

Isolated horizons: Hamiltonian evolution and the first law

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(Received 24 May 2000; published 27 October 2000)

A framework was recently introduced to generalize black hole mechanics by replacing stationary event horizons with isolated horizons. That framework is significantly extended. The extension is nontrivial in that not only do the boundary conditions now allow the horizon to be distorted and rotating, but also the subsequent analysis is based on several new ingredients. Specifically, although the overall strategy is closely related to that in the previous work, the dynamical variables, the action principle and the Hamiltonian framework are all quite different. More importantly, in the nonrotating case, the first law is shown to arise as a *necessary and sufficient condition for the existence of a consistent Hamiltonian evolution*. Somewhat surprisingly, this consistency condition in turn leads to new predictions even for *static* black holes. To complement the previous work, the entire discussion is presented in terms of tetrads and associated (real) Lorentz connections.

PACS number(s): 04.70.Bw

I. INTRODUCTION

The zeroth and first laws of black hole mechanics refer to equilibrium situations and small departures therefrom. The standard treatments [1–5] restrict themselves to stationary space-times admitting event horizons and small perturbations from stationarity. While this simple idealization is a natural starting point, from physical considerations it seems overly restrictive. (See [6,7] and especially [8] for a detailed, critical discussion.) A framework which is tailored to more realistic physical situations was introduced in [6] and the zeroth and first laws were extended to it in [7–9]. This analysis generalizes black hole mechanics in two directions. First, the notion of event horizons is replaced by that of “isolated horizons.” While the former can only be defined retroactively, requiring access to the entire space-time history, the latter can be defined quasilocally. Second, the underlying space-time need not admit *any* Killing field; isolated horizons need not be Killing horizons. The static event horizons normally used in black hole mechanics [1–3,10] and the cosmological horizons in de Sitter space-times [11] are all special cases of isolated horizons. Furthermore, since space-times can now admit gravitational and matter radiation, there is a large class of other examples.

The framework developed in [8] for generalizing black hole mechanics was based on two restrictive assumptions. First, only undistorted, nonrotating horizons were considered. That is, the boundary conditions used in [8] implied that the *intrinsic* 2-metric of the horizon is spherically

symmetric¹ and that the imaginary part of the Weyl tensor component Ψ_2 —which encodes the angular momentum—vanishes. Second, while a rather general class of matter fields was allowed, it was assumed that the only relevant charges—i.e., hair—are the standard electromagnetic ones. The second assumption was weakened in [9,12] which allowed dilaton couplings and Yang-Mills fields. In this paper, we allow for distortion *and* more general matter sources. Distortion plays an important role in several astrophysical situations, e.g., in problems involving black holes immersed in external fields or surrounded by matter rings, and especially in the problem of black hole collisions. Post-Newtonian considerations suggest that, during black hole coalescence, individual horizons are distorted due to the Coulomb attraction even in the regime in which the black holes are sufficiently far from each other for the gravitational radiation falling into their horizons to be negligible. This phenomenon is also seen in numerical simulations.

The extensions [9,12] which incorporated dilatonic and Yang-Mills charges did not involve a significant generalization of the basic framework developed in [8]. The present paper, on the other hand, does. We begin with substantially weaker boundary conditions [formulated in terms of (real) tensor fields rather than the spinors used in [8]], and show that they imply constancy of surface gravity (and electrostatic potential) on the horizon. This property turns out to be *necessary and sufficient* for the usual action principle of tetrad gravity to continue to be valid in presence of isolated horizons. The action leads to a covariant phase space, con-

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[†]Email address: fairhurst@phys.psu.edu[‡]Email address: krishnan@phys.psu.edu¹However, it allowed space-times, such as the Robinson-Trautman solutions, in which there is no space-time Killing field whatsoever in any neighborhood of the isolated horizon [15].

structed from solutions to the field equations. Reference [8], by contrast, used the canonical phase space based on spinorial variables which is tailored for quantization but which contains technical complications that are unnecessary to the classical mechanics of isolated horizons. *Up to this point, distortion and rotation are both incorporated.* However, in the last step, i.e., in the discussion of the first law, we restrict ourselves to nonrotating horizons. (Rotation is incorporated in [13].)

To formulate the first law one must first define the energy E_Δ associated with any isolated horizon Δ . Since there can be radiation in the spacetime outside isolated horizons, the Arnowitt-Deser-Misner (ADM) energy E_{ADM} is not a good measure of E_Δ [7,8]. Instead, as in the work of Brown and York [14], the strategy is to define the energy of the horizon using a Hamiltonian framework. Experience with the phase space formulation of general relativity suggests that, in the presence of boundaries, the Hamiltonian H_t generating time translation along a suitable vector field t^a acquires surface terms. The idea is to *define E_Δ as the surface term at Δ in the Hamiltonian.* The key issue then is that of selecting the ‘‘appropriate’’ time translation t^a . Since one expects the volume term in the expression of H_t to be a linear combination of constraints and thus vanish when evaluated on solutions, the problem reduces to that of specifying the boundary values of t^a (or, equivalently, of the lapse and shift fields). The conditions at infinity are obvious and, in any case, will not affect the surface term at Δ . Thus, we need to focus only on the boundary value of t^a on Δ .

In the nonrotating case it is clear that, at the horizon, t^a should be proportional to the null normal to Δ . However, our boundary conditions do not select the null normal uniquely; there is a freedom to rescale² the normal by a constant (on Δ) which *can vary from one space-time to another.* This freedom is physically important because, among other things, the surface gravity is sensitive to it. Suppose we fix this freedom by tying the boundary value of t^a to fields on the horizon, e.g., by demanding that surface gravity be a *specific* function of the area and charges. We can then ask whether the time evolution generated by this t^a preserves the symplectic structure. It turns out that the answer is not always in the affirmative. On the horizon, the evolution vector field t^a and the electromagnetic potential A_a have to be tied to the horizon parameters appropriately. These conditions impose a constraint on the surface gravity $\kappa_{(t)}$ and the electric potential $\Phi_{(t)}$. Somewhat surprisingly, the constraint is precisely the first law $\delta E_\Delta^t = (\kappa_{(t)}/8\pi G) \delta a_\Delta + \Phi_{(t)} \delta Q_\Delta$. *Thus the evolution defined by t^a is Hamiltonian if and only if the first law*

²This is not surprising since this freedom exists already on Killing horizons. If the space-time is asymptotically flat and admits a static Killing field globally, one can eliminate this freedom by restricting oneself to that Killing field which is unit at infinity. However, this strategy is not available if there is radiation in the exterior region, or, as in the static solutions representing distorted black holes, the metric fails to be asymptotically flat.

*holds.*³ In this sense, the first law is even more fundamental than it is generally taken to be. Conceptually, this is perhaps the most striking feature of the present framework.

The requirement that the first law hold is not sufficient to fix t^a uniquely. Although every t^a must be a null normal to Δ , the rescaling of t^a from one spacetime to another can depend on the horizon parameters and the first law does not fully determine this parameter dependence. There is an infinite family of parameter-dependent vector fields t^a each defining a consistent Hamiltonian evolution and a horizon energy E_Δ^t . By contrast, at infinity all these vector fields must tend to a time translation of a universal flat metric, used in the construction of the phase space. Hence, there is only the familiar 3-parameter freedom in the definition of E_{ADM}^t , associated with the choice of a rest frame. Furthermore, in any one space-time, we can eliminate it by simply going to the rest frame and thus extracting the total mass M_{ADM} of the system. Although not necessary for mechanics, it is natural to ask if one can define a similar notion of mass of isolated horizons. The answer is in the affirmative in Einstein-Maxwell theory. Let us require that t^a should not only lead to a consistent Hamiltonian evolution but also agree, on static solutions, with the static Killing field which is *unit* at infinity. Then the horizon value of t^a is uniquely determined for *all* space-times in the phase space. There is a preferred notion of time translation, say t_0^a . We can set $M_\Delta = E_\Delta^{t_0}$ and regard M_Δ as the mass of an isolated horizon Δ . In the earlier work on nondistorted horizons [7–9,12], the discussion of the first law was carried out only in the context of these preferred evolution vector fields t_0^a . That derivation is more closely tied to the traditional discussion of the laws in the static context but is not necessary from the more general perspective of isolated horizons. Nonetheless, the availability of a canonical definition of the mass M_Δ is useful for other applications of this framework, e.g., to numerical relativity.

The paper is organized as follows. In Sec. II, we specify the boundary conditions defining general isolated horizons, allowing both distortion and rotation. We explain the role of these conditions, compare them with those used in [6–9,12] and work out their consequences, including the zeroth law. In Sec. III we introduce the Lagrangian framework based on tetrads and (real) Lorentz connections and, in Sec. IV, the covariant phase space. The first law is discussed in Sec. V. Up to this point, the focus is on Einstein-Maxwell theory (although incorporation of the dilaton is straightforward). In Sec. VI we extend the framework to incorporate Yang-Mills fields. The horizon mass is introduced in Sec. VII and subtleties associated with the dilaton and Yang-Mills fields are discussed. For the convenience of readers who may not be familiar with distorted black holes, Appendix A presents a variety of examples and, for convenience of readers who

³In the undistorted context, while this role of the first law was known to the authors of [8], its importance was not fully appreciated. The importance was noticed independently in [16] and used effectively in [12] to extract physical information on spherical black holes with Yang-Mills hair.

work in the Newman-Penrose formalism, Appendix B summarizes the structure of isolated horizons in that framework.

We have attempted to make this paper self-contained in terms of methodology and technical details. However, the motivation behind isolated horizons and certain properties of our Hamiltonian are the same as those discussed in detail in [8]. Since the inclusion of distortion does not add anything substantial to these issues, we have refrained from repeating that discussion in this paper.

II. STRUCTURE OF ISOLATED HORIZONS AND THE ZEROETH LAW

In this section, we will introduce the basic definitions of isolated horizons and analyze their immediate consequences. The definitions will become progressively stronger. However, even the strongest boundary conditions are significantly weaker than requiring the horizon to be a Killing horizon for a local Killing vector field. By proceeding in steps, we will be able to keep track of the precise assumptions that are needed to obtain various results. Also, the availability of a hierarchy of definitions will be useful in other applications—such as numerical relativity and quantum gravity—which lie beyond the scope of the present paper.

Let us begin by introducing some notation. Throughout this paper we assume that all manifolds and fields under consideration are smooth. Let \mathcal{M} be a 4-manifold equipped with a metric g_{ab} of signature $(-, +, +, +)$. Let Δ be a null hypersurface of (\mathcal{M}, g_{ab}) . A *future directed* null normal to Δ will be denoted by l . (In this paper, the term “null normal” will always refer to a future directed null normal.) Let $q_{ab} \triangleq g_{ab}$ be the degenerate intrinsic metric on Δ .⁴ A tensor q^{ab} on Δ will be called an “inverse” of q_{ab} if it satisfies $q^{ab}q_{ac}q_{bd} \triangleq q_{cd}$. Thus q^{ab} is unique only up to addition of terms of the form $l^{(a}V^{b)}$ for some vector field V tangential to Δ . The expansion $\theta_{(l)}$ of a specific null normal l is defined by $\theta_{(l)} = q^{ab}\nabla_a l_b$, where ∇_a is the derivative operator compatible with g_{ab} . It is straightforward to check that $\theta_{(l)}$ is independent of the choice of q^{ab} . With this structure at hand, we can now introduce our first definition.

A. Nonexpanding horizons

Definition 1. A three-dimensional submanifold Δ of a space-time (\mathcal{M}, g_{ab}) is said to be a *nonexpanding horizon* if it satisfies the following conditions:

- (i) Δ is topologically $S^2 \times \mathbb{R}$ and null.
- (ii) The expansion $\theta_{(l)}$ of l vanishes on Δ for any null normal l .
- (iii) All equations of motion hold at Δ and the stress-energy tensor T_{ab} of matter fields at Δ is such that $-T_b^a l^b$ is future directed and causal for any future directed null normal l .

Note that if conditions (ii) and (iii) hold for one null normal l , they hold for all.

The role of these conditions is as follows. The restriction on topology is geared to the structure of horizons that result from gravitational collapse. However, it can be weakened. One can retain the requirement that the horizon have compact cross sections but replace S^2 by a manifold with higher genus. Our main analysis will extend to this case in a straightforward manner. More generally, we can allow Δ to have noncompact cross sections, as for example in the case of certain acceleration horizons. The results presented in this section, including our derivation of the zeroth law, will go through. However, since such horizons extend to infinity, our Hamiltonian framework will have to be modified appropriately. Finally, one could envisage incorporation of Newman-Unti-Tamburino (NUT) charge. This extension would be even more subtle because, if all fields are smooth, Δ would be topologically S^3 and l would provide a Hopf fibration. In this case, Δ would not admit *any* cross sections which are everywhere transverse to l . This extension will be discussed elsewhere.

Requirement (iii) is analogous to the dynamical conditions one imposes at infinity. While at infinity one requires that the metric (and other fields) approach a specific solution to the field equations (the “classical vacuum”), at the horizon we only ask that the field equations be satisfied. The energy condition involved is very weak; it is implied by the (much stronger) dominant energy condition that is typically used. Thus, the first and the last conditions are quite tame.

The key condition is (ii). It implies, in particular, that *the horizon area is constant “in time”* and thus incorporates the idea that the horizon is isolated without having to assume the existence of a Killing field. We will denote the area by a_Δ and refer to R_Δ defined by $a_\Delta = 4\pi R_\Delta^2$ as the *horizon radius*. All these conditions are satisfied on any Killing horizon (with 2-sphere sections) if gravity is coupled to physically reasonable matter (including perfect fluids, Klein-Gordon fields, Maxwell fields possibly with dilatonic coupling, Yang-Mills fields).

Although the conditions in the definition are quite weak, they have surprisingly rich consequences. We will now discuss them in detail. In some of this analysis it is convenient to use a null tetrad and the associated Newman-Penrose quantities (see Appendix B and references therein). Given a specific null normal field l^a to Δ , we can introduce a complex null vector field m^a tangential to Δ and a real, future directed null field n^a transverse to Δ so that the following relations hold: $n \cdot l = -1$, $m \cdot \bar{m} = 1$ and all other scalar products vanish. The quadruplet (l, n, m, \bar{m}) constitutes a null tetrad. There is of course an infinite number of null tetrads compatible with a given l , related to one another by restricted Lorentz rotations. Our conclusions will not be sensitive to this gauge freedom.

(a) *Properties of l .* Since l^a is a null normal to Δ , it is automatically twist free and geodesic. We will denote the acceleration of l^a by $\kappa_{(l)}$:

$$l^a \nabla_a l^b \triangleq \kappa_{(l)} l^b. \quad (2.1)$$

⁴Equalities which hold only at Δ will be denoted by “ \triangleq ” and the pullback of a covariant index will be denoted by an arrow under that index: e.g., ω_a will denote the pullback of the 1-form ω_a to Δ .

Note that the acceleration is a property not of the horizon Δ itself, but of a specific null normal to it: if we replace l by $l' = fl$, then the acceleration changes via $\kappa_{(l')} = f\kappa_{(l)} + \mathcal{L}_l f$.

Since the twist of l vanishes, the Raychaudhuri equation implies

$$\mathcal{L}_l \theta_{(l)} \triangleq \kappa_{(l)} \theta_{(l)} - \frac{1}{2} \theta_{(l)}^2 - \sigma \bar{\sigma} - R_{ab} l^a l^b$$

where $\sigma = m^a m^b \nabla_a l_b$ is the shear of l in the given null tetrad. Since $\theta_{(l)}$ vanishes on Δ , we conclude that $\sigma \bar{\sigma} + R_{ab} l^a l^b \triangleq 0$. The condition on the stress-energy tensor ensures that $R_{ab} l^a l^b = 8\pi G T_{ab} l^a l^b$ is non-negative on Δ . Hence, we conclude

$$\sigma \triangleq 0 \quad \text{and} \quad R_{ab} l^a l^b \triangleq 0. \quad (2.2)$$

Thus, in particular, every null normal l is free of expansion, twist and shear.

(b) *Conditions on the Ricci tensor.* The second equation in Eqs. (2.2) implies that the vector $-R^a{}_b l^b$ is tangential to Δ . The energy condition and the field equations imply that this vector must also be future causal. This means that $R^a{}_b l^b$ must be proportional to l^a and, hence, $R_{ab} l^a l^b \triangleq 0$. In the Newman-Penrose formalism this condition translates to

$$\Phi_{00} = \frac{1}{2} R_{ab} l^a l^b \triangleq 0 \quad \text{and} \quad \Phi_{01} = \bar{\Phi}_{10} = \frac{1}{2} R_{ab} l^a m^b \triangleq 0. \quad (2.3)$$

Since this statement is equivalent to $R_{ab} l^b \triangleq 0$, it is gauge invariant: i.e., it does not depend upon the specific choice of null normal l and m .

(c) *A natural connection 1-form on Δ .* Since l is expansion, shear and twist free, there exists a one-form ω_a intrinsic to Δ such that

$$\nabla_a l^b \triangleq \omega_a l^b \quad (2.4)$$

which in turn implies

$$\mathcal{L}_l q_{ab} \triangleq 2 \nabla_a l_b \triangleq 0.$$

Thus, every null normal l is a ‘‘Killing field’’ of the degenerate metric on Δ . Furthermore, we will now show that

$${}^2\epsilon := im \wedge \bar{m} \quad (2.5)$$

is also invariantly defined. Since $\mathcal{L}_l q_{ab} = 0$, the space \mathcal{S} of integral curves of l is naturally equipped with a nondegenerate metric q_{ab} (so that q_{ab} on Δ is the pullback of q_{ab}). Denote by $\underline{\epsilon}_{ab}$ the unique (up to orientation) unit alternating tensor on (\mathcal{S}, q_{ab}) . Here ${}^2\epsilon$ is the pullback to Δ of $\underline{\epsilon}$. Although l is a ‘‘Killing field’’ of the intrinsic horizon geometry, the space-time metric g_{ab} need *not* admit a Killing field in any neighborhood of Δ . Robinson-Trautman metrics [15] and the Kastor-Traschen solutions [17] provide explicit examples of this type.

The 1-form ω will play an important role throughout this paper. It has an interesting geometrical interpretation. We can regard ω as a connection on the line bundle $T\Delta^\perp$ over Δ whose fibers are the one-dimensional null normals to Δ . Under the rescalings $l \mapsto \tilde{l} = fl$, of the null normal l , it transforms via

$$\omega_a \mapsto \tilde{\omega}_a = \omega_a + \nabla_a \ln f. \quad (2.6)$$

(d) *Induced connection on Δ .* Each metric submanifold M of \mathcal{M} admits a natural connection—one which is torsion free and compatible with the induced metric on M . This connection is also canonically induced by the space-time connection ∇ . However, since the induced metric q_{ab} on Δ is degenerate, there exist infinitely many connections compatible with it. A general null submanifold inherits a unique (torsion-free) derivative operator \mathcal{D} from ∇ if and only if its null normal l satisfies $\nabla_a l_b = 0$. Therefore, the conditions imposed in definition 1 guarantee that every nonexpanding horizon has a unique intrinsic derivative operator \mathcal{D} . The action of \mathcal{D} on a vector field X^a tangent to Δ and on a 1-form η_a intrinsic to Δ is given by

$$\mathcal{D}_a X^b \triangleq \nabla_a \tilde{X}^b \quad \text{and} \quad \mathcal{D}_a \eta_b \triangleq \nabla_a \tilde{\eta}_b$$

where \tilde{X}^b and $\tilde{\eta}_b$ are arbitrary extensions of X^b and η_b to the full space-time manifold \mathcal{M} . It is easy to show that \mathcal{D} is independent of the extensions.

The 1-form ω captures only part of the information in \mathcal{D} . The full connection \mathcal{D} on Δ plays an important role in extracting physics in the strong field regime near Δ [18]. However, it is not essential to the discussion of isolated horizon mechanics.

(e) *Conditions on the Weyl tensor.* Let us begin with the definition of the Riemann tensor, $[\nabla_a \nabla_b - \nabla_b \nabla_a] X^c = -R_{abd}{}^c X^d$. If we set $X^c = l^c$ and pull back the indices a and b , then using Eq. (2.4), we obtain

$$[\mathcal{D}_a \omega_b - \mathcal{D}_b \omega_a] l^c \triangleq -R_{abd}{}^c l^d \triangleq -C_{abd}{}^c l^d \quad (2.7)$$

where C_{abcd} is the Weyl tensor. The last equality follows from $R_{ab} l^b \triangleq 0$. Thus, if v is any 1-form on Δ satisfying $v \cdot l \triangleq 0$, contracting the previous equation with v_c we get

$$C_{abd}{}^c v_c l^d \triangleq 0.$$

Let us choose a null tetrad and set v to be m or \bar{m} . Then

$$\begin{aligned} \Psi_0 &:= C_{abcd} l^a m^b l^c m^d \triangleq 0, \\ \Psi_1 &:= C_{abcd} l^a m^b l^c \bar{m}^d \\ &= C_{abcd} l^a m^b \bar{m}^c m^d \triangleq 0, \end{aligned} \quad (2.8)$$

where we have used the trace-free property of the Weyl tensor in the second equation. It is also clear that Eqs. (2.8) are independent of which null normal l and vector fields m and \bar{m} we choose to construct the null tetrad; Eqs. (2.8) are gauge invariant.

(f) *Curvature of ω .* Let us contract Eq. (2.7) with n_c and use $l^a n_a = -1$. Then we have

$$2\mathcal{D}_{[a}\omega_{b]}\triangleq C_{abcd}l^d n_c \triangleq C_{abcd}l^c n^d. \quad (2.9)$$

Expanding the Weyl tensor in terms of the Ψ 's, one obtains

$$\begin{aligned} C_{abcd}l^c n^d \triangleq & 4(\operatorname{Re}[\Psi_2])n_{[a}l_{b]} + 2\Psi_3l_{[a}m_{b]} + 2\bar{\Psi}_3l_{[a}\bar{m}_{b]} \\ & - 2\bar{\Psi}_1n_{[a}m_{b]} - 2\Psi_1n_{[a}\bar{m}_{b]} \\ & + 4i(\operatorname{Im}[\Psi_2])m_{[a}\bar{m}_{b]} \end{aligned} \quad (2.10)$$

where

$$\Psi_2 \triangleq C_{abcd}l^a m^b \bar{m}^c n^d, \quad \Psi_3 \triangleq C_{abcd}l^a n^b \bar{m}^c n^d. \quad (2.11)$$

Substituting this expression into Eq. (2.9), pulling back on the two free indices and taking into account Eqs. (2.8) and (2.5), we obtain

$$d\omega \triangleq 2(\operatorname{Im}[\Psi_2])^2 \epsilon. \quad (2.12)$$

This relation will play an important role in what follows. Note that, because Ψ_0 and Ψ_1 vanish on Δ , Ψ_2 is gauge invariant.

Remark. It is interesting to compare the structure of Δ with that of null infinity \mathcal{I} (in the usual conformal gauge, in which the conformal factor is chosen such that the null normal to \mathcal{I} is divergence free). Both are null surfaces and can be regarded as ‘‘line bundles’’ over a base space \mathcal{S} of the integral curves of null normals. (For brevity, we will ignore a caveat concerning completeness of fibers.) The null normals are Killing fields of the intrinsic degenerate metric so that this metric is the pullback to the 3-surface of a positive-definite metric on \mathcal{S} . In both cases, the space-time connection induces an intrinsic connection on the 3-surface [19]. These connections capture physically important information. However, there are a number of differences as well. Since \mathcal{I} is constructed by a conformal completion, the conformal freedom permeates all geometric structures at \mathcal{I} . In particular, given a physical space-time, the intrinsic metric and the derivative operator are known only up to conformal transformations. On the other hand, since \mathcal{I} is at infinity, in some ways its structure is both more rigid and simpler. First, without loss of generality, we can assume that the metric on \mathcal{S} is a 2-sphere metric; the issue of distortion is physically irrelevant at \mathcal{I} . Second, the Weyl tensor vanishes identically at \mathcal{I} and the curvature of the intrinsic connection captures non-trivial information about the next order space-time curvature. By contrast, at Δ only four components of the Weyl curvature vanish and four other components are coded in the curvature of the intrinsic connection on Δ . In spite of these differences, one can carry over some techniques from null infinity to extract physical information about isolated horizons. In particular, using the analogues of techniques which have been successful at \mathcal{I} , one can introduce preferred cross sections of and Bondi-type expansions near Δ [18].

This concludes our analysis of the consequences of the boundary conditions defining nonexpanding horizons. Note that, even though l is a Killing field for the intrinsic, degenerate metric q_{ab} on Δ , it is not an infinitesimal symmetry for other geometrical fields such as the intrinsic connection \mathcal{D} or components of the curvature tensor. In the next subsection, we will make the structure more rigid by suitably restricting the choice of l .

B. Weakly isolated horizons

The time independence of the intrinsic metric q_{ab} captures the idea that Δ is isolated in a suitable sense. While this condition has rich consequences, the resulting structure is still not sufficient for physical applications. In particular, since l can be rescaled by an arbitrary positive definite function, the acceleration $\kappa_{(l)}$ is not necessarily constant on Δ . Therefore, we need to impose additional restrictions on the physical fields at Δ to establish the zeroth law. Since l is already a symmetry of the intrinsic metric, it is natural to require it also be a symmetry of the ‘‘extrinsic curvature.’’ However, the standard definition of the extrinsic curvature is not applicable to null surfaces. Nonetheless, given a null normal l , we can construct a tensor field $K_a{}^b := \mathcal{D}_a l^b$, defined intrinsically on Δ , which can be thought of as the analogue of the extrinsic curvature.⁵ Indeed, on a metric submanifold, if we replace l by the *unit* normal, $K_a{}^b$ is precisely the extrinsic curvature. It is then natural to demand that, on an isolated horizon, $K_a{}^b$ also be time independent: $\mathcal{L}_l K_a{}^b \triangleq 0$. As a consequence of Eq. (2.4), this is equivalent to imposing $\mathcal{L}_l \omega \triangleq 0$.

Let us examine the above condition. As we will show at the end of this section, given a nonexpanding horizon we can always find a null normal l^a which satisfies $\mathcal{L}_l \omega \triangleq 0$. The behavior of this condition under rescalings of l is complicated by the fact that ω itself depends upon the choice of null normal [see Eq. (2.6)]. However, under a constant rescaling $l \mapsto \tilde{l} = cl$, the connection 1-form ω is unchanged. Therefore, if l satisfies the condition $\mathcal{L}_l \omega \triangleq 0$, so does any \tilde{l} related to l by constant rescaling. This suggests we introduce an equivalence relation: *Two future-directed null normals l and \tilde{l} belong to the same equivalence class $[l]$ if and only if $\tilde{l} = cl$ for some positive constant c .*

The above considerations lead us to the following definition:

Definition 2. A weakly isolated horizon $(\Delta, [l])$ consists of a nonexpanding horizon Δ , equipped with an equivalence

⁵We are grateful to Thibault Damour for pointing out that $K_a{}^b$ is called the Weingarten map and is analogous to extrinsic curvature. This comment suggested the above motivation for our condition on the connection 1-form ω . For an alternate, and in a sense weaker, condition see the remark at the end of Sec. II D. From the viewpoint of intrinsic structures on Δ discussed in Sec. II A, it is perhaps more natural to ask that l be a symmetry of the *full* intrinsic connection \mathcal{D} (see Sec. II D and [18]). However, this stronger condition is not necessary for the laws of mechanics discussed here.

class $[l]$ of null normals to it satisfying

$$\mathcal{L}_l \omega \triangleq 0 \quad \text{for all } l \in [l]. \quad (2.13)$$

As pointed out above, if this last equation holds for one l , it holds for all l in $[l]$.

A Killing horizon (with 2-sphere cross sections) is automatically a weakly isolated horizon (provided the matter fields satisfy the energy condition of definition 1). Given a nonexpanding horizon Δ , one can always find an equivalence class $[l]$ of null normals such that $(\Delta, [l])$ is a weakly isolated horizon. However, condition (2.13) does not by itself single out the appropriate equivalence class $[l]$. As indicated in Sec. IID, one *can* further strengthen the boundary conditions and provide a specific prescription to select the equivalence class $[l]$ uniquely. However, for mechanics of isolated horizons, these extra steps are unnecessary. In particular, our analysis will not depend on how the equivalence class $[l]$ is chosen. The adverb “weakly” in definition 2 emphasizes this point.

Condition (2.13) has several consequences which are relevant for this paper.

(a) *Surface gravity.* In the case of Killing horizons Δ_K , surface gravity is defined as the acceleration of the Killing field ξ normal to Δ_K . However, if Δ_K is a Killing horizon for ξ , it is also a Killing horizon for $c\xi$ for any positive constant c . Hence, surface gravity is not an intrinsic property of Δ_K , but depends also on the choice of a specific Killing field ξ . (Of course the result that the surface gravity is constant on Δ_K is insensitive to this rescaling freedom.) In asymptotically flat space-times admitting global Killing fields, this ambiguity is generally resolved by selecting a preferred normalization in terms of the structure at infinity. For example, in the static case, one requires the Killing field ξ to be unit at infinity. However, in absence of a *global* Killing field or asymptotic flatness, this strategy does not work and we simply have to accept the constant rescaling freedom in the definition of surface gravity. In the context of isolated horizons, then, it is natural to keep this freedom.

A weakly isolated horizon is similarly equipped with a preferred family $[l]$ of null normals, unique up to constant rescalings. Therefore, it is natural to interpret $\kappa_{(l)}$ as the surface gravity associated with l . Under the permissible rescalings $l \rightarrow \tilde{l} = cl$, the surface gravity transforms via $\kappa_{(\tilde{l})} = c\kappa_{(l)}$. Thus, while ω is insensitive to the rescaling freedom in $[l]$, $\kappa_{(l)}$ captures this freedom fully. One can, if necessary, select a specific l in $[l]$ by demanding that $\kappa_{(l)}$ be a specific function of the horizon parameters which are insensitive to this freedom, e.g., by setting $\kappa_{(l)} = 1/2R_\Delta$, where R_Δ is the horizon radius (related to the horizon area a_Δ via $a_\Delta = 4\pi R_\Delta^2$).

(b) *Zeroth law.* The boundary conditions of definition 2 allow us to define surface gravity $\kappa_{(l)}$ of a weakly isolated horizon $(\Delta, [l])$. We will now show that the surface gravity is constant on Δ . In other words, the zeroth law holds for weakly isolated horizons.

Recall from Eq. (2.12) that on a nonexpanding horizon, $d\omega \triangleq 2 \text{Im}[\Psi_2]^2 \epsilon$ for any choice of null normal l . Since ${}^2\epsilon$ is

the pullback to Δ of the alternating tensor ϵ on the space Σ (of orbits of l), clearly $l \cdot {}^2\epsilon \triangleq 0$. Therefore,

$$l \cdot d\omega \triangleq 0$$

for every null normal l . In particular, on a weakly isolated horizon this equation holds for any $l \in [l]$. Moreover, each of these restricted null normals also satisfies

$$0 \triangleq \mathcal{L}_l \omega \triangleq d(l \cdot \omega) + l \cdot d\omega.$$

Hence, we conclude

$$d(l \cdot \omega) \triangleq d(\kappa_{(l)}) \triangleq 0,$$

where we have used the definition (2.1) of $\kappa_{(l)}$. Thus, surface gravity is constant on Δ .

Although this proof of the zeroth law appears extremely simple, the argument is not as trivial as it might first appear since we have used a number of consequences of the boundary conditions derived in Sec. II A. In contrast to earlier derivations [2,10] we do not require the presence of a Killing field even in a neighborhood of Δ . Therefore the proof applies also to space-times such as the Robinson-Trautman solutions [15] which do not admit a Killing field. Also, Δ need not be “complete”—it may be of finite affine length with respect to any l —and may not admit the analogue of a “bifurcate surface” on which the Killing field vanishes. Finally, the field equations are used rather weakly; we only need to assume that $-(R^a{}_b - \frac{1}{2}R\delta^a{}_b)l^b$ is a future directed causal vector.

Surface gravity does not have a definite value on a weakly isolated horizon. The value of $\kappa_{(l)}$ depends upon the choice of null normal $l \in [l]$. Since all the normals l to Δ are future directed, the rescaling constant c is necessarily positive. Therefore, if the surface gravity is nonzero (respectively, zero) with respect to one l , it is nonzero (respectively, zero) with respect to any other $l \in [l]$. This rescaling freedom is the same as the one discussed above in the context of Killing horizons.

We will conclude this subsection with three remarks.

(i) *Freedom in the choice of $[l]$.* Given a nonexpanding horizon Δ , it is natural to ask if one can always select an equivalence class $[l]$ of null normals such that $(\Delta, [l])$ is a weakly isolated horizon. As indicated earlier in this section, the answer is in the affirmative and, furthermore, there is a considerable freedom in the choice of $[l]$. Let us examine this issue in some detail.

Since $l \cdot d\omega \triangleq 0$ for any null normal l to a nonexpanding horizon, it follows that a null normal l satisfies $\mathcal{L}_l \omega \triangleq 0$ if and only if $d\kappa_{(l)} \triangleq 0$. Thus, to find a family $[l]$ required in the definition of weakly isolated horizons, it is necessary and sufficient to find a null normal l for which the surface gravity is constant. On a nonexpanding horizon, the surface gravity transforms as follows:

$$\text{If } l \rightarrow \tilde{l} \triangleq fl, \quad \text{then } \kappa_{(\tilde{l})} \triangleq f\kappa_{(l)} + \mathcal{L}_l f.$$

Hence, starting with any l , we can simply solve for f by requiring that $\kappa_{(\tilde{l})}$ be constant on Δ . The solution is not

unique. If κ_l is constant, given any nonzero function g satisfying $\mathcal{L}_l g \triangleq 0$ and a constant $\tilde{\kappa}$, let us set

$$f \triangleq g e^{-\kappa_{(l)} v} + \frac{\tilde{\kappa}}{\kappa_{(l)}}$$

where v satisfies $\mathcal{L}_l v \triangleq 1$. Then, we obtain an $\tilde{l} \in [l]$ for which $\kappa_{(\tilde{l})} \triangleq \tilde{\kappa}$. This is the only freedom if both κ_l and $\kappa_{(\tilde{l})}$ are to be constant. Thus, each nonexpanding horizon gives rise to an infinite family of weakly isolated horizons. Put differently, although one can easily obtain weakly isolated horizons from nonexpanding ones by choosing appropriate null normals $[l]$, a specific weakly isolated horizon carries much more information than the nonexpanding horizon it comes from. At the end of Sec. IID, we will indicate how one can further strengthen the boundary conditions to give a prescription for selecting a specific $[l]$. However, the analysis of this paper does *not* depend on how this selection is made.

(ii) *How does definition 2 compare with that used in the undistorted, nonrotating case?* As one would expect, the definition given in [6–8] is significantly stronger. Furthermore, it was tied to a foliation from the beginning. More precisely, it assumed that there exists a foliation to which ω is normal, with $\omega = -\kappa n$, and it required that the expansion $\text{Re}[\mu]$ of the null normal n to the leaves of the foliation be constant on Δ (see Appendix B for definitions of the NP spin coefficients). Although it was shown that the foliation is unique if it exists, the heavy reliance on the foliation right from the beginning made that definition less elegant and its invariant content less transparent. Also, since we now allow $d\omega$ to be nonzero and impose no conditions on $\text{Re}[\mu]$, we can now incorporate rotation and distortion.

(iii) *Alternate boundary conditions.* In the definition of isolated horizons, we required $\mathcal{L}_l \omega \triangleq 0$, which in particular implies $\mathcal{L}_l \kappa_{(l)} \equiv \mathcal{L}_l(l \cdot \omega) \triangleq 0$. Thus, the definition itself assured us that $\kappa_{(l)}$ is “time independent” and to prove the zeroth law we had to show that it is also independent of “angles.” Could we have used another definition in which the “time dependence” of $\kappa_{(l)}$ was not explicitly required but followed from other conditions? The answer is in the affirmative: In place of $\mathcal{L}_l \omega \triangleq 0$, we could have required that Δ admit a foliation on which the expansion $\text{Re}[\mu]$ of n and the Newman-Penrose spin coefficient π which carries the angular momentum information are “time independent” [13]. Then the field equations would have *implied* that $\kappa_{(l)}$ is time independent. Furthermore, all results of this paper go through (and were in fact first obtained) with these modified boundary conditions. Note however that the new condition is neither weaker nor stronger than the one we used. Both require that π be time independent. In addition, the present definition of isolated horizons requires that $\kappa_{(l)}$ be time independent while the alternative definition would have required, instead, that $\text{Re}[\mu]$ be time independent. In this paper, we chose the present definition because it can be stated without reference to a foliation.

C. Electromagnetic field

We shall now describe the form of the electromagnetic field at an isolated horizon and introduce a partial gauge fixing at the horizon which will allow us to introduce the notion of an electric potential. In the next three sections—where we discuss the action, phase space, and first law—we will assume that the only matter fields present *at* the isolated horizon are Maxwell fields. However, as our discussion will make clear, this restriction is made primarily for simplicity. The overall framework is rather general and can accommodate matter for which there exists a well-defined action principle and a (covariant) Hamiltonian framework. In particular, in Sec. VI, we describe how to extend the formalism to include Yang-Mills fields.

The isolated horizon boundary conditions restrict matter primarily through conditions on the stress-energy tensor T_{ab} . Let us begin with $T_{ab} l^a l^b \triangleq 0$, a direct consequence of the boundary conditions and Raychaudhuri equation. (This restriction arises due to the fact that Δ is a nonexpanding horizon; the subsequent stronger boundary conditions do not further constrain \mathbf{F} .) Although this condition is weak, it turns out to have interesting consequences on the form of the electromagnetic field, \mathbf{F} , at Δ . The stress-energy tensor for electromagnetism is given in terms of the field strength \mathbf{F} as

$$\mathbf{T}_{ab} = \frac{1}{4\pi} \left[\mathbf{F}_{ac} \mathbf{F}_b{}^c - \frac{1}{4} g_{ab} \mathbf{F}_{cd} \mathbf{F}^{cd} \right]. \quad (2.14)$$

Let us contract this expression with $l^a l^b$ and examine the consequences for \mathbf{F} . This gives

$$0 \triangleq \mathbf{T}_{ab} l^a l^b \triangleq |l^a m^b \mathbf{F}_{ab}|^2, \quad (2.15)$$

where, to obtain the last expression, we have used the anti-symmetry of \mathbf{F} and the fact that the metric at the horizon can be expressed in terms of a null tetrad as $g_{ab} = -2l_{(a} n_{b)} + 2m_{(a} \bar{m}_{b)}$. An immediate consequence of Eq. (2.15) is that

$$\underline{l^a \mathbf{F}_{ab} \triangleq 0}. \quad (2.16)$$

In order to obtain a similar expression for ${}^* \mathbf{F}$ recall that the stress energy tensor can be rewritten as $\mathbf{T}_{ab} = (-1/4\pi) [{}^* \mathbf{F}_{ac} {}^* \mathbf{F}_b{}^c - \frac{1}{4} g_{ab} {}^* \mathbf{F}_{cd} {}^* \mathbf{F}^{cd}]$. Applying the same argument which led to Eq. (2.16), we obtain a similar restriction on ${}^* \mathbf{F}$, namely

$$\underline{l^a {}^* \mathbf{F}_{ab} \triangleq 0}. \quad (2.17)$$

These two restrictions tell us there is no flux of electromagnetic radiation across the horizon.

It is straightforward to show that Eqs. (2.16),(2.17) and the form of the metric at Δ place further restrictions on \mathbf{T}_{ab} :

$$\mathbf{T}_{ab} l^a m^b \triangleq 0, \quad \mathbf{T}_{ab} l^a \bar{m}^b \triangleq 0,$$

$$\mathbf{T}_{ab} m^a m^b \triangleq 0, \quad \mathbf{T}_{ab} \bar{m}^a \bar{m}^b \triangleq 0.$$

The first two equations contain no new information since we already knew from general arguments [see Eq. (2.3)] that $\Phi_{10} \triangleq 0$ and $\Phi_{01} \triangleq 0$. However, the last two equations do

place further restrictions on the stress-energy tensor. Since the equations of motion are enforced at the boundary, we see immediately that they are equivalent to further restricting the Ricci tensor at the horizon by requiring:

$$\Phi_{02} := \frac{1}{2} R_{ab} m^a m^b \triangleq 0, \quad \Phi_{20} := \frac{1}{2} R_{ab} \bar{m}^a \bar{m}^b \triangleq 0. \quad (2.18)$$

This result need not hold for general matter fields; it relies on the properties of the electromagnetic stress-energy tensor (2.14).

The next task is to define the electric and magnetic charges of the horizon. Since the horizon is an inner boundary of spacetime, the normal to a 2-sphere cross section of the horizon will naturally be inward pointing. Bearing this in mind, we define the electric and magnetic charges of the horizon as

$$Q_\Delta := -\frac{1}{4\pi} \oint_{S_\Delta} \star \mathbf{F}, \quad P_\Delta := -\frac{1}{4\pi} \oint_{S_\Delta} \mathbf{F}. \quad (2.19)$$

For these definitions to be meaningful, the values of Q_Δ and P_Δ should be independent of the cross section of the horizon S_Δ . We will now show that Δ , being a nonexpanding horizon, guarantees this is the case. Let us first evaluate

$$\mathcal{L}_l \mathbf{F} \triangleq l \cdot d\mathbf{F} + d(l \cdot \mathbf{F}).$$

The first term on the right hand side vanishes due to Maxwell's equations on Δ , while the second term is zero due to the previous restriction on \mathbf{F} , Eq. (2.16). Therefore we conclude that \mathbf{F} is Lie dragged by l . An identical argument for $\star \mathbf{F}$ leads to the analogous conclusion. Therefore we obtain

$$\mathcal{L}_l \mathbf{F} \triangleq 0 \quad \text{and} \quad \mathcal{L}_l \star \mathbf{F} \triangleq 0. \quad (2.20)$$

This result, along with Eqs. (2.16) and (2.17), guarantees that Q_Δ and P_Δ are independent of the choice of cross section S_Δ of the horizon. Note that this result was obtained using only the boundary conditions; equations of motion in the bulk are not needed.

Finally, let us examine the remaining freedom in the electromagnetic field. The boundary conditions do not restrict $\mathbf{F}_{ab} n^a m^b$ and $\star \mathbf{F}_{ab} n^a m^b$ at all. These components describe the electromagnetic radiation flowing along the horizon. Therefore, isolated horizon boundary conditions allow electromagnetic radiation arbitrarily close to—and even at—the horizon, provided none crosses it.

So far we have confined ourselves to the field strengths \mathbf{F} and $\star \mathbf{F}$. However, in the action principle and the Hamiltonian framework we have to consider also the Maxwell potential \mathbf{A} . Now, if the magnetic charge is nonzero, either one has to allow “wire singularities” in the vector potentials or regard \mathbf{A} as a connection on a nontrivial $U(1)$ bundle. (If we regard it as a connection on a \mathbb{R}^+ bundle, the magnetic charge is necessarily zero.) Since we wish to deal only with smooth fields, we will not allow “wire singularities” in the potentials. If we work with bundles, the magnetic charge is quantized whence the space of histories has several discon-

nected components. Thus, in the first law, we will not be able to consider variations δ of fields with $\delta P \neq 0$. As far as mechanics of isolated horizons is concerned, there is essentially no loss of generality if we restrict ourselves to the case $P_\Delta = 0$. Therefore, in the next three sections, while working with Maxwell fields, we will do so. As usual, our *final* results can be formally extended to the case of nonvanishing magnetic charge by performing a duality rotation on \mathbf{F} . (As discussed in Sec. VI, the situation is rather different in the case of Yang-Mills fields.)

Recall that the first law in the Einstein-Maxwell case involves the electrostatic potential Φ . In static space-times, one typically sets $\Phi = -\xi^a \mathbf{A}_a$ where ξ is the static Killing field and the gauge is chosen such that the vector potential \mathbf{A} tends to zero at infinity and satisfies $\mathcal{L}_\xi \mathbf{A} = 0$ *everywhere in space-time*. We now need a strategy to define the electric potential Φ *without* reference to a Killing field or infinity. To this end, we introduce the following definition:

Definition 3. The electromagnetic potential \mathbf{A} will be said to be in a *gauge adapted to the weakly isolated horizon* $(\Delta, [l])$ if it satisfies

$$\mathcal{L}_l \mathbf{A} \triangleq 0. \quad (2.21)$$

Mathematically, this restriction is analogous to this condition $\mathcal{L}_l \omega \triangleq 0$ imposed on the gravitational field in definition 2. However, while the condition on ω is a physical restriction on the form of the gravitational field at Δ , the condition on \mathbf{A} is a gauge choice; it can always be imposed without physically constraining the electromagnetic field strength.

Given an electromagnetic potential \mathbf{A} in a gauge adapted to $(\Delta, [l])$, we can now define the scalar potential $\Phi_{(l)}$ at the horizon in an obvious fashion:

$$\Phi_{(l)} := -l \cdot \mathbf{A}.$$

The key question now is whether our boundary conditions are strong enough to ensure that $\Phi_{(l)}$ is constant on Δ . Only then can we hope to use this notion of the scalar potential in the first law. Note that this question is rather similar to the one we asked of surface gravity $\kappa_{(l)}$ in Sec. II B. By using arguments completely analogous to those that led us to the zeroth law, we will now show that the answer to the present question is also in the affirmative. In a gauge adapted to the horizon,

$$0 \triangleq \mathcal{L}_l \mathbf{A} \triangleq l \cdot \mathbf{F} - d\Phi_{(l)}.$$

As we saw above, the boundary conditions imply $\mathcal{L}_l \mathbf{F} \triangleq 0$ [Eq. (2.16)]. Hence, it follows immediately that $\Phi_{(l)}$ is constant on the horizon. We can regard this result as the “electromagnetic part” of the zeroth law of isolated horizon mechanics.

We will see in the next section that these zeroth laws play a key role in making the gravitational and the electromagnetic action principles well defined in the presence of isolated horizons. As with surface gravity $\kappa_{(l)}$, the functional dependence of $\Phi_{(l)}$ on the horizon parameters varies with the

choice of $l \in [l]$. We will see in Sec. V that the Hamiltonian framework constrains these dependences in an interesting fashion.

D. Other definitions and remarks

In this subsection, we introduce two new definitions which are important to the general framework of isolated horizons.

The first is concerned with rotation. From one's experience with the Newman-Penrose framework, one expects the gravitational contribution to angular momentum to be coded in the imaginary part of Ψ_2 . This expectation will be shown to be correct in [13]. Therefore, in the Einstein-Maxwell theory, we introduce the following definition:

Definition 4. A weakly isolated horizon $(\Delta, [l])$ will be said to be *nonrotating* if $\text{Im}[\Psi_2]$ vanishes on Δ .

If $(\Delta_K, [\xi])$ is a Killing horizon and ξ is a *hypersurface orthogonal*, timelike vector field near Δ_K , on physical grounds one would expect the horizon to be nonrotating. Is this expectation compatible with our definition? The answer is in the affirmative. For, in this case, one can show that $B_{ab} := C_{abcd} \xi^c \xi^d$ vanishes in the region where ξ^a is timelike. Hence, by continuity, it also vanishes on Δ forcing $\text{Im}[\Psi_2]$ to vanish there. Similarly, if the space-time admits a *hypersurface orthogonal*, rotational Killing field φ^a in a neighborhood of Δ , $\text{Im}[\Psi_2]$ again vanishes on Δ . The definition is again compatible with one's intuition that the horizon should be nonrotating in this case. In this paper, while we allow for presence of rotation in the first four sections, we will restrict ourselves to nonrotating horizons in the proof of the first law in Sec. V.

Finally, for completeness, let us introduce a stronger notion of "isolation" by strengthening the boundary conditions of definition 2.

Definition 5. A weakly isolated horizon $(\Delta, [l])$ is said to be *isolated* if

$$[\mathcal{L}_l, \mathcal{D}]V \triangleq 0 \quad (2.22)$$

for all vector fields V tangential to Δ and all $l \in [l]$.

As before, if any one l satisfies this condition, so do all $l \in [l]$. However, unlike Eq. (2.13), condition (2.22) is a *genuine* restriction in the sense that it cannot always be met by a judicious choice of null normals. Generically it does suffice to single out the equivalence class $[l]$ uniquely [18]. In particular, in the Kerr family, the only $[l]$ which satisfies Eq. (2.22) is the one containing constant multiples of the globally defined Killing field which is orthogonal to the horizon. Every Killing horizon is of course an isolated horizon. Thus, even though Eq. (2.22) is a stronger condition than Eq. (2.13), it is still *very* weak compared to conditions normally imposed. For most physical applications, e.g., to numerical relativity, it is appropriate to work with isolated horizons. For mechanics of isolated horizons, however, we can—and will—work with the larger class of weakly isolated horizons.

The following consequences of the boundary conditions defining weakly isolated horizons, derived in this section, will play an important role in the subsequent discussion:

- (a) $\nabla_a l_b \triangleq \omega_a l^b$.
- (b) $\kappa_{(l)} \triangleq l^a \omega_a$, the surface gravity defined by the null normal l^a , is constant on Δ .
- (c) There is a natural (area) 2-form ${}^2\epsilon$ on Δ satisfying $\mathcal{L}_l {}^2\epsilon \triangleq 0$ and ${}^2\epsilon_{ab} l^b \triangleq 0$.
- (d) The electromagnetic potential \mathbf{A} is chosen to satisfy $\mathcal{L}_l \mathbf{A} \triangleq 0$ and in this gauge the scalar potential $\Phi_{(l)} := -l^a \mathbf{A}_a$ is constant on Δ .
- (e) The electromagnetic field satisfies $l^a \mathbf{F}_{ab} \triangleq 0$; and $l^a {}^* \mathbf{F}_{ab} \triangleq 0$.

III. ACTION

In this paper we use the first order formulation of general relativity in terms of tetrads and connections. Since tetrads are essential to incorporate spinorial matter, it is natural to base the framework on tetrads from the beginning. The use of a first order formalism, on the other hand, is motivated primarily by mathematical simplicity. In the first order framework, the action and the Hamiltonians can be expressed entirely in terms of differential forms which significantly simplify the variational calculations. The previous paper [8] which dealt with undistorted horizons used spinors and self-dual connections, while here we choose to use orthonormal tetrads and real, Lorentz connections. For analyzing mechanics of isolated horizons, there are two advantages to this. First, the Hamiltonian and symplectic structure are now manifestly real which simplifies evaluation of the boundary terms at the horizon. Second, the analysis can now be extended to other space-time dimensions in a straightforward manner. However, these simplifications come with a price. Since, at present, the self-dual variables appear to be indispensable for nonperturbative quantization, the results obtained here will have to be reexpressed in terms of self-dual variables in order to extend the analysis [20] of the quantum horizon geometry and black hole entropy to include rotation.

A. Preliminaries

Let us begin with the first order action for Einstein-Maxwell theory on a four-dimensional manifold \mathcal{M} which is topologically $M \times \mathbb{R}$, where M is an oriented Riemannian 3-manifold *without boundary* (the complement of a compact set of) which is diffeomorphic to (the complement of a compact set of) \mathbb{R}^3 . Thus, topological complications of M , if any, are confined to a compact set. In this subsection we shall only give the relevant formulas. For details, see e.g. [21]. Our basic fields will consist of a triplet $(e_a^I, A_{aI}^J, \mathbf{A}_a)$ defined on \mathcal{M} where e_a^I denotes a co-tetrad, A_{aI}^J the gravitational (Lorentz) connection, and \mathbf{A}_a the electromagnetic connection. Here, lowercase latin letters refer to the tangent space of \mathcal{M} while the uppercase letters I, J , etc. refer to an internal four-dimensional vector space V with a fixed metric η_{IJ} of signature $(-, +, +, +)$. The co-tetrad e_a^I is an isomorphism between the tangent space $T_p(\mathcal{M})$ at any point p and the internal space V . Using it, we define a metric on \mathcal{M} by

$g_{ab} := e_a^I e_b^J \eta_{IJ}$ which also has signature $(-, +, +, +)$. The Lorentz connection A_{aI}^J acts only on internal indices and defines a derivative operator

$$D_a k_I := \partial_a k_I + A_{aI}^J k_J,$$

where ∂ is a fiducial derivative operator which, as usual, will be chosen to be flat and torsion free. Finally, \mathbf{A}_a is the U(1) electromagnetic connection 1-form on \mathcal{M} . (As noted in Sec. II C, we will assume that the magnetic charge is zero.) All fields will be assumed to be smooth and satisfy the standard asymptotic conditions at infinity.

The 2-forms Σ^{IJ}

$$\Sigma_{IJ} := \frac{1}{2} \epsilon_{IJKL} e^K \wedge e^L$$

constructed from the co-tetrads will play an important role throughout our calculations. In particular, the action for an asymptotically flat space-time (with no internal boundary) is given by (see e.g. [21])

$$\begin{aligned} S(e, A, \mathbf{A}) &= \frac{-1}{16\pi G} \int_{\mathcal{M}} \Sigma^{IJ} \wedge F_{IJ} \\ &+ \frac{1}{16\pi G} \int_{\tau_\infty} \Sigma^{IJ} \wedge A_{IJ} - \frac{1}{8\pi} \int_{\mathcal{M}} \mathbf{F} \wedge \star \mathbf{F}. \end{aligned} \quad (3.1)$$

Here F and \mathbf{F} are the curvatures of the gravitational and electromagnetic connections A and \mathbf{A} , respectively:

$$F_I^J = dA_I^J + A_I^K \wedge A_K^J, \quad \mathbf{F} = d\mathbf{A},$$

$\star \mathbf{F}_{ab} = \frac{1}{2} \epsilon_{ab}{}^{cd} \mathbf{F}_{cd}$ is the dual of \mathbf{F} defined using e_a^I , and τ_∞ is the timelike cylinder at infinity. The boundary term at τ_∞ ensures the differentiability of the action.

Let us briefly examine the equations of motion arising from the action. Varying the action with respect to the connection, one obtains

$$D\Sigma = 0.$$

This condition implies that the connection D defined by A has the same action on internal indices as the unique connection ∇ compatible with the co-tetrad, i.e., satisfying $\nabla_a e_b^I = 0$. When this equation of motion is satisfied, the curvature F is related to the Riemann curvature R of ∇ by

$$F_{ab}{}^{IJ} = R_{ab}{}^{cd} e_c^I e_d^J.$$

Varying the action with respect to e_a^I and taking into account the above relation between curvatures, one obtains Einstein's equations

$$G_{ab} = 8\pi G T_{ab} \equiv 2G \left(\mathbf{F}_{ac} \mathbf{F}_{bd} g^{cd} - \frac{1}{4} g_{ab} \mathbf{F}_{cd} \mathbf{F}^{cd} \right)$$

where G_{ab} is the Einstein tensor and T_{ab} the electromagnetic stress energy tensor. Finally, variation with respect to the electromagnetic connection, \mathbf{A} , yields Maxwell's equation

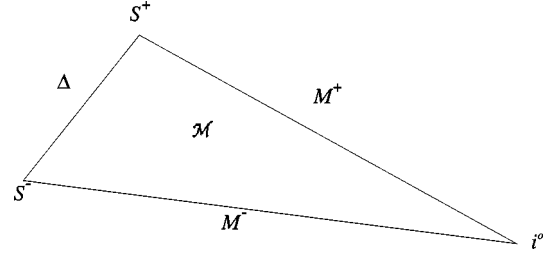


FIG. 1. The region of space-time \mathcal{M} under consideration has an internal boundary Δ and is bounded by two partial Cauchy surfaces M^\pm which intersect Δ in the 2-spheres S^\pm and extend to spatial infinity i^0 .

$$d\star \mathbf{F} = 0.$$

B. Internal boundary Δ

Let us now consider the variational principle for asymptotically flat histories which admit a weakly isolated horizon Δ as their internal boundary. The manifold \mathcal{M} under consideration has an internal boundary Δ , topologically $S^2 \times \mathbb{R}$. As before, \mathcal{M} is topologically $M \times \mathbb{R}$, where M is now an oriented manifold with an internal, 2-sphere boundary, whose topological complications are again confined to a compact region. Space-time is bounded to the future and past by two (partial Cauchy) surfaces M^\pm , extending to spatial infinity (see Fig. 1).

Following definition 2 of weakly isolated horizons, we will equip Δ with a fixed equivalence class of vector fields $[l]$ which are transversal to its 2-sphere cross sections (where, as before, $l \sim l'$ if and only if $l \triangleq c l'$ for a constant c). It is also convenient to fix an *internal* null tetrad $(l^I, n^I, m^I, \bar{m}^I)$ on Δ , each element of which is annihilated by the fiducial, flat internal connection ∂ .

The permissible histories consist of smooth triplets (e, A, \mathbf{A}) on \mathcal{M} satisfying boundary conditions at infinity and on Δ . The boundary conditions at infinity are, as before, the standard ones which ensure asymptotic flatness. Since the asymptotic behavior and boundary integrals at infinity play only a secondary role in our analysis, we shall not spell out the precise fall off requirements. At Δ , the histories are subject to three conditions: (i) the tetrads e should be such that the vector field $l^a := l^I e_I^a$ defined by each history belongs to the equivalence class $[l]$ fixed on Δ , (ii) the tetrad e and the gravitational connection A should be such that $(\Delta, [l])$ is a weakly isolated horizon for the history, and (iii) the electromagnetic potential \mathbf{A} is in a gauge adapted to the horizon, i.e., $\mathcal{L}_l \mathbf{A} \triangleq 0$.

Remark. In space-time, we have the freedom to perform a local, internal Lorentz rotation on the tetrad e_I^a (and the gravitational connection A_{aI}^J). All these tetrads define the same Lorentzian metric g_{ab} . Since l^a is required to be a null normal to Δ , the permissible gauge rotations are reduced on Δ to the subgroup $(\mathbb{R}^+ \times E^2)_{\text{loc}}$ of local null rotations preserving the null direction field l . (Here \mathbb{R}^+ is the group of rescalings of l^a , n^a which leaves m^a fixed and E^2 is the three-dimensional Euclidean group consisting of rotations in the l - m , l - \bar{m} , and m - \bar{m} planes.) Condition (i) above—

dictated by the existence of a preferred equivalence class $[l]$ in definition 2—further reduces the internal gauge freedom to $\mathbb{R}^+ \times (E_{\text{loc}}^2)$, i.e., reduces the group $\mathbb{R}_{\text{loc}}^+$ of local l - n rescalings to the group \mathbb{R}^+ of global rescalings. Thus, while any one space-time (\mathcal{M}, g_{ab}) , still defines infinitely many histories due to the freedom of tetrad rotations, this freedom is somewhat reduced at Δ because of the structure fixed by the boundary conditions.⁶

Given any tetrad e_I^a , the internal null vectors $(l^I, n^I, m^I, \bar{m}^I)$ fixed on Δ trivially provide a null tetrad $(l^a, n^a, m^a, \bar{m}^a)$. In terms of these vectors, we can express $\underline{\Sigma}^{IJ}$ as

$$\underline{\Sigma}^{IJ} \triangleq 2l^I n^J \epsilon + 2n \wedge (iml^I \bar{m}^J - i\bar{m}l^I m^J), \quad (3.2)$$

where, as before, ${}^2\epsilon = im \wedge \bar{m}$ is the pullback to Δ of the natural alternating tensor on the 2-sphere \mathcal{S} of integral curves of l^a associated with the given history. The weak isolation of $(\Delta, [l])$ restricts the form of the connection A at Δ . To see this, recall that one of the equations of motion requires the connection D defined by A to have the same action on internal indices as ∇ . Hence, $\nabla_a l_I \triangleq \partial_a l_I + A_{aI}{}^J l_J \triangleq A_{aI}{}^J l_J$ where, in the second step we have used the fact that the flat derivative operator, ∂ , has been chosen to annihilate the internal tetrad on Δ . Since $\nabla_a e_b^I = 0$ by definition of ∇ , and $\nabla_a l^b \triangleq \omega_a l^b$ [see Eq. (2.4)] it follows that $\underline{A}_{IJ} l^J \triangleq \omega l_I$. Hence, on Δ , \underline{A} has the form

$$\underline{A}_{IJ} \triangleq 2l_I n_J \omega + C_{IJ}, \quad (3.3)$$

where the 1-form C_{IJ} satisfies $C_{IJ} l^J \triangleq 0$.

With this background material at hand, we are now ready to consider variations of the action (3.1) in the presence of an inner boundary representing a weakly isolated horizon. A key question is whether a new surface term at the horizon is necessary to make the variational principle well defined. We will show that, thanks to the zeroth law, such a term is not needed.

A simple calculation yields

$$\begin{aligned} \delta S(e, A, \mathbf{A}) &= \int_{\mathcal{M}} (\text{equations of motion}) \times \delta \phi \\ &\quad - \frac{1}{16\pi G} \int_{\Delta} \underline{\Sigma}^{IJ} \wedge \delta A_{IJ} - \frac{1}{4\pi} \int_{\Delta} \delta \mathbf{A} \wedge \star \mathbf{F} \end{aligned} \quad (3.4)$$

where, in the first term, ϕ stands for the basic fields (e, A, \mathbf{A}) in the action. Note that, as in the case without an internal

⁶Nonetheless, from a space-time perspective, the multiplicity of histories can still be rather surprising. For example, if g_{ab} is the Schwarzschild metric with mass $M > 0$, there is a history in which the surface gravity $\kappa_{(l)}$ is positive and another in which it is zero. This redundancy can be eliminated by working with isolated, rather than weakly isolated horizons.

boundary, the variation of the boundary term at infinity precisely cancels the contribution arising from the variation of the bulk terms.

In order to show that the action principle is viable, it is necessary to show that the terms at the horizon vanish due to the boundary conditions imposed there. Let us begin with the gravitational term. Using Eqs. (3.2) and (3.3) it can be reexpressed as

$$- \frac{1}{8\pi G} \int_{\Delta} \delta \omega \wedge {}^2\epsilon. \quad (3.5)$$

Since ${}^2\epsilon$ is the pullback to Δ of the alternating tensor on the 2-sphere \mathcal{S} of integral curves of l , it follows that $\mathcal{L}_l {}^2\epsilon \triangleq 0$. Furthermore, the weak isolation of the horizon ensures $\mathcal{L}_l \omega \triangleq 0$ and, since the null normal l^a defined by *any* tetrad belongs to the fixed equivalence class $[l]$ at the horizon, we have $\delta l \triangleq c_\delta l$ for some constant c_δ . These two facts imply $\mathcal{L}_l \delta \omega \triangleq 0$. Thus the entire integrand is Lie dragged by l . In the variational principle, however, all fields are fixed on the initial and final hypersurfaces, say M^\pm . In particular, $\delta \omega$ necessarily vanishes on the initial and final cross sections of the horizon. Therefore, the integrand Eq. (3.5) vanishes on the initial and final cross sections *and* is Lie dragged by l . This immediately implies that Eq. (3.5) is zero.

Let us now consider the electromagnetic term. Since every \mathbf{A} is in a gauge adapted to the isolated horizon, $\mathcal{L}_l \mathbf{A} \triangleq 0$. Furthermore, $\delta l^a \triangleq c_\delta l^a$, so we conclude $\mathcal{L}_l \delta \mathbf{A} \triangleq 0$. Next, Eq. (2.20) ensures that $\mathcal{L}_l \star \mathbf{F} \triangleq 0$. Thus, the integrand of the electromagnetic surface term is Lie dragged by l^a . An identical argument to the one presented above implies that the electromagnetic surface term Eq. (3.4) also vanishes. Therefore, the variation of the action (3.1) continues to yield Einstein-Maxwell equations in spite of the presence of an inner boundary representing a weakly isolated horizon.

It is instructive to reexamine the key step in the above argument. Suppose we only had a nonexpanding horizon. Then, the gravitational surface term could still be reduced Eq. (3.5), and ${}^2\epsilon$ and $\star \mathbf{F}$ would still be Lie dragged by l . However, in this case, we could not argue that ω and \mathbf{A} are also Lie dragged. As we saw in Secs. IIB and IIC, these conditions are equivalent, respectively, to the constancy of the surface gravity $\kappa_{(l)}$ and the electromagnetic potential $\Phi_{(l)}$ on Δ . In this sense, given a nonexpanding horizon as the inner boundary, the gravitational and electromagnetic zeroth laws are the *necessary and sufficient conditions* one must impose for the viability of the standard, first order, tetrad action principle.

Remark. Note that Eq. (3.1) is not the unique viable action for the problem: as usual, there is freedom to add suitable boundary terms without affecting the viability. Specifically, we are free to add any horizon boundary term which is composed entirely of fields which are Lie dragged by l , for example the intrinsic horizon metric q_{ab} and fields ω , ${}^2\epsilon$, and \underline{A} . Then, as a result of the argument given above, the new action would also be viable. However, as usual, this freedom will not affect the definition of the symplectic structure which underlies the Hamiltonian treatment of the next section.

IV. COVARIANT PHASE SPACE

Let us now construct the phase space of space-times containing weakly isolated horizons. In the next section, we will use this framework to construct Hamiltonians generating suitable time translations and define the energy of an isolated horizon. In [8], the phase space was constructed by performing a Legendre transform of the action. This procedure leads to a ‘‘canonical’’ framework in which the phase space consists of configuration and momentum variables defined on a spatial hypersurface. With the self-dual connections used in [8], the gravitational configuration variable turns out to be a connection and its conjugate momentum a 2-form so that the Hamiltonian description can again be given in terms of forms. With the full Lorentz connections now under consideration, the situation turns out to be more complicated. Specifically, one encounters certain second class constraints and, when these are solved, one ends up with the same canonical phase space that one would have obtained through a second order formalism. In the Hamiltonian framework, then, the simplicity we encountered in Sec. III is lost. More specifically, constraint functions and Hamiltonians now contain terms involving *second* derivatives of the basic canonical variables which make variations rather complicated. (For details, see Chaps. 3 and 4 in [21].) Therefore, in this section we will not use a Legendre transform. Instead, we will construct the ‘‘covariant phase space’’ from the space of solutions to field equations (see, e.g., [22]). As in Sec. III, all expressions will now involve only the basic form fields and their exterior derivatives and variational calculations will continue to be simple.

To specify the phase space, let us begin as in Sec. III by fixing a manifold \mathcal{M} with an internal boundary Δ (see Fig. 1). As before, we will equip Δ with an equivalence class $[l]$ of vector fields transverse to its 2-sphere cross sections. To evaluate the symplectic structure and Hamiltonians, we will often use a partial Cauchy surface M in the interior of \mathcal{M} which intersects Δ in a 2-sphere S . Points of the covariant phase space Γ will consist of histories considered in Sec. III *which satisfy field equations*. More explicitly, Γ consists of asymptotically flat solutions (e, A, \mathbf{A}) to the field equations on \mathcal{M} such that (i) the vector field $l^a := l^I e_I^a$ belongs to the equivalence class $[l]$ fixed on Δ , (ii) in each solution, $(\Delta, [l])$ is a weakly isolated horizon, and (iii) the electromagnetic potential \mathbf{A} is in a gauge adapted to the horizon, i.e., $\mathcal{L}_l \mathbf{A} \triangleq 0$.

Our next task is to use the action (3.1) to define the symplectic structure Ω on Γ . It is convenient to make a brief detour and first introduce two new fields which can be regarded as ‘‘potentials’’ for the surface gravity $\kappa_{(l)}$ and the electric potential $\Phi_{(l)}$. Given any point (e, A, \mathbf{A}) in the phase space Γ , let us define scalar fields ψ and χ on Δ as follows:

- (i) $\mathcal{L}_l \psi \triangleq (l \cdot \omega) \triangleq \kappa_{(l)}$ and $\mathcal{L}_l \chi \triangleq (l \cdot \mathbf{A}) \triangleq -\Phi_{(l)}$.
- (ii) ψ and χ vanish on S^- , the intersection of M^- with Δ .⁷

⁷Condition (ii) serves only to fix the freedom to add constants to ψ and χ . One could envisage replacing it by a different condition. Our results will be insensitive to this choice.

Note that these conditions associate with each point of Γ a unique pair (ψ, χ) on Δ and in the ‘‘extremal’’ case $\kappa_{(l)} = 0$, ψ vanishes identically.

We wish to use the standard procedure [22] involving second variations of the action to define the symplectic structure.⁸ Let us recall the main steps of this procedure. One first constructs the symplectic current J : Given a point γ in the phase space Γ and two tangent vectors δ_1 and δ_2 at that point, J provides a closed 3-form $J(\gamma; \delta_1, \delta_2)$ on \mathcal{M} . Integrating dJ over the part $\tilde{\mathcal{M}}$ of space-time under consideration, one obtains

$$0 = \int_{\tilde{\mathcal{M}}} dJ(\gamma; \delta_1, \delta_2) = \oint_{\partial \tilde{\mathcal{M}}} J.$$

Now, if there is no internal boundary, one can choose $\tilde{\mathcal{M}}$ to be a region bounded by any two Cauchy surfaces M_1 and M_2 so that the boundary is given by $\partial \tilde{\mathcal{M}} = M_1 \cup M_2 \cup \tau_\infty$, where τ_∞ is the timelike ‘‘cylinder at infinity.’’ In simple cases, the asymptotic conditions ensure that the integral $\int_{\tau_\infty} J(\gamma; \delta_1, \delta_2)$ vanishes. Then, taking orientations into account, it follows that $\int_M J(\gamma; \delta_1, \delta_2)$ is independent of the choice of Cauchy surface M . One then sets the symplectic structure to be

$$\Omega|_\gamma(\delta_1, \delta_2) = \int_M J(\gamma; \delta_1, \delta_2).$$

In our case, the second variation of the action (3.1) yields the following symplectic current:

$$J(\gamma; \delta_1, \delta_2) = \frac{-1}{16\pi G} [\delta_1 \Sigma^{IJ} \wedge \delta_2 A_{IJ} - \delta_2 \Sigma^{IJ} \wedge \delta_1 A_{IJ}] - \frac{1}{4\pi} [\delta_1^* \mathbf{F} \wedge \delta_2 \mathbf{A} - \delta_2^* \mathbf{F} \wedge \delta_1 \mathbf{A}]. \quad (4.1)$$

Using the fact that the fields $\gamma \equiv (e, A, \mathbf{A})$ satisfy the field equations and the tangent vectors δ_1, δ_2 satisfy the linearized equations off γ , one can directly verify that $J(\gamma; \delta_1, \delta_2)$ is in fact closed as guaranteed by the general procedure involving second variations. It is now natural to choose $\tilde{\mathcal{M}}$ to be a part of our space-time \mathcal{M} bounded by two partial Cauchy surfaces M_1, M_2 , the timelike cylinder τ_∞ and a part $\tilde{\Delta}$ of the isolated horizon bounded by M_1 and M_2 . Again, the asymptotic conditions ensure that the integral of J over τ_∞ vanishes. Hence,

⁸Actually this procedure provides a presymplectic structure, i.e., a closed 2-form on the phase-space which, however, is generally degenerate. The vectors in its kernel represent infinitesimal ‘‘gauge transformations.’’ The physical phase space is obtained by quotienting the space of solutions by gauge transformations and inherits a true symplectic structure from the presymplectic structure on the space of solutions. The 2-form Ω introduced below is indeed degenerate. However, for simplicity, we will abuse the notation somewhat and refer to Ω as the symplectic structure.

$$\left(\int_{M_1} + \int_{M_2} + \int_{\tilde{\Delta}} \right) J(\gamma; \delta_1, \delta_2) = 0.$$

However, this does not immediately provide us the conserved symplectic structure because the integral of J over $\tilde{\Delta}$ does not vanish in general. Since the isolation of the horizon implies that there are no fluxes of physical quantities across Δ , one might expect that, although nonzero, the integral over $\tilde{\Delta}$ would be ‘‘controllable.’’ This is indeed the case. Using the forms (3.2) and (3.3) of $\underline{\Sigma}$ and \underline{A} on Δ and the definitions of the potentials ψ and χ , it is easy to verify that the pullback of the symplectic current to Δ is itself exact:

$$\underline{J}(\gamma; \delta_1, \delta_2) \triangleq dj(\gamma, \delta_1, \delta_2)$$

where the 2-form j on Δ is given by

$$\begin{aligned} \Omega|_{\gamma}(\delta_1, \delta_2) = & \frac{-1}{16\pi G} \int_M [\delta_1 \Sigma^{IJ} \wedge \delta_2 A_{IJ} - \delta_2 \Sigma^{IJ} \wedge \delta_1 A_{IJ}] + \frac{1}{8\pi G} \oint_S [\delta_1(^2\epsilon) \delta_2 \psi - \delta_2(^2\epsilon) \delta_1 \psi] \\ & - \frac{1}{4\pi} \int_M [\delta_1^* \mathbf{F} \wedge \delta_2 \mathbf{A} - \delta_2^* \mathbf{F} \wedge \delta_1 \mathbf{A}] + \frac{1}{4\pi} \oint_S [\delta_1^* \mathbf{F} \delta_2 \chi - \delta_2^* \mathbf{F} \delta_1 \chi]. \end{aligned} \quad (4.2)$$

Again, using field equations one can directly verify that the right side Eq. (4.2) is independent of the choice of the partial Cauchy surface; the symplectic structure is ‘‘conserved.’’ We will use (Γ, Ω) as our covariant phase space.

Note that, even though the action did not contain a surface term at the horizon, the symplectic structure does. So the overall situation is the same as in the undistorted, nonrotating case considered in [8]. Finally, our discussion of the action principle and our construction of the covariant phase space is applicable to *all* weakly isolated horizons Δ ; nowhere did we have to restrict ourselves to the nonrotating case.

V. HAMILTONIAN EVOLUTION AND THE FIRST LAW

To discuss the first law, we must first define horizon energy, which in turn requires a time evolution field t^a on \mathcal{M} . Given a vector field t^a satisfying appropriate boundary conditions, $\delta_t := (\mathcal{L}_t e, \mathcal{L}_t A, \mathcal{L}_t \mathbf{A})$ satisfies the linearized equations for any $\gamma := (e, A, \mathbf{A})$ in Γ and thus defines a vector field on Γ .⁹ This δ_t can be interpreted as the infinitesimal generator of time evolution on the covariant phase space. It is then natural to ask if this vector field is a phase space symmetry, i.e., if $\mathcal{L}_{\delta_t} \Omega$ vanishes everywhere on Γ . The necessary and

⁹In the Lie derivatives, the internal indices are treated as scalars; thus $\mathcal{L}_t e'_a = t^b \partial_b e'_a + e'_b \partial_a t^b$. To make δ_t a well-defined vector field on Γ , we now exclude M^\pm from \mathcal{M} and let \mathcal{M} and Δ be without future and past boundaries. Whenever needed, these boundaries, M^\pm and S^\pm , can be added by taking the obvious closure of \mathcal{M} .

$$\begin{aligned} j(\gamma; \delta_1, \delta_2) = & \frac{1}{8\pi G} [\delta_1 \psi \delta_2(^2\epsilon) - \delta_2 \psi \delta_1(^2\epsilon)] \\ & + \frac{1}{4\pi} [\delta_1 \chi \delta_2^* \mathbf{F} - \delta_2 \chi \delta_1^* \mathbf{F}]. \end{aligned}$$

Hence, if M_1 and M_2 intersect $\tilde{\Delta}$ in 2-spheres S_1 and S_2 respectively, we have

$$\int_{\tilde{\Delta}} J(\gamma; \delta_1, \delta_2) = - \left(\int_{S_1} + \int_{S_2} \right) j(\gamma; \delta_1, \delta_2).$$

The negative sign appearing in the above expression is due to the choice of orientation of S_Δ , which is induced from M rather than from Δ . Using these results we can define the symplectic structure as

sufficient condition for this to happen is that there exist a function H_t —the Hamiltonian generating the t evolution—such that

$$\delta H_t = \Omega(\delta, \delta_t) \quad (5.1)$$

for all vector fields δ to Γ . On general grounds, one expects H_t to contain a surface term E_{ADM}^t at infinity representing the total (i.e., ADM) energy, and a surface term E_Δ^t at the horizon which can be interpreted as the horizon energy, both tied to the evolution field t^a .

A key question then is to specify the appropriate boundary conditions on t^a . It is clear that, at infinity, t^a should be an asymptotic time translation, i.e., should approach a time-translation Killing field of the flat metric used to specify the boundary conditions. At the horizon, on the other hand, the metric is *not* universal and the space-time defined by a generic point γ of the covariant phase does not admit *any* Killing field near Δ . Therefore, specification of the boundary conditions at Δ is not as straightforward as that at infinity. It is for this reason that we now assume that $(\Delta, [l])$ is a *nonrotating*, weakly isolated horizon for all points $\gamma \equiv (e, A, \mathbf{A})$ of the phase space. The problem of specifying the appropriate boundary conditions on t^a in the rotating case is more complicated. However, it has been addressed successfully and will be discussed in [13].

Recall that the internal boundary Δ of \mathcal{M} is equipped with a *specific* equivalence class $[l]$ of vector fields. As discussed in Sec. II, these l are the isolated horizon analogues of constant multiples of Killing fields on the Killing horizons in static space-times. Therefore, in the *nonrotating* case, it is

natural to demand that, on Δ , the evolution vector field t^a should belong to the equivalence class $[l]$. This automatically ensures that $(\mathcal{L}_t e, \mathcal{L}_t A, \mathcal{L}_t \mathbf{A})$ satisfy the appropriate boundary conditions to define a tangent vector at each point of the phase space Γ . However, unlike at infinity, the geometry at the horizon is *not* fixed once and for all. Therefore, it is natural to allow the precise value of the evolution vector field t^a on Δ to vary from one point of the phase space to another. In more familiar terms, this corresponds to allowing the (boundary values of) lapse and shift fields to depend on dynamical fields (e, A, \mathbf{A}) themselves, a procedure routinely used in numerical relativity and gauge-fixed calculations in canonical gravity. Following the current terminology in numerical relativity, we will refer to such t^a as *live* evolution vector fields. The use of live fields turns out to be necessary to ensure that δ_t is a phase space symmetry, i.e., yields a Hamiltonian evolution on (Γ, Ω) .

Let us fix a live t^a whose restriction to the horizon belongs to the equivalence class $[l]$ at all points of the phase space. To analyze if δ_t is a Hamiltonian vector field, it is simplest to compute the 1-form X_t on Γ defined by

$$X_t(\delta) = \Omega(\delta, \delta_t). \quad (5.2)$$

Now δ_t is Hamiltonian—i.e., $\mathcal{L}_{\delta_t} \Omega = 0$ on Γ —if and only if X_t is closed, i.e.,

$$dX_t = 0$$

where d denotes the exterior derivative on (the infinite dimensional) phase space Γ . If this is the case then, up to an additive constant, the Hamiltonian is given by

$$dH_t = X_t.$$

To calculate the right side of Eq. (5.2), it is useful to note the following identities from differential geometry:

$$\begin{aligned} \mathcal{L}_t A &= t \cdot F + D(t \cdot A), \\ \mathcal{L}_t \Sigma &= t \cdot D\Sigma + D(t \cdot \Sigma) - [(t \cdot A), \Sigma], \\ \mathcal{L}_t \mathbf{A} &= t \cdot \mathbf{F} + d(t \cdot \mathbf{A}), \\ \mathcal{L}_t \star \mathbf{F} &= t \cdot (d\star \mathbf{F}) + d(t \cdot \star \mathbf{F}). \end{aligned} \quad (5.3)$$

Using these, the field equations satisfied by (e, A, \mathbf{A}) and the linearized field equations for δ , we obtain the required expression of X_t :

$$\begin{aligned} X_t(\delta) &:= \Omega(\delta, \mathcal{L}_t) \\ &= \frac{-1}{16\pi G} \int_{\partial M} \text{Tr}[(t \cdot A) \delta \Sigma - (t \cdot \Sigma) \wedge \delta A] \\ &\quad - \frac{1}{4\pi} \int_{\partial M} (t \cdot \mathbf{A}) \delta(\star \mathbf{F}) - (t \cdot \star \mathbf{F}) \wedge \delta \mathbf{A}. \end{aligned} \quad (5.4)$$

Note that the expression involves integrals *only* over the 2-sphere boundaries S_∞ and S_Δ of M , the partial Cauchy surface used in the evaluation of the symplectic structure; there is no volume term.

The integrals at infinity can be evaluated easily by making use of the falloff conditions. As one would expect, the electromagnetic term vanishes (because \mathbf{A} falls off at least as $1/r$ while \mathbf{F} falls off as $1/r^2$) while the gravitational term yields precisely the ADM energy E_{ADM}^t associated with the asymptotic time translation defined by t^a . At the horizon, we can use Eqs. (3.2) and (3.3) to show that $t \cdot \Sigma$ contracted on internal indices with δA vanishes and Eq. (2.17) implies $t \cdot \star \mathbf{F} = 0$, leaving

$$\begin{aligned} X_t(\delta) &= -\frac{1}{8\pi G} \int_{S_\Delta} (t \cdot \omega) \delta(2\epsilon) \\ &\quad - \frac{1}{4\pi} \int_{S_\Delta} (t \cdot \mathbf{A}) \delta(\star \mathbf{F}) + \delta E_{\text{ADM}}^t \\ &= -\frac{1}{8\pi G} \kappa_{(t)} \delta a_\Delta - \Phi_{(t)} \delta Q_\Delta + \delta E_{\text{ADM}}^t \end{aligned} \quad (5.6)$$

where, in the last step, we have used the fact that both $t \cdot \omega = \kappa_{(t)}$ and $t \cdot \mathbf{A} = -\Phi_{(t)}$ are constant on the horizon and the definition (2.19) of electric charge.

The necessary and sufficient condition for the existence of a Hamiltonian is that X_t be closed. Clearly, this is equivalent to

$$\frac{1}{8\pi G} d\kappa_{(t)} \wedge da_\Delta + d\Phi_{(t)} \wedge dQ_\Delta = 0, \quad (5.7)$$

where \wedge denotes the antisymmetric tensor product on Γ . Now, Eq. (5.7) trivially implies that the surface gravity $\kappa_{(t)}$ and the electric potential $\Phi_{(t)}$ at the horizon defined by t^a can depend *only* upon the area and charge of the horizon. Other factors, such as the ‘‘shape’’ of the distorted horizon, cannot affect the values of $\kappa_{(t)}$ or $\Phi_{(t)}$. Finally, Eq. (5.7) is the necessary and sufficient condition that there exist a function E_Δ^t , also only of a_Δ and Q_Δ such that

$$\delta E_\Delta^t = \frac{1}{8\pi G} \kappa_{(t)} \delta a_\Delta + \Phi_{(t)} \delta Q_\Delta. \quad (5.8)$$

Since E_Δ^t is a function *only* of a_Δ and Q_Δ , it is a function of fields defined *locally* at the horizon. As noted before, it is natural to interpret E_Δ^t as the horizon energy defined by the time translation t^a . The total Hamiltonian is given by

$$H_t = E_{\text{ADM}}^t - E_\Delta^t. \quad (5.9)$$

Let us summarize. Equation (5.8) is a necessary and sufficient condition for the 1-form X_t to be closed. Therefore, the vector field δ_t on Γ defined by the space-time evolution field t^a is Hamiltonian *if and only if* the first law (5.8) holds. Thus Eq. (5.8) is a restriction on the choice of the *live* vector

field t^a : While any t^a (which preserves the boundary conditions) defines an evolution flow on the phase space, it is only when

$$\frac{1}{8\pi G} \kappa_{(t)} da_\Delta + \Phi_{(t)} dQ_\Delta$$

is an exact 1-form on Γ that this flow is Hamiltonian (i.e., preserves the symplectic structure). At first, this restriction seems somewhat surprising because, in absence of internal boundaries, *every* vector field t^a (which tends to a fixed Killing field of the flat metric at infinity) defines a Hamiltonian evolution. However, even in this context, there is no *a priori* reason to expect this tight correspondence to hold if one allows general, live vector fields t^a whose boundary values at infinity can change from one space-time to another. Finally, we will see in Sec. VII that every space-time belonging to the phase space Γ admits an infinite family of vector fields t^a for which X_t is closed. Therefore, in particular, the first law does *not* restrict the ‘‘background’’ space-times (or the variations δ) in any way. Indeed, for any space-time in our phase space, there is an infinite family of first laws, one associated with each permissible t^a .

We will conclude this section with a few remarks.

(i) *Form of H_t* . The Hamiltonian (5.9) contains only surface terms. This may seem surprising because, in the canonical framework, the familiar Hamiltonian contains a volume integral consisting of a linear combination of constraints. While the volume term vanishes ‘‘on shell’’ and does not contribute to numerical value of the canonical Hamiltonian on physical states, it is nonetheless crucial for obtaining the correct evolution equations since derivatives of the Hamiltonian transverse to the constraint surface are needed to construct the Hamiltonian vector field. The covariant phase space, by contrast, consists only of solutions to the field equations whence the issue of taking ‘‘off shell’’ derivatives never arises. In diffeomorphism invariant theories, the Hamiltonian on the covariant phase space is always made of surface terms.¹⁰ If space-time has several asymptotic regions, the boundary term in each region defines the standard energy corresponding to that region. Therefore, in the present case, it was natural to interpret E_Δ^t as the horizon energy defined by the t^a evolution. Finally, we should emphasize that we used a covariant phase space only for simplicity. The final results go through (and, in fact, were first obtained) in a canonical framework as well.

(ii) *Comparison*. As noted in the Introduction, all treatments of the first law for nonrotating but possibly distorted horizons available in the literature refer to static space-times. The isolated horizon framework, by contrast, does not refer to a Killing field at all and thus allows a significantly larger class of physically interesting situations. On the other hand,

since it relies on a Hamiltonian framework, we cannot incorporate phenomenological matter if it does not admit a phase space description. Other treatments based on Hamiltonian methods generally restrict themselves to static space-times with a nonzero surface gravity. This assumption is essential there because those treatments use ‘‘bifurcate’’ surfaces in an important way and these do not exist in the extremal static solutions where the surface gravity vanishes. In contrast, the results of this section do not refer to a bifurcate surface and go through irrespective of whether $\kappa_{(t)}$ is nonzero or zero. In a realistic collapse, the physical space-time is *not* expected to have the bifurcate surface. The present analysis uses only the portion of the physical space-time in which the horizon has settled down with no further in-going radiation, rather than an analytical continuation of the near horizon geometry used in certain approaches. Finally, in contrast to other treatments, we have an infinite family of first laws, one for each evolution field t^a for which δ_t is a phase space symmetry.

(iii) *Nonuniqueness of energy*. Each permissible, live t^a defines a horizon energy E_Δ^t . At first it seems surprising that there is so much freedom in the notion of energy. Let us compare the situation at \mathcal{I}^+ , which, like Δ , is null. There, we only have a 3-parameter freedom which, furthermore, can be eliminated simply by fixing a rest frame. How does this difference arise? Recall that energy is (the numerical value of) the generator of a *unit* time translation. At infinity, all 4-metrics in the phase space approach the *same* flat metric. Hence, we can simply fix a unit time-translation Killing field t_0^a of that flat metric near infinity and use its restriction to \mathcal{I}^+ as the unit time translation for *all* metrics in the phase space. By contrast, there is no fixed 4-metric near Δ to which all the metrics in our phase space approach. Hence, we do not have the analogue of t_0^a ; only the equivalence class $[l]$ is now common to all the metrics. If, for a given metric \bar{g}_{ab} in our collection, we select the time translation represented by a specific \bar{l}^a in $[l]$, *a priori* we do not know which vector field l^a in $[l]$ would represent the ‘‘same’’ time translation for another geometry g_{ab} . One might imagine using the seemingly simplest strategy: just fix a l_0^a in $[l]$ and demand that t^a approach that l_0^a for *all* points γ in the phase space. Unfortunately, the strategy is not viable because such a t^a fails to define a Hamiltonian evolution in Γ .¹¹ Finally, if we restrict ourselves to *globally* static space-times, we can overcome this difficulty by always working with the Killing field which is unit at infinity. However, in absence of global Killing fields, the behavior of the evolution vector field t^a near the horizon is unrelated to its behavior near infinity. Nonetheless, as we shall show in Sec. VII, if one has sufficient control on the space of static solutions of the theory under consideration, it is possible to select a preferred energy function on the phase space and use it as the mass of the isolated

¹⁰In particular, therefore, Hamiltonians generating diffeomorphism which have support away from the boundaries vanish identically. Unlike their counterparts on the canonical phase space, the infinitesimal phase-space motions induced by such space-time vector fields are in the kernel of the covariant symplectic structure.

¹¹As we saw above, a necessary condition for δ_t to be a Hamiltonian vector field on Γ is that $\kappa_{(t)}$ be a function *only* of a_Δ and Q_Δ . Therefore, if we can find a tangent vector δ in the phase space with $\delta a_\Delta = \delta Q_\Delta = 0$ but $\delta \kappa_{(t)} \neq 0$, t^a cannot define a Hamiltonian evolution. It is easy to find such a tangent vector δ for this t^a .

horizon. In the Einstein-Maxwell case, all static solutions with horizons are explicitly known whence the strategy is viable.

VI. YANG-MILLS FIELD

In the previous three sections we restricted our attention to Einstein-Maxwell theory. We will now indicate how Yang-Mills fields can be included. This section is divided into three parts. In the first, we discuss restrictions on the Yang-Mills fields due to the horizon boundary conditions and introduce the notion of a ‘‘Yang-Mills gauge adapted to the horizon.’’ In the second part, we discuss the action principle and construct the covariant phase space for Einstein-Yang-Mills theory. Using this formalism, in the third subsection, we introduce a Hamiltonian generating time evolution and extend the first law to the Yang-Mills case.

A. Preliminaries

We will restrict ourselves to compact gauge groups G and Yang-Mills connections defined on trivial bundles. Since the bundle is trivial, the connection gives rise to a smooth, globally defined Lie algebra valued 1-form, \mathbf{A} . As usual, the Yang-Mills derivative operator \mathbf{D} will be defined as $\mathbf{D}\lambda = \partial\lambda + [\mathbf{A}, \lambda]$, where ∂ is a flat Yang-Mills connection, and the field strength, \mathbf{F} , via

$$\mathbf{F} := d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}. \quad (6.1)$$

The stress energy tensor, \mathbf{T} , is given in terms of the field strength as

$$\mathbf{T}_{ab} = \frac{1}{4\pi} \left[\mathbf{F}_{ac}{}^i \mathbf{F}_b{}^c{}_i - \frac{1}{4} g_{ab} F_{cd}{}^i F^{cd}{}_i \right], \quad (6.2)$$

where the label i runs over the internal indices in the Lie algebra of the group G .

Let us begin by examining how the isolated horizon boundary conditions restrict the form of the field strength, \mathbf{F} , on Δ . Since the Yang-Mills stress energy tensor has the same form as the Maxwell one, Eq. (2.14), the analysis is completely analogous to that of Sec. II C. Therefore, we shall not include derivations of the results, but instead highlight the differences.

Recall that, on a nonexpanding horizon, $R_{ab} l^a l^b \triangleq 0$, whence $T_{ab} l^a l^b \triangleq 0$. This has several consequences for the Yang-Mills field. In particular, one concludes

$$\overleftarrow{l^a \mathbf{F}_{ab}} \triangleq 0 \quad \text{and} \quad \overleftarrow{l^a \mathbf{F}_{ab}} \triangleq 0. \quad (6.3)$$

These two restrictions guarantee that there is no flux of Yang-Mills field across the horizon. Making use of the specific form of the stress-energy tensor, we also conclude

$$\Phi_{02} = \frac{1}{2} R_{ab} m^a m^b \triangleq 0, \quad \Phi_{20} = \frac{1}{2} R_{ab} \bar{m}^a \bar{m}^b \triangleq 0. \quad (6.4)$$

Our next task is to define the Yang-Mills equivalents of the electric and magnetic charges of the horizon. Naively, one might consider integrating \mathbf{F} and \mathbf{F} over a 2-sphere

cross section of the horizon as was done in the Maxwell theory. However, these 2-forms now have a free internal index and are only gauge covariant rather than gauge invariant. Since there is no preferred internal basis at the horizon, the integrals would fail to be well defined. Therefore, we must look for 2-forms which are gauge invariant. A natural quantity to consider is the norm of \mathbf{F} , defined by the Killing-Cartan form K_{ij} on the Lie algebra of G and the (contravariant) ${}^2\epsilon$ on the horizon:

$$|\mathbf{F}| := [({}^2\epsilon \cdot \mathbf{F})^i ({}^2\epsilon \cdot \mathbf{F})^j K_{ij}]^{1/2}. \quad (6.5)$$

[Although the contravariant ${}^2\epsilon$ is ambiguous up to terms of the type $l^a V^b$ where V^a is any vector field tangential to Δ , this ambiguity does not affect $|\mathbf{F}|$ because of Eq. (6.3).] The norm of \mathbf{F} is defined analogously. These two quantities are gauge invariant and allow us to define the electric and magnetic Yang-Mills charges of the horizon:

$$Q_{\Delta}^{YM} \triangleq -\frac{1}{4\pi} \oint_{S_{\Delta}} |{}^*\mathbf{F}|^2 \epsilon, \quad P_{\Delta}^{YM} \triangleq -\frac{1}{4\pi} \oint_{S_{\Delta}} |\mathbf{F}|^2 \epsilon. \quad (6.6)$$

Recall that the unusual signs in the definitions of the charges arise due to the orientation of the S_{Δ} —the normal to the sphere is inward pointing. In Maxwell theory, the magnetic charge is zero unless we consider either connections on non-trivial bundles or allow ‘‘wire singularities.’’ As is well known, this is not true for Yang-Mills theory: the magnetic charge can be nonzero even if we restrict attention to smooth fields on a trivial bundle.

We would now like to verify that the charges defined in Eqs. (6.6) are independent of the cross section of the horizon S_{Δ} on which the integration is performed. The isolated horizon boundary conditions guarantee this is the case. First, recall the geometric identity

$$\mathcal{L}_l \mathbf{F} = l \cdot \mathbf{D}\mathbf{F} - [(l \cdot \mathbf{A}), \mathbf{F}] + \mathbf{D}(l \cdot \mathbf{F}). \quad (6.7)$$

A similar expression for \mathbf{F} is also true. The first term on the right hand side vanishes due to the field equations and the third term is zero due to the previous restriction on \mathbf{F} , Eq. (6.3). Therefore at the isolated horizon,

$$\mathcal{L}_l \underline{\mathbf{F}} \triangleq -[(l \cdot \mathbf{A}), \underline{\mathbf{F}}] \quad \text{and} \quad \mathcal{L}_l \underline{\mathbf{F}} \triangleq -[(l \cdot \mathbf{A}), \underline{\mathbf{F}}]. \quad (6.8)$$

In the Maxwell case, \mathbf{F} and \mathbf{F} are Lie dragged by l . However, for non-Abelian fields, this is not a gauge invariant statement; the terms on the right hand sides of Eqs. (6.8) are necessary for gauge invariance. Although the field strength and its dual are not Lie dragged along l^a , recalling that $\mathcal{L}_l {}^2\epsilon \triangleq 0$ and using the cyclic property of the trace, it is straightforward to demonstrate that their norms are

$$\mathcal{L}_l |\mathbf{F}| \triangleq 0 \quad \text{and} \quad \mathcal{L}_l |{}^*\mathbf{F}| \triangleq 0. \quad (6.9)$$

This result, along with Eq. (6.3), guarantees that the charges Q_{Δ}^{YM} and P_{Δ}^{YM} are independent of the choice of cross section S_{Δ} of the horizon.

Let us consider the remaining components of the Yang-Mills field. The boundary conditions place no restrictions on $\mathbf{F}_{ab}n^am^b$ and ${}^*\mathbf{F}_{ab}n^am^b$ at all. As in the electromagnetic case, these components describe the radiation flowing along the horizon. The isolated horizon boundary conditions allow radiation arbitrarily close to—and even at—the horizon, provided none crosses it.

We have so far restricted our attention to the field strength and its dual. However, in the action principle and phase space, the basic variable will be the Yang-Mills connection \mathbf{A} . Let us begin with the definition of the Yang-Mills equivalent of the electric potential. Recall that, given an l , the electric potential was defined in Sec. II C as $\Phi_{(l)} \triangleq -(l \cdot \mathbf{A})$. This definition is not appropriate in the Yang-Mills case since the resulting potential has a free internal index and is therefore not gauge invariant. Instead, we define the Yang-Mills potential, $\Phi_{(l)}^{YM}$ to be negative the norm of $(l \cdot \mathbf{A})$:

$$\Phi_{(l)}^{YM} \triangleq -|(l \cdot \mathbf{A})|. \quad (6.10)$$

This gives us a gauge invariant potential at the horizon.

As in Maxwell theory, we need to constrain the form of \mathbf{A} at the horizon. Several considerations motivate our choice of these boundary conditions. First they must be chosen so that the action principle is well defined. Second, if the gauge group is $U(1)$, the boundary conditions should reduce to those given in Sec. II C for the electromagnetic field. Finally, we should be able to show that the Yang-Mills electric potential is constant on the horizon. These considerations suggest the following definition:

Definition 6. The connection \mathbf{A} will be said to be in a *gauge adapted to the isolated horizon* $(\Delta, [l])$ if it satisfies the following two conditions:

- (i) The Yang-Mills potential is constant on the horizon,

$$d\Phi_{(l)}^{YM} \triangleq 0. \quad (6.11)$$

- (ii) The dual of the field strength (${}^*\mathbf{F}$) and $(l \cdot \mathbf{A})$ point in the same Lie algebra direction,

$$(l \cdot \mathbf{A})^i \propto ({}^2\epsilon \cdot {}^*\mathbf{F})^i \quad (6.12)$$

on the horizon.

These boundary conditions satisfy the requirements discussed above. First, it is straightforward to show that in the $U(1)$ case, condition (i) is equivalent to requiring $\mathcal{L}_l \mathbf{A} \triangleq 0$ and (ii) is redundant. Second, as we shall see in Sec. VI B, these boundary conditions are also sufficient to make the variational principle well defined.

It is not difficult to show that these conditions can always be satisfied. The remaining gauge freedom is simply $\mathbf{A} \rightarrow g^{-1} \mathbf{A} g + g^{-1} \partial g$, where g satisfies $\mathcal{L}_l g \triangleq 0$.

B. Action and phase space

In this section we will consider the first order action for Einstein-Yang-Mills theory on the manifold \mathcal{M} described in Sec. III B. The basic fields will consist of the triplet $(e_a^I, A_{aI}^J, \mathbf{A}_a^i)$, where e_a^I and A_{aI}^J are the tetrad and Lorentz

connection and \mathbf{A}_a^i is the Yang-Mills connection. The gravitational fields, e_a^I and A_{aI}^J satisfy the same boundary conditions (at Δ and infinity) as in Sec. III B. Furthermore, we require the Yang-Mills fields to be in a gauge adapted to the horizon and assume they fall off sufficiently fast at infinity.¹² The Einstein-Yang-Mills action is

$$\begin{aligned} S(e, A, \mathbf{A}) = & \frac{-1}{16\pi G} \int_{\mathcal{M}} \Sigma^{IJ} \wedge F_{IJ} \\ & + \frac{1}{16\pi G} \int_{\tau_{\infty}} \Sigma^{IJ} \wedge A_{IJ} \\ & - \frac{1}{8\pi} \int_{\mathcal{M}} \text{Tr}[\mathbf{F} \wedge {}^*\mathbf{F}]. \end{aligned} \quad (6.13)$$

The gravitational part of the action has previously been discussed in Sec. III; therefore we shall only consider in detail the Yang-Mills terms and verify the variational principle is well defined. Taking into account the results of Sec. III, a variation of the action can be expressed as

$$\begin{aligned} \delta S(e, A, \mathbf{A}) = & \int_{\mathcal{M}} (\text{equations of motion}) \delta \phi \\ & - \frac{1}{4\pi} \int_{\Delta} \text{Tr}[\delta \mathbf{A} \wedge {}^*\mathbf{F}]. \end{aligned} \quad (6.14)$$

We must demonstrate the boundary term at Δ vanishes due to the conditions imposed on the Yang-Mills fields. Using Eqs. (6.13) and (6.12), one can show that the trace in Eq. (6.14) can be replaced by a product of norms:

$$\text{Tr}[\delta \mathbf{A} \wedge {}^*\mathbf{F}] = (\delta \Phi_{(l)}^{YM} - c_{\delta} \Phi_{(l)}^{YM}) |{}^*\mathbf{F}| \Delta \epsilon,$$

where $\Delta \epsilon = n \wedge {}^2\epsilon$ is the volume form on Δ and $\delta l = c_{\delta} l$.

In the action principle, variations are performed keeping data fixed at the initial and final slices. In particular $\delta \Phi_{(l)}^{YM}$ and c_{δ} vanish there. However, the boundary conditions guarantee that $\Phi_{(l)}^{YM}$, and hence its variation, is constant on Δ . Since $\delta \Phi_{(l)}^{YM}$ and c_{δ} vanish on the initial cross section of the horizon and are constant, it follows that $\delta \Phi_{(l)}^{YM} \triangleq 0$ and $c_{\delta} \triangleq 0$. Therefore, the Yang-Mills horizon boundary term vanishes; the action principle is well defined in the presence of Yang-Mills fields. As in the Einstein-Maxwell case, boundary conditions played a crucial role in demonstrating the viability of the action.

¹²More specifically we require the falloff conditions on the Yang-Mills connection to be such that all integrals, in particular the symplectic structure, be finite and yet the asymptotic electric and magnetic charges are not forced to vanish. While it is not trivial to meet these conditions (for examples the conditions used in [23,12] appear not to lead to a well-defined symplectic structure) they can be met. However, as in the rest of the paper, for brevity, we will not spell out the boundary conditions at infinity in detail.

We now wish to construct the covariant phase space and symplectic structure. As before, points in covariant phase space Γ will consist of histories which satisfy the Einstein-Yang-Mills field equations, appropriate falloff conditions at infinity and the isolated horizon boundary conditions at Δ . Before proceeding further, we shall once again need to introduce an additional field at the horizon. This can be regarded as a ‘‘potential’’ for the Yang-Mills potential Φ^{YM} . Given any point (e, A, \mathbf{A}) in the phase space Γ , let us define the scalar field χ on Δ as follows: (i) $\mathcal{L}_l \chi = -\Phi_{(l)}^{YM}$ and (ii) χ vanishes on S^- , the intersection of M^- with Δ . These conditions are identical to those imposed in the Maxwell case.

Once again, we take second variations of the action in order to obtain a symplectic structure. Since the gravitational terms are exactly the same in Sec. III, we shall only describe in detail the Yang-Mills part of the symplectic structure. The second variation of the action (6.13) yields the following symplectic current:

$$J(\gamma; \delta_1, \delta_2) = J_{\text{grav}} - \frac{1}{4\pi} \text{Tr}[\delta_1 \star \mathbf{F} \wedge \delta_2 \mathbf{A} - \delta_2 \star \mathbf{F} \wedge \delta_1 \mathbf{A}]. \quad (6.15)$$

Using the fact that the field equations and linearized field equations are satisfied, one can directly verify that $J(\gamma; \delta_1, \delta_2)$ is indeed a closed 3-form. We again choose the spacetime region of interest, $\tilde{\mathcal{M}}$, to be that part of the spacetime \mathcal{M} bounded by M_1 , M_2 , infinity and a portion $\tilde{\Delta}$ of the isolated horizon. Integrating dJ over $\tilde{\mathcal{M}}$ and using asymptotic falloff conditions, we obtain

$$\left(\int_{M_1} + \int_{M_2} + \int_{\tilde{\Delta}} \right) J(\gamma; \delta_1, \delta_2) = 0.$$

The integral of J over $\tilde{\Delta}$ does not vanish but, as in Sec. III, the pullback of J to Δ is exact. Therefore, we can express the integral over $\tilde{\Delta}$ of J as integrals over the initial and final 2-spheres S_1 and S_2 . Using these results, and keeping track of orientations, we obtain the symplectic structure

$$\begin{aligned} \Omega|_{\gamma}(\delta_1, \delta_2) = & -\frac{1}{16\pi G} \int_M [\delta_1 \Sigma^{IJ} \wedge \delta_2 A_{IJ} - \delta_2 \Sigma^{IJ} \wedge \delta_1 A_{IJ}] \\ & + \frac{1}{8\pi G} \oint_S [\delta_1 ({}^2\epsilon) \delta_2 \psi - \delta_2 ({}^2\epsilon) \delta_1 \psi] \\ & - \frac{1}{4\pi} \int_M [\delta_1 \star \mathbf{F} \wedge \delta_2 \mathbf{A} - \delta_2 \star \mathbf{F} \wedge \delta_1 \mathbf{A}] \\ & + \frac{1}{4\pi} \oint_S [\delta_1 (|\star \mathbf{F}|^2 \epsilon) \delta_2 \chi - \delta_2 (|\star \mathbf{F}|^2 \epsilon) \delta_1 \chi]. \end{aligned} \quad (6.16)$$

Full use of the isolated horizon boundary conditions has been made in obtaining this symplectic structure. In particular, to obtain the given form of the Yang-Mills surface term, we have used the fact that $(l \cdot \mathbf{A})$ and $({}^2\epsilon \cdot \star \mathbf{F})$ point in the same direction in the Lie algebra. Using field equations one can

directly verify that the right side of Eq. (6.16) is independent of the choice of the partial Cauchy surface; the symplectic structure is ‘‘conserved.’’ We will use (Γ, Ω) as our covariant phase space.

C. Hamiltonian and the first law

In this subsection we will generalize the arguments of Sec. V to obtain an expression for the energy of the horizon in Einstein-Yang-Mills theory. To do so, we must specify a time evolution vector field t^a . As before, we require t^a to be a member of the preferred equivalence class $[l^a]$ at the horizon (this again requires restriction to nonrotating isolated horizons) and approach unit time translation asymptotically. Given t^a we can calculate the infinitesimal generator of time evolution, $\delta_t = (\mathcal{L}_t e, \mathcal{L}_t A, \mathcal{L}_t \mathbf{A})$, and determine whether it is Hamiltonian. Recall that δ_t is Hamiltonian if and only if the 1-form X_t on the phase space defined by

$$X_t(\delta) := \Omega(\delta, \delta_t) \quad (6.17)$$

is closed. Let us calculate X_t . The gravitational part will be identical to the expression obtained in Sec. V; therefore we shall concentrate on the Yang-Mills terms. As with the Maxwell field, the Lie derivatives of the Yang-Mills fields can be reexpressed using the following identities:

$$\begin{aligned} \mathcal{L}_t \mathbf{A} &= t \cdot \mathbf{F} + \mathbf{D}(t \cdot \mathbf{A}), \\ \mathcal{L}_t \star \mathbf{F} &= t \cdot (\mathbf{D} \star \mathbf{F}) - [l \cdot \mathbf{A}, \star \mathbf{F}] + \mathbf{D}(t \cdot \star \mathbf{F}). \end{aligned} \quad (6.18)$$

Making use of these expressions, the field equations satisfied by (e, A, \mathbf{A}) and the linearized field equations for δ , we obtain the required expression for X_t :

$$\begin{aligned} X_t(\delta) = & \frac{-1}{16\pi G} \int_{\partial M} \text{Tr}[(t \cdot A) \delta \Sigma - (t \cdot \Sigma) \wedge \delta A] \\ & - \frac{1}{4\pi} \int_{\partial M} \text{Tr}[(t \cdot \mathbf{A}) \delta(\star \mathbf{F}) - (t \cdot \star \mathbf{F}) \wedge \delta \mathbf{A}], \end{aligned} \quad (6.19)$$

As before the expression involves integrals *only* over the 2-sphere boundaries S_∞ and S_Δ of M ; there is no volume term. The gravitational terms yield $\delta E'_{\text{ADM}}$ at infinity and $-(1/8\pi G) \kappa_{(l)} \delta a_\Delta$ at the horizon. The Yang-Mills term at infinity vanishes due to falloff conditions: therefore we need only calculate the Yang-Mills contribution at the horizon. This is composed of two terms, the second of which vanishes due to the restriction (6.3) which guarantees $l \cdot \star \mathbf{F} \triangleq 0$. Since we are in a gauge adapted to the horizon, $(l \cdot \mathbf{A})$ and $\star \mathbf{F}$ point in the same internal direction. This allows us to replace the trace in the first term by norms,

$$\text{Tr}[(t \cdot \mathbf{A}) \delta(\star \mathbf{F})] = -\Phi_{(t)}^{YM} \delta(|\star \mathbf{F}|^2 \epsilon),$$

and guarantees that $\Phi_{(t)}^{YM}$ is constant on the horizon. Making use of the definition of Yang-Mills electric charge Eq. (6.6), we obtain

$$X_t(\delta) = -\frac{1}{8\pi G} \kappa_{(t)} \delta a_\Delta - \Phi_{(t)}^{YM} \delta Q_\Delta^{YM} + \delta E_{ADM}^t.$$

Recall that the necessary and sufficient condition for the existence of a Hamiltonian is that X_t be closed. Clearly, this is equivalent to

$$\frac{1}{8\pi G} \mathbb{d}\kappa_{(t)} \wedge \mathbb{d}a_\Delta + \mathbb{d}\Phi_{(t)}^{YM} \wedge \mathbb{d}Q_\Delta^{YM} = 0. \quad (6.20)$$

Once again, we conclude that the surface gravity $\kappa_{(t)}$ and the Yang-Mills potential $\Phi_{(t)}^{YM}$ at the horizon defined by t^a can depend *only* upon the area a_Δ and charge Q_Δ^{YM} of the horizon. Finally, Eq. (6.20) is also the necessary and sufficient condition that there exist a function E_Δ^t , also only of a_Δ and Q_Δ^{YM} such that

$$\delta E_\Delta^t = \frac{1}{8\pi G} \kappa_{(t)} \delta a_\Delta + \Phi_{(t)}^{YM} \delta Q_\Delta^{YM}. \quad (6.21)$$

As before, E_Δ^t is interpreted as the horizon energy defined by the time translation t^a . We conclude that the vector field δ_t is Hamiltonian if and only if the first law, Eq. (6.21), holds.

We will conclude this section with a few remarks.

(i) The derivation of the first law and its final form are completely analogous to those in the Einstein-Maxwell theory. By contrast, in the discussion of the first law for undistorted isolated horizons of [12], certain restrictions were imposed on the permissible variations δ in the Einstein-Yang-Mills case. In our treatment, subtleties arise only in the definition of a canonical horizon mass (see Sec. VII C) rather than the discussion of the first law itself.

(ii) Although the Yang-Mills magnetic charge P_Δ^{YM} will generically not be zero, no term involving δP_Δ^{YM} arises in the first law.

(iii) How does our result compare with those previously available [23,24]? In [24] the first law for Yang-Mills fields is proved for globally stationary spacetimes and small perturbations from one such space-time to another. Assuming the Yang-Mills fields fall off sufficiently fast at infinity, (in the nonrotating case) the first law of [24] then reads

$$\delta M = \frac{1}{8\pi G} \kappa \delta a + \int_{\text{for}} \text{Tr}[\phi \delta^* \mathbf{F}].$$

Here, M is the ADM mass evaluated at infinity, while all terms on the right hand side are evaluated at the horizon. Because of a different gauge choice at the horizon, the authors define a Lie algebra valued potential ϕ and leave the “ $\Phi \delta Q$ ” term inside an integral. However, the general form of this first law is the same as ours. In this sense, our framework generalizes the results of [24] to nonstatic contexts.

In [23], the first law is proved for globally stationary space-times and arbitrary small departures therefrom. However, there are a number of important differences between these results and the ones obtained in this paper. In the nonrotating case, the first law of [23] reads

$$\delta M_{ADM} + V \delta Q = \frac{1}{8\pi G} \kappa \delta a$$

where V and Q are the Yang-Mills potential and charge evaluated at *infinity* while κ and a are of course evaluated at the horizon. Because of the non-Abelian nature of the Yang-Mills field, unlike in the Maxwell case, the charge Q evaluated at infinity is now quite different from the charge evaluated at the horizon and, as in the Maxwell theory, the potential V evaluated at infinity has no direct bearing on the potential at the horizon. Furthermore, that calculation makes an essential use of the bifurcation 2-sphere and all fields are required to be smooth there. This restriction implies that the Yang-Mills potential at the horizon *vanishes*. (The same is true if one restricts the analysis of [23] to the Maxwell case.)

The first law derived in the isolated horizon framework is valid also in presence of radiation in the exterior space-time region and makes no reference to the bifurcation 2-sphere. (Although we restricted ourselves to the nonrotating case, rotation has been incorporated in this framework in [13].) Furthermore, it has the aesthetically pleasing feature that all quantities that appear in Eq. (6.21)—including the energy E_Δ^t , the potential $\Phi_{(t)}^{YM}$, and the charge Q_Δ^{YM} —are evaluated at the horizon. In particular, one can now meaningfully consider the physical process version in which one does an experiment *at* the horizon by dropping a test particle and field and changing the *horizon charge* infinitesimally. More generally Eq. (6.21) is genuinely a law governing the mechanics of the *horizon*.

VII. HORIZON MASS

For notational simplicity, we will say that a (live) vector field t^a is *permissible* if it gives rise to a Hamiltonian evolution. We saw in Secs. V and VI that each permissible vector field t^a defines a horizon energy E_Δ^t . In the phase space framework, E_Δ^t has a direct interpretation: it is the surface term at the horizon in the expression of the Hamiltonian generating the t^a evolution. However, in many physical applications—such as the study of black hole mergers—one is interested in properties of a *specific* space-time, rather than the full phase space. Then, it is useful to have at one’s disposal a canonical notion of energy, the analog of the ADM energy in the rest frame at infinity. This quantity could then be interpreted as the horizon mass. In this section, we will introduce this notion in detail. The discussion is divided into three parts. In the first, we consider the Einstein-Maxwell theory; in the second, we discuss dilatonic couplings [9]; and, in the third, we analyze the Einstein-Yang-Mills system.

A. Einstein-Maxwell theory

In Sec. V we showed that t^a is permissible if and only if Eq. (5.7) holds on the phase space. We will now construct a large family of permissible evolution fields t^a . Fix any regular function κ_0 of two variables a_Δ and Q_Δ . Then, given any point $\gamma \equiv (e, A, \mathbf{A})$ of Γ , we define (the boundary value of) the vector field t^a as follows. Consider the vector field l^a on Δ defined by the tetrad, $l^a = e_l^a l^l$, and denote by $\kappa_{(l)}$ the

surface gravity associated with it. Then, $\kappa_0 = c \kappa_{(t)}$ for some constant c . Let us set $t^a = c l^a$. Repeating this procedure at each phase-space point γ , we obtain a live vector field t^a with $\kappa_{(t)} = \kappa_0$. (The resulting c will be constant on Δ but a function on the phase space.) Next, consider the electromagnetic potential, which is guaranteed to be constant on Δ by our boundary conditions but whose value at any phase space point is so far completely free. We will now use Eq. (5.7) to fix it. Equation (5.7) implies

$$\frac{\partial \kappa_{(t)}}{\partial Q_\Delta} = \frac{\partial \Phi_{(t)}}{\partial a_\Delta}.$$

Since $\kappa_{(t)} = \kappa_0$ is known, we can simply integrate the equation for $\Phi_{(t)}$ as a function of a_Δ and Q_Δ . Furthermore, the solution is unique if we impose the physical condition that $\Phi_{(t)}$ should vanish whenever $Q_\Delta = 0$. Thus, starting from any regular function κ_0 of a_Δ and Q_Δ , we have obtained a permissible evolution field t^a . Conversely, it is easy to verify that every permissible vector field arises via this construction. There is clearly a very large family of such live vector fields.

An obvious question is if there is a ‘‘canonical’’ or ‘‘natural’’ choice of t^a ? We will now show that the answer is in the affirmative. Recall that, in the Einstein-Maxwell theory, there is precisely a 2-parameter family of *globally static* solutions admitting horizons: the Reissner-Nordström family. [Since \mathbf{A} is required to be a globally defined connection on a trivial $U(1)$ bundle, the magnetic charge is zero on the entire phase space.] Let us focus on this family. Denote by ξ^a the static Killing field which is *unit* at infinity. Its surface gravity is a specific function of a_Δ and Q_Δ :

$$\kappa_{(\xi)} = \frac{1}{2R_\Delta} \left(1 - \frac{GQ_\Delta^2}{R_\Delta^2} \right).$$

As before, R_Δ is the horizon radius, defined by $a_\Delta = 4\pi R_\Delta^2$. We can therefore use $\kappa_{(\xi)}$ in place of κ_0 in the above construction. The resulting permissible, live vector field t_0^a agrees with ξ^a on the horizon of every static solution. This property is satisfied *only if* we set $\kappa_0 = \kappa_{(\xi)}$.

Next, we can ‘‘integrate’’ Eq. (5.8) to obtain the horizon energy $E_\Delta^{t_0}$. Although *a priori* there is the freedom to add a constant, we can fix it by requiring that the energy vanish as a_Δ and Q_Δ tend to zero. Indeed, we have no choice in this since one cannot construct a quantity with dimensions of mass from the fundamental constants that appear in the Einstein-Maxwell theory. (Einstein-Yang-Mills theory does admit such a constant and we will see in Sec. VII C that it leads to an interesting modification of the situation discussed here.) Let us define the horizon mass via

$$M_\Delta = E_\Delta^{t_0}.$$

To justify this definition, let us begin by restricting ourselves to static solutions. In each static solution, we are free to extend t_0^a such that it coincides with the Killing field ξ^a . General symplectic arguments imply that, on any connected component of the space of static solutions, the numerical

value of the total Hamiltonian, generating evolution along ξ^a , must be constant (see, e.g., [8]). In the Einstein-Maxwell case, there is a single connected component and, by the dimensional argument given above, the numerical value of the Hamiltonian must vanish on it. Hence, from Eq. (5.9) it follows that, on any static solution,

$$H_{t_0} = M_{\text{ADM}} - M_\Delta = 0.$$

On a general solution, of course, M_{ADM} would be greater than M_Δ , the difference being equal to the energy in radiation. If the horizon is complete in the future and timelike infinity i^+ satisfies certain regularity conditions, as in [8] one can argue that the difference is precisely the total energy radiated across \mathcal{I}^+ and hence M_Δ equals the future limit of the Bondi mass. These considerations support our interpretation of M_Δ as the horizon mass.

Finally, since we now have a canonical evolution field t_0^a , we can drop the suffix t on surface gravity and electromagnetic potential and write the first law (5.8) in the more familiar form

$$\delta M_\Delta = \frac{1}{8\pi G} \kappa \delta a_\Delta + \Phi \delta Q_\Delta.$$

In contrast to treatments based on static space-times, the quantities that enter this law are all defined *at* the horizon. Therefore, as pointed out in [8], it is now possible to interpret this law also in the ‘‘active’’ sense where one considers physical processes which increase the area and the charge of a given horizon. To our knowledge, the standard proofs of this physical version [5,10] are not applicable to processes in which the background has nonzero electric charge and the process changes it infinitesimally.

We will conclude with a few remarks.

(i) In the above discussion, the permissible evolution field t_0^a was constructed by setting $t_0^a = c l^a$ where c is given by $\kappa_0 \equiv (1/2R_\Delta)[1 - G(Q_\Delta/R_\Delta)^2] = c \kappa_{(l)}$. For c to be well defined, it is necessary that κ_0 vanishes whenever $\kappa_{(l)}$ does. Therefore, for the mass to be well defined, we must excise those points from the phase space at which $\kappa_{(l)}$ vanishes but κ_0 does not. However, this is not a serious limitation. In particular, we still retain all static solutions *including the extremal ones* at which κ_0 vanishes.

(ii) Since we have a specific κ_0 , we can use Eq. (5.7) to obtain the corresponding electrostatic potential: $\Phi_0 = Q_\Delta/R_\Delta$. Furthermore, by integrating Eq. (5.8) it is easy to express M_Δ explicitly in terms of the horizon parameters:

$$M_\Delta = \frac{1}{4\pi G} \kappa a_\Delta + \Phi Q_\Delta = \frac{R_\Delta}{2G} \left(1 + \frac{GQ_\Delta^2}{R_\Delta^2} \right). \quad (7.1)$$

Thus, the functional dependence of M_Δ on the horizon parameters at any point of the phase space is the same as in static space-times. Note that this is a *result* of the framework, not an assumption. Its derivation involved two distinct steps. First, and most importantly, the first law (5.8) arose as a necessary and sufficient condition for the existence of a consistent Hamiltonian framework. Second, the freedom in t^a

was exploited in order to construct the preferred, permissible evolution field t_0^a . It is quite significant that M_Δ can be expressed so simply using just the parameters defined *locally* at the horizon even when there is radiation arbitrarily close to it. This fact is likely to play an important role in the problem of extracting physics in the strong field regimes from numerical simulations of black hole collisions [18]. It is important to notice that although we made use of our knowledge of static solutions to arrive at a canonical t_0^a and the mass function M_Δ , the final result (7.1) *makes no reference* to these solutions. M_Δ is a simple function of the parameters which can be directly computed from the geometry of any one isolated horizon.

(iii) In the earlier work [8] on undistorted horizons, one restricted oneself to the preferred evolution field t_0^a from the very beginning (although this vector field was selected using a different but equivalent procedure).¹³ The *a priori* freedom in the choice of a permissible t^a was not discussed and the first law appeared only in the more familiar form, given above.

B. Dilatonic coupling

The Einstein-Maxwell-dilaton system was studied in some detail in the undistorted case in [9]. We will revisit it here in the more general context considered in this paper because it brings out a subtlety in the definition of the horizon mass M_Δ and the associated first law.

The dilaton is a scalar field ϕ which can couple to the Maxwell field in a nonstandard fashion. The coupling is governed by a constant α . If $\alpha=0$, one obtains the standard Einstein-Maxwell-Klein-Gordon theory and the situation then is completely analogous to the Einstein-Maxwell theory considered above. If $\alpha=1$, the theory represents the low energy limit of string theory. In this case, there are some interesting differences from the Einstein-Maxwell theory considered in this paper. To bring out these differences, in this subsection we will set $\alpha=1$. (The situation for a general value of α is discussed in [9] where one can also find details on the material summarized below.)

In the standard formulation, the theory has three charges, all defined at infinity; the ADM mass M_{ADM} , the usual electric charge Q_∞ , and another charge \tilde{Q}_∞ :

$$Q_\infty = \frac{1}{4\pi} \oint_{S_\infty} \star \mathbf{F} \quad \text{and} \quad \tilde{Q}_\infty = \frac{1}{4\pi} \oint_{S_\infty} e^{-2\phi} \star \mathbf{F}.$$

¹³This strategy seems to have generated a misunderstanding (see, e.g., [29]) that the first law was obtained in [8,9,12] merely by identifying the parameters labeling a general isolated horizon with those of static horizons and then using the Smarr formulas available in the static context. This was not the case. Rather, static solutions were used *only* to select the appropriate normalization of the evolution vector field t^a at the horizon. The Hamiltonian framework was then used to define the horizon mass without any reference to Smarr formulas. As in this section, the mass was then *shown* to reproduce the Smarr-type formulas on general horizons.

\tilde{Q} is conserved in space-time (i.e., its value does not change if the 2-sphere of integration is deformed) while Q is not. From the perspective of the isolated horizons, it is more useful to use $a_\Delta, Q_\Delta, \tilde{Q}_\Delta$ as the basic charges:¹⁴

$$Q_\Delta = \frac{1}{4\pi} \oint_{S_\Delta} \star \mathbf{F}, \quad \tilde{Q}_\Delta = \frac{1}{4\pi} \oint_{S_\Delta} e^{-2\phi} \star \mathbf{F}.$$

Although the standard electric charge is not conserved in space-time, it *is* conserved along Δ , whence Q_Δ is well defined.

It is straightforward to extend the construction of the phase space to include the dilaton. The only difference is that the charge Q in Eqs. (5.6)–(5.8) is replaced by \tilde{Q} . With this minor change, the discussion of the first part of Sec. VII A is also unaffected. Thus, given any function κ_0 of a_Δ and \tilde{Q}_Δ , we can construct a permissible, (live) evolution field t^a .

The difference arises in the next step where we constructed a preferred t_0^a . With the dilatonic coupling, the theory has a unique [25] *three* parameter family of static solutions which can be labeled by $(a_\Delta, Q_\Delta, \tilde{Q}_\Delta)$. As in the Reissner-Nordström family, these solutions are spherically symmetric. In terms of these parameters, the surface gravity $\kappa_{(\xi)}$ of the static Killing field which is unit at infinity is given by

$$\kappa_{(\xi)} = \frac{1}{2R_\Delta} \left[1 + 2G \frac{Q_\Delta \tilde{Q}_\Delta}{R_\Delta^2} \right] \left[1 - 2G \frac{Q_\Delta \tilde{Q}_\Delta}{R_\Delta^2} \right]^{-1/2}.$$

The problem in the construction of the preferred t_0^a is that we need a function κ_0 which depends only on a_Δ and \tilde{Q}_Δ . Therefore, we can no longer set $\kappa_0 = \kappa_{(\xi)}$ on the entire phase space because $\kappa_{(\xi)}$ depends on all three horizon parameters.

To extract the mass function M_Δ on the phase space, we can proceed as follows. Let us foliate Γ by $Q_\Delta = \text{const}$ surfaces. On each leaf, $\kappa_{(\xi)}$ trivially depends only on a_Δ and \tilde{Q}_Δ and so we can set $\kappa_0 = \kappa_{(\xi)}$. Therefore, by the procedure outlined in Sec. VII A, we obtain a (live) vector field t_0^a and can define the mass $M_\Delta(\gamma) = E_\Delta^{t_0}(\gamma)$ for all points γ on this leaf. Repeating this procedure for each leaf, we obtain a live vector field t_0^a and a mass function M_Δ everywhere on Γ . However, the surface gravity $\kappa_{(t_0)}$ now depends on all three parameters, rather than just a_Δ and \tilde{Q}_Δ . Therefore, the first law (5.8) cannot hold for arbitrary variations δ and consequently δ_{t_0} fails to be a Hamiltonian vector field. Put differently, although there is a multitude of permissible, live vector fields, each leading to a first law, none of them can coincide with the Killing field ξ^a (which is unit at infinity) on *all* static solutions. This is a significant departure from the Einstein-Maxwell case considered above.

Nonetheless (modulo the caveat discussed in the first remark at the end of Sec. VII A), the above procedure does

¹⁴In the undistorted case, the dilaton is constant on Δ and hence we can replace Q_Δ by ϕ_Δ as in [9].

provide us with a well-defined mass function M_Δ on the entire phase space which can be expressed in terms of the horizon parameters as

$$M_\Delta = \frac{R_\Delta}{2G} \left[1 - 2G \frac{Q_\Delta \tilde{Q}_\Delta}{R_\Delta^2} \right]^{-1/2}.$$

It equals M_{ADM} in static space-times and has other properties which motivated our interpretation of M_Δ as the horizon mass in the Einstein-Maxwell case. Since this function is well defined on the entire phase space, we can simply vary it and express the result in terms of the horizon parameters. The result is

$$\delta M_\Delta = \frac{1}{8\pi G} \kappa \delta a_\Delta + \hat{\Phi} \delta \hat{Q}_\Delta$$

where $\kappa = \kappa_{(t_0)}$, $\hat{\Phi}^2 = (Q_\Delta \tilde{Q}_\Delta / R_\Delta^2)$, and $\hat{Q}_\Delta^2 = Q_\Delta \tilde{Q}_\Delta$. Thus, although there is still a first law in terms of t_0^a and M_Δ , it does not have the canonical form (5.8) because t_0^a is not a permissible vector field. More generally, in theories with multiple scalar fields [26], if one focuses only on static sectors, one obtains similar ‘‘nonstandard’’ forms of the first law with work terms involving scalar fields. This reflects the fact that there is no permissible vector field t^a , defined for all points of the phase space, which coincides with the properly normalized Killing field on *all* static solutions. In the undistorted case, the analysis was carried out only in terms of the vector field t_0^a and the horizon mass M_Δ [9]. The resulting first law had the above form.

Alternatively, one can restrict oneself to variations $\bar{\delta}$ which are tangential to the leaves of the phase space foliation constructed above. Since t_0^a is a permissible vector field for any one leaf, we obtain the standard first law

$$\bar{\delta} M_\Delta = (1/8\pi G) \kappa_{(t_0)} \bar{\delta} a_\Delta + \bar{\Phi}_{(t_0)} \bar{\delta} \tilde{Q}_\Delta$$

for the restricted variations. The idea of using such restricted variations was suggested in [12] in the context of Yang-Mills fields (although the foliations and other details were not spelled out there).

To summarize, because there is now a three parameter family of static solutions rather than two—or, more precisely, because the standard surface gravity $\kappa_{(\xi)}$ in static space-times depends on a_Δ , \tilde{Q}_Δ , and Q_Δ —a canonical, permissible evolution field is no longer available. However, there is still a multitude of permissible evolution fields and corresponding first laws. Furthermore, one can still define a canonical mass function M_Δ on the entire phase space.

C. Yang-Mills fields

In Einstein-Maxwell theory, with and without the dilaton, there is no way to construct a quantity with the dimensions of mass from the fundamental constants in the theory. The situation is different for Einstein-Yang-Mills theory because the coupling constant g has dimensions $(LM)^{-1/2}$. The exist-

ence of a quantity with units of mass has interesting consequences which we will now discuss.

Let us begin with a summary of the known static solutions in Yang-Mills theory. First, the Reissner-Nordström family constitutes a continuous two parameter set of static solutions of the Einstein-Yang-Mills theory, labeled by $(a_\Delta, Q_\Delta^{YM})$. In addition, there is a one-parameter family of ‘‘embedded Abelian solutions’’ with (a fixed) magnetic charge P_Δ^0 , labeled by (a_Δ, P_Δ^0) . Finally, there are families of ‘‘genuinely non-Abelian solutions.’’ For these, the analogue of the Israel theorem for Einstein-Maxwell theory fails to hold [27]; the theory admits static solutions which need not be spherically symmetric. In particular, an infinite family of solutions labeled by two integers (n_1, n_2) is known to exist. All static, spherically symmetric solutions are known and they correspond to the infinite subfamily $(n_1, n_2 = 0)$, labeled by a single integer. However, the two parameter family is obtained using a specific ansatz, so there may well exist other static solutions. Although the available information on static solutions is quite rich, in contrast to the Einstein-Maxwell-dilaton system, one is still rather far from having complete control of the static sector of the Einstein-Yang-Mills theory.

The zeroth and first laws do hold in the Einstein-Yang-Mills case. At present, however, we can only hope to repeat the strategy used in the last two subsections to define a canonical mass function M_Δ on portions of the phase space. In order to define it on the full phase space, the ‘‘uniqueness’’ and ‘‘completeness’’ conjectures of [12] will have to hold (possibly with a suitable modification).¹⁵ Nevertheless, new insight into the static solutions can be obtained by restricting our attention to certain leaves of the phase space. The basic idea is taken from [12] but applied in a slightly different manner to the more general context of distorted horizons.

Consider a connected component of the known static solutions, labeled by $\vec{n} \equiv (n_1, n_2)$. This is a one-dimensional subspace of the phase space which we denote $S_{\vec{n}}$. Each point in $S_{\vec{n}}$ can be labeled by the value of the horizon area a_Δ . Calculate the surface gravity $\kappa_{(\xi)}$ for this family, where ξ^a is the static Killing field which is unit at infinity, and set $\kappa_0 = \kappa_{(\xi)}$ in the construction sketched in Sec. VII A. We then obtain a live vector field t_0^a , and the corresponding first law

$$\delta E_\Delta^{t_0} = \frac{1}{8\pi G} \kappa_{(t_0)} \delta a_\Delta$$

on the full phase space.

When restricted to $S_{\vec{n}}$, we can interpret $E_\Delta^{t_0}$ as the horizon mass $M_\Delta^{(\vec{n})}$ and replace $\kappa_{(t_0)}$ by the function $\beta_{\vec{n}}(a_\Delta)$ used in the literature: $\beta_{\vec{n}} = 2\kappa_{(t_0)} R_\Delta$. Then, by integrating the first law along $S_{\vec{n}}$, one obtains

¹⁵For example it may be appropriate to restrict oneself to the class of space-times admitting isolated horizons which are complete in the future. Physically, this is the most interesting case since such horizons would represent the asymptotic geometry resulting from a gravitational collapse or black hole mergers.

$$M_{\Delta}^{(\vec{n})} = \frac{1}{2G} \int_0^{R_{\Delta}} \beta_{\vec{n}}(x) dx$$

where we have used the fact that, since $E_{\Delta}^{(t_0)}$ is a surface integral at Δ , it vanishes as the horizon area goes to zero. Thus, the horizon mass is completely determined by $\beta_{\vec{n}}(a_{\Delta})$.

Next, we use the fact that the Hamiltonian given by $H_{t_0} = E_{\text{ADM}}^{t_0} - E_{\Delta}^{t_0}$ [see Eq. (5.9)] is constant on each connected, static sector if t_0^a coincides with the static Killing field on the entire sector. By construction, our t_0^a has this property. In the Einstein-Maxwell case, since there is no constant with the dimension of energy, it follows that the restriction of $H_{\text{ADM}}^{t_0}$ to the static sector must vanish. The situation is quite different in Einstein-Yang-Mills theory where the Yang-Mills coupling constant g provides a scale. In $c=1$ units, $(g\sqrt{G})^{-1} \sim \text{mass}$. Therefore, we can only conclude

$$M_{\text{ADM}}^{(\vec{n})} = M_{\Delta}^{(\vec{n})} + (g\sqrt{G})^{-1} C^{(\vec{n})}$$

for some \vec{n} -dependent constant $C^{(\vec{n})}$. As the horizon radius shrinks to zero, the static solution [27,28] under consideration tends to the solitonic solution with the same ‘‘quantum numbers’’ \vec{n} . Hence, by taking this limit, we conclude $(g\sqrt{G})^{-1} C^{(\vec{n})} = M_{\text{ADM}}^{\text{soliton},(\vec{n})}$. Therefore, we have the following interesting relation between the black hole and solitonic solutions:

$$M_{\text{ADM}}^{\text{BH},(\vec{n})} = \frac{1}{2G} \int_0^{R_{\Delta}} \beta_{\vec{n}}(x) dx + M_{\text{ADM}}^{\text{soliton},(\vec{n})}$$

where the integral of $\beta_{\vec{n}}$ is evaluated on the one-dimensional ‘‘parameter space’’ of $S_{\vec{n}}$ (given by the horizon radius). Furthermore, as is clear from the above discussion, the ADM mass of the soliton is a multiple of $(g\sqrt{G})^{-1}$. Thus, somewhat surprisingly, the derivation of the first law in the isolated horizon framework has led to an interesting relation between the ADM masses of black holes and their solitonic analogs in the *static sector*.

VIII. DISCUSSION

In the first part of this paper, we introduced the notions of weakly isolated and isolated horizons. In contrast with earlier work [8,9,12], the definitions allow for the possible presence of distortion and rotation at the horizon. In addition, the present definitions are more geometric and intrinsic; in particular, they never refer to a foliation.

The notion of an isolated horizon, unlike that of an event horizon, is completely quasilocal. One can test if a given 3-surface in space-time is (weakly) isolated or not simply by examining space-time geometry at the surface. Furthermore, space-times admitting an isolated horizon Δ need not admit *any* Killing field even in a neighborhood of Δ . In particular, they can admit radiation in the exterior region. Therefore, such space-times can serve as more realistic models of late stages of a gravitational collapse or black hole merger. In

[18] the near Δ geometry of vacuum solutions is examined in detail using similar techniques to those used at null infinity. The resulting structure—the presence of a preferred ‘‘rest frame,’’ constraints on possible isometrics, Bondi-type expansions of the metric—should be useful to extract physics in the strong field regime of general relativity, especially in the problem of binary black hole collisions.

This paper, however, focused on another aspect of isolated horizons: extensions of the zeroth and first laws of black hole mechanics. All previous discussions of these laws were restricted to perturbations of stationary black holes. Using Lagrangian and Hamiltonian frameworks, we extended these laws to arbitrary space-times admitting nonrotating isolated horizons in Einstein-Maxwell-dilaton and Einstein-Yang-Mills theory. Furthermore, the analysis suggests that it should be rather easy to incorporate other forms of matter, provided they admit Lagrangian and Hamiltonian descriptions.

The generalization of black hole mechanics presented in this paper has several interesting features. First, all quantities that enter the first laws are defined *locally* at the horizon Δ . In standard treatments, some quantities such as area and surface gravity are defined at the horizon. Others, like energy and sometimes [23] even the Yang-Mills–Maxwell charge and potential, are evaluated at infinity. In part because of this ‘‘mismatch,’’ to our knowledge the ‘‘physical process version’’ of the first law [10] had not previously been established for processes which change the charge of the black hole. Since all quantities in the present treatment are defined locally at the horizon, it is now straightforward to establish the law for such processes [8]. Second, other treatments based on a Hamiltonian framework [10,32,23] often critically use the bifurcate 2-surface which does not exist in the extremal case. Therefore, extremal black holes are often excluded from the first law. The present analysis never makes reference to bifurcate surfaces (which do not exist in physical space-times resulting from gravitational collapse). Therefore, our discussion of the first law holds also in the extremal case. Third, with obvious modifications of boundary conditions at infinity, our analysis includes cosmological horizons where thermodynamic considerations are also applicable [11].

Finally, and perhaps most importantly, our analysis sheds new light on the ‘‘origin’’ of the first law: it arose as a necessary and sufficient condition for the existence of a Hamiltonian generating time evolution. A new feature of our framework is the existence of an infinite family of first laws corresponding to the infinite family of ‘‘permissible’’ vector fields t^a . (A vector field t^a is permissible if it is Hamiltonian, that is, induces canonical transformations on the phase space.) In theories where we have sufficient control on the space of static solutions, such as Einstein-Maxwell, one can select a natural evolution field t_0^a . Corresponding to evolution along this t_0^a , there is a canonical notion of energy which can be interpreted as the mass of the isolated horizon. There exist also preferred values of surface gravity and electric potential and a canonical first law. This additional structure is extremely useful in other applications of the framework, such as extraction of physical information from numerical

simulations of black-hole collisions. However, it is not essential to the discussion of mechanics: our derivation of the first law in Secs. V and VI does *not* require any knowledge of the static sector of the theory.

The Hamiltonian approach to black hole mechanics has appeared in the literature before, most notably in the work of Brown and York [14]. The spirit of the Brown-York approach is similar to ours. In particular, they do not restrict themselves to stationary situations. However, in that work, the focus is on an outer, timelike boundary whereas our focus is on the inner, null boundary representing the isolated horizon. Conserved quantities in presence of internal boundaries were recently discussed also by Julia and Silva [30] in a more general context of theories with gauge symmetries. As in our framework, their treatment exploits the simplifications that occur in a first order formalism and the final surface-integral expressions of conserved charges are dictated by the precise boundary conditions imposed at the internal boundaries. Their treatment is based on superpotentials and thus complements the Hamiltonian methods used here and in [14].

In this paper, the Lagrangian and Hamiltonian frameworks are based on real tetrads and Lorentz connections. It is therefore quite straightforward to extend our analysis to any space-time dimension. Indeed, it has already been extended to 2 + 1 dimensions in [31]. However, our phase space—and especially the explicit symplectic structure used here—is tailored to the Einstein-matter system. While it should be possible to extend it to higher derivative theories of gravity as in [32], that task would not be as simple.

ACKNOWLEDGMENTS

We are grateful to Chris Beetle and Jerzy Lewandowski for countless discussions. We thank Alejandro Corichi and Daniel Sudarsky for stimulating correspondence, Jiri Bicak and Werner Israel for pointing out references on distorted black holes, and Thibault Damour, Sean Hayward, and Bob Wald for helpful comments. This work was supported in part by the NSF grants PHY94-07194, PHY95-14240, INT97-22514 and by the Eberly research funds of Penn State.

APPENDIX A: EXAMPLES OF DISTORTED HORIZONS

Because of the no hair theorems in Einstein-Maxwell theory, distorted horizons have received a rather limited attention in the literature. Therefore, in this appendix we will discuss a few explicit examples in Einstein-Maxwell theory. For a general construction and an existence result, see [18,33].

To obtain explicit solutions, one has to impose symmetries. All solutions considered in this section will be static and axisymmetric. As one would expect from the no-hair theorems, they fail to be asymptotically flat, whence they fail to represent isolated black holes in the standard sense. Nonetheless, they all satisfy the isolated horizon boundary conditions. That framework also serves to “explain” the otherwise surprising feature that the surface gravity of these solutions depends only on the area and the charge and is

insensitive to the distortion parameters.

In the literature on static, distorted black holes, it is generally assumed that the solution is valid only in a finite region around the horizon and its distant behavior is suitably modified by the far-away matter which causes the distortion. For undistorted isolated horizons, Robinson-Trautman space-time [15] offer interesting examples of vacuum, asymptotically flat solutions which admit isolated horizons but no Killing fields whatsoever. Distorted analogues of these solutions are not known but presumably exist. It would be interesting to find them.

1. Black hole in a magnetic universe

Let us begin with a simple example: an uncharged black hole in an “external magnetic field” which distorts the horizon. The specific solution we wish to consider is static and axisymmetric and was first obtained in the Ernst-potential framework [34]. The magnetic field is nonzero on the horizon. Thus, one has to consider the full set of Einstein-Maxwell equations on the horizon.

The space-time has topology $S^2 \times \mathbb{R}^2$ and the metric is given by

$$ds^2 = F^2 \left[- \left(1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\theta^2 \right] + \frac{r^2 \sin^2 \theta}{F^2} d\phi^2 \quad (\text{A1})$$

where

$$F = 1 + \frac{1}{4} B_0^2 r^2 \sin^2 \theta.$$

B_0 is a constant and on the axis the magnetic field is given by $B = B_0 dr$. Because F diverges at infinity, the metric fails to be asymptotically flat. For $M = 0$, the metric reduces to that of the Melvin universe and for $M \neq 0$ it admits a Killing horizon at $r = 2M$. To examine the behavior at the horizon, let us first cast the metric in the Eddington-Finkelstein coordinates (v, r, θ, ϕ) where $dv = dt + (1 - 2M/r)^{-1} dr$:

$$ds^2 = -F^2 \left(1 - \frac{2M}{r} \right) dv^2 + 2F^2 dv dr + F^2 r^2 d\theta^2 + \frac{r^2 \sin^2 \theta}{F^2} d\phi^2. \quad (\text{A2})$$

Since the metric is not asymptotically flat, the standard procedure of normalizing the Killing field to be unit at infinity is not applicable. Thus, we only have an equivalence class $[l]$ of (preferred) null normals to the horizon, $l \propto \partial/\partial v$. Let Δ be the Killing horizon and assume the Killing field $\partial/\partial v$ is a member of the equivalence class $[l]$. It follows trivially that $(\Delta, [l])$ is a nonrotating isolated horizon. If $B_0 \neq 0$, the scalar curvature 2R of the horizon 2-metric has θ dependence; the horizon is distorted. However, an explicit calculation shows that, as in the Schwarzschild space-time, the surface gravity κ is given by $1/2r \equiv 1/2R_\Delta$ and the electrostatic potential Φ

vanishes on Δ . At first, it is quite surprising that while the presence of distortion affects μ , Φ_{11} , Ψ_2 , and 2R , it does not affect κ or Φ . However, as we saw in Sec. V, this result is to be expected from the general Hamiltonian considerations.

2. Distorted black holes as special cases of Weyl solutions

In this subsection, we will review the construction of a large family of distorted black holes starting from Weyl solutions [35] and a recent generalization of these results to include electric charge [36].

A general class of static, axisymmetric spacetimes was found by Weyl in 1917 [37]. The metric for such a spacetime can be cast in the following form:

$$ds^2 = -e^{2\psi} dt^2 + e^{2(\gamma-\psi)}(d\rho^2 + dz^2) + e^{-2\psi} \rho^2 d\phi^2 \quad (\text{A3})$$

where ψ and γ are smooth functions of ρ and z . Einstein's vacuum equations expressed in terms of ψ and γ take a particularly simple form. The equation for ψ ,

$$\psi_{,\rho\rho} + \frac{\psi_{,\rho}}{\rho} + \psi_{,zz} = 0, \quad (\text{A4})$$

is simply the Laplace equation in *flat space* with cylindrical coordinates (ρ, z, ϕ) . (In addition, ψ has to be independent of the angular coordinate ϕ .) Given a solution for ψ , the function γ can be determined by simple integration:

$$\begin{aligned} \gamma_{,\rho} &= \rho[\psi_{,\rho}^2 - \psi_{,z}^2] \\ \gamma_{,z} &= 2\rho[\psi_{,\rho}\psi_{,z}]. \end{aligned} \quad (\text{A5})$$

The Schwarzschild metric is of course a particular solution to these equations and corresponds to choosing for ψ and γ :

$$\psi = \psi_S := \frac{1}{2} \ln\left(\frac{L-M}{L+M}\right), \quad \gamma = \gamma_S := \frac{1}{2} \ln\left(\frac{L^2 - M^2}{L^2 - \eta^2}\right), \quad (\text{A6})$$

where $L = \frac{1}{2}(l_+ + l_-)$, $\eta = \frac{1}{2}(l_+ - l_-)$ with $l_+ = \sqrt{\rho^2 + (z+M)^2}$ and $l_- = \sqrt{\rho^2 + (z-M)^2}$ and M is the mass of the Schwarzschild solution. Note that ψ is just the Newtonian potential due to a rod of length $2M$ placed symmetrically about the origin on the z axis. Both ψ_S and γ_S diverge logarithmically in the limit $\rho \rightarrow 0$ (for $|z| \leq M$). In order to recast this solution in the standard Schwarzschild form, one must transform from (z, ρ) to the Schwarzschild coordinates (r, θ) by

$$z = (r-M)\cos\theta, \quad \rho^2 = r^2(1-2M/r)\sin^2\theta. \quad (\text{A7})$$

This coordinate transformation shows that the horizon, $r = 2M$, corresponds to the line segment $\rho = 0$, $|z| \leq M$ in Weyl coordinates. Therefore, the Weyl coordinates cover only the exterior of the horizon.

Now, the key point is that Eq. (A4), the only field equation one has to solve, is linear. Hence we can ‘‘distort’’ the

Schwarzschild solution simply by adding to ψ_S any solution ψ_D of the flat space Laplace equation which is regular along the z axis [35]. Thus, we can set

$$\psi = \psi_S + \psi_D, \quad \gamma = \gamma_S + \gamma_D. \quad (\text{A8})$$

Substituting these expressions into Eqs. (A4) and (A5) and using the forms of the Schwarzschild functions ψ_S and γ_S , one can show that, at $\rho = 0$,

$$\gamma_D|_{\rho=0} \triangleq 2\psi_D|_{\rho=0}. \quad (\text{A9})$$

This fact plays an important role in analyzing the horizon structure.

In Schwarzschild coordinates, the distorted metric takes the form

$$\begin{aligned} ds^2 &= -e^{2\psi_D}(1-2M/r)dt^2 + \frac{e^{2(\gamma_D-\psi_D)}}{(1-2M/r)}dr^2 \\ &+ e^{2(\gamma_D-\psi_D)}r^2d\theta^2 + r^2\sin^2\theta e^{-2\psi_D}d\phi^2. \end{aligned} \quad (\text{A10})$$

As usual, the metric has a coordinate singularity at $r = 2M$. Let us therefore introduce the Eddington-Finkelstein coordinate v as before. The metric can be reexpressed in (v, r, θ, ϕ) coordinates as

$$\begin{aligned} ds^2 &= -e^{2\psi_D}(1-2M/r)dv^2 + (1-2M/r)^{-1} \\ &\times e^{2\psi_D}(e^{2(\gamma_D-2\psi_D)} - 1)dr^2 + 2e^{2\psi_D}dvdr \\ &+ e^{2(\gamma_D-\psi_D)}r^2d\theta^2 + e^{-2\psi_D}r^2\sin^2\theta d\phi^2. \end{aligned} \quad (\text{A11})$$

Using condition (A9) it is not difficult to show that the coefficient of dr^2 in the metric is regular at $r = 2M$ [35].

It is immediately obvious from Eq. (A11) that the $r = 2M$ surface is a Killing horizon of $\partial/\partial v$. However, we cannot select a preferred normalization for this vector field since the metric is not asymptotically flat. As in the last subsection, let Δ be the Killing horizon and choose $l \propto \partial/\partial v$. Then, it is straightforward to verify that $(\Delta, [l])$ is a complete, nonrotating isolated horizon. Let us calculate the value of surface gravity for $l \triangleq \partial/\partial v$. We obtain

$$\kappa \triangleq (e^{2\psi_D-\gamma_D}) \frac{1}{2r} \triangleq \frac{1}{2r} \quad (\text{A12})$$

where we arrived at the last expression by using Eq. (A9). Again, while the spin coefficient $\text{Re}[\mu]$, the Weyl component Ψ_2 , and the scalar curvature 2R of the horizon metric all depend on the distortion function ψ_D , somewhat surprisingly the surface gravity $\kappa_{(l)}$ does not.

The natural question is whether the above framework can be extended to obtain distorted Reissner-Nordström solutions. This turns out to be nontrivial because the key equation (A4) now acquires a source term from the electromagnetic field and this field itself depends nontrivially on ψ through the Maxwell equations. At first, the coupled system appears to be hopelessly difficult. However, there exists a prescription [38] for defining a new potential $\tilde{\psi}$ in terms of ψ and the electromagnetic field such that $\tilde{\psi}$ satisfies the flat

space Laplacian (A4). Using this method, the known distorted black hole solutions were recently generalized to the charged case [36]. The distorted Reissner-Nordström solution is given by the metric

$$ds^2 = -(1 - 2M/r + Q^2/r^2)e^{2\psi_D} dt^2 + (1 - 2M/r + Q^2/r^2)^{-1} e^{2(\gamma_D - \psi_D)} dr^2 + e^{2(\gamma_D - \psi_D)} r^2 d\theta^2 + e^{-2\psi_D} r^2 \sin^2 \theta d\phi^2. \quad (\text{A13})$$

The forms of ψ_D and γ_D are now substantially more complicated than in the uncharged case. Nonetheless, it is still possible to show that Eq. (A9) continues to hold. As before this equality implies that the apparent singularity at $r_H^2 - 2Mr_H + Q^2 = 0$ is only a coordinate singularity. The surface defined by $r = r_H$ is a Killing horizon of $\partial/\partial t$. There is once again, no natural way to normalize the Killing field, so we only have an equivalence class $[l^a]$ of null normals to the Killing horizon. $(\Delta, [l])$ is a nonrotating isolated horizon.

The surface gravity of $\partial/\partial t$ is given by

$$\kappa = \frac{1}{2r_H} \left(1 - \frac{Q^2}{r_H^2} \right). \quad (\text{A14})$$

Again, the surface gravity is independent of the distortion of the horizon and has the same dependence on the horizon radius R_Δ (which turns out to be equal to r_H) and charge Q

as in Reissner-Nordström space-time. Considerations of Sec. V suggest that this peculiar behavior of κ in all these examples is not accidental but can be ‘‘explained’’ from general Hamiltonian considerations which led us to the first law.

APPENDIX B: THE NEWMAN-PENROSE FORMALISM

1. Notation and conventions

Let us begin with a summary of the Newman-Penrose formalism (see [39] or [40–42] for a complete account). Apart from the spacetime signature which we take to be $(-, +, +, +)$, we will follow the conventions used in [42]. Consider a tetrad of null vectors n , l , m , and \bar{m} (n and l are real while m is complex) which satisfy

$$\begin{aligned} n \cdot l &= -1, & n \cdot m &= 0, & n \cdot \bar{m} &= 0, \\ l \cdot m &= 0, & l \cdot \bar{m} &= 0, & m \cdot \bar{m} &= 1. \end{aligned} \quad (\text{B1})$$

The directional derivatives along the basis vectors are denoted by

$$D = l^a \nabla_a, \quad \Delta = n^a \nabla_a, \quad \delta = m^a \nabla_a, \quad \bar{\delta} = \bar{m}^a \nabla_a. \quad (\text{B2})$$

The full information contained in the connection is expressed in terms of twelve complex scalars called the Newman-Penrose spin coefficients defined as follows:

$$\begin{aligned} \kappa &= -m^a l^b \nabla_b l_a, & \epsilon &= \frac{1}{2} (\bar{m}^a l^b \nabla_b m_a - n^a l^b \nabla_b l_a), & \pi &= \bar{m}^a l^b \nabla_b n_a, \\ \sigma &= -m^a m^b \nabla_b l_a, & \beta &= \frac{1}{2} (\bar{m}^a m^b \nabla_b m_a - n^a m^b \nabla_b l_a), & \mu &= \bar{m}^a m^b \nabla_b n_a, \\ \rho &= -m^a \bar{m}^b \nabla_b l_a, & \alpha &= \frac{1}{2} (\bar{m}^a \bar{m}^b \nabla_b m_a - n^a \bar{m}^b \nabla_b l_a), & \lambda &= \bar{m}^a \bar{m}^b \nabla_b n_a, \\ \tau &= -m^a n^b \nabla_b l_a, & \gamma &= \frac{1}{2} (\bar{m}^a n^b \nabla_b m_a - n^a n^b \nabla_b l_a), & \nu &= \bar{m}^a n^b \nabla_b n_a. \end{aligned} \quad (\text{B3})$$

It is sometimes more useful to express these definitions in terms of covariant derivatives of the basis vectors:

$$\begin{aligned} Dl &= (\epsilon + \bar{\epsilon})l - \bar{\kappa}m - \kappa\bar{m}, & Dn &= -(\epsilon + \bar{\epsilon})n + \pi m + \bar{\pi}\bar{m}, \\ \Delta l &= (\gamma + \bar{\gamma})l - \bar{\tau}m - \tau\bar{m}, & \Delta n &= -(\gamma + \bar{\gamma})n + \nu m + \bar{\nu}\bar{m}, \\ \delta l &= (\bar{\alpha} + \beta)l - \bar{\rho}m - \sigma\bar{m}, & \delta n &= -(\bar{\alpha} + \beta)n + \mu m + \bar{\lambda}\bar{m}, \\ Dm &= \bar{\pi}l - \kappa n + (\epsilon - \bar{\epsilon})m, & \Delta m &= \bar{\nu}l - \tau n + (\gamma - \bar{\gamma})m, \\ \delta m &= \bar{\lambda}l - \sigma n + (\beta - \bar{\alpha})m, & \bar{\delta} m &= \bar{\mu}l - \rho n + (\alpha - \bar{\beta})m. \end{aligned} \quad (\text{B4})$$

The ten independent components of the Weyl tensor are expressed in terms of five complex scalars Ψ_0 , Ψ_1 , Ψ_2 , Ψ_3 , and Ψ_4 . The ten components of the Ricci tensor are defined in terms of four real and three complex scalars Φ_{00} , Φ_{11} , Φ_{22} , Λ , Φ_{10} , Φ_{20} , and Φ_{21} . These scalars are defined as follows:

$$\begin{aligned} \Psi_0 &= C_{abcd} l^a m^b l^c m^d, & \Phi_{01} &= \frac{1}{2} R_{ab} l^a m^b, & \Phi_{10} &= \frac{1}{2} R_{ab} l^a \bar{m}^b, \\ \Psi_1 &= C_{abcd} l^a m^b l^c n^d, & \Phi_{02} &= \frac{1}{2} R_{ab} m^a m^b, & \Phi_{20} &= \frac{1}{2} R_{ab} \bar{m}^a \bar{m}^b, \\ \Psi_2 &= C_{abcd} l^a m^b \bar{m}^c n^d, & \Phi_{21} &= \frac{1}{2} R_{ab} \bar{m}^a n^b, & \Phi_{12} &= \frac{1}{2} R_{ab} m^a n^b, \end{aligned} \quad (\text{B5})$$

$$\begin{aligned}\Psi_3 &= C_{abcd} l^a n^b \bar{m}^c n^d, & \Phi_{00} &= \frac{1}{2} R_{ab} l^a l^b, & \Phi_{11} &= \frac{1}{4} R_{ab} (l^a n^b + m^a \bar{m}^b), \\ \Psi_4 &= C_{abcd} \bar{m}^a n^b \bar{m}^c n^d, & \Phi_{22} &= \frac{1}{2} R_{ab} n^a n^b, & \Lambda &= R/24.\end{aligned}$$

The six components of the electromagnetic-field 2-form \mathbf{F}_{ab} can be defined in terms of three complex scalars:

$$\begin{aligned}\phi_0 &= -l^a m^b \mathbf{F}_{ab}, \\ \phi_1 &= -\frac{1}{2} (l^a n^b - m^a \bar{m}^b) \mathbf{F}_{ab}, \\ \phi_2 &= n^a \bar{m}^b \mathbf{F}_{ab}.\end{aligned}\quad (\text{B6})$$

The eight real Maxwell equations $d\mathbf{F}=0$ and $d^*\mathbf{F}=0$ can be written as a set of four complex equations:

$$D\phi_1 - \bar{\delta}\phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2, \quad (\text{B7})$$

$$D\phi_2 - \bar{\delta}\phi_1 = -\lambda_0 + 2\pi\phi_1 + (\rho - 2\epsilon)\phi_2, \quad (\text{B8})$$

$$\Delta\phi_0 - \delta\phi_1 = (2\gamma - \mu)\phi_0 - 2\tau\phi_1 + \sigma\phi_2, \quad (\text{B9})$$

$$\Delta\phi_1 - \delta\phi_2 = \nu\phi_0 - 2\mu\phi_1 + (2\beta - \tau)\phi_2. \quad (\text{B10})$$

2. Boundary conditions

In this section we describe the isolated horizon boundary conditions in the Newman-Penrose formalism. We will restrict ourselves to Einstein-Maxwell theory with zero cosmological constant.

In a null-tetrad adapted to the null hypersurface Δ , take l to be a null normal, m and \bar{m} tangent to Δ , and n transverse to Δ . Since l is hypersurface orthogonal and null, it is geodesic. This implies that¹⁶ $\kappa_{NP} \triangleq 0$ and $\text{Im}[\rho] \triangleq 0$. Thus

$$Dl^b := l^a \nabla_a l^b \triangleq (\epsilon + \bar{\epsilon})l^b. \quad (\text{B11})$$

The surface gravity is therefore given by $\kappa_{(l)} = \epsilon + \bar{\epsilon}$ and the expansion of l is $\theta_{(l)} \triangleq 2\text{Re}[\rho]$.

For a nonexpanding horizon Δ , the conditions on l imply $\rho \triangleq 0$ and the Raychaudhuri equation then implies $\sigma \triangleq 0$ and $\Phi_{00} = \frac{1}{2} R_{ab} l^a l^b \triangleq 0$. Furthermore, from Eq. (2.16) (which is a consequence of the energy condition), it follows that $\phi_0 \triangleq 0$. This leads to the following conditions on the Ricci tensor at the horizon:

$$\begin{aligned}\Phi_{00} &\triangleq 0, & \Phi_{01} &\triangleq 0, & \Phi_{10} &\triangleq 0, \\ \Phi_{02} &\triangleq 0, & \Phi_{20} &\triangleq 0, & \Phi_{11} &\triangleq -2G\phi_1 \bar{\phi}_1.\end{aligned}\quad (\text{B12})$$

The first Maxwell equation (B7) gives

$$D\phi_1 \triangleq 0 \quad \text{which implies} \quad D\Phi_{11} \triangleq 0. \quad (\text{B13})$$

¹⁶We will denote the NP spin coefficient κ by κ_{NP} to distinguish it from the surface gravity $\kappa_{(l)}$.

Also, as shown in Eq. (2.8)

$$\Psi_0 \triangleq 0, \quad \Psi_1 \triangleq 0. \quad (\text{B14})$$

The intrinsically defined one-form ω_a defined in Eq. (2.4) is given by

$$\omega_a = -\kappa_{(l)} n_a + (\alpha + \bar{\beta}) m_a + (\bar{\alpha} + \beta) \bar{m}_a. \quad (\text{B15})$$

It is often convenient to choose the null tetrad such that $\underline{dn} = 0$ which implies

$$\mu \triangleq \bar{\mu}, \quad \pi \triangleq \alpha + \bar{\beta}. \quad (\text{B16})$$

In this case we get a foliation of Δ spanned by m and \bar{m} . Furthermore, by an appropriate spin transformation, we can choose ϵ to be real so that $\epsilon \triangleq \bar{\epsilon}$ and thus the foliation is Lie dragged along l :

$$\mathcal{L}_l m^a = (\epsilon - \bar{\epsilon}) m^a \triangleq 0. \quad (\text{B17})$$

The one-form ω now becomes

$$\omega_a = -\kappa_{(l)} n_a + \pi m_a + \bar{\pi} \bar{m}_a. \quad (\text{B18})$$

Let us consider a weakly isolated horizon $(\Delta, [l])$. The condition $\mathcal{L}_l \omega = 0$ is equivalent to requiring

$$\mathcal{L}_l \pi \triangleq 0, \quad \mathcal{L}_l \kappa_{(l)} \triangleq 0 \quad (\text{B19})$$

and as we proved in Sec. II B, these conditions imply that the surface gravity $\kappa_{(l)}$ is constant on Δ .

As mentioned in Sec. II B, a weakly isolated horizon with nonzero surface gravity admits a natural foliation. In the Newman-Penrose framework this foliation can be characterized as follows: It is the unique foliation on each leaf of which the pullback of the 1-form $\pi m_a + \bar{\pi} \bar{m}_a$ is divergence free. This condition was first introduced by Hájíček [43] in the context of stationary spacetimes.

Finally, since our boundary conditions require that $\mathcal{L}_l \kappa_{(l)} \triangleq 0$, in a sense, a part of the zeroth law is simply assumed. As mentioned in Sec. II B, we could have used a slightly different set of boundary conditions which make no direct requirement on $\kappa_{(l)}$ and yet lead to the zeroth law (as well as the results of Secs. III–VII).

Let $(\Delta, [l])$ be a nonexpanding horizon, equipped with an equivalence class $[l]$ of null normals to be related to each other by constant positive rescalings. As above, introduce a null tetrad where l is an element of $[l]$, m and \bar{m} are tangent to the foliation, n is curl free, and ϵ is real. In place of definition 2, let us assume that Δ admits a foliation by a family S_Δ of 2-spheres transverse to $[l]$ such that the NP spin coefficients in an associated null tetrad satisfy

$$\mathcal{L}_l \mu \triangleq 0, \quad \mathcal{L}_l \pi \triangleq 0. \quad (\text{B20})$$

These conditions now replace the requirement $\mathcal{L}_l \omega \triangleq 0$ used in the definition of a weakly isolated horizon. We can prove the zeroth law from these conditions as follows. First, consider the definition $2\nabla_{[a}\nabla_{b]}\xi_c = R_{abc}{}^d \xi_d$ of the Riemann tensor. In the NP formalism, these are written as a set of 18 complex equations known as the ‘‘field equations.’’ For our purposes, we need only three of these equations [42]:

$$D\alpha - \bar{\delta}\epsilon = (\rho + \bar{\epsilon} - 2\epsilon)\alpha + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda$$

$$- \bar{\kappa}\gamma + (\epsilon + \rho)\pi + \Phi_{10},$$

$$D\beta - \delta\epsilon = (\alpha + \pi)\sigma + (\bar{\rho} - \bar{\epsilon})\beta$$

$$- (\mu + \gamma)\kappa - (\bar{\alpha} - \bar{\pi})\epsilon + \Psi_1,$$

$$D\mu - \delta\pi = (\bar{\rho} - \epsilon - \bar{\epsilon})\mu + \sigma\lambda + (\bar{\pi} - \bar{\alpha} + \beta)\pi$$

$$- \nu\kappa + \Psi_2 + 2\Lambda.$$

Adding the first equation to the complex conjugate of the second equation and imposing our boundary conditions gives

$$\delta(\epsilon + \bar{\epsilon}) \triangleq \delta\kappa_{(l)} \triangleq 0 \quad (\text{B21})$$

while the third equation reduces to

$$\Psi_2 \triangleq (\epsilon + \bar{\epsilon})\mu. \quad (\text{B22})$$

Equation (B21) tells us that surface gravity is constant on each leaf of the foliation. It now only remains to show that it is also constant along l . To show this we turn to the Bianchi

identity $\nabla_{[a}R_{bc]de} = 0$. In the NP formalism, this is written as a set of nine complex and two real equations. We shall need only two of these equations [42]:

$$\begin{aligned} D\Psi_2 - \bar{\delta}\Psi_1 + \Delta\Phi_{00} - \bar{\delta}\Phi_{01} + 2D\Lambda \\ = -\lambda\Psi_0 + 2(\pi - \alpha)\Psi_1 + 3\rho\Psi_2 \\ - 2\kappa\Psi_3 + \bar{\sigma}\Phi_{02} \\ + (2\gamma + 2\bar{\gamma} - \bar{\mu})\Phi_{00} - 2(\alpha + \bar{\tau})\Phi_{01} \\ - 2\tau\Phi_{10} + 2\rho\Phi_{11}, \end{aligned}$$

$$\begin{aligned} D\Phi_{11} - \delta\Phi_{10} + \Delta\Phi_{00} - \bar{\delta}\Phi_{01} + 3D\Lambda \\ = (2\gamma + 2\bar{\gamma} - \mu - \bar{\mu})\Phi_{00} + (\pi - 2\alpha - 2\bar{\tau})\Phi_{01} \\ + (\bar{\pi} - 2\bar{\alpha} - 2\tau)\Phi_{10} + 2(\rho + \bar{\rho})\Phi_{11} \\ + \bar{\sigma}\Phi_{02} + \sigma\Phi_{20} - \bar{\kappa}\Phi_{12} - \kappa\Phi_{21}. \end{aligned} \quad (\text{B23})$$

Subtracting these equations, imposing our boundary conditions, and using $\Lambda = 0$, we get $D\Psi_2 \triangleq 0$. Combining this result with Eq. (B22) gives $D(\epsilon + \bar{\epsilon}) \triangleq 0$. This completes the proof of the zeroth law within the alternate definition of weak isolation. Most of the results of this paper were first obtained using that definition. However, since that notion is tied so heavily to the presence of a foliation, its intrinsic meaning is somewhat obscure. Therefore, it was then replaced by definition 2 used in the main body of the paper.

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