B_{\gamma\delta}(\boldsymbol{q}_1,\boldsymbol{q}_2)]=0 \quad (p_i^+,q_i^+>0). \quad (3.14)$$

Note also that the state annihilated by $B(\boldsymbol{p},\boldsymbol{q})$ is identified with the original Fock vacuum:

$$B(\boldsymbol{p},\boldsymbol{q})|0\rangle=0. \quad (3.15)$$

More detailed discussions about the boson expansion method applied to LF field theories are found in Ref. [15] and Appendix D of the present paper.

C. Solution to the bilocal fermionic constraint

We are ready to solve the bilocal fermionic constraint using the $1/N$ expansion. As we commented before, it is convenient to solve the equations for $\mathcal{T}(\boldsymbol{p},\boldsymbol{q})$ and $\mathcal{U}(\boldsymbol{p},\boldsymbol{q})$ [see Eqs. (D1) and (D2) in Appendix D for their explicit forms]. Once we know $\mathcal{T}(\boldsymbol{p},\boldsymbol{q})$ and $\mathcal{U}(\boldsymbol{p},\boldsymbol{q})$, then it is straightforward to obtain $\mathcal{T}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})$ and $\mathcal{U}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})$.

Expanding $\mathcal{T}(\boldsymbol{p},\boldsymbol{q})$ and $\mathcal{U}(\boldsymbol{p},\boldsymbol{q})$ as

$$\mathcal{T}(\boldsymbol{p},\boldsymbol{q})=N\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{N}}\right)^n t^{(n)}(\boldsymbol{p},\boldsymbol{q}), \quad (3.16)$$

$$\mathcal{U}(\boldsymbol{p},\boldsymbol{q})=N\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{N}}\right)^n u^{(n)}(\boldsymbol{p},\boldsymbol{q}), \quad (3.17)$$

and inserting them into the fermionic constraints, we find for the lowest order $\mathcal{O}(N)$

$$\begin{pmatrix} t^{(0)}(\boldsymbol{p},\boldsymbol{q}) \\ u^{(0)}(\boldsymbol{p},\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} m_0\dfrac{\epsilon(p^+)}{q^+}\delta^{(3)}(\boldsymbol{p}+\boldsymbol{q}) \\ 0 \end{pmatrix} - g_0^2\frac{\epsilon(p^+)}{q^+}\int_{-\infty}^{\infty}\frac{d^3\boldsymbol{k}}{(2\pi)^3}\begin{pmatrix} t^{(0)}(\boldsymbol{k},\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k}) \\ u^{(0)}(\boldsymbol{k},\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k}) \end{pmatrix}, \quad (3.18)$$

where $g_0^2=g^2N$. Since there are no operators in these equations, $t^{(0)}$ and $u^{(0)}$ are c numbers. Nonzero solutions give the leading-order contribution to $\langle\bar{\Psi}\Psi\rangle$ and $\langle\bar{\Psi}i\gamma_5\Psi\rangle$:

$$\langle 0|\bar{\Psi}\Psi|0\rangle=N\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}f_t(\boldsymbol{p})+\cdots, \quad (3.19)$$

$$\langle 0|\bar{\Psi}i\gamma_5\Psi|0\rangle=N\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}f_u(\boldsymbol{p})+\cdots, \quad (3.20)$$

where $t^{(0)}(\boldsymbol{p},\boldsymbol{q})=f_t(\boldsymbol{p})\,\delta^{(3)}(\boldsymbol{p}+\boldsymbol{q})$ and $u^{(0)}(\boldsymbol{p},\boldsymbol{q})=f_u(\boldsymbol{p})\,\delta^{(3)}(\boldsymbol{p}+\boldsymbol{q})$. As Eq. (3.18) with $m_0=0$ is invariant under the chiral rotation, we can always take $u^{(0)}(\boldsymbol{p},\boldsymbol{q})=0$. For the massive case, we also take $u^{(0)}(\boldsymbol{p},\boldsymbol{q})=0$ and $t^{(0)}(\boldsymbol{p},\boldsymbol{q})\neq 0$. Now let us introduce a quantity M, which corresponds to the dynamical mass of fermion:

$$M=m_0-g^2\langle\bar{\Psi}\Psi\rangle. \quad (3.21)$$

Then, to obtain $t^{(0)}(\boldsymbol{p},\boldsymbol{q})$ is equivalent to determining M, viz.

$$t^{(0)}(\boldsymbol{p},\boldsymbol{q})=-M\frac{\epsilon(p^+)}{p^+}\delta^{(3)}(\boldsymbol{p}+\boldsymbol{q}). \quad (3.22)$$

In terms of M, the leading-order fermionic constraint (3.18) is rewritten as

$$\frac{M-m_0}{M}=g_0^2\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\epsilon(p^+)}{p^+}. \quad (3.23)$$

Physically this equation should be interpreted as a gap equation. This is clarified in the next subsection.

Similarly higher-order fermionic constraints are solved order by order. This is because the fermionic constraints for $t^{(n)}(\boldsymbol{p},\boldsymbol{q})$ and $u^{(n)}(\boldsymbol{p},\boldsymbol{q})$ are linear equations with respect to the highest order. The $n=1,2$ solutions are important for giving a nontrivial Hamiltonian and so on. More details are discussed in Appendix D.

D. Gap equation

Now let us discuss the physics meaning of Eq. (3.23). As we mentioned above, this equation should be regarded as a gap equation for chiral condensate. In several previous studies of ours, we have seen essentially the same kind of equations [9,7,4]. Indeed, in Ref. [7] it was pointed out that Eq. (3.23) itself is the gap equation. Also in the chiral Yukawa model [4], the zero-mode constraint for the scalar field reduced to the above equation and was interpreted as a gap equation. Since this identification is an indispensable step for our framework, let us again explain it within the NJL model.

First of all, consider a naive massless limit $m_0\to 0$ of Eq. (3.23):

$$M\left(1-g_0^2\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\epsilon(p^+)}{p^+}\right)=0.$$

Thus we find two possibilities: the first is $M=0$ and the second is

$$1-g_0^2\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\epsilon(p^+)}{p^+}=0 \tag{3.24}$$

but M is arbitrary. Of course the first case is not interesting because it corresponds to the symmetric phase. On the other hand, the second case with nonzero M does not immediately mean the existence of the broken phase. Since Eq. (3.24) is independent of M as it is, the dynamical mass M is left undetermined, which is not a physically acceptable situation. However, this observation is not correct because the divergent integral in Eq. (3.24) is not regularized. Indeed, we can identify Eq. (3.24) with the gap equation only after we carefully treat the infrared (IR) divergence.

To see this, let us put an IR cutoff. First consider the same cutoff schemes as in the equal-time formulation, such as the covariant four-momentum cutoff. We can easily translate it into a cutoff on the light-cone momentum k^+ and $k_\perp^i$ and obtain the correct gap equation. Indeed, in Ref. [8], a noncovariant (rotationally invariant) three-momentum cutoff was performed to obtain the known result. But such a cutoff is artificial as a light-front theory, and we here adopt another cutoff scheme, *the parity invariant cutoff*. Usually, it is natural and desirable to choose a cutoff so as to preserve symmetry of a system as much as possible. For the LF coordinates $x^\pm$ and $x_\perp^i$, it would be natural to consider parity transformation ($x^+\leftrightarrow x^-$, $x_\perp^i\to -x_\perp^i$) and two-dimensional rotation in the transverse plane. In the usual canonical formulation where x^+ is treated separately, the parity invariance is not manifest. However, we find it useful for obtaining the gap equation. In momentum space, the parity transformation is exchange of k^+ and k^- and replacement $k_\perp^i\to -k_\perp^i$. Therefore the parity invariant cutoff is given by $k^\pm<\Lambda$ and $k_\perp^2<\Lambda'^2$. Using the dispersion relation $2k^+k^- - k_\perp^2 = M^2$, we find that the parity invariant regularization inevitably relates the ultraviolet (UV) and IR cutoffs:

$$\frac{k_\perp^2+M^2}{2\Lambda}<k^+<\Lambda. \tag{3.25}$$

This also implies the planar rotational invariance $k_\perp^2<2\Lambda^2-M^2=\Lambda'^2$. What is important here is the use of constituent mass M in the dispersion relations. Physically it corresponds to imposing *self-consistency conditions*. Since the IR cutoff includes M, the rhs of Eq. (3.23) has nontrivial dependence on M:

$$\frac{M-m_0}{M}=\frac{g_0^2\Lambda^2}{4\pi^2}\left\{2-\frac{M^2}{\Lambda^2}\left(1+\ln\frac{2\Lambda^2}{M^2}\right)\right\}. \tag{3.26}$$

This is the gap equation and is equivalent to that of the previous result in the chiral Yukawa model [4]. It has a nonzero solution $M\neq 0$ even in the $m_0\to 0$ limit. The somewhat unfamiliar equation (3.26) of the NJL model exhibits the same property as the standard gap equations of the equal-time quantization. For example, when $m_0=0$, there is a critical coupling $g_{\rm cr}^2=2\pi^2/\Lambda^2$, above which $M\neq 0$.

The essential and inevitable step to obtain the gap equation is the inclusion of mass information as the regularization rather than the fact that the UV and IR cutoffs are related to each other. If we regulate the divergent integral without mass information (e.g., introducing the UV and IR cutoffs independently), we cannot reproduce the gap equation. The loss of mass information is closely related to the fundamental problem of the LF formalism [17], and the parity invariant regularization can be considered as one of the prescriptions for it. Reference [17] discussed within scalar theory that *the light-front quantization gives a mass-independent two-point function* (at equal LF time), which contradicts the result from general arguments concerning the spectral representation. We have been encountered with the same problem in Eq. (3.23) because the integral is regarded as a naive estimation of $\langle\bar{\Psi}\Psi\rangle/M$ by using fermion with mass M. And also the origin of mass-independent result can be traced back to the mode expansion (3.2). Even if we include the wave function for free fermion field, we do not have any mass dependence on the mode expansion [10].

Let us give a brief comment on the chiral Gross-Neveu model [18]. Of course the important difference of the (1+1)-dimensional case is the renormalizability. Ignoring the transverse directions in the above calculation, we easily find the gap equation $(M-m_0)/M=g_0^2/(2\pi)\ln(2\Lambda^2/M^2)$ where the parity-invariant cutoff $M^2/2\Lambda<k^+<\Lambda$ was used. Though it explicitly depends on the cutoff Λ and is divergent as $\Lambda\to\infty$, we can remove the divergence by coupling constant renormalization [7].

E. Hamiltonian

Having the solution to the bilocal fermionic constraint, we can rewrite the fermion bilinear operators in terms of the bilocal bosons. Of special importance is the (Hermitian) Hamiltonian, which is easily expressed by $\mathcal{T}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})$ and $\mathcal{U}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})$ as follows:

$$\begin{aligned}H=P^-&=\frac{1}{2\sqrt{2}}\int d^3x[(\psi^\dagger\sigma^i\partial_i\chi+\partial_i\chi^\dagger\sigma^i\psi)\\&\quad+m_0(\psi^\dagger\chi+\chi^\dagger\psi)]\\&=\frac{1}{4}\int d^3\boldsymbol{p}\ d^3\boldsymbol{q}\,\delta^{(3)}(\boldsymbol{p}+\boldsymbol{q})\,i q_\perp^i\sigma^i_{\alpha\beta}\\&\quad\times[(\mathcal{T}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})+\mathcal{T}_{\beta\alpha}(\boldsymbol{q},\boldsymbol{p}))\\&\quad+i(\mathcal{U}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})-\mathcal{U}_{\beta\alpha}(\boldsymbol{p},\boldsymbol{q}))]\\&\quad+\frac{m_0}{2}\int d^3\boldsymbol{p}d^3\boldsymbol{q}\,\mathcal{T}(\boldsymbol{p},\boldsymbol{q})\,\delta^{(3)}(\boldsymbol{p}+\boldsymbol{q}).\end{aligned} \tag{3.27}$$

Apparently this Hermitian version of the Hamiltonian seems equivalent to the free Hamiltonian, but the information of interaction enters through the bad component χ. We find the $1/N$ expansion of the Hamiltonian

$$H=N\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{N}}\right)^{n}h^{(n)}, \tag{3.28}$$

by substituting the solutions of the fermionic constraints into the Hamiltonian. The zeroth-order contribution is just a divergent constant and we discard it. The first order is strictly zero. Nontrivial contribution comes from the order $\mathcal{O}(N^0)$,

$$\begin{aligned}h^{(2)}=&\int[d\boldsymbol{p}][d\boldsymbol{q}]\left(\frac{p_\perp^2+M^2}{2p^+}+\frac{q_\perp^2+M^2}{2q^+}\right)B^\dagger_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})B_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})\\&+\frac{g_0^2}{(2\pi)^3}\int[d\boldsymbol{p}][d\boldsymbol{q}][d\boldsymbol{k}][d\boldsymbol{l}]\delta^{(3)}(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k}-\boldsymbol{l})\\&\times\alpha(p^++q^+)[\Sigma^{\alpha\beta}_{\gamma\delta}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{k},\boldsymbol{l})-\Pi^{\alpha\beta}_{\gamma\delta}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{k},\boldsymbol{l})]\\&\times B^\dagger_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})B_{\gamma\delta}(\boldsymbol{k},\boldsymbol{l})+c\ \text{ number},\end{aligned} \tag{3.29}$$

where $\alpha(p^++q^+)$ is defined by Eq. (D13) and ''kernels'' of the interaction terms are

$$\Sigma^{\alpha\beta}_{\gamma\delta}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{k},\boldsymbol{l})\equiv[\mathcal{S}(-\boldsymbol{p})-\mathcal{S}(-\boldsymbol{q})]_{\alpha\beta}[\mathcal{S}(\boldsymbol{k})-\mathcal{S}(\boldsymbol{l})]_{\delta\gamma}, \tag{3.30}$$

$$\Pi^{\alpha\beta}_{\gamma\delta}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{k},\boldsymbol{l})\equiv[\mathcal{P}(-\boldsymbol{p})-\mathcal{P}(\boldsymbol{q})]_{\alpha\beta}[\mathcal{P}(-\boldsymbol{k})-\mathcal{P}(\boldsymbol{l})]_{\delta\gamma}, \tag{3.31}$$

with

$$\mathcal{S}_{\alpha\beta}(\boldsymbol{p})=\left(\frac{ip^i\sigma^i-M}{2p^+}\right)_{\alpha\beta},\quad \mathcal{P}_{\alpha\beta}(\boldsymbol{p})=\left(\frac{ip^i\sigma^i-M}{2p^+}\sigma^3\right)_{\alpha\beta}. \tag{3.32}$$

As is evident from the explicit forms of the kernels (3.30) and (3.31), they originate from the scalar interaction $(\bar{\Psi}\Psi)^2$ and the pseudoscalar one $(\bar{\Psi}i\gamma_5\Psi)^2$, respectively. If we substitute a nontrivial (trivial) solution of the gap equation (3.26) into the above Hamiltonian, then it governs the dynamics of the broken (symmetric) phase. The first term of $h^{(2)}$ corresponds to a free part with mass M and the second term to an interaction part. In the broken phase, M is the dynamical mass and the Hamiltonian suggests a constituent picture.

As we mentioned before, the Hermite Hamiltonian of the chiral Gross-Neveu model has only an m_0-dependent term. Neglecting the transverse coordinates in Eq. (3.27), we have

$$P^-_{\rm GN}=\frac{m_0}{2\sqrt{2}}\int dx^-(\psi^\dagger\chi+\chi^\dagger\psi).$$

Furthermore, the classical solution for the bad spinor component χ is proportional to m_0. Therefore the naive $m_0\to 0$ limit gives just a zero Hamiltonian. However, if we solve the gap equation and substitute the nontrivial solution into the Hamiltonian, the resulting Hamiltonian turns out to be proportional to M^2 and survives even in the chiral limit. This is easily seen from the Hamiltonian of the NJL model (3.29). The (constituent) mass term in Eq. (3.29) comes from the bare mass term, whose factor m_0 cancels with a factor M^2/m_0 in the second-order solution $\int d^3\boldsymbol{p}t^{(2)}(\boldsymbol{p},-\boldsymbol{p})$. Of course this is not reached if we set $m_0=0$ from the beginning. Therefore, inclusion of the bare mass term is necessary to obtain a correct (constituent) mass term of the Hamiltonian.

IV. PHYSICS IN THE BROKEN PHASE

By solving the fermionic constraint, we acquired the necessary ingredients for discussing physics consequences of the chiral symmetry breaking. Basic quantities such as $\bar{\Psi}\Psi$, $\bar{\Psi}i\gamma_5\Psi$, and the null-plane chiral charge (2.13) are expressed in terms of the bilocal bosons $B_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})$ and $B^\dagger_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})$ as

$$\begin{aligned}\bar{\Psi}\Psi(x)&=\mathcal{T}(\boldsymbol{x},\boldsymbol{x})\\&=\frac{N}{g_0^2}(m_0-M)+\sqrt{N}\int\frac{d^3\boldsymbol{p}d^3\boldsymbol{q}}{(2\pi)^3}\ t^{(1)}(\boldsymbol{p},\boldsymbol{q})\\&\quad\times e^{i(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{x}}+\mathcal{O}(N^0),\end{aligned} \tag{4.1}$$

$$\begin{aligned}\bar{\Psi}i\gamma_5\Psi(x)&=\mathcal{U}(\boldsymbol{x},\boldsymbol{x})\\&=\sqrt{N}\int\frac{d^3\boldsymbol{p}d^3\boldsymbol{q}}{(2\pi)^3}\ u^{(1)}(\boldsymbol{p},\boldsymbol{q})e^{i(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{x}}+\mathcal{O}(N^0),\end{aligned} \tag{4.2}$$

$$\begin{aligned}Q_5^{\rm LF}&=\int d^3\boldsymbol{p}\ \ \sigma^3_{\alpha\beta}{:}\mathcal{M}_{\alpha\beta}(\boldsymbol{p},-\boldsymbol{p}){:}\\&=\int d^3\boldsymbol{p}\ \ \sigma^3_{\alpha\beta}\mu^{(2)}_{\alpha\beta}(\boldsymbol{p},-\boldsymbol{p})+\mathcal{O}(N^{1/2}),\end{aligned} \tag{4.3}$$

where $t^{(1)}(\boldsymbol{p},\boldsymbol{q})$ and $u^{(1)}(\boldsymbol{p},\boldsymbol{q})$ are given in Appendix D. Now that these are given as functions of the bilocal bosons at the operator level, all the calculation is done with the commutators (3.13) and (3.14).

A. Chiral transformation and nonconservation of chiral charge

Why could we obtain a nonzero fermion condensate? To understand this, let us rewrite the fermionic constraint (2.1) as

$$i\partial_-\chi_a=\frac{1}{\sqrt{2}}(-\sigma^i\partial_i+m_0)\psi_a-\frac{g^2}{\sqrt{2}}(\psi_a\mathcal{T}(\boldsymbol{x},\boldsymbol{x})+i\sigma^3\psi_a\mathcal{U}(\boldsymbol{x},\boldsymbol{x})),$$

and substitute Eqs. (4.1) and (4.2) into $\mathcal{T}(\boldsymbol{x},\boldsymbol{x})$ and $\mathcal{U}(\boldsymbol{x},\boldsymbol{x})$, respectively.[3] Then the leading-order equation turns out to be equivalent to the constraint equation for a free fermion with mass M,

$$i\partial_-\chi_a=\frac{1}{\sqrt{2}}(-\sigma^i\partial_i+M)\psi_a .$$

Also at the same order, the equation of motion for the good component ψ says that the fermion acquires a mass M. This means that the operator structure of the bad spinor χ changes depending on which solutions of the gap equation (3.23) is selected. For a massive fermion, the fermion condensate $\langle\bar{\Psi}\Psi\rangle$ is no longer zero even if the vacuum is trivial. One can find an analogy between the chiral Yukawa model and the NJL model because in the chiral Yukawa model, the operator structure of the longitudinal zero modes and subsequently of the bad spinor component changes depending on the phases.

One thing to be noted is the peculiarity of the mode expansion (3.2). It is evident that the mode expansion has no mass dependence in it. This caused the problem of identifying the lowest fermionic constraint with the gap equation. We had to supply mass information properly when we regularize the IR divergence. On the other hand, such independence of mass, in turn, implies that our mode expansion allows fermions with *any value of mass*. In other words, the LF vacuum does not distinguish the mass of the fermion. Therefore, we can regard the vacuum for massless fermion as that for massive one. The mass of the fields is determined by the Hamiltonian. This is the reason why we can live with the trivial vacuum while having a nonzero fermion condensate. This fact is not a limited phenomenon for our specific mode expansion but a common one for light-front field theories. Indeed, even if we expand a fermion field with free spinor wave functions, $u(p)$ and $v(p)$, we have no mass dependence [10].

The fact that the operator structure changes depending on the phases, also resolves a seeming contradiction between the triviality of the null-plane chiral charge and the nonzero chiral condensate $\langle 0|\bar{\Psi}\Psi|0\rangle\neq 0$. In general, it is known that a null-plane charge always annihilates the vacuum irrespective of the presence of symmetry. This can be checked explicitly by expression (4.3), viz.

$$Q_5^{\rm LF}|0\rangle=0. \tag{4.4}$$

However, the triviality of $Q_5^{\rm LF}$ in the presence of the chiral condensate immediately leads to a contradiction if an equation $[Q_5^{\rm LF},\bar{\Psi}i\gamma_5\Psi]=-2i\bar{\Psi}\Psi$ could hold in the broken phase. In the previous analysis of the chiral Yukawa model [4], we were faced with exactly the same problem and resolved it by recognizing that in the broken phase the chiral transformation of *dependent* variables are different from the usual one simply because their operator structure changes. This is of course true of the NJL model. First of all, as we saw above, if we select the nontrivial solution of the gap equation, the fermion is no longer a massless fermion even in the chiral limit. Second, we can explicitly show that the usual transformation law $[Q_5^{\rm LF},\bar{\Psi}i\gamma_5\Psi]=-2i\bar{\Psi}\Psi$ holds only in the symmetric phase ($M=0$). In the broken phase a simple calculation [up to $\mathcal{O}(N^{1/2})$] leads to

$$\begin{aligned}[][Q_5^{\rm LF},\bar{\Psi}i\gamma_5\Psi(x)]&=-2i\bar{\Psi}\Psi(x)+2i\frac{N}{g_0^2}(m_0-M)\\&\quad+2i\sqrt{N}M\int\frac{d\boldsymbol{p}d\boldsymbol{q}}{(2\pi)^3}e^{i(\boldsymbol{p}+\boldsymbol{q})x}\Bigg[\frac{\mu^{(1)}_{\alpha\alpha}(\boldsymbol{p},\boldsymbol{q})}{q^+}\\&\quad-g_0^2\frac{\epsilon(p^+)}{q^+}\alpha(p^++q^+)\\&\quad\times\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{\mu^{(1)}_{\alpha\alpha}(\boldsymbol{k},\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k})}{p^++q^+-k^+}\Bigg]+\mathcal{O}(N^0).\end{aligned} \tag{4.5}$$

Even if we take the chiral limit $m_0\to 0$, the extra term proportional to M survives nonzero. This also implies that if we put $M=0$, the usual relation holds. The unusual chiral transformation, however, is consistent with the triviality of $Q_5^{\rm LF}$ because $\langle 0|[Q_5^{\rm LF},\bar{\Psi}i\gamma_5\Psi]|0\rangle=0$.

A similar situation occurs for the Hamiltonian. Nonconservation of the null-plane chiral charge has been pointed out by several people as a characteristic feature of the chiral symmetry breaking on the LF [19,12]. They discussed it under the assumption of the PCAC relation, but we can check it explicitly by using the broken Hamiltonian. After lengthy calculation, we find the commutator $[Q_5^{\rm LF},H]$ is really nonzero and again proportional to the dynamical mass M:

[3] In the leading order, this procedure corresponds to the mean-field approximation done by Heinzl *et al.* [8]. They solved the fermionic constraint by simply linearizing the interaction parts as $-g^2/\sqrt{2}\psi_a\langle\mathcal{T}(\boldsymbol{x},\boldsymbol{x})\rangle$. By evaluating the vacuum expectation value $\langle\mathcal{T}(\boldsymbol{x},\boldsymbol{x})\rangle$ self-consistently with the dynamical fermion mass $M=m_0-g^2\langle\mathcal{T}(\boldsymbol{x},\boldsymbol{x})\rangle$, they obtained the gap equation. If one uses the parity-invariant cutoff, the result coincides with ours.

$$[Q_5^{\rm LF},H]=M\frac{g_0^2}{16\pi^3 i}\int[d\boldsymbol{p}][d\boldsymbol{q}][d\boldsymbol{k}][d\boldsymbol{l}]\delta^{(3)}(\boldsymbol{k}+\boldsymbol{l}-\boldsymbol{p}-\boldsymbol{q})\alpha(p^++q^+)B^\dagger_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})B_{\gamma\delta}(\boldsymbol{k},\boldsymbol{l})$$
$$\times\Bigg[\left(\frac{p_\perp^i}{p^+}-\frac{q_\perp^i}{q^+}\right)\sigma^i_{\alpha\beta}\left(\frac{1}{k^+}+\frac{1}{l^+}\right)\sigma^3_{\delta\gamma}-\left(\frac{1}{p^+}-\frac{1}{q^+}\right)\delta_{\alpha\beta}\left(\frac{k_\perp^i}{k^+}-\frac{l_\perp^i}{l^+}\right)(\sigma^i\sigma^3)_{\delta\gamma}$$
$$-\left(\frac{p_\perp^i}{p^+}-\frac{q_\perp^i}{q^+}\right)(\sigma^i\sigma^3)_{\alpha\beta}\left(-\frac{1}{k^+}+\frac{1}{l^+}\right)\delta_{\delta\gamma}+\left(\frac{1}{p^+}+\frac{1}{q^+}\right)\sigma^3_{\alpha\beta}\left(\frac{k_\perp^i}{k^+}-\frac{l_\perp^i}{l^+}\right)\sigma^i_{\delta\gamma}\Bigg]+\mathcal{O}(N^{-1/2}). \quad (4.6)$$

Therefore, the LF chiral charge is not conserved even in the chiral limit. In our framework it would be more understandable to mention that *the Hamiltonian is not invariant under the LF chiral transformation in the broken phase*. The broken phase Hamiltonian (3.29) has three terms: M independent, linearly dependent, and quadratically dependent terms. The quadratically dependent term, as well as the M-independent one, does not break the LF chiral symmetry. It is the term proportional to the dynamical fermion mass M which breaks the LF chiral symmetry. And also, since Eq. (4.6) is proportional to g_0^2, the symmetry-breaking term purely comes from the interaction.[4]

This result should be consistent with the current divergence relation Eq. (2.16). Integrating it over the space, we have

$$\partial_+ Q_5^{\rm LF}=\frac{1}{i}[Q_5^{\rm LF},H]=2m_0\int dx^-d^2x_\perp\overline{\Psi}i\gamma_5\Psi. \quad (4.7)$$

Therefore, if the LF chiral charge is not conserved in the chiral limit, the rhs must show a singular behavior

$$\int dx^-d^2x_\perp\overline{\Psi}i\gamma_5\Psi\propto\frac{1}{m_0}. \quad (4.8)$$

This can be verified directly by using the solution of the fermionic constraint. Indeed we find that $\int dx^-d^2x_\perp\overline{\Psi}i\gamma_5\Psi=\int d\boldsymbol{p}\;\; u^{(2)}(\boldsymbol{p},-\boldsymbol{p})$ is proportional to M/m_0 and gives exactly the same result as Eq. (4.6). The importance of such singular behavior for making the Nambu-Goldstone boson meaningful was stressed by Tsujimaru *et al.* in scalar theories [12]. Assuming the PCAC relation, they showed that the zero mode of the Nambu-Goldstone boson has a singularity of $m_{\rm NG}^{-2}$ where $m_{\rm NG}$ is an explicit symmetry-breaking mass. Our result (4.8) is consistent with theirs because the operator $\overline{\Psi}i\gamma_5\Psi$ is directly related to the Nambu-Goldstone boson. Later, we will prove that the PCAC relation is derived from the current divergence relation (2.16).

[4]For a massive free fermion, we have $[Q_5^{\rm LF},H]=0$.

B. LF bound-state equations for mesons and their solutions

1. Single bosonic state as a fermion-antifermion state

In our formulation with the boson expansion method, any bosonic excited state is described by the Fock states of the bilocal bosons constructed on the trivial vacuum:

$$\prod_i B^\dagger_{\alpha_i\beta_i}(\boldsymbol{p}_i,\boldsymbol{q}_i)|0\rangle. \quad (4.9)$$

Since the Hamiltonian (3.29) is quadratic with respect to the bilocal bosons, the first excited state is given by a single bosonic state. In fermionic degrees of freedom, the one boson state corresponds to the leading contribution (of $1/N$ expansion) of a fermion-antifermion state. To see this, let us write a mesonic state only with a ''color'' singlet fermion-antifermion Fock component:

$$|\text{meson};P^+,P_\perp\rangle=\frac{1}{\sqrt{N}}\int_0^{P^+}dk^+\times\int_{-\infty}^{\infty}d^2k_\perp\Phi^{\alpha\beta}(\boldsymbol{k})b_\alpha^{a\dagger}(\boldsymbol{k})d_\beta^{a\dagger}(\boldsymbol{P}-\boldsymbol{k})|0\rangle, \quad (4.10)$$

where the LC wave function $\Phi^{\alpha\beta}(\boldsymbol{k})$ is normalized so as to satisfy the condition

$$\langle\text{meson};\boldsymbol{P}|\text{meson};\boldsymbol{Q}\rangle=(2\pi)^3 2P^+\delta^{(3)}(\boldsymbol{P}-\boldsymbol{Q}), \quad (4.11)$$

or equivalently,

$$\int_0^1 dx\int\frac{d^2k_\perp}{16\pi^3}\sum_{\alpha\beta}|\Phi^{\alpha\beta}(\boldsymbol{k})|^2=1. \quad (4.12)$$

According to the Holstein-Primakoff type expansion (D8), the fermion-antifermion operator $b_\alpha^\dagger d_\beta^\dagger$ can be equivalently rewritten as

$$b^{a\dagger}_{\alpha}(\boldsymbol{k})d^{a\dagger}_{\beta}(\boldsymbol{P}-\boldsymbol{k})=:\mathcal{M}^{--}_{\alpha\beta}(-\boldsymbol{k},-\boldsymbol{P}+\boldsymbol{k}):$$
$$=\sqrt{N}B^{\dagger}_{\alpha\beta}(\boldsymbol{k},\boldsymbol{P}-\boldsymbol{k})$$
$$-\frac{1}{2\sqrt{N}}\int[d\boldsymbol{q}][d\boldsymbol{q}']B^{\dagger}_{\gamma\beta}(\boldsymbol{q},\boldsymbol{P}-\boldsymbol{k})$$
$$\times B^{\dagger}_{\alpha\delta}(\boldsymbol{k},\boldsymbol{q}')B_{\gamma\delta}(\boldsymbol{q},\boldsymbol{q}')+\cdots. \quad (4.13)$$

Therefore at the leading order of the $1/N$ expansion, the mesonic state is described as a single (bilocal) boson state,

$$|\text{meson};P^+,P_\perp\rangle=\int_0^{P^+}dk^+\int_{-\infty}^{\infty}d^2k_\perp\Phi^{\alpha\beta}(\boldsymbol{k})B^{\dagger}_{\alpha\beta}(\boldsymbol{k},\boldsymbol{P}-\boldsymbol{k})|0\rangle+\mathcal{O}(N^{-1/2}). \quad (4.14)$$

Besides this, it is evident from the normalization condition (4.11), a local operator $a^\dagger(\boldsymbol{P})=\int d^3k\Phi^{\alpha\beta}(\boldsymbol{k})B^\dagger_{\alpha\beta}(\boldsymbol{k},\boldsymbol{P}-\boldsymbol{k})$ satisfies the usual bosonic commutators.

The LC wave function $\Phi^{\alpha\beta}(\boldsymbol{k})$ and the mass of a meson M_{meson} is determined by solving the light-front eigenvalue equation:

$$h^{(2)}|\text{meson};P^+,P_\perp=0\rangle=\frac{M^2_{\text{meson}}}{2P^+}|\text{meson};P^+,P_\perp=0\rangle, \quad (4.15)$$

where we set $P^i_\perp=0$, for simplicity.

2. Scalar and pseudoscalar mesons

In the leading order of $1/N$ expansion, the Hamiltonian has only quadratic terms of the bosonic operators. Therefore, diagonalization of the Hamiltonian, or equivalently, solving the light-cone bound-state equation (4.15) is straightforward. First of all, if one notices the orthogonal property $[\mathcal{P}(-\boldsymbol{k})-\mathcal{P}(\boldsymbol{l})]_{\alpha\beta}[\mathcal{S}(\boldsymbol{k})-\mathcal{S}(\boldsymbol{l})]_{\beta\alpha}=0$ where $\boldsymbol{k}=(xP^+,k^i_\perp)$ and $\boldsymbol{l}=\boldsymbol{P}-\boldsymbol{k}=[(1-x)P^+,-k^i_\perp]$, one can easily find the spinor structure for scalar (σ) and pseudoscalar (π) states should be

$$|\pi;P^+,P_\perp=0\rangle=P^+\int_0^1dx\int d^2k_\perp\phi_\pi(x,k^i_\perp)\{(ik^i_\perp\sigma^i+M)\sigma^3\}_{\alpha\beta}B^\dagger_{\alpha\beta}(\boldsymbol{k},\boldsymbol{l})|0\rangle, \quad (4.16)$$

$$|\sigma;P^+,P_\perp=0\rangle=P^+\int_0^1dx\int d^2k_\perp\phi_\sigma(x,k^i_\perp)\{ik^i_\perp\sigma^i+(1-2x)M\}_{\alpha\beta}B^\dagger_{\alpha\beta}(\boldsymbol{k},\boldsymbol{l})|0\rangle. \quad (4.17)$$

These two states are orthogonal to each other. Somewhat nonstandard spinor structure of the mesonic states is due to our specific choice of the mode expansion Eq. (3.2) and the representation for γ matrices Eq. (1.4). For example, if one rewrites the pseudoscalar field $\overline{\Psi}i\gamma_5\Psi$ in terms of the bilocal bosons, one finds the same spinor structure as that of Eq. (4.16). Note also that $\{\gamma_5(\gamma^i_\perp k^i_\perp+M)\}_{\alpha\beta}=\{(ik^i_\perp\sigma^i+M)\sigma^3\}_{\alpha\beta}$ for $\alpha,\beta=1,2$ in our two-component representation for the γ matrices.

Spinor independent parts of the LC wave functions $\phi_{\pi,\sigma}(x,k^i_\perp)$ are given as solutions of the following integral equations:

$$m^2_\pi\phi_\pi(x,k^i_\perp)=\frac{k^2_\perp+M^2}{x(1-x)}\phi_\pi(x,k^i_\perp)-\frac{g^2_0\alpha}{(2\pi)^3}\frac{1}{x(1-x)}\times\int_0^1dy\int d^2l_\perp\frac{l^2_\perp+M^2}{y(1-y)}\phi_\pi(y,l^i_\perp), \quad (4.18)$$

$$m^2_\sigma\phi_\sigma(x,k^i_\perp)=\frac{k^2_\perp+M^2}{x(1-x)}\phi_\sigma(x,k^i_\perp)-\frac{g^2_0\alpha}{(2\pi)^3}\frac{1}{x(1-x)}\times\int_0^1dy\int d^2l_\perp\frac{l^2_\perp+(1-2y)^2M^2}{y(1-y)}\phi_\sigma(y,l^i_\perp). \quad (4.19)$$

Here the factor $\alpha=\alpha(P^+)$ defined by Eq. (D13) is given as a result of the gap equation,

$$\alpha=\left(\frac{m_0}{M}+\frac{2g^2_0}{(2\pi)^3}\int d^2q_\perp\int_0^1\frac{dx}{x}\right)^{-1}. \quad (4.20)$$

Since these integral equations are separable ones, solutions are easily found

$$\phi_\pi(x,k^i_\perp)=-C_\pi\frac{g^2_0}{(2\pi)^3}\frac{M}{m_0}\frac{1}{x(1-x)-(k^2_\perp+M^2)/m^2_\pi}, \quad (4.21)$$

$$\phi_\sigma(x,k^i_\perp)=-C_\sigma\frac{g^2_0}{(2\pi)^3}\frac{M}{m_0}\left(\frac{m^2_\sigma-4M^2}{m^2_\sigma}\right)\times\frac{1}{x(1-x)-(k^2_\perp+M^2)/m^2_\sigma}, \quad (4.22)$$

where C_π and C_σ are constants $C_{\pi,\sigma}=\int_0^1dx\int d^2k_\perp\phi_{\pi,\sigma}(x,k^i_\perp)$.

Equations for m_π and m_σ are derived from the normalization condition for the LC wave functions, viz.

$$1=g^2_0\frac{M}{m_0}\int_0^1dx\int\frac{d^2k_\perp}{(2\pi)^3}\frac{m^2_\pi}{k^2_\perp+M^2-m^2_\pi x(1-x)}, \quad (4.23)$$

$$1=g^2_0\frac{M}{m_0}\int_0^1dx\int\frac{d^2k_\perp}{(2\pi)^3}\frac{m^2_\sigma-4M^2}{k^2_\perp+M^2-m^2_\sigma x(1-x)}. \quad (4.24)$$

These are exactly equivalent to the corresponding equations in the previous work on the chiral Yukawa model [4], where we obtained them by calculating pole masses of the scalar and pseudoscalar bosons. If one uses the same cutoff scheme (extended parity-invariant cutoff) as in Ref. [4]:

$$\frac{k_\perp^2+M^2}{x}+\frac{k_\perp^2+M^2}{1-x}<2\Lambda^2, \tag{4.25}$$

the pseudoscalar mass for small bare mass m_0 is

$$m_\pi^2=\frac{m_0 N}{g_0^2 M}Z_\pi+\mathcal{O}(m_0^2), \tag{4.26}$$

where a cutoff dependent factor

$$Z_\pi=\frac{1}{N}\left[\frac{1}{8\pi^2}\ln\left(\frac{1+\sqrt{1-2M^2/\Lambda^2}}{1-\sqrt{1-2M^2/\Lambda^2}}\right)-\frac{\sqrt{1-2M^2/\Lambda^2}}{4\pi^2}\right]^{-1} \tag{4.27}$$

is related to normalization of a pseudoscalar state [see Eq. (4.35)]. Clearly m_π vanishes in the chiral limit $m_0\to 0$ and the pseudoscalar state is identified with the Nambu-Goldstone boson. In Eq. (4.18), the first term corresponds to a kinetic energy part of the fermion and antifermion with the constituent mass M and the second term, a potential energy part. The masslessness of the pseudoscalar state in the chiral limit is realized by the exact cancellation between the kinetic energy and the potential energy. Indeed, if we integrate Eq. (4.18) over x and $k_\perp^i$, we find

$$m_\pi^2\int_0^1 dx\int d^2k_\perp\,\phi_\pi(x,k_\perp^i)$$
$$=\left(1-\frac{2g_0^2\int_0^1\frac{dx}{x}\int\frac{d^2q_\perp}{(2\pi)^3}}{\frac{m_0}{M}+2g_0^2\int_0^1\frac{dx}{x}\int\frac{d^2q_\perp}{(2\pi)^3}}\right)$$
$$\times\int_0^1 dy\int d^2l_\perp\frac{l_\perp^2+M^2}{y(1-y)}\phi_\pi(y,l_\perp^i)\to 0\quad (m_0\to 0). \tag{4.28}$$

Therefore, $m_\pi=0$ is fulfilled in the chiral limit even though the fermion has the constituent mass.

On the other hand, the squared mass of the scalar state for small m_0 is

$$m_\sigma^2=4M^2+\mathcal{O}(m_0). \tag{4.29}$$

At a first glance, the result $m_\sigma=2M$ in the chiral limit seems to suggest a static picture of a fermion and an antifermion, but actually the mass of the scalar meson comes from a part of the potential energy. The kinetic energy cancels with the rest of the potential energy.

Equations (4.21) and (4.22) have the same functional form with respect to the variables x and $k_\perp^i$. However, the difference between m_π and m_σ greatly affects the shape of the LC wave functions. This is most remarkable in the chiral limit: As $m_0\to 0$, Eq. (4.21) becomes independent of x:

$$\phi_\pi(x,k_\perp^i)\to -i\sqrt{N}\sqrt{Z_\pi}\frac{1}{k_\perp^2+M^2}, \tag{4.30}$$

where the constant C_π was evaluated as $C_\pi\to -i(2\pi)^3(NZ_\pi)^{-1/2}$. On the other hand, Eq. (4.22) shows a narrow peak at $x=1/2$. Therefore, the pseudoscalar state is a highly collective state, while the scalar state shows an approximate constituent picture.

Now let us compare our result Eq. (4.21) with those of the literature [10,11]. First of all, equivalence with the result of Ref. [11] is easily verified. As we commented before, the unfamiliar spinor structure in Eq. (4.16) is due to our specific choice of the mode expansion and the representation for the γ matrices. If one uses the following mode expansion for the good component of the fermion:

$$\psi_+(x)=\sum_\lambda\int_{-\infty}^{\infty}\frac{d^2k_\perp}{2\pi}\int_0^\infty\frac{dk^+}{\sqrt{2\pi k^+}}[\tilde{b}(\boldsymbol{k},\lambda)u_+(\boldsymbol{k},\lambda)e^{i\boldsymbol{kx}}$$
$$+\tilde{d}^\dagger(\boldsymbol{k},\lambda)v_+(\boldsymbol{k},\lambda)e^{-i\boldsymbol{kx}}],$$

one can obtain the same spinor structure as that of Ref. [11]. Of course the two LC wavefunctions should coincide with each other for observable quantities. Indeed, both give the same (quark) distribution function $q(x)=\int d^2k_\perp/(2\pi)^3\Sigma_{\alpha,\beta}|\Phi_{\alpha\beta}(\boldsymbol{k})|^2$.

On the other hand, the result of Ref. [10] seems different from ours Eq. (4.21). The possible origin of the discrepancy might be attributed to the following two points. First of all, the author of Ref. [10] considered the Melosh transformation [22] which relates the LF spinor and the usual spinor in the equal-time quantization. Such nonstatic spin effects might be important when we discuss phenomenological aspects of light mesons (for example, see Ref. [23]). However, even if we take it into account, it is hard to see the coincidence. Secondary, but most importantly, he derived the pion LC wave function by projecting the Bethe-Salpeter amplitude on the equal LC time plane. Though this procedure should give the same result as that of the LF bound-state equation as far as we are considering only the ladder ($1/N$ leading) contribution, equivalence of the two is a highly nontrivial problem in our complicated analysis.

C. The Gell-Mann–Oakes–Renner and PCAC relations

Now that we have the LC wave function for the pseudoscalar meson, it is straightforward to obtain the decay constant f_π:

$$iP^\mu f_\pi=\langle 0|j_5^\mu(0)|\pi;\boldsymbol{P}\rangle. \tag{4.31}$$

For actual calculation, it is safer and easier to treat the plus component. If we use the extended parity-invariant cutoff, the result is

$$f_\pi = 2MZ_\pi^{-1/2} + \mathcal{O}(N^0). \tag{4.32}$$

Together with the pseudoscalar mass (4.26) in the chiral limit, we find the Gell-Mann–Oakes–Renner relation,

$$m_\pi^2 f_\pi^2 = -4m_0\left(-\frac{NM}{g_0^2}\right) = -4m_0\langle 0|\bar{\Psi}\Psi|0\rangle. \tag{4.33}$$

The PCAC relation is also checked by using the state $|\pi;\boldsymbol{P}\rangle$. If we normalize the pseudoscalar field $\pi_{\rm n}(x) \propto \bar{\Psi}(x)i\gamma_5\Psi(x)$ as

$$\langle 0|\pi_{\rm n}(0)|\pi;\boldsymbol{P}\rangle = 1, \tag{4.34}$$

we find that $Z_\pi^{-1/2}$ given in Eq. (4.27) is the normalization factor

$$\pi_{\rm n}(x) = Z_\pi^{-1/2} g^2 \bar{\Psi}(x) i\gamma_5 \Psi(x), \tag{4.35}$$

where we have used the gap equation. Therefore, we arrive at the PCAC relation

$$\partial_\mu j_5^\mu = 2m_0 \bar{\Psi}(x) i\gamma_5\Psi(x) = m_\pi^2 f_\pi \pi_{\rm n}(x). \tag{4.36}$$

Note that the decay constant (4.32) and the normalization factor (4.35) are equivalent to the previous results [Eqs. (5.25) and (5.28) in Ref. [4]] in the infinitely heavy mass limit of bosons $\mu\to\infty$.

D. Symmetric phase

Here we consider the symmetric phase in the chiral limit $m_0=0$. When $g_0^2 < g_{\rm cr}^2 = 2\pi^2/\Lambda^2$, the gap equation (3.26) has only a trivial solution $M=0$. A quantity which should be zero in the broken phase is now estimated as

$$1 - g_0^2\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{\epsilon(k^+)}{k^+} = 1 - \frac{g_0^2}{g_{\rm cr}^2} \neq 0. \tag{4.37}$$

Subsequently the factor α defined by Eq. (D13) is different from that of the broken phase [Eq. (4.20)],

$$\alpha^{-1} \equiv \alpha_{\rm sym}^{-1} = 1 - \frac{g_0^2}{g_{\rm cr}^2} + \frac{2g_0^2}{(2\pi)^3}\int_0^1 \frac{dx}{x}\int d^2q_\perp. \tag{4.38}$$

Then, both of the LF bound-state equations for the scalar and pseudoscalar states are given by

$$m_{\rm sym}^2 \phi_{\rm sym}(x,k_\perp^i) = \frac{k_\perp^2}{x(1-x)}\phi_{\rm sym}(x,k_\perp^i) - \frac{g_0^2\alpha_{\rm sym}}{(2\pi)^3}\frac{1}{x(1-x)}\int_0^1 dy\int d^2 l_\perp \times \frac{l_\perp^2}{y(1-y)}\phi_{\rm sym}(y,l_\perp^i). \tag{4.39}$$

The solution to the bound-state equation is

$$\phi_{\rm sym}(x,k_\perp^i) = -C_{\rm sym}\frac{g_0^2}{(2\pi)^3}\left(1-\frac{g_0^2}{g_{\rm cr}^2}\right)^{-1} \times \frac{1}{x(1-x)-k_\perp^2/m_{\rm sym}^2}, \tag{4.40}$$

where $C_{\rm sym}$ is a normalization constant and $m_{\rm sym}=m_\pi=m_\sigma$ is given as a solution of the equation

$$\frac{1}{g_0^2} - \frac{1}{g_{\rm cr}^2} = \int_0^1 dx\int\frac{d^2k_\perp}{(2\pi)^3}\frac{m_{\rm sym}^2}{k_\perp^2 - m_{\rm sym}^2 x(1-x)}. \tag{4.41}$$

Again this is equal to the previous result of the chiral Yukawa model with $\mu^2\to\infty$ [Eq. (5.26) in Ref. [4]] and therefore if we use the same cutoff as before, we obtain the same result for $m_{\rm sym}$. Moreover, though the above calculation was intended only to $g_0^2 < g_{\rm cr}^2$ case, if we increase the coupling constant over its critical value $g_{\rm cr}^2$, we find a negative solution $m_{\rm sym}^2 < 0$. This implies that the symmetric solution causes instability when $g_0^2 > g_{\rm cr}^2$ and thus we must choose the broken solution.

V. SUMMARY AND CONCLUSION

We have investigated a description of DχSB on the LF in the NJL model. The importance of solving the fermionic constraint for the bad spinor component was greatly stressed in analogy with the zero-mode constraint of scalar models. The exact classical solution enabled us to check the properties of the LF chiral transformation. Though the chiral transformation is differently introduced on the LF, we finally found the equivalence to the usual chiral transformation.

For a description of DχSB of LF NJL model, it was very important to solve the fermionic constraint nonperturbatively in quantum treatment. To do so, we introduced a bilocal formulation and solved the bilocal fermionic constraint with a fixed operator ordering by the $1/N$ expansion. Systematic $1/N$ expansion of the fermion bilocal operator is realized by the boson expansion method as the Holstein-Primakoff expansion. The leading bilocal fermionic constraint was identified with the gap equation for the chiral condensate after we took care of the infrared divergence. If we choose a nontrivial solution of the gap equation, we have a Hamiltonian in the broken phase but with a trivial vacuum.

The physical role of the fermionic constraint in the LF

NJL model is very similar to that of the zero-mode constraint for scalar models. We have seen a close parallel between these two constraints. Especially it should be noted that the gap equation came from the longitudinal zero mode of the bilocal fermionic constraint.

It is very natural that we can reach the broken phase by solving the quantum fermionic constraint by $1/N$ expansion because the fermionic constraint is originally a part of the Euler-Lagrange equation and thus must include relevant information of dynamics. What we did is very similar to the usual mean-field approximation for the Euler-Lagrange equations. Indeed the leading order in the $1/N$ expansion corresponds to the mean-field approximation. However, our way of solving the fermionic constraint with the boson expansion method can easily go beyond the mean-field level. Such a higher-order calculation enabled us to derive a correct broken Hamiltonian and to show the divergent behavior of the (spatial integration of) pseudoscalar field.

Independence of mass from the mode expansion has both desirable and undesirable aspects. The inclusion of correct mass dependence into the IR divergent integral was required when we identify the lowest fermionic constraint with the gap equation. This is the point we must always take into account. On the other hand, the Fock vacuum is defined independent of the value of mass. Due to this fact, it is enough to have only one vacuum, namely, the Fock vacuum even in the chirally broken phase. This is the favorable aspect. However, the cost of such a simple vacuum was payed by, for example, unusual chiral transformation of fields such as $[Q_5^{\rm LF},\bar{\Psi}i\gamma_5\Psi]\neq -2i\bar{\Psi}\Psi$ and nonvanishing of the LF chiral charge $[Q_5^{\rm LF},H]\neq 0$ in the broken phase. We found that both effects are proportional to the dynamical fermion mass M. We also insisted the necessity of a bare mass term which accurately produced the constituent mass term. Although the special role of the fermionic constraint might be restricted to the LF NJL model, the unusual chiral transformation and the nonconservation of the chiral charge are general features of the chiral symmetry breaking on the LF. This is because they are natural consequences of the coexistence of the chiral symmetry breaking and the Fock vacuum.

The leading-order eigenvalue equation for a single bosonic state is equivalent to the leading-order fermion-antifermion bound-state equation. The bound-state equations were solved analytically for scalar and pseudoscalar mesons and we obtained their light-cone wave functions and masses. The meson masses, the decay constant, and so on were fairly consistent with those of our previous analysis on the chiral Yukawa model. The leading-order calculation was limited only to two-body sector (fermion and antifermion). If we consider the higher-order Hamiltonian such as $h^{(3)}$ or $h^{(4)}$, we will be able to discuss four- or six-body sectors. In other words, since we have bosonic meson states, we can expand the Fock space in terms of the mesonic degrees of freedom. Then, for example, we will be able to discuss the mixing of scalar state and two pseudoscalar fields (π-π mixing with σ).

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APPENDIX A: CONVENTIONS

We follow the Kogut-Soper convention [20]. First of all, the light-front coordinates are defined as

$$x^{\pm}=\frac{1}{\sqrt{2}}(x^0\pm x^3),\quad x_\perp^i=x^i\quad (i=1,2),\tag{A1}$$

where we treat x^+ as ''time.'' The spatial coordinates x^- and $x_\perp$ are called the longitudinal and transverse directions, respectively. Derivatives in terms of $x^\pm$ are defined by $\partial_\pm=\partial/\partial x^\pm$. It is useful to introduce projection operators $\Lambda_\pm$ defined by

$$\Lambda_\pm=\frac{1}{2}\gamma^\mp\gamma^\pm=\frac{1}{\sqrt{2}}\gamma^0\gamma^\pm.\tag{A2}$$

Indeed $\Lambda_\pm$ satisfy the projection properties $\Lambda_\pm^2=\Lambda_\pm$, $\Lambda_+ + \Lambda_-=1$, etc. Splitting the fermion field by the projectors as

$$\Psi^a=\psi_+^a+\psi_-^a,\quad \psi_\pm^a\equiv\Lambda_\pm\Psi^a,\tag{A3}$$

we find that for any fermion on the LF, the ψ_- component is a dependent degree of freedom. ψ_+ and ψ_- are called the ''good component'' and the ''bad component,'' respectively.

As was noted in the text, for practical calculation, we use the two-component representation for the gamma matrices. The two-component representation is characterized by a specific form of the projectors (1.2). Then the projected fermions $\psi_\pm$ have only two components. There are many possibilities which realize Eq. (1.2). For example, a specific representation

$$\gamma^0=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \gamma^3=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\quad \gamma^i=\begin{pmatrix}-i\sigma^i & 0\\ 0 & i\sigma^i\end{pmatrix}\tag{A4}$$

is used in Ref. [21]. In this paper, however, we choose a representation (1.4) from which it is easy to extract information of the (1+1)-dimensional results. Two-component spinors ψ and χ are defined by Eq. (1.3). Results of the chiral Gross-Neveu model can be easily obtained if we make a replacement for the Pauli matrices $\sigma_3\to 1$ and $\sigma^i\to 0$, and regard ψ and χ as one-component spinors.

Using this representation, the Lagrangian density of the NJL model is written as

$$\mathcal{L}=i\psi^\dagger\partial_+\psi+i\chi^\dagger\partial_-\chi-\frac{1}{\sqrt{2}}(\psi^\dagger\sigma^i\partial_i\chi-\chi^\dagger\sigma^i\partial_i\psi)$$
$$-\frac{m_0}{\sqrt{2}}(\psi^\dagger\chi+\chi^\dagger\psi)+\frac{g^2}{4}$$
$$\times\{(\psi^\dagger\chi+\chi^\dagger\psi)^2-(\psi^\dagger\sigma^3\chi-\chi^\dagger\sigma^3\psi)^2\}. \qquad \text{(A5)}$$

APPENDIX B: CLASSICAL SOLUTIONS TO THE FERMIONIC CONSTRAINTS

To solve the fermionic constraints *classically* means that we treat all the fermion fields (both good and bad components) as Grassmann numbers and neglect all the c numbers which will emerge in quantum theory under the exchange of variables.

Before discussing a complicated equation of the NJL model, it would be better to go first with the chiral Gross-Neveu model. We solve the fermionic constraint of the chiral Gross-Neveu model with the antiperiodic boundary condition:

$$\{i\partial_-+g^2a(x^-)\}\chi=\frac{m_0}{\sqrt{2}}\;\psi, \qquad \text{(B1)}$$

$$\chi_a(x^-=-\infty)=-\chi_a(x^-=\infty), \qquad \text{(B2)}$$

where we used a matrix notation with a matrix $a_{ab}(x^-)\equiv\psi_a(x^-)\psi_b^\dagger(x^-)$. The solution to this equation is given by

$$\chi(x^-)=\int_{-\infty}^{\infty}dy^-G_{\rm GN}(x^-,y^-)\frac{m_0}{\sqrt{2}}\psi(y^-), \qquad \text{(B3)}$$

where $G_{\rm GN}(x^-,y^-)$ is the Green function satisfying

$$\{i\partial_-^x+g^2a(x^-)\}G_{\rm GN}(x^-,y^-)=\delta(x^--y^-), \qquad \text{(B4)}$$

$$G_{\rm GN}(x^-=-\infty,y^-)=-G_{\rm GN}(x^-=\infty,y^-). \qquad \text{(B5)}$$

Due to Eq. (B5), the solution of course satisfies the antiperiodic boundary condition. Equation (B4) is solved as

$$G_{\rm GN}(x^-,y^-)=G_{\rm GN}^{(0)}(x^-)\left[\frac{1}{2i}\epsilon(x^--y^-)+C\right]G_{\rm GN}^{(0)-1}(y^-), \qquad \text{(B6)}$$

$$G_{\rm GN}^{(0)}(x^-)=Pe^{ig^2\int_{-\infty}^{x^-}a(y^-)dy^-}, \qquad \text{(B7)}$$

$$C=-\frac{1}{2i}\,\frac{G_{\rm GN}^{(0)}(\infty)-G_{\rm GN}^{(0)}(-\infty)}{G_{\rm GN}^{(0)}(\infty)+G_{\rm GN}^{(0)}(-\infty)}, \qquad \text{(B8)}$$

where $G_{\rm GN}^{(0)}(x^-)$ is a solution of a homogeneous equation $\{i\partial_-^x+g^2a(x^-)\}G_{\rm GN}^{(0)}(x^-)=0$ and the integral constant C has been determined so that $G_{\rm GN}(x^-,y^-)$ satisfies the antiperiodic boundary condition. When $N=1$, the solution (B3) is equivalent to Domokos' solution to the Thirring model on the light front [13].

In two dimensions, the LF chiral transformation is not distinguishable with the $U(1)$ transformation. Indeed, the "LF chiral transformation" on the good component is $\psi\to e^{i\theta}\psi$ and equivalent to the $U(1)$ transformation. And also, the solution (B3) implies that the bad component rotates just the same way as the good component $\chi\to e^{i\theta}\chi$.

Next let us turn to the NJL model. If we explicitly write all the indices, the fermionic constraint (2.1) is

$$i\partial_-\begin{pmatrix}\chi_{1a}\\ \chi_{2a}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}m_0\psi_{1a}-\partial_z\psi_{2a}\\ -\partial_{\bar z}\psi_{1a}+m_0\psi_{2a}\end{pmatrix}-g^2\begin{pmatrix}\psi_{1a}\psi_{1b}^\dagger\chi_{1b}-\psi_{1a}\psi_{2b}\chi_{2b}^\dagger\\ \psi_{2a}\psi_{2b}^\dagger\chi_{2b}-\psi_{2a}\psi_{1b}\chi_{1b}^\dagger\end{pmatrix}, \qquad \text{(B9)}$$

where $\partial_z=\partial_1-i\partial_2$ and $\partial_{\bar z}=\partial_1+i\partial_2$. Since the equation for χ_1 (or χ_2) includes χ_1 and $\chi_2^\dagger$ (or χ_2 and $\chi_1^\dagger$), it is useful to introduce a constraint equation for $-\chi_2^\dagger$ instead of χ_2. Then we have a more tractable equation

$$i\partial_-\begin{pmatrix}\chi_{1a}\\ -\chi_{2a}^\dagger\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}m_0\psi_{1a}-\partial_z\psi_{2a}\\ -\partial_z\psi_{1a}^\dagger+m_0\psi_{2a}^\dagger\end{pmatrix}-g^2\begin{pmatrix}\psi_{1a}\psi_{1b}^\dagger & \psi_{1a}\psi_{2b}\\ \psi_{2a}^\dagger\psi_{1b}^\dagger & \psi_{2a}^\dagger\psi_{2b}\end{pmatrix}\begin{pmatrix}\chi_{1b}\\ -\chi_{2b}^\dagger\end{pmatrix}. \qquad \text{(B10)}$$

As in the (1+1)-dimensional case, the solution is immediately given if we find the Green function $G(x^-,y^-,x_\perp)$ which satisfies

$$\{i\partial_-^x+g^2\mathcal{A}(x^-)\}G(x^-,y^-,x_\perp)=\delta(x^--y^-), \qquad \text{(B11)}$$

with a matrix $\mathcal{A}_{ijab}(x)$ defined by Eq. (2.5). The result is very similar to the two-dimensional result and is given by Eqs. (2.2) and (2.3) in the text.

APPENDIX C: PROBLEM OF OPERATOR ORDERING

Here we consider the problem of operator ordering within the chiral Gross-Neveu model with $N=1$. Following the standard procedure, the Dirac brackets are calculated as

$$\{\psi(x),\psi^\dagger(y)\}_{\rm D}=-i\,\delta(x^--y^-), \qquad \text{(C1)}$$

$$\{\chi(x),\psi^\dagger(y)\}_{\rm D}=-iG_{\rm GN}(x,y)\left(\frac{m_0}{\sqrt{2}}-g^2\psi^\dagger(y)\chi(y)\right), \qquad \text{(C2)}$$

$$\{\chi(x),\psi(y)\}_{\rm D}=-iG_{\rm GN}(x,y)g^2\psi\chi(y), \qquad \text{(C3)}$$

where $G_{\rm GN}(x,y)$ is the Green function (B6) for the $N=1$ case. To quantize the system we simply replace the Dirac bracket $\{A,B\}_{\rm D}$ by the anticommutation relation $-i\{A,B\}$. This procedure has the ambiguity of the operator ordering.

The operator ordering we took for the fermionic constraint (2.1) in the NJL model corresponds to the following one in the chiral Gross-Neveu model:

$$i\partial_-\chi+g^2\psi\psi^\dagger\chi=\frac{m_0}{\sqrt{2}}\psi. \tag{C4}$$

We can easily find its *quantum* solution due to $[\psi\psi^\dagger(x),\psi\psi^\dagger(y)]=0$. The solution is

$$\chi_{\rm sol}(x^-)=\int_{-\infty}^{\infty}dy^-G_{\rm GN}(x,y)\frac{m_0}{\sqrt{2}}\psi(y), \tag{C5}$$

where G is again the Green function (B6) with $N=1$.

Now let us consider the consistency for the anticommutator $\{\chi,\psi^\dagger\}$. It can be calculated two different ways: (i) from the solution $\chi_{\rm sol}$ of the fermionic constraint, and (ii) from the Dirac bracket (C2). We fix the operator ordering of the fermionic constraint by Eq. (C4) and check whether the Dirac bracket can produce the same anticommutator or not.

Instead of the anticommutator itself, we present here the calculation of a quantity $iD_-^x\{\chi(x),\psi^\dagger(y)\}$ where $iD_-^x=i\partial_-+g^2\psi\psi^\dagger$. Using the solution (C5), we have

$$iD_-^x\{\chi_{\rm sol}(x),\psi^\dagger(y)\}=\delta(x^--y^-)\left(\frac{m_0}{\sqrt{2}}-g^2\psi^\dagger\chi\right). \tag{C6}$$

On the other hand, if we take the simplest ordering in the rhs of the Dirac bracket (C2), we obtain

$$\begin{aligned}iD_-^x\{\chi(x),\psi^\dagger(y)\}&=iD_-^xG_{\rm GN}(x,y)\left(\frac{m_0}{\sqrt{2}}-g^2\psi^\dagger\chi(y)\right)\\&=\delta(x^--y^-)\left(\frac{m_0}{\sqrt{2}}-g^2\psi^\dagger\chi(y)\right).\end{aligned} \tag{C7}$$

This is identical with the result (C6). Therefore we find our ordering Eq. (C4) is consistent with the anticommutation relation

$$\{\chi(x),\psi^\dagger(y)\}=G_{\rm GN}(x,y)\left(\frac{m_0}{\sqrt{2}}-g^2\psi^\dagger\chi\right). \tag{C8}$$

Of course if we take other operator ordering, the two results do not coincide. We expect that even in the NJL model, we can select the rhs of the Dirac brackets so that they coincide with the direct result with the ordering defined by Eq. (2.1).

APPENDIX D: BILOCAL FERMIONIC CONSTRAINTS AND THEIR SOLUTIONS BY THE BOSON EXPANSION METHOD

It is tractable to solve the equations for $\mathcal{T}(\boldsymbol{x},\boldsymbol{y})$ and $\mathcal{U}(\boldsymbol{x},\boldsymbol{y})$ rather than $\mathcal{T}_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})$ and $\mathcal{U}_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})$. In momentum representation, the fermionic constraints for $\mathcal{T}$ and $\mathcal{U}$ are

$$\begin{aligned}q^+\mathcal{T}(\boldsymbol{p},\boldsymbol{q})=&\frac{1}{2}(-iq_\perp^i\sigma^i+m_0)_{\alpha\beta}(\mathcal{M}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})-\mathcal{M}_{\alpha\beta}(\boldsymbol{q},\boldsymbol{p}))-\frac{g^2}{2}\int\frac{d^3\boldsymbol{p}'d^3\boldsymbol{q}'}{(2\pi)^3}\{\mathcal{M}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q}-\boldsymbol{p}'-\boldsymbol{q}')(\delta_{\alpha\beta}\mathcal{T}(\boldsymbol{p}',\boldsymbol{q}')\\&+\sigma^3_{\alpha\beta}i\mathcal{U}(\boldsymbol{p}',\boldsymbol{q}'))-(\delta_{\alpha\beta}\mathcal{T}(\boldsymbol{p}',\boldsymbol{q}')-\sigma^3_{\alpha\beta}i\ \ \mathcal{U}(\boldsymbol{p}',\boldsymbol{q}'))\mathcal{M}_{\alpha\beta}(\boldsymbol{q}-\boldsymbol{p}'-\boldsymbol{q}',\boldsymbol{p})\},\end{aligned} \tag{D1}$$

$$\begin{aligned}q^+i\mathcal{U}(\boldsymbol{p},\boldsymbol{q})=&\frac{1}{2}[\{\sigma^3(-iq_\perp^i\sigma^i+m_0)\}_{\alpha\beta}\mathcal{M}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})+\{(-iq_\perp^i\sigma^i+m_0)\sigma^3\}_{\alpha\beta}\mathcal{M}_{\alpha\beta}(\boldsymbol{q},\boldsymbol{p})]-\frac{g^2}{2}\int\frac{d^3\boldsymbol{p}'d^3\boldsymbol{q}'}{(2\pi)^3}\\&\times\{\mathcal{M}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q}-\boldsymbol{p}'-\boldsymbol{q}')(\sigma^3_{\alpha\beta}\mathcal{T}(\boldsymbol{p}',\boldsymbol{q}')+\delta_{\alpha\beta}i\mathcal{U}(\boldsymbol{p}',\boldsymbol{q}'))+(\sigma^3_{\alpha\beta}\mathcal{T}(\boldsymbol{p}',\boldsymbol{q}')-\delta_{\alpha\beta}i\mathcal{U}(\boldsymbol{p}',\boldsymbol{q}'))\mathcal{M}_{\alpha\beta}(\boldsymbol{q}-\boldsymbol{p}'-\boldsymbol{q}',\boldsymbol{p})\}.\end{aligned} \tag{D2}$$

In place of the quantization condition (3.1), the system with bilocal operators can be characterized by the following algebra:

$$\begin{aligned}[:\mathcal{M}_{\alpha\beta}(\boldsymbol{p}_1,\boldsymbol{p}_2):,\quad :\mathcal{M}_{\gamma\delta}(\boldsymbol{q}_1,\boldsymbol{q}_2):]=&:\mathcal{M}_{\alpha\delta}(\boldsymbol{p}_1,\boldsymbol{q}_2):\delta_{\beta\gamma}\delta^{(3)}(\boldsymbol{p}_2+\boldsymbol{q}_1)-:\mathcal{M}_{\gamma\beta}(\boldsymbol{q}_1,\boldsymbol{p}_2):\delta_{\alpha\delta}\delta^{(3)}(\boldsymbol{p}_1+\boldsymbol{q}_2)\\&+N\delta_{\alpha\delta}\delta_{\beta\gamma}\delta^{(3)}(\boldsymbol{p}_1+\boldsymbol{q}_2)\delta^{(3)}(\boldsymbol{p}_2+\boldsymbol{q}_1)(\theta(p_1^+)\theta(p_2^+)\theta(-q_1^+)\theta(-q_2^+)\\&-\theta(-p_1^+)\theta(-p_2^+)\theta(q_1^+)\theta(q_2^+)),\end{aligned} \tag{D3}$$

where the normal order of $\mathcal{M}$ was defined with respect to the Fock vacuum (3.3)

$$:\mathcal{M}_{\alpha\beta}^{+-}(\boldsymbol{p},\boldsymbol{q}):|0\rangle=:\mathcal{M}_{\alpha\beta}^{-+}(\boldsymbol{p},\boldsymbol{q}):|0\rangle=:\mathcal{M}_{\alpha\beta}^{++}(\boldsymbol{p},\boldsymbol{q}):|0\rangle=0. \tag{D4}$$

The upper indices stand for signs of the longitudinal momenta.

The complicated structure of the algebra for the bilocal operators which originates from the fermion statistics, is greatly reduced if one introduces the boson expansion method. We can represent the operators $:\mathcal{M}:$ in terms of bilocal boson operators $B(\boldsymbol{p},\boldsymbol{q})$ of order $\mathcal{O}(N^0)$ so that they fulfill the original algebra (D3). Since the algebra has a bosonic feature in the large-N limit,

$$[:\mathcal{M}^{++}_{\alpha\beta}(\boldsymbol{p}_1,\boldsymbol{p}_2):,\ :\mathcal{M}^{--}_{\gamma\delta}(\boldsymbol{q}_1,\boldsymbol{q}_2):] \to N\delta_{\alpha\delta}\delta_{\beta\gamma}\delta^{(3)}(\boldsymbol{p}_1+\boldsymbol{q}_2)\,\delta^{(3)}(\boldsymbol{p}_2+\boldsymbol{q}_1),$$

it would be better to choose a representation which satisfies this.[5] The Holstein-Primakoff-type expansion satisfies the requirement.

Physically this procedure corresponds to extracting purely bosonic degrees of freedom from a fermion-antifermion system, i.e., a mesonic system. The power of the boson expansion method in the light-front field theories was first demonstrated by one of the authors [15]. He applied the Holstein-Primakoff type expansion to (1+1)-dimensional QCD and derived an effective Hamiltonian for mesons as local bosons whose masses are given by the 't Hooft equation. Using the effective Hamiltonian, we can in principle study, say, scattering of mesons as $q\bar{q}$ bound states.

Since the essential structure of the algebra (D3) is determined only by the longitudinal momentum, it is straightforward to apply the Holstein-Primakoff expansion discussed in Ref. [15] to four-dimensional fermionic theories. Indeed the operators $:\mathcal{M}:$ are represented as follows:

$$:\mathcal{M}^{-+}_{\alpha\beta}(\boldsymbol{p}_1,\boldsymbol{p}_2):=\int[d\boldsymbol{q}]\sum_\gamma B^\dagger_{\alpha\gamma}(-\boldsymbol{p}_1,\boldsymbol{q})B_{\beta\gamma}(\boldsymbol{p}_2,\boldsymbol{q}) \equiv A_{\beta\alpha}(\boldsymbol{p}_2,-\boldsymbol{p}_1), \tag{D5}$$

$$:\mathcal{M}^{+-}_{\alpha\beta}(\boldsymbol{p}_1,\boldsymbol{p}_2):=-\int[d\boldsymbol{q}]\sum_\gamma B^\dagger_{\gamma\beta}(\boldsymbol{q},-\boldsymbol{p}_2)B_{\gamma\alpha}(\boldsymbol{q},\boldsymbol{p}_1), \tag{D6}$$

$$:\mathcal{M}^{++}_{\alpha\beta}(\boldsymbol{p}_1,\boldsymbol{p}_2):=\int[d\boldsymbol{q}]\sum_\gamma(\sqrt{N-A})_{\beta\gamma}(\boldsymbol{p}_2,\boldsymbol{q})B_{\gamma\alpha}(\boldsymbol{q},\boldsymbol{p}_1), \tag{D7}$$

$$:\mathcal{M}^{--}_{\alpha\beta}(\boldsymbol{p}_1,\boldsymbol{p}_2):=\int[d\boldsymbol{q}]\sum_\gamma B^\dagger_{\gamma\beta}(\boldsymbol{q},-\boldsymbol{p}_2)\times(\sqrt{N-A})_{\gamma\alpha}(\boldsymbol{q},-\boldsymbol{p}_1). \tag{D8}$$

These give the $1/N$ expansion of $\mathcal{M}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})$. The first few terms are shown in the text [Eqs. (3.10)–(3.12)].

If we expand the bilocal operators $\mathcal{T}(\boldsymbol{p},\boldsymbol{q})$, $\mathcal{U}(\boldsymbol{p},\boldsymbol{q})$, and $\mathcal{M}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})$, the equation for order n can be written in a compact form:

$$\begin{pmatrix} t^{(n)}(\boldsymbol{p},\boldsymbol{q}) \\ u^{(n)}(\boldsymbol{p},\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} F^{(n)}(\boldsymbol{p},\boldsymbol{q}) \\ G^{(n)}(\boldsymbol{p},\boldsymbol{q}) \end{pmatrix} - g_0^2\frac{\epsilon(p^+)}{q^+}\int_{-\infty}^{\infty}\frac{d^3\boldsymbol{k}}{(2\pi)^3}\begin{pmatrix} t^{(n)}(\boldsymbol{k},\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k}) \\ u^{(n)}(\boldsymbol{k},\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k}) \end{pmatrix}, \tag{D9}$$

where quantities $F^{(n)}(\boldsymbol{p},\boldsymbol{q})$ and $G^{(n)}(\boldsymbol{p},\boldsymbol{q})$ are generally complicated functions of bilocal operators except for the lowest order [see Eq. (3.18)]. For example, $F^{(1)}$ and $G^{(1)}$ are

$$F^{(1)}(\boldsymbol{p},\boldsymbol{q}) = \frac{1}{2q^+}(-iq^i_\perp\sigma^i+M)_{\alpha\beta}[\mu^{(1)}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})-\mu^{(1)}_{\alpha\beta}(\boldsymbol{q},\boldsymbol{p})], \tag{D10}$$

$$G^{(1)}(\boldsymbol{p},\boldsymbol{q}) = \frac{-i}{2q^+}[\{\sigma_3(-iq^i_\perp\sigma^i+M)\}_{\alpha\beta}\mu^{(1)}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q}) + \{(-iq^i_\perp\sigma^i+M)\sigma_3\}_{\alpha\beta}\mu^{(1)}_{\alpha\beta}(\boldsymbol{q},\boldsymbol{p})], \tag{D11}$$

where $\mu^{(1)}_{\alpha\beta}(\boldsymbol{p},\boldsymbol{q})$ is given by the boson expansion method Eq. (3.11). Since all of the orders of the operators are less than n, we can solve this equation order by order. The solution of this integral equation is

$$\begin{pmatrix} t^{(n)}(\boldsymbol{p},\boldsymbol{q}) \\ u^{(n)}(\boldsymbol{p},\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} F^{(n)}(\boldsymbol{p},\boldsymbol{q}) \\ G^{(n)}(\boldsymbol{p},\boldsymbol{q}) \end{pmatrix} - g_0^2\frac{\epsilon(p^+)}{q^+}\alpha(p^++q^+) \times\int_{-\infty}^{\infty}\frac{d^3\boldsymbol{k}'}{(2\pi)^3}\begin{pmatrix} F^{(n)}(\boldsymbol{k}',\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k}') \\ G^{(n)}(\boldsymbol{k}',\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k}') \end{pmatrix}, \tag{D12}$$

where

$$\alpha(P^+) = \left(1+g_0^2\int_{-\infty}^{\infty}\frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{\epsilon(k^+)}{P^+-k^+}\right)^{-1}. \tag{D13}$$

The quantities $t^{(2)}(\boldsymbol{p},\boldsymbol{q})$ and $u^{(2)}(\boldsymbol{p},\boldsymbol{q})$ are necessary for obtaining a correct Hamiltonian of the system.

[5] Actually there are many possibilities to express Eq. (D3) in terms of bosonic operators, corresponding to various ''local expansions'' of the Grassmannian manifold of the bilocal operators.

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