Three dimensional gravity from $ISO(2,1)$ coset models

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Starting from a WZWN action in the $ISO(2,1)$ Poincaré group which describes a bosonized spinning string in 2+1 Minkowski space-time, we show that a sequence of non-trivial compactifications leads to the description of a spinless string which moves in a (linear dilaton) vacuum, AdS_3 or BTZ black hole background. Other solutions are also obtained and their T duals analyzed.

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I. INTRODUCTION

The observational evidence for the existence of black holes in nature is now very strong. The data support the existence of both supermassive black holes at the centers of galaxies and smaller (a few solar masses up to a few tens of solar masses) black holes in binary systems [1]. The best candidate for a unified theory of all the physical phenomena observed so far, including black holes, is string theory [2] and, indeed, several black *p*-brane solutions have been found in various space-time dimensions in the low energy limit of this theory (for a review see, e.g., Ref. [3]). However, only one black hole [4] is known to exist in the three-dimensional low energy limit of string theory and it coincides with the only known black hole in three-dimensional Einstein gravity [5]: the black hole of Bañados, Teitelboim and Zanelli (BTZ) [6] (see also Ref. [7]).

Although the BTZ black hole is not useful as a global description of real black holes (for example, the curvature of the BTZ black hole is constant and there are no gravitational waves in three dimensions), it does provide a manageable model of string propagation on a black background in which an infinite number of propagating modes is present. The Green's function for this black hole can be constructed, and the quantum stress tensor can be calculated from it [8]. This system has also been used to study such problems as the quantization of a string on a black hole background (see [9] and references therein).

Recently, the theoretical interest in the BTZ black hole has also been raised by the conjectured AdS-conformal field theory (CFT) correspondence [10], according to which all the relevant quantities of the gravitational field theory in the bulk of the anti–de Sitter (AdS) space-time (or any space-time with a time-like boundary) can be described in terms of a conformal field theory (CFT) on the boundary. Thus, by applying this conjecture to the black *p*-branes there is some hope of describing the complete evolution of a black hole, from its formation [11] to the evaporation [12], and solve the riddle of its final fate (see Ref. [13] for a list of still unanswered questions). However, it is not clear whether the AdS-CFT correspondence extends beyond perturbation theory on a given background manifold [14] as the solution of the black hole problem would require in order to compute the back reaction of the evaporation radiation on the geometry [15].

Because of the usefulness of three-dimensional black holes as prototypes for four-dimensional black holes in string theory, a search for a second exact, three-dimensional black hole in string theory would seem to be a worthwhile pursuit, especially if one could be found which has non-negative curvature. This work describes our attempt to obtain such a solution, starting from a Wess-Zumino-Witten-Novikov (WZWN) model in the Poincaré group $ISO(2,1)$ [16]. Our procedure for obtaining a three-dimensional metric is to promote the six parameters of the $ISO(2,1)$ group to space-time variables and then to reduce space-time to three dimensions by various compactifications. After each compactification we investigate the symmetries of the resulting model.

A partial result has been obtained, since we can now show that the string theory we start from can be compactified in such a way as to yield either a (linear dilaton) vacuum or AdS_3 (the BTZ black hole). We also obtain other solutions which contain a non-trivial dilaton field and, thus, might be of interest for studying evaporation.

In Sec. II we review the WZWN Poincaré action in three dimensions, its coset descendants $ISO(2,1)/\mathbb{R}^n$ [17,18] and specialize to the case when $\mathbb{R}$ is a translation in the time direction. In Sec. III, we further compactify to three-dimensional space-time in which we recover the AdS (and BTZ) manifold. In Sec. IV we describe other solutions and their T duals, and finally comment on our results in Sec. V. For the metric and other geometrical quantities we follow the convention of Ref. [19].

II. $ISO(2,1)$ WZWN MODELS

The WZWN construction starts with the σ-model action at level κ [20]:

$$S_\sigma=\frac{\kappa}{4}\int_{\partial\mathcal{M}} d^2\sigma\,\mathrm{Tr}\,(g^{-1}\,\partial_+ g\,g^{-1}\,\partial_- g)$$
$$-\frac{\kappa}{4}\int_{\mathcal{M}} d^3\zeta\,\mathrm{Tr}\,(g^{-1}\,\partial g\wedge g^{-1}\,\partial g\wedge g^{-1}\,\partial g),\quad(2.1)$$

where in the present case g is an element of the Poincaré group $ISO(2,1)$ and $\sigma^{\pm}=\tau\pm\sigma$ are light-cone coordinates on the boundary $\partial\mathcal{M}$ of the three-dimensional manifold $\mathcal{M}$.

The elements of $ISO(2,1)$ can be written using the notation $g=(\Lambda,v)$, where $\Lambda\in SO(2,1)$ and $v\in\mathbb{R}^3$. Given the map g: $\mathcal{M}=D^2\times R\mapsto ISO(2,1)$ from the two-dimensional disc $\times$ time to $ISO(2,1)$, the action can be written entirely on the boundary $\mathbb{R}\times S^1$ and it describes a closed bosonized spinning string moving in 2+1 Minkowski space-time with coordinates v^i [16]:

$$S=-\frac{\kappa}{4}\int_{\partial\mathcal{M}} d^2\sigma\,\epsilon^{ijk}\,(\partial_+\Lambda\,\Lambda^{-1})_{ij}\,\partial_- v_k\,, \tag{2.2}$$

where ϵ^{ijk} is the Levi-Civita symbol in three dimensions and the metric tensor is $\eta_{ij}=\mathrm{diag}[-1,+1,+1]$ $(i,j,\ldots=0,1,2)$.

The basic property of the action S is that it is invariant under

$$g\mapsto g_{_L}(\sigma^+)\,g\,g_{_R}^{-1}(\sigma^-), \tag{2.3}$$

where $g_{_{L/R}}\in ISO(2,1)$, and also under the left and right action of the group of diffeomorphisms of the world sheet [16]. Starting from this observation, the canonical structure of the model can be computed by reverting to the ''chiral'' version of Eq. (2.2), which is obtained by formally replacing $\sigma^+\to\tau\in\mathbb{R}$ and $\sigma^-\to\sigma\in(0,2\pi)$ [21]. One then finds two sets of conserved current densities, the first of which is given by

$$P^i(\sigma)=\frac{\kappa}{2}\,\epsilon^{ijk}\,(\Lambda^{-1}\,\partial_\sigma\Lambda)_{jk} \tag{2.4}$$

$$J^i(\sigma)=\kappa\,(\Lambda^{-1}\,\partial_\sigma v)^i, \tag{2.5}$$

with Poisson brackets [16]

$$\{P^i(\sigma),P^j(\sigma')\}=0 \tag{2.6}$$

$$\{J^i(\sigma),J^j(\sigma')\}=-\epsilon_{ijk}\,J_k(\sigma)\,\delta(\sigma-\sigma') \tag{2.7}$$

$$\{J^i(\sigma),P^j(\sigma')\}=-\epsilon^{ijk}\,P_k(\sigma)\,\delta(\sigma-\sigma')+\kappa\,\eta^{ij}\frac{\partial}{\partial\sigma}\,\delta(\sigma-\sigma'), \tag{2.8}$$

and generate $L^*ISO(2,1)$, the Poincaré loop group with the central extension given by the last term in Eq. (2.8). This is the algebra of the right transformations in Eq. (2.3), since in the chiral picture $g_{_R}(\sigma^-)\to g_{_R}(\sigma)$ has become a space-dependent transformation of the field g on the world sheet. The (time-dependent) left chiral transformation, $g_{_L}(\sigma^+)\to g_{_L}(\tau)$, in Eq. (2.3) is now an $ISO(2,1)$ invariance generated by the zero Fourier modes of the second set of (weakly vanishing) current densities

$$\overline{P^i}=\frac{\kappa}{2}\,\epsilon^{ijk}\,(\partial_\sigma\Lambda\,\Lambda^{-1})_{jk} \tag{2.9}$$

$$\overline{J^i}=\kappa\,[\Lambda\,\partial_\sigma(\Lambda^{-1}\,v)]^i. \tag{2.10}$$

The latter commute with P^i and J^i and have Poisson brackets among themselves given by Eqs. (2.6), (2.7) and (2.8) with a central extension opposite in sign [16]. One then concludes that (half [21]) the (classical) gauge invariant phase space of the model is $L^*ISO(2,1)/ISO(2,1)$.

One would expect that the Fourier modes of P_i and J_i (the Kac-Moody generators) yield a Virasoro algebra (for each chiral sector) with generators L_n. However, the standard highest weight construction [2], which would give a central charge $c=\dim ISO(2,1)=6$, fails to deliver unitary representations (the conditions $\hat{L}_n|phys\rangle=0$, $n\geqslant 0$, does not suppress all negative norm states). Spaces of positive norm states can instead be obtained by employing the method of induced representations which yields a central charge $c=0$ for each chiral sector [16]. In either case, the total central charge of the model, after adding the ghost contribution [22,2], is $c_T=c-26$ and one must eventually add $26-c$ bosonic degrees of freedom in order to have a quantum model which is free of anomaly.

The action (2.2) is one of the two exceptional cases described in Ref. [18], where it was shown that, if one considers all parameters of the six-dimensional Poincaré group as space-time coordinates, then S describes a spinless string moving on a curved background with six-dimensional metric. It was also proved that this action is unique in the sense that no generalization of the kind studied in Refs. [23,18] exists for the Poincaré group in three dimensions.

A. Coset models

The action (2.2) is not invariant under the local action of any subgroup H of $ISO(2,1)$ given by

$$h\cdot g: g\mapsto h_{_L}(\sigma^-,\sigma^+)\,g\,h_{_R}^{-1}(\sigma^-,\sigma^+), \tag{2.11}$$

where now $h_{_{L/R}}=(\theta_{_{L/R}},y_{_{L/R}})\in H$, due to the dependence of h_L on σ^- and of h_R on σ^+. However, H can in general be promoted to a gauge symmetry of the action by introducing suitable gauge fields $A_\pm=(\omega_\pm,\xi_\pm)$ belonging to the Lie algebra of H, and the corresponding covariant derivatives $D_\pm=\partial_\pm+A_\pm$.

In order that $ISO(2,1)/H$ be a coset, H must be normal, $H\cdot g=g\cdot H$, under the action defined in Eq. (2.11). This means that, for all $g\in ISO(2,1)$ and $h\in H$, there must exist an $\overline{h}\in H$ such that $hgh^{-1}=\overline{h}^{-1}g\,\overline{h}$, and we thus find that the only possible choices are subgroups of the translation group $\mathbb{R}^3$, that is $h_{_{L/R}}=(\mathbb{1},y^{\overline{n}}_{_{L/R}})$, where $\overline{n}$ runs in a subset of $\{0,1,2\}$ and $\mathbb{1}$ is the identity in $SO(2,1)$. In this case, by inspecting the action (2.2) one argues that $\omega_\pm\equiv\xi_+\equiv 0$, and $\xi^i_-\equiv 0$ iff the translation in the i direction is not included in H. The gauged action finally reads [17]

$$S_g=-\frac{\kappa}{4}\int d^2\sigma\,\epsilon^{ijk}\,(\partial_+\Lambda\,\Lambda^{-1})_{ij}\,(\partial_- v+\xi_-)_k\,. \tag{2.12}$$

For the ungauged action in Eq. (2.2), variation with respect to v leads to the conservation of the six momentum currents on the light cone of the string world sheet,

$$\partial_- P^i_+ = \partial_+ P^i_- = 0, \tag{2.13}$$

where P^i_+ is given by $\overline{P^i}$ in Eq. (2.9) with $\sigma \to \sigma^+$ and P^i_- by P^i in Eq. (2.4) with $\sigma \to \sigma^-$. In the gauged case this variation must be supplemented by the condition that the gauge field varies under an infinitesimal H transformation, $\xi^{\bar{n}}_- \to \xi^{\bar{n}}_- - \partial_-(\delta v^{\bar{n}})$, and one obtains

$$\partial_- P^{i\neq\bar{n}}_+ = 0, \tag{2.14}$$

so that only the currents $P^{i\neq\bar{n}}_+$ are still conserved.

Similarly, varying the action S with respect to Λ leads to the conservation of the six angular momentum currents $J^i_- = J^i$ in Eq. (2.5) with $\sigma \to \sigma^-$ and $J^i_+ = \overline{J^i}$ in Eq. (2.10) with $\sigma \to \sigma^+$. When interpreted as components of the string angular momentum in the target space-time, these currents are shown to include a contribution of intrinsic (non orbital) spin [16]. In the gauged case, one obtains

$$\partial_+ J^i_- = -\kappa\, \partial_+ (\Lambda\, \xi_-)^i, \tag{2.15}$$

so that the currents J^i_- couple to the gauge field.

Since the gauge field is not dynamical, we are now free to choose $\dim H$ gauge conditions to be satisfied by the elements of $ISO(2,1)/H$. A natural choice is $\xi^{\bar{n}}_- = -\partial_- v^{\bar{n}}$, so that the previous equations of motion become the same as those obtained by varying the effective action

$$S^{(\bar{n})}_{eff} = \int d^2\sigma \sum_{k\neq\bar{n}} P^k_+\, \partial_- v_k, \tag{2.16}$$

where the sum runs over only the indices corresponding to the translations not included in H.

An explicit form for the effective action (2.16) can be obtained by writing an $SO(2,1)$ matrix as a product of two rotations (of angles α and γ) and a boost (β) [17], which yields

$$P^0_+ = \frac{\kappa}{2}(\partial_+\alpha + \cosh\beta\, \partial_+\gamma)$$

$$P^1_+ = \frac{\kappa}{2}(\cos\alpha\, \partial_+\beta + \sin\alpha \sinh\beta\, \partial_+\gamma) \tag{2.17}$$

$$P^2_+ = \frac{\kappa}{2}(\sin\alpha\, \partial_+\beta - \cos\alpha \sinh\beta\, \partial_+\gamma).$$

B. Gauging the time translations

We gauge the subgroup $H = \{(\mathbb{1}, y^0)\}$ of the translations in the time direction. This choice is peculiar, since no derivative of α occurs in P^1_+ and P^2_+, and we can then rotate the variables v^1 and v^2 by an angle $-\alpha$ [17],

$$\begin{bmatrix} \partial_-\tilde{v}^1 \\ \partial_-\tilde{v}^2 \end{bmatrix} \equiv \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \partial_- v^1 \\ \partial_- v^2 \end{bmatrix}. \tag{2.18}$$

This can be considered as an internal symmetry of the effective theory which is used to further simplify the action in Eq. (2.16) with $\bar{n} = 0$ to the form [24]

$$S^{(0)}_{eff} = \frac{\kappa}{2}\int d^2\sigma\, [\partial_+\beta\, \partial_-\tilde{v}^1 - \sinh\beta\, \partial_+\gamma\, \partial_-\tilde{v}^2]. \tag{2.19}$$

In the following we shall find it more convenient to regard β, γ, $\tilde{v}^2$ and $\tilde{v}^1$ as canonical (field) variables by foliating the closed string world sheet with circles of constant time τ [25]. Their conjugate momenta are then given by

$$P^1 \equiv \frac{\delta S^{(0)}_{eff}}{\delta\partial_\tau\beta} = \frac{\kappa}{2}\partial_-\tilde{v}^1$$

$$P^2 \equiv \frac{\delta S^{(0)}_{eff}}{\delta\partial_\tau\gamma} = -\frac{\kappa}{2}\sinh\beta\, \partial_-\tilde{v}^2 \tag{2.20}$$

$$P^3 \equiv \frac{\delta S^{(0)}_{eff}}{\delta\partial_\tau\tilde{v}_2} = -\frac{\kappa}{2}\sinh\beta\, \partial_+\gamma = P^2_+(\alpha=0)$$

$$P^4 \equiv \frac{\delta S^{(0)}_{eff}}{\delta\partial_\tau\tilde{v}_1} = \frac{\kappa}{2}\partial_+\beta = P^1_+(\alpha=0).$$

The above relations can be inverted to express the velocities in terms of the momenta. This signals the fact that all (explicit) symmetries of the original model have been ''gauge fixed'' and $\tilde{v}^1$, $\tilde{v}^2$, β and γ are physical degrees of freedom which we are allowed to consider as target space-time coordinates for the compactified string.

In target space-time coordinates $X^1 = \beta$, $X^2 = \gamma$, $X^3 = \tilde{v}^2$ and $X^4 = \tilde{v}^1$, the action (2.19) can be written as [18]

$$\begin{aligned} S^{(0)}_{eff} &= -\frac{\kappa}{2}\int d^2\sigma\, (\eta^{ab} + \epsilon^{ab})\, [\partial_a X^1\, \partial_b X^4 \\ &\quad - F(X^1)\, \partial_a X^2\, \partial_b X^3] \\ &= -\frac{1}{2}\int d^2\sigma\, [\eta^{ab}\, G^{(4)}_{\mu\nu}\, \partial_a X^\mu\, \partial_b X^\nu \\ &\quad + \epsilon^{ab}\, B^{(4)}_{\mu\nu}\, \partial_a X^\mu\, \partial_b X^\nu], \end{aligned} \tag{2.21}$$

where $F \equiv \sinh X^1$, η^{ab} and ϵ^{ab} are, respectively, the Minkowski tensor and the Levi-Civita symbol in two dimensions and $\mu, \nu, \ldots = 1, \ldots, 4$. By interpreting $\kappa = l_s^2$ as the square of the fundamental (string) length, the symmetric tensor $\mathbf{G}^{(4)}$ in the chosen reference frame has components

$$G^{(4)}_{\mu\nu}=l_s^2\begin{bmatrix}0 & 0 & 0 & -1\\ 0 & 0 & F & 0\\ 0 & F & 0 & 0\\ -1 & 0 & 0 & 0\end{bmatrix} \tag{2.22}$$

and is the space-time metric with signature $2+2$ and $G^{(4)}\equiv\det\mathbf{G}^{(4)}=l_s^8\,F^2$. The antisymmetric tensor $\mathbf{B}^{(4)}$ has components

$$B^{(4)}_{\mu\nu}=l_s^2\begin{bmatrix}0 & 0 & 0 & -1\\ 0 & 0 & F & 0\\ 0 & -F & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix} \tag{2.23}$$

and is the axion potential.

The Euler-Lagrange equations of motion can be written as

$$\begin{aligned}
\frac{\delta S^{(0)}_{eff}}{\delta\beta}&=-\frac{\kappa}{2}[\partial_+\partial_-X^4+\sqrt{1-F^2}\,\partial_+X^2\,\partial_-X^3]=0\\
\frac{\delta S^{(0)}_{eff}}{\delta\gamma}&=-\frac{\kappa}{2}\,\partial_+(F\,\partial_-X^3)=-\partial_+P^2=0\\
\frac{\delta S^{(0)}_{eff}}{\delta\tilde{v}^2}&=\frac{\kappa}{2}\,\partial_-(F\,\partial_+X^2)=-\partial_-P^3=0\\
\frac{\delta S^{(0)}_{eff}}{\delta\tilde{v}^1}&=-\frac{\kappa}{2}\,\partial_-\partial_+X^1=-\partial_-P^4=0,
\end{aligned} \tag{2.24}$$

from which one sees that three of the canonical momenta (P^2, P^3 and P^4) are conserved along (one of the two) null directions of the world sheet. We also note that X^1 is a ‘‘flat’’ target space direction, since the fourth of Eqs. (2.24) is the free wave equation whose general solution is given by

$$X^1=X^1_L+X^1_R, \tag{2.25}$$

where the arbitrary functions $X^\mu_L=X^\mu_L(\sigma^+)$ stand for left-moving and $X^\mu_R=X^\mu_R(\sigma^-)$ for right-moving waves.

The system of Eqs. (2.24) considerably simplifies for zero canonical momentum modes along X^2 ($X^3=X^3_L$) or X^3 ($X^2=X^2_R$), in which cases $X^4=X^4_L+X^4_R$. When both P^2 and P^3 vanish one then has the simple solution

$$\begin{aligned}
X^1&=X^1_L+X^1_R\\
X^2&=X^2_R\\
X^3&=X^3_L\\
X^4&=X^4_L+X^4_R,
\end{aligned} \tag{2.26}$$

which describes free wave modes in all of the four space-time directions. More general solutions would instead describe wave modes which propagate along a direction but couple with modes propagating in (some of) the other directions.

III. THE FINAL COMPACTIFICATION

Upon using the fact that $X^1=\beta$ is a flat direction for the propagating string, we impose a further compactification condition in order to eliminate $X^4=\tilde{v}^1$,

$$e^{2\lambda}\,\partial_\pm x^1=\partial_\pm X^4, \tag{3.1}$$

where λ is, at present, an arbitrary function of $x^1\equiv X^1$. We also define the two coordinates x^0 and x^2 according to

$$\begin{aligned}
\partial_+X^2&=e^\rho\,(c_1\,\partial_+x^0+c_2^{-1}\,\partial_+x^2)\\
\partial_+X^3&=e^\rho\,(c_1^{-1}\,\partial_-x^0-c_2\,\partial_-x^2),
\end{aligned} \tag{3.2}$$

with $\rho=\rho(x^1)$ and c_1 and c_2 are non-zero real constants. This reduces the action (2.21) to

$$\begin{aligned}
S_3&=\frac{\kappa}{2}\int d^2\sigma\,[e^{2\lambda}\,\partial_+x^1\,\partial_-x^1-e^{2\rho}\,F\,(\partial_+x^0\,\partial_-x^0\\
&\quad-\partial_+x^2\,\partial_-x^2)+c_1\,c_2\,e^{2\rho}\,F\,(\partial_+x^0\,\partial_-x^2-\partial_+x^2\,\partial_-x^0)]\\
&=-\frac{1}{2}\int d^2\sigma\,[\,\eta^{ab}\,G^{(3)}_{ij}\,\partial_ax^i\,\partial_bx^j+\epsilon^{ab}\,B^{(3)}_{ij}\,\partial_ax^i\,\partial_bx^j],
\end{aligned} \tag{3.3}$$

where now the three-metric $\mathbf{G}^{(3)}$ has components

$$G^{(3)}_{ij}=l_s^2\,\mathrm{diag}[-e^{2\rho}\,F\,,\,e^{2\lambda}\,,\,e^{2\rho}\,F], \tag{3.4}$$

and signature $2+1$. Further, the only non-vanishing independent component of the axion potential $\mathbf{B}^{(3)}$,

$$B^{(3)}_{02}=l_s^2\,c_1\,c_2\,e^{2\rho}\,F, \tag{3.5}$$

does not depend on λ.

We then observe that the axion field strength in three dimensions must be proportional to the Levi-Civita (pseudo)tensor,

$$H_{ijk}=\sqrt{-G^{(3)}}\,\epsilon_{ijk}\,\mathcal{H}, \tag{3.6}$$

where $\mathcal{H}=\mathcal{H}(x^i)$ is a function of the space-time coordinates to be determined from the field equations and $\sqrt{-G^{(3)}}\equiv\sqrt{-\det\mathbf{G}^{(3)}}$ is the volume element. In the present case we have

$$H_{012}=\partial_1B^{(3)}_{20}=-l_s^2\,c_1\,c_2\,e^{2\rho}\,(\sqrt{1-F^2}+2\,F\,\rho'), \tag{3.7}$$

which yields

$$\mathcal{H}=-\frac{c_1\,c_2}{l_s}\,e^{-\lambda}\left(\sqrt{\frac{1-F^2}{F}}+2\rho'\right). \tag{3.8}$$

With respect to the particular solution (2.26), we note that the condition (3.1) can be safely imposed only if $X_L^4=0$, and then the compactification condition becomes

$$e^{2\lambda(x_R^1)}\,\partial_- x_R^1=\partial_- X_R^4, \tag{3.9}$$

or $X_R^4=0$ and

$$e^{2\lambda(x_L^1)}\,\partial_+ x_L^1=\partial_+ X_L^4. \tag{3.10}$$

The functions ρ, λ and the constants c_1, c_2 can then be determined by noting that the low energy string action in three dimensions is given by (we set $l_s=1$ henceforth) [2]

$$S_{low}=\int d^3x\,\sqrt{-G^{(3)}}\times e^{-2\phi}\left[R+\frac{4}{k}+4\,\nabla_k\phi\,\nabla^k\phi-\frac{1}{12}H_{ijk}H^{ijk}\right], \tag{3.11}$$

where $4/k$ is a cosmological constant, R the scalar curvature, ∇ the covariant derivative with respect to the metric $\mathbf{G}^{(3)}$ and ϕ the dilaton. On varying S_{low} one obtains the field equations

$$R_{ij}+2\,\nabla_i\nabla_j\phi-\frac{1}{4}H_{ikl}H_j{}^{kl}=0 \tag{3.12}$$

$$\nabla_k(e^{-2\phi}H^k{}_{ij})=0 \tag{3.13}$$

$$4\,\nabla_k\nabla^k\phi-4\,\nabla_k\phi\,\nabla^k\phi+\frac{4}{k}+R-\frac{1}{12}H_{ijk}H^{ijk}=0, \tag{3.14}$$

which must be satisfied by the metric (3.4) and the axion obtained from the potential (3.5).

A. Linear dilaton vacuum

First we observe that for

$$e^{-2\rho}=\pm F, \tag{3.15}$$

the metric (3.4) becomes the flat Minkowski metric

$$ds^2=\mp(dx^0)^2+(dz_\pm)^2\pm(dx^2)^2, \tag{3.16}$$

where the upper signs correspond to $x^1>0$ $(F>0)$ and the lower signs to $x^1<0$ $(F<0)$ and the new coordinate z is determined by

$$dz_\pm=e^\lambda\,dx^1=\frac{dx^1}{\sqrt{\pm F}}. \tag{3.17}$$

Correspondingly $\mathbf{B}^{(3)}$ is constant and the axion vanishes; thus the field equation (3.13) is trivially satisfied.

The remaining Eqs. (3.12) and (3.14) yield the following expression for the dilaton field:

$$\phi=a+\frac{x^0}{b}+\frac{z_\pm}{c}+\frac{x^2}{d}, \tag{3.18}$$

where the constant a is arbitrary and the constants b, c and d must satisfy

$$\pm\frac{1}{b^2}-\frac{1}{c^2}\mp\frac{1}{d^2}=-\frac{1}{k}. \tag{3.19}$$

This solution represents a *linear dilaton vacuum*. When $k\to\infty$ one of course obtains the trivial form for such a vacuum with $\phi=a$.

We finally observe that along $x^1=0$ there occurs a signature flip, so that the roles of x^0 and x^2 as, respectively, a time coordinate and a spatial coordinate are exchanged. We shall find the same feature again in the following.

B. Recovering AdS_3 and BTZ

If we assume

$$\mathcal{H}=-\frac{2}{l}, \tag{3.20}$$

where l is a constant, then the field equations (3.12)–(3.14) are satisfied by choosing $\rho=0$, $c_1c_2=1$ and

$$e^{2\lambda}=\frac{l^2}{4}\coth^2 x^1, \tag{3.21}$$

which yields

$$X^4=\frac{l}{2}(x^1-\coth x^1)+X_0^4, \tag{3.22}$$

with X_0^4 an integration constant. It then follows that the compactification we are employing is indeed singular, since $X^4\sim\mp 1/x^1$ for $x^1\to 0^\pm$, which means that we are mapping vanishing boosts along v^1 (parameterized by β) into infinite translations along $\tilde{v}^1$. For this reason we tentatively consider the range of $x^1=\beta$ as divided into the two (disjoint) half lines $x^1>0$ and $x^1<0$.

It is indeed possible to show that this partition of the range of β has a natural interpretation in terms of the spacetime manifold. In fact, the choice (3.21) reduces Eq. (3.3) to the action for a string propagating in the three-dimensional AdS space-time. This can be seen, e.g., by defining new (dimensionless) coordinates $r_\pm\in\mathbb{R}$ such that

$$r_+=\ln(+\sinh x^1)\quad\text{for}\ \ x^1>0$$
$$r_-=\ln(-\sinh x^1)\quad\text{for}\ \ x^1<0. \tag{3.23}$$

The metric

$$ds^2=\sinh x^1\,[(dx^2)^2-(dx^0)^2]+\frac{l^2}{4}\coth^2 x^1\,(dx^1)^2 \tag{3.24}$$

then becomes

$$ds^2 = \pm\, e^{r_\pm}\,[(dx^2)^2 - (dx^0)^2] + \frac{l^2}{4}(dr_\pm)^2, \qquad (3.25)$$

where the equality holds with the plus sign for $x^1 > 0$ and with the minus sign for $x^1 < 0$. The expression in Eq. (3.25) is one of the standard forms for AdS_3 with x^0 (or x^2) playing the role of time and r_+ and x^2 (or r_- and x^0) of spatial coordinates. This can perhaps be more easily recognized if one defines a coordinate

$$z = \exp\left(-\frac{r_\pm}{2}\right), \qquad (3.26)$$

and obtains

$$ds^2 = \frac{\pm[(dx^2)^2 - (dx^0)^2] + l^2\,(dz)^2}{z^2}, \qquad (3.27)$$

which describes the half of (one of the two) AdS_3 with $z > 0$ (the half $z < 0$ being given by a definition of z with the opposite sign). The BTZ black hole is then obtained by the usual periodicity condition imposed on the coordinates [6,7].

It then follows that the metric (3.25) we have found simultaneously describes two copies of (half of) AdS_3 and $x^1 = 0$ again plays the role of a boundary at $r_+ = r_- = -\infty$ across which the signature of the metric flips. In fact this is the standard AdS horizon at $|z| = +\infty$, while the time-like infinity is at $z = 0$, and the scalar curvature,

$$R = -\frac{6}{l^2}, \qquad (3.28)$$

is a negative (regular) constant everywhere. Of course, the metrics (3.25) and (3.27) solve the field equations (3.12)–(3.14) provided ϕ is also a constant and $k = l^2$ [4].

We conclude this part by noting that the solution (2.26) places further restrictions on the propagating modes, since F can then be a function of either x^1_L or x^1_R, but not of both [see Eqs. (3.9) and (3.10)]. This selects out a subclass of solutions with only left- (or right-) movers along x^1 and both kinds of waves along x^0 and x^3.

IV. OTHER SOLUTIONS

Various other solutions to the field equations (3.12)–(3.14) can be found for a non-constant dilaton.

A. First example

Let us consider the metric

$$G^{(3)}_{ij} = \mathrm{diag}\,[-F\,,\, e^{2\lambda}\,,\, F], \qquad (4.1)$$

where now $\lambda = \lambda(x^2)$ and similarly for the dilaton $\phi = \phi(x^2)$. This metric can be obtained from the action (2.21) by applying again a nonlinear transformation of the form (3.1),(3.2),

$$\partial_- X^4 = e^{2\lambda}\,\partial_- x^1$$

$$\partial_+ X^2 = e^{\rho}\,[c_1\,\partial_+ x^0 + c_2^{-1}\,\partial_+ x^2]$$

$$\partial_- X^3 = e^{\rho}\,[c_1^{-1}\,\partial_- x^0 - c_2\,\partial_- x^2], \qquad (4.2)$$

where c_1 and c_2 are constants and $\rho = \rho(x^1)$. If we choose ρ such that

$$e^{-2\rho} = F, \qquad (4.3)$$

the sinh(x^1) term in Eq. (2.19) is canceled, and the axion potential $\mathbf{B}^{(3)}$ is constant in our model, so the axion field strength $\mathcal{H}$ is zero. Substituting this form for $\mathbf{G}^{(3)}$ into the field equations (with $k=1$) allows us to determine λ and leads to the invariant line element

$$ds^2 = -(dx^0)^2 + \coth^2(x^2)\,(dx^1)^2 + (dx^2)^2. \qquad (4.4)$$

The dilaton in this case is

$$\phi = C_\phi - \ln(\sinh(x^2)), \qquad (4.5)$$

where C_ϕ is an integration constant and the domain of validity of the solution is $x^2 > 0$ (which we call region II). This metric is ''asymptotically flat,'' in the sense that it converges to the Minkowski metric for $x^2 \to \infty$. The Ricci scalar is negative and diverges at the origin (i.e., for $x^2 = 0$),

$$R = -\frac{4}{\sinh^2(x^2)}. \qquad (4.6)$$

The change of variables [27]

$$t = x^0, \quad r = \coth(x^2), \quad \theta = x^1, \qquad (4.7)$$

brings the invariant line element into the form

$$ds^2 = -dt^2 + \frac{dr^2}{(r^2-1)^2} + r^2\,d\theta^2, \qquad (4.8)$$

which shows explicitly the cylindrical symmetry. The radial coordinate is understood to be $r > 1$, according to the above definition of region II, and the dilaton is written as

$$\phi_{II} = C_\phi + \frac{1}{2}\ln(r^2 - 1). \qquad (4.9)$$

In the form (4.8), the metric can also be extended to the region $0 \le r < 1$ (region I), where the Ricci scalar,

$$R = 4\,(1 - r^2), \qquad (4.10)$$

is positive and regular everywhere and the dilaton becomes

$$\phi_I = C_\phi + \frac{1}{2}\ln(1 - r^2). \qquad (4.11)$$

A plot of an angular sector of the ''lifted surface'' [26], adapted so as to include the origin of region I, is shown in Fig. 1. With this choice a new singularity appears at $r = 1$, where the surface has diverging slope and would extend to

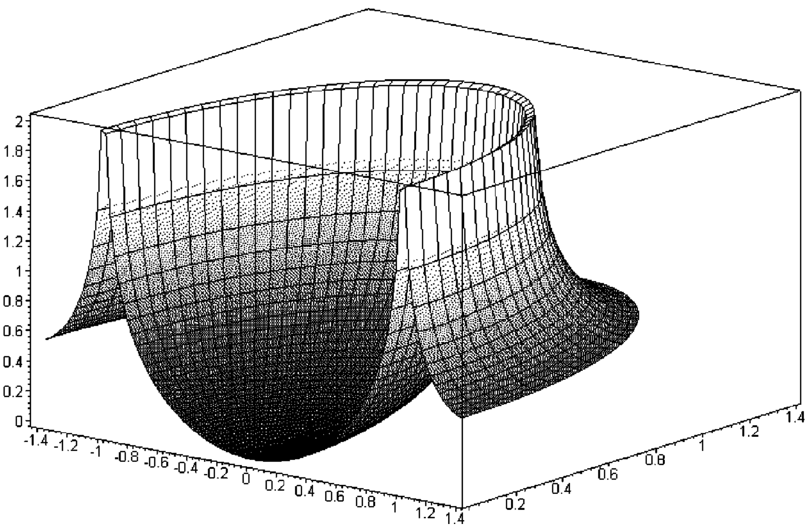

FIG. 1. The sector $0 \leqslant \theta \leqslant \pi$ of the lifted surface for the metric (4.8) extended to all positive values of r.

unlimited height (the plot is of course truncated along the vertical axis). This simply represents the fact that the asymptotically flat region ($r \to 1$) is at an infinite proper distance ($\sim \ln|r-1|$) both from the origin $r=0$ of region I and from the singularity $r \to \infty$ ($x^2=0$) of region II. In fact, the Ricci scalar actually vanishes along $r=1$ and the only real singularity is at $r \to \infty$.

The real singularity in region II is not accessible from region I. In particular, by solving the equation of radial null geodesics,

$$\frac{d^2 r}{d\tau^2} - \frac{2\,r}{r^2-1}\left(\frac{dr}{d\tau}\right)^2 = 0, \tag{4.12}$$

where τ is an affine parameter, one finds (near $r=1$ and with $\tau \gg 0$)

$$r \sim 1 \pm e^{-\tau}, \tag{4.13}$$

where the minus (plus) sign is for geodesics starting in region I (II). Such trajectories define the light cones in regions I and II and therefore show that the two regions are causally disconnected.

B. Second example

Let us now consider the metric

$$G_{ij}^{(3)} = \mathrm{diag}\,[-e^{-2\rho}, e^{2\rho}, e^{2\rho}], \tag{4.14}$$

where now $\rho = \rho(x^1)$ and $\phi = \phi(x^1)$. This metric results from the nonlinear coordinate transformation

$$\partial_- X^4 = e^{2\rho}\,\partial_- x^1$$

$$\partial_+ X^2 = F^{-1/2}[c_1\,e^{-\rho}\,\partial_+ x^0 + c_2^{-1}\,e^{\rho}\,\partial_+ x^2]$$

$$\partial_- X^3 = F^{-1/2}[c_1^{-1}\,e^{-\rho}\,\partial_- x^0 - c_2\,e^{\rho}\,\partial_- x^2]. \tag{4.15}$$

The transformation of coordinates used to obtain this form for $\mathbf{G}^{(3)}$ in our model insures that the axion potential is again constant. Eliminating x^1 in favor of ρ and solving the field equations again with $k=1$, we find for the invariant line element

$$ds^2 = -e^{-2\rho}(dx^0)^2 + \frac{4\,d\rho^2}{\sinh^2(\sqrt{2}\,\rho)} + e^{2\rho}(dx^2)^2, \tag{4.16}$$

and the dilaton is

$$\phi = C_\phi - \frac{1}{2}\ln(\sinh(\sqrt{2}\,\rho)), \tag{4.17}$$

with C_ϕ the usual integration constant.

The change of variables [27]

$$t = x^0, \quad r = e^{\rho}, \quad \theta = x^2, \tag{4.18}$$

which is well defined for $r>0$, gives the invariant line element the manifestly cylindrically symmetric form

$$ds^2 = -\frac{dt^2}{r^2} + \frac{dr^2}{r^2\,(r^{\sqrt{2}} - r^{-\sqrt{2}})^2} + r^2\,d\theta^2, \tag{4.19}$$

and the dilaton can now be written as

$$\phi = C_\phi - \frac{1}{2}\ln(r^{\sqrt{2}} - r^{-\sqrt{2}}). \tag{4.20}$$

The Ricci scalar is everywhere negative,

$$R = -(r^{\sqrt{2}} - r^{-\sqrt{2}})^2, \tag{4.21}$$

has essential singularities at both $r=0$ and $r \to \infty$ and vanishes along the circle $r=1$. This implies a similarity with the previous metric (4.8), namely one can define a region I for $0<r<1$ and a region II for $r>1$. The main difference is then that region I also contains a real singularity at $r=0$.

C. *T*-dual solutions

We conclude this section by noting that in both solutions above, there are two isometric coordinates, to wit t and θ. Therefore, one can generate new solutions by employing T duality [28,18]. In particular, we shall T dualize with respect to one coordinate at a time, and denote the fields of the dual solutions with a tilde. We also denote by B the only non-vanishing component of the axion potential, $B^{(3)}_{02}$, which is constant in all solutions considered.

For the solution in Sec. IV A, the non-vanishing component of the axion potential in the coordinate system (t,r,θ) is given by

$$B^{(3)}_{tr}=\frac{B}{1-r^2}. \tag{4.22}$$

On dualizing the metric (4.8) with respect to t then yields the non-diagonal line element

$$\tilde{d}s^2=-dt^2+\frac{1-B^2}{(r^2-1)^2}dr^2+\frac{B\,dt\,dr}{r^2-1}+r^2\,d\theta^2, \tag{4.23}$$

which solves the field equations with a vanishing axion potential, $\tilde{\mathbf{B}}^{(3)}=0$, and an unchanged dilaton field, $\tilde{\phi}=\phi$ [28].

Dualizing with respect to θ instead leaves the metric unaffected, as can be seen by switching to the new radial coordinate $R=r^{-1}$ after applying the dual relations [28], but yields $\tilde{\mathbf{B}}^{(3)}=0$ and a shifted dilaton field, $\tilde{\phi}=\phi+\ln(R)$.

The duals of the metric (4.19) and of the axion potential $B^{(3)}_{t\theta}=B$ of Sec. IV B with respect to t are given by

$$\tilde{d}s^2=-r^2\,dt^2+\frac{dr^2}{r^2\,(r^{\sqrt{2}}-r^{-\sqrt{2}})^2}-B\,r^2\,dt\,d\theta+r^2\,(1-B^2)\,d\theta^2 \tag{4.24}$$

and $\tilde{\mathbf{B}}^{(3)}=0$, with the dilaton $\tilde{\phi}=\phi+\ln(r)$. The metric above represents a rotating space-time, since the off-diagonal term $\tilde{G}^{(3)}_{t\theta}\neq 0$ (for $B\neq 0$).

Dualizing with respect to θ and defining $R=r^{-1}$ gives

$$\tilde{d}s^2=-R^2\,(1-B^2)^2\,dt^2+\frac{dR^2}{R^2\,(R^{\sqrt{2}}-R^{-\sqrt{2}})^2}-B\,R^2\,dt\,d\theta+R^2\,d\theta^2, \tag{4.25}$$

$\tilde{\mathbf{B}}^{(3)}=0$ and $\tilde{\phi}=\phi+\ln(R)$. Again this represents a rotating space-time.

In three out of four cases above the presence of a non-vanishing axion potential, although it corresponds to zero field strength, affects the metric field in a non-trivial manner. The axion potential is in fact always absorbed into the dual metric and dilaton fields and sometimes generates off-diagonal terms and rotation.

V. CONCLUSIONS

Starting from the six parameter group $ISO(2,1)$ we have, by using various types of compactification (gauge fixing, internal symmetries and coordinate identification), reduced the original action, which describes a spinless string moving on a curved six-dimensional background, to a string propagating on either a flat (Minkowski) background with a linear dilaton or on AdS space-time with a constant dilaton field. If the fields satisfying the equations obtained from the low energy effective string action are restricted to be functions of a single variable (in our case one of the boost parameters from the original Poincaré group), the fields are so tightly constrained that there are apparently only two possible solutions with a trivial dilaton.

The original goal of this work was to find a three-dimensional black hole other than the BTZ black hole by starting from a model of string propagation on a group manifold different from the $SL(2,\mathbb{R})$ manifold. This goal has not been realized, but the tactic has resulted in a relatively simple form for the compactified Lagrangian, allowing us to recover the space-time of AdS_3 (and BTZ) and to obtain solutions of the field equations we might not otherwise have been able to attain.

The fact that AdS_3 can be related to the (non-semisimple) three-dimensional Poincaré group might be surprising at first sight. However, one can consider the following general argument: The natural group of symmetry of AdS_3, that is the semisimple group $SL(2,\mathbb{R})$, is contained within $SL(2,C)$ which, in turn, is isomorphic to $SO(3,1)$. Moreover, the Lie algebra of the group $ISO(2,1)$ can be reached from the Lie algebra of the (semisimple) group $SO(3,1)$ by means of a transformation called *contraction* (see, e.g., Ref. [29]). One can therefore conclude that the sequence of operations we have performed reproduces the (local) effect of an *expansion* (roughly, the opposite of a contraction [29]) from the coset $ISO(2,1)/\mathbb{R}$ to $SL(2,\mathbb{R})$.

Other such formal constructions can be envisioned, and might turn out to be useful in the search for new solutions, as we have shown in Sec. IV. Whether or not the BTZ black hole is the only one in three-dimensional space-time remains an open question, and so, therefore, is the question of whether or not another exact three-dimensional black hole solution to string theory exists.

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