

Type II critical collapse of a self-gravitating nonlinear σ model

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We report on the existence and phenomenology of type II critical collapse within the one-parameter family of SU(2) σ models coupled to gravity. Numerical investigations in spherical symmetry show discretely self-similar (DSS) behavior at the threshold of black hole formation for values of the dimensionless coupling constant η ranging from 0.2 to 100; at 0.18 we see small deviations from DSS. While the echoing period Δ of the critical solution rises sharply towards the lower limit of this range, the characteristic mass scaling has a critical exponent γ which is almost independent of η , asymptoting to 0.1185 ± 0.0005 at large η . We also find critical scaling of the scalar curvature for near-critical initial data. Our numerical results are based on an outgoing-null-cone formulation of the Einstein-matter equations, specialized to spherical symmetry. Our numerically computed initial-data critical parameters p^* show second order convergence with the grid resolution, and after compensating for this variation in p^* , our individual evolutions are uniformly second order convergent even very close to criticality.

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I. INTRODUCTION

Since the numerical investigation of dynamical behavior of a massless scalar field under the influence of its gravitational forces by Choptuik [1], critical behavior has been observed in a number of different matter models coupled to gravity. In the context of type II critical collapse, these models have in common that at the threshold of black hole formation their dynamics show a universal characteristic approach to either a discretely (DSS) or continuously (CSS) self-similar solution.

Nonlinear σ fields provide particularly interesting models to study the dynamics of gravitating self-interacting matter in general relativity. In addition to their applications in physics (see e.g. Ref. [2]), they have a simple geometrical interpretation as harmonic maps, which have been extensively studied in the mathematical literature (see e.g. Refs. [3,4]).

Recently Bizoń *et al.* [5,6] and also independently Liebling *et al.* [7] have observed critical (threshold) behavior for non-gravitating systems: the transition between globally regular time evolution and singularity formation for the SU(2) σ model on Minkowski background. It was shown by Bizoń [5] that this system admits a countably infinite family of CSS solutions. The stable ground state is the end point of singular evolution for supercritical initial data, while the first excitation, which has one unstable mode, plays the role of the critical (CSS) solution.

The interesting question arises of what happens if gravity is added to this system. The gravitating SU(2) σ model is a family of theories with a *dimensionless parameter* η , which acts as a coupling constant (for $\eta=0$ gravity decouples from the field). It was argued in Ref. [6] that the singularity formation in flat space might not be relevant for black hole formation when gravity is active, since the CSS blowup excludes the concentration of energy at the singularity. Since no asymptotically flat solitonic configurations exist [8], this suggests that the only alternative to dispersion or collapse to

a black hole is the formation of a naked singularity. Here we focus on critical phenomena at the threshold of black hole formation. As Bizoń *et al.* have pointed out [9], criticality is expected to depend on the coupling constant η . If so, does the system show discrete or continuous self-similarity? And in which way do critical phenomena depend on the coupling?

In this paper we present results from a numerical study of the dynamical evolution for the SU(2) nonlinear σ model coupled to gravity in spherical symmetry. Our code uses a characteristic formulation, specialized to the spherical symmetry. Initial data are specified on an outgoing null cone with vertex at the center of symmetry. The discretized field equations are used to evolve the matter field and the geometry to future outgoing null cones, using a nonuniformly spaced set of grid points which follow ingoing null geodesics.

We find critical behavior at the boundary between black hole formation and dispersion for values of the coupling constant η in the range of 0.18–100. The critical solution is DSS, with the echoing period Δ strongly depending on η : As η tends to 0.18 from above, Δ rises sharply. Moreover, we observe small deviations from exact DSS at this smallest η value. This leads us to conjecture that DSS ceases to be a critical solution for still smaller values of the coupling constant.

The organization of this paper is as follows: In Sec. II we review the basic properties of the SU(2) σ model in spherical symmetry and discuss the system of field equations. We present our main physical results in Sec. III, and end the main body of the paper with some conclusions in Sec. IV. In Appendix A we discuss our numerical methods, which are based on previous work of Goldwirth, Ori, and Piran [10,11], Garfinkle [12], and Gómez and Winicour [13–16]. Finally, in Appendix B we discuss the convergence of our numerical evolutions to the continuum limit as the grid resolution is increased, including both uniform convergence of accuracy diagnostics within a single evolution, and also convergence

of the numerically computed critical parameter p^* itself.

Conventions are chosen as follows: spacetime indices are Greek letters, SU(2) indices are uppercase Latin letters, the spacetime signature is $(-, +, +, +)$, the Ricci tensor is defined as $R_{\mu\nu} = R_{\mu\lambda\nu}{}^\lambda$ with the sign convention of Ref. [17], and the speed of light is set to unity, $c = 1$.

II. SU(2) σ MODEL IN SPHERICAL SYMMETRY

Nonlinear σ models are special cases of harmonic maps from a spacetime $(\mathbf{M}, g_{\mu\nu})$ into some target manifold (\mathbf{N}, G_{AB}) (see, e.g., Ref. [2]). Harmonic maps $X^A(x^\mu)$ are defined as the extrema of the simple geometric action

$$S = -\frac{f_\pi^2}{2} \int_{\mathbf{M}} d^4x \sqrt{|g|} g^{\mu\nu} \partial_\mu X^A \partial_\nu X^B G_{AB}(X). \quad (2.1)$$

If the spacetime metric is dynamically coupled to the matter fields X^A , then Eq. (2.1) must be supplemented by the Einstein-Hilbert action.

Variation of the total action with respect to the σ field X^A and the metric $g_{\mu\nu}$ yields the coupled Einstein σ field equations. The stress-energy tensor resulting from Eq. (2.1) obeys the weak, strong and dominant energy conditions [18]. The coupling constant f_π^2 and the gravitational constant G enter the equations only in the dimensionless product $\eta \equiv 4\pi G f_\pi^2$, thereby defining a one-parameter family of distinct gravitating matter models. The field equations are scale invariant.

For the SU(2) σ model, the target manifold is taken as S^3 with G_{AB} the ‘‘round’’ metric of constant curvature. Note that the coupling η may be interpreted as the inverse of the scalar curvature of the target manifold. In the limit $\eta \rightarrow \infty$ our model thus corresponds to the σ model with 3-dimensional flat target manifold. [This is also easily checked by rescaling the field $\phi \rightarrow \phi/\sqrt{\eta}$ and performing the limit $\eta \rightarrow \infty$ in Eqs. (2.10) and (2.12)–(2.14).] We restrict ourselves to spherically symmetric harmonic maps coupled to gravity, which implies that the base space (spacetime) must share this symmetry.

We introduce a Bondi coordinate system $\{u, r, \theta, \phi\}$ on spacetime based upon outgoing null hypersurfaces $u = \text{constant}$, with the line element

$$ds^2 = -e^{2\beta(u,r)} du \left(\frac{V(u,r)}{r} du + 2dr \right) + r^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (2.2)$$

and assume that spacetime admits a regular center $r=0$ of spherical symmetry. This requires the metric functions near the origin to behave at fixed retarded time u_0 like

$$\beta(u_0, r) = O(r^2), \quad (2.3)$$

$$V(u_0, r) = r + O(r^3), \quad (2.4)$$

where the gauge has been fixed such that the family of outgoing null cones emanating from the center is parametrized

by the proper time u at the center. Radial ingoing null geodesics are obtained by integrating the equation

$$\frac{d}{du} r(u) = -\frac{V(u, r(u))}{2r(u)}. \quad (2.5)$$

In spherical symmetry the null expansions Θ_\pm of inward and outward directed null rays emanating from $r = \text{const}$ surfaces, can be defined as $\Theta_\pm = 2(\mathcal{L}_\pm r)/r$, where \mathcal{L}_\pm is the Lie derivative along the null directions $l_+ = e^{-2\beta} \partial_r$ and $l_- = 2\partial_u - (V/r)\partial_r$. Thus we have

$$\Theta_+ = \frac{2}{r} e^{-2\beta}, \quad \Theta_- = -\frac{2}{r} \left(\frac{V}{r} \right). \quad (2.6)$$

Whenever Θ_+ vanishes on some 2-sphere $r = \text{const}$, this sphere is marginally outer trapped. Since this means diverging β , the Bondi-like coordinate system (2.2) cannot penetrate a marginally outer trapped surface—in particular an apparent horizon.

We introduce polar coordinates (ϕ, Θ, Φ) on the target manifold (S^3, G) , and write the SU(2) line element as

$$dS^2 = d\phi^2 + \sin^2\phi (d\Theta^2 + \sin^2\Theta d\Phi^2). \quad (2.7)$$

We focus on a particular spherically symmetric harmonic map (a corotational equivariant map) obtained via the well-known hedgehog ansatz:

$$\phi(x^\mu) = \phi(u, r), \quad \Theta(x^\mu) = \theta, \quad \Phi(x^\mu) = \varphi. \quad (2.8)$$

With this ansatz two of the three coupled fields are determined and only one field $\phi(u, r)$ enters the equations. Regularity at the origin forces the σ field ϕ to vanish at $r=0$, so the origin is always mapped to one of the poles of S^3 , defined by the choice of coordinates (2.7). As ϕ represents the ‘‘areal coordinate’’ of the polar coordinate system (2.7) on the target manifold, its regularity behavior near the origin is the same as that of the areal coordinate r :

$$\phi(u_0, r) = O(r). \quad (2.9)$$

The matter field equations are then reduced to the single nonlinear wave equation

$$\square \phi = \frac{\sin(2\phi)}{r^2}, \quad (2.10)$$

where \square is the wave operator $g^{\mu\nu} \nabla_\mu \nabla_\nu$:

$$\square = e^{-2\beta} \left\{ \left[\frac{2V}{r^2} + \left(\frac{V}{r} \right)' \right] \partial_r - \frac{2}{r} \partial_u - 2\partial_u \partial_r + \frac{V}{r} \partial_{rr} \right\}. \quad (2.11)$$

The nontrivial Einstein equations split up into the hypersurface equations (the $\{rr\}$ and $\{ur\} - (V/2r)\{rr\}$ components of $G_{\mu\nu} = \kappa T_{\mu\nu}$)

$$\beta' = \frac{\eta}{2} r (\phi')^2, \quad (2.12a)$$

$$V' = e^{2\beta} [1 - 2\eta \sin(\phi)^2], \quad (2.12b)$$

the subsidiary equation ($r^2[\{uu\} - (V/r)\{ur\}]$)

$$\dot{V} - 2V\dot{\beta} = 2\eta \left[(r\dot{\phi})^2 - \frac{V}{r} (r\phi') (r\dot{\phi}) \right], \quad (2.13)$$

and the redundant equation ($\{\theta\theta\}$)

$$\begin{aligned} V(r\beta'' - \beta') + r\beta' V' + \frac{1}{2} r V'' - 2r^2 \partial_{ur} \beta \\ = \eta r \phi' (-V\phi' + 2r\partial_u \phi). \end{aligned} \quad (2.14)$$

The combination of the hypersurface equations (2.12) and the matter field equation (2.10) suffices to evolve all the dynamical fields V/r , β , and ϕ . Assuming these equations to be satisfied, the redundant equation (2.14) then holds identically, and if the subsidiary equation (2.13) is satisfied on some $r = \text{const}$ surface (this is assured for $r=0$ by the regularity conditions there), then it too must hold everywhere.

In view of this, we construct initial data on a $u = \text{constant}$ slice by choosing ϕ as free data on the slice, then integrating the hypersurface equations (2.12) to obtain the metric coefficients V/r and β on the slice. To evolve these data to future $u = \text{const}$ slices, we simultaneously integrate the hypersurface equations (2.12) and the matter field equation (2.10). Throughout the initial data construction and the evolution, we use the subsidiary equation (2.13) and the redundant equation (2.14) solely to check the accuracy of our numerical computations. We discuss our numerical treatment of all these equations in Appendix A.

In our coordinates, the Misner-Sharp mass function [19–21] can be written directly in terms of the metric,

$$m(u, r) \equiv m_{\text{MS}}(u, r) = \frac{r}{2} \left(1 - \frac{V}{r} e^{-2\beta} \right), \quad (2.15)$$

or by using the Einstein equations, rewritten as a radial integral within a single slice,

$$m(u, r) \equiv m_\rho(u, r) = \int_0^r m'(u, \tilde{r}) d\tilde{r}, \quad (2.16a)$$

where

$$m'(u, r) = \frac{\eta}{2} r^2 \left(\frac{V}{r} e^{-2\beta} (\phi')^2 + 2 \frac{\sin^2 \phi}{r^2} \right). \quad (2.16b)$$

Since our coordinates would be singular on an apparent horizon, we have designed our numerical evolution scheme to slow down as an apparent horizon is approached, in such a manner that the evolution only asymptotes to the apparent horizon (cf. Appendix A 1).

In other words, none of our numerically computed slices ever actually contain an apparent horizon. Thus strictly speaking we can never *measure* a black hole mass, but only *estimate* what the mass *will* be when (if) a black hole even-

tually forms. To do this, at each numerical time step we compute the Misner-Sharp mass function m_{MS} , and look for regions of the numerical grid which are almost at the critical density for black hole formation, i.e. where $2m_{\text{MS}}/r$ is almost 1. More precisely, if anywhere in the grid $2m_{\text{MS}}/r$ exceeds a specified threshold,¹ then we estimate that a black hole will form, with a final mass m_{BH} given by the mass function m_{MS} at the outermost such grid point. In general this mass estimate changes during the evolution; we use the last value before a numerical evolution terminates as our overall estimate for the black hole mass.

It is also of interest to compute the total mass m_{total} within the outer grid boundary.² This gives an upper bound for our final estimate m_{BH} .

III. RESULTS

For each value of the coupling constant η , we consider a 1-parameter family of initial data $\phi = \phi_p(u_0, r)$, such that (say) for small values of p this initial data eventually disperses without forming a black hole, while for large values of p it eventually forms a black hole. By using a binary search in p , we can find (a numerical approximation to) the critical value $p = p^*$ which defines the threshold of black hole formation.

We have studied the Einstein- σ -model system in this manner over the range of coupling constants $0.18 \leq \eta \leq 100$, using several different initial-data families. Here we present results using the Gaussian-like initial data family

$$\phi(u_0, r) = A r^2 \exp \left[- \left(\frac{r - r_0}{\sigma} \right)^2 \right] \quad (3.1)$$

with the ‘‘amplitude’’ A as the parameter p (holding σ and r_0 constant for a given critical search), and also using the ‘‘derivative of 4th-power pseudo-Gaussian’’ family

$$\phi(u_0, r) = -4A r^2 \left(\frac{r - r_0}{\sigma} \right)^3 \exp \left[- \left(\frac{r - r_0}{\sigma} \right)^4 \right] \quad (3.2)$$

with the ‘‘width’’ σ as the parameter p (holding A and r_0 constant for a given critical search).³ All the results reported here used a ‘‘position’’ $r_0 = 5$ and an initial-slice outer boundary of $r_{\text{outer}} = 30$. Table I shows some near-critical initial data parameters.

We have also carried out a number of convergence tests of our numerical scheme, both for single evolutions and for entire critical searches. We discuss these in Appendix B.

¹0.995 for all results reported here.

²To be precise, we use m_ρ at the outermost grid point, not m_{MS} , since m_{MS} is numerically somewhat ill conditioned in the outer part of the grid, whereas m_ρ is well conditioned everywhere.

³For this latter case the relative sign of p is reversed with respect to black hole formation; i.e. for large p the initial data eventually disperses, while for small p it eventually forms a black hole.

TABLE I. This table shows two families of near-critical initial data parameters for various coupling constants η . For the Gaussian-like initial data family (3.1), we use the ‘‘amplitude’’ A as the parameter p (at a fixed ‘‘width’’ $\sigma=1$), with a numerical grid of 16 000 grid points. For the ‘‘derivative of 4th-power pseudo-Gaussian’’ initial data family (3.2), we use the ‘‘width’’ σ as the parameter p (with different ‘‘amplitudes’’ A for different coupling constants), with 8000 grid points. For each coupling constant and each family, the table also shows the max $2m/r$ echoing period $\Delta/2$ of the near-critical evolution, and the mass-scaling-law critical exponent γ determined for the entire critical search.

η	Initial Data Family (3.1)			Initial Data Family (3.2)			
	Parameter is A			Parameter is σ			
	A^*	$\Delta/2$	γ	A	σ^*	$\Delta/2$	γ
0.18	0.019 523 015	0.5522	0.1063	0.003	1.083 153 54	0.5478	0.1028
0.2	0.018 942 512	0.4367	0.1091	0.002	0.615 317 49	0.4327	0.1150
0.225	0.018 241 056	0.3464	0.1207	0.002	0.651 519 42	0.3472	0.1169
0.25	0.017 578 042	0.3043	0.1173	0.002	0.688 851 73	0.3046	0.1173
0.3	0.016 392 639	0.2668	0.1152	0.002	0.766 003 44	0.2675	0.1146
0.4	0.014 534 866	0.2452	0.1132	0.002	0.929 746 89	0.2445	0.1139
0.5	0.013 167 548	0.2386	0.1152	0.0015	0.707 335 37	0.2386	0.1130
1	0.009 528 975 1	0.2314	0.1163	0.0015	1.210 138 07	0.2313	0.1155
2	0.006 809 778 3	0.2295	0.1179	0.0010	1.064 744 72	0.2305	0.1167
5	0.004 333 205 6	0.2304	0.1183	0.0005	0.734 344 76	0.2308	0.1178
10	0.003 070 144 2	0.2293	0.1186	0.0005	1.318 800 46	0.2312	0.1182
100	0.000 972 589 54	0.2302	0.1187	0.0001	0.631 472 58	0.2311	0.1182

A. DSS echoing

Discrete self-similarity is defined by the existence of a discrete diffeomorphism Φ_Δ such that for some fixed $\Delta \in \mathbb{R}$:

$$(\Phi_\Delta^*)^n g = e^{2n\Delta} g \quad \forall n \in \mathbb{N}. \quad (3.3)$$

In adapted coordinates $\tau = -\ln[(u^* - u)/u^*]$ and $\rho = r/(u^* - u)$, where u^* is a real number which denotes the accumulation time of DSS, we have

$$Z(\tau + n\Delta, \rho) = Z(\tau, \rho) \quad \forall n \in \mathbb{N}, \quad (3.4)$$

where Z denotes β , V/r , ϕ , or any combination of these, e.g. $2m/r$. In addition the σ field ϕ satisfies the stronger condition

$$(\Phi_{\Delta/2}^*)^n \phi = (-1)^n \phi, \quad (3.5)$$

so that fields even in ϕ (e.g. β , V/r , and quantities constructed from them) are actually periodic in τ with period

$\Delta/2$. As a DSS diagnostic, we typically look for $(\Delta/2)$ periodicity in the black hole formation diagnostic $\max 2m/r$, where the maximum is taken over r within each $u = \text{constant}$ slice.

We have clear evidence for the existence of a type II critical collapse with a DSS critical solution. Figure 1 shows examples of this for two values of the coupling constant. Since $\max 2m/r$ periodicity is only a necessary condition for DSS, we have also explicitly verified that the matter field ϕ at selected times u coincides with its image under the DSS diffeomorphism Φ . Figure 2 shows an example of this.

We find that the self-similarity echoing period $\Delta/2$ varies strongly with the coupling constant η . Table I gives some numerical data showing this, and Fig. 3 shows this same data graphically. At large η , $\Delta/2$ asymptotes to 0.2300 ± 0.0003 . As η decreases towards the lower limit of the data in Table I, $\eta=0.18$, $\Delta/2$ rises sharply. At the very smallest coupling constant $\eta=0.18$, but not at $\eta=0.20$ or any larger value, the critical solution shows small deviations from exact DSS: the

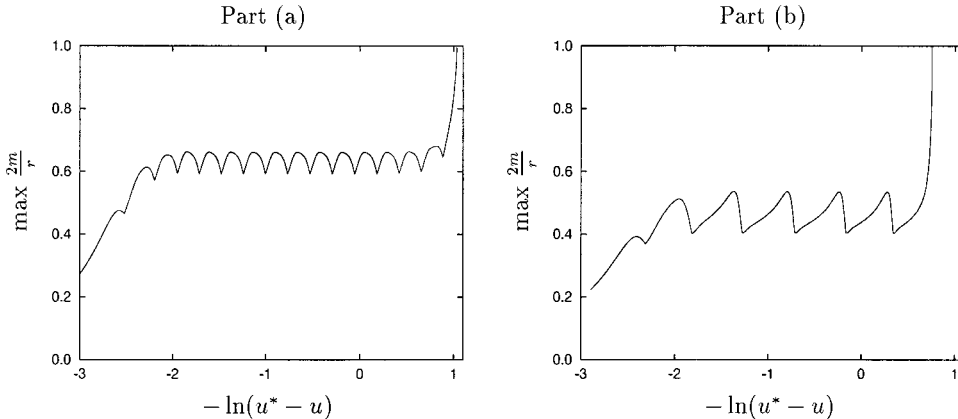


FIG. 1. This figure shows DSS echoing behavior in the black hole formation diagnostic $\max 2m/r$ in near-critical (in this case slightly supercritical) evolutions for coupling constants $\eta=0.5$ [part (a)] and 0.18 [part (b)]. Notice the much longer period $\Delta/2$ of the echoes at $\eta=0.18$. Although it is not apparent to the eye at the scale of this figure, the 0.18 echoes are not *exactly* identical: they vary in period and shape by 5–10%.

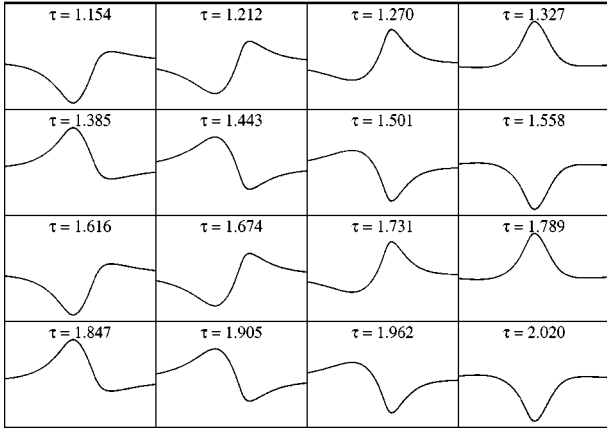


FIG. 2. Snapshots of a near-critical evolution of the $SU(2)$ ϕ field as a function of $\ln r$ for $\eta=1.0$. The frames are evenly spaced in $\tau = -\ln[(u^* - u)/u^*]$ (this increases towards the accumulation time u^*). τ increases to the right within each row, then downwards between successive rows. Observe that ϕ is the same in frames in the same column but 2 rows apart; this indicates that ϕ is periodic in τ with period $\Delta=0.46$. Also notice that ϕ is negated between frames in the same column of adjacent rows; i.e. it satisfies the half-period self-similarity condition (3.5).

periods and shapes of the individual max $2m/r$ oscillations deviate by 5–10% from the best-fitting DSS prediction. The physical significance of this is not yet clear.

B. Scaling and universality

In the presence of DSS, the black hole mass m_{BH} of slightly supercritical evolutions shows a universal scaling law (Refs. [22–25])

$$\ln m_{\text{BH}} = \gamma \ln(p - p^*) + \Psi(\ln(p - p^*)) + \text{const} \quad (3.6)$$

where γ sets the overall slope of the scaling law, and the function Ψ is periodic with period $\frac{1}{2}\Delta/\gamma$ in $\ln(p - p^*)$. For

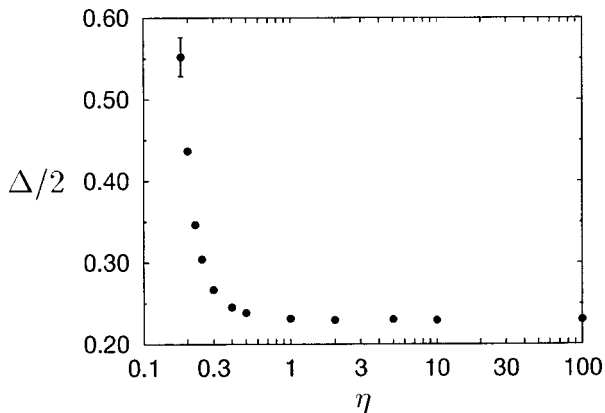


FIG. 3. This figure shows the variation of the near-critical max $2m/r$ echoing period $\Delta/2$ with the coupling constant η , for the Gaussian-like initial data family given in Table I. Notice the rapid rise in $\Delta/2$ at small η . The error bar for the $\eta=0.18$ point is estimated from the dispersion in $\Delta/2$ values when fitting different subsets of echoes in Fig. 1(b); for larger values of η this dispersion is negligible.

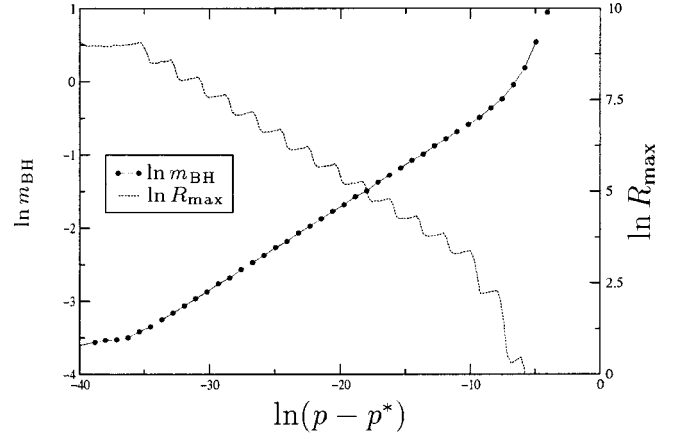


FIG. 4. Supercritical scaling of the black hole mass m_{BH} , and of R_{max} , the maximum (over retarded time u within a single evolution) of the scalar curvature at the origin. The slopes of the masses and R_{max} are $+\gamma$ and -2γ respectively. The scaling fine structure is clearly visible for R_{max} . Its period is found to be 2.099 which is very close to the value $\frac{1}{2}\Delta/\gamma=2.097$ predicted by perturbation theory and computed from known values of the critical exponents. This series of evolutions was done for $\eta=0.5$ using a resolution of 2000 grid points.

slightly subcritical evolutions, the maximum (taken over u within each evolution) of the 4-Ricci scalar evaluated at the origin, R_{max} , also shows a similar scaling law, but with slope -2γ [26]. This is also true for supercritical evolutions, with R_{max} now defined by taking the maximum in u within each evolution only until a (null) slice reaches the apparent horizon. (Our actual evolutions terminate slightly before the apparent horizon, but R_{max} does not change significantly in this interval.)

We have investigated these scaling laws using a sequence of supercritical evolutions with varying $\ln(p - p^*)$. We extract γ by least-squares fitting $\ln m_{\text{BH}}$ as a linear function of $\ln(p - p^*)$; after subtracting this fit from $\ln m_{\text{BH}}$, we are left with the periodic fine structure. Figure 4 shows a typical supercritical scaling law and Fig. 5 its fine structure.

Since the numerical resolution of our code is limited by the use of IEEE double precision floating point numbers, we expect the errors to blow up for $p - p^* \leq 10^{-16}$ which corresponds to $\ln(p - p^*) \leq 35$. This can be seen in Figs. 4 and 5, and also in Fig. 8(b) (discussed in the next section). For $p - p^* \geq -10$ deviations from the scaling laws are also apparent, demarcating the range of validity of linear perturbation theory.

We find that the mass scaling exponent γ varies by at most 5% over the range of η we have studied, asymptoting to $\gamma=0.1185 \pm 0.0005$ at large η . (The error is estimated from the dispersion in γ values between fits to critical searches with different initial-data families and/or finite difference grid resolutions.)

The periodicity present in the fine structure of the scaling law (Figs. 4 and 5) can be measured directly. Figure 6 shows a comparison of the measured periods with the perturbation-theory prediction $\frac{1}{2}\Delta/\gamma$ in $\ln(p - p^*)$. The agreement is excellent.

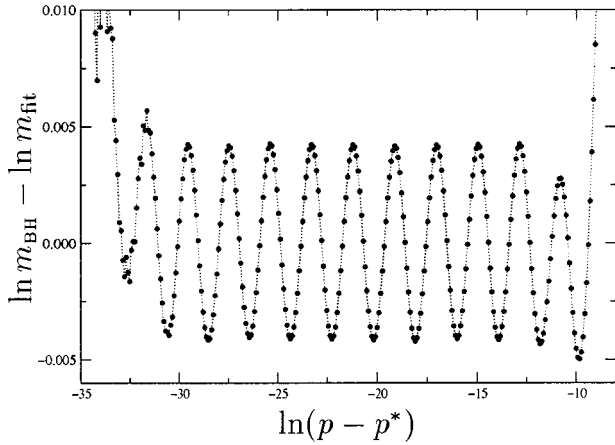


FIG. 5. This figure shows the fine-scale structure in m_{BH} (shown in Fig. 4) after subtracting a linear fit. For these evolutions we disabled the “ $2m_{\text{MS}}/r > 0.995$ detected for N time steps” stopping criterion in our code (cf. Appendix), running each evolution until $\Delta u < 10^{-15}$; in this case our final slices’ outer grid boundaries almost touched the apparent horizon, so m_{BH} and m_{total} were essentially identical (within $\lesssim 10^{-10}$ of each other).

Comparing results for different one-parameter families of initial data, we find that the critical behavior is universal at all coupling constants η : The critical exponent γ and the echoing period $\Delta/2$ are the same for all critical searches at a given coupling constant, regardless of which initial data family is used. For example, Table I shows that γ and $\Delta/2$ are the same (to within numerical errors) for even the very different initial data families (3.1) and (3.2).

IV. CONCLUSIONS

In this paper we have presented a detailed numerical analysis of $\text{SU}(2)$ σ models coupled to gravity in spherical symmetry for a wide range of the coupling constant η . For

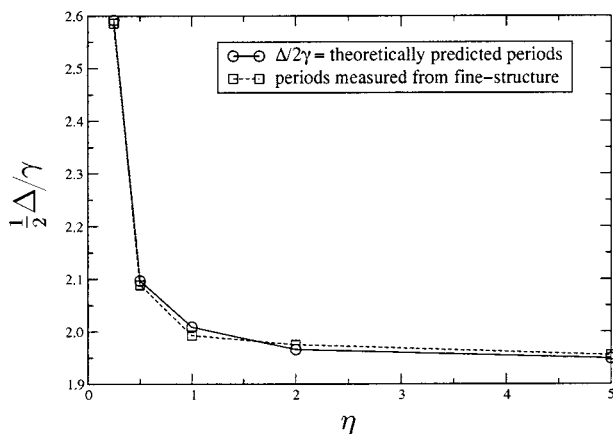


FIG. 6. This figure compares the quantity $\frac{1}{2}\Delta/\gamma$, as computed from the echoing in $\max 2m/r$ and the mass-scaling law, to the period of the oscillations present in the fine structure of the mass-scaling law, which is predicted by perturbation theory to be $\frac{1}{2}\Delta/\gamma$. This has been carried out for coupling constants ranging from 0.25 up to 5. All evolutions were done with 2000 grid points radial resolution.

$0.18 \leq \eta \leq 100$ we have evidence of universal critical type II collapse behavior. The critical solution is DSS. We have observed both the typical mass scaling at the threshold of black hole formation of supercritical initial data and the corresponding scaling of the scalar curvature for both sub- and supercritical evolutions.

Our numerical results are based on an outgoing-null-cone formulation of the Einstein-matter equations, specialized to spherical symmetry (our numerical methods are discussed in detail in Appendix A). We have carried out thorough convergence tests to ensure the validity of our results (see Appendix B). Notably, we have demonstrated second order uniform-in- r convergence of the error diagnostic δm (measuring finite differencing errors in the Misner-Sharp mass function) for even very nearly critical spacetimes. We have also demonstrated second order convergence for the initial data’s critical parameter p^* . To our knowledge this is the first time the latter has been reported.

In the limit of large couplings our model corresponds to the σ model with 3-dimensional flat target manifold. This model has already been studied by Liebling [27], where he considered an additional potential. As this potential does not play a role for criticality we should observe the same critical solution for large couplings. In fact our results for both the echoing period $\Delta=0.4604$ and the scaling exponent $\gamma=0.1187$ are in good agreement with the results reported in [27].

While we observe at most a small variation of the critical exponent γ over the range of coupling constants studied, the period Δ of the DSS depends strongly on the value of the coupling constant: as η tends to 0.18 from above the period increases by more than a factor of 2 in the narrow range of $0.18 \leq \eta \leq 0.3$. Also, close to the lower limit we observe small deviations from exact self-similarity.

These observations seem to signal a transition region around the value of $\eta=0.18$. From results of the work on the σ model in flat space [5–7] it is known that there exists a critical (threshold) CSS solution. In a recent paper, Bizoń and Wasserman [28] have shown numerically that this solution persists when gravity is turned on, at least up to a certain value of the coupling constant. Whether or not the CSS solution plays a role at the threshold of black hole formation for small couplings is under current investigation.

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APPENDIX A: NUMERICAL METHODS

1. Overview

We discretize the coupled Einstein-matter equations using second order finite differencing in r within each $u=\text{const}$

slice and in u along ingoing null geodesics. Our grid points are generically distributed non-uniformly within each slice: On the initial slice we place them equidistantly in r between the origin and some finite maximum radius r_{outer} , but thereafter they freefall in towards the origin along ingoing null geodesics. We always maintain a grid point at the origin $r = 0$; when another grid point reaches the origin we drop the point previously at the origin from the grid.

The choice of freely falling grid points provides some degree of adaptive grid refinement by the focusing of geodesics towards regions of strong curvature. Following Garfinkle [12], we also gain additional resolution at late times by explicitly refining our grid by a factor of 2 everywhere in the slice, each time we have lost half of the remaining grid points. Again following Ref. [12], for some runs we also manually fine-tune the position of the outermost grid point on the initial slice (r_{outer}), so that this grid point will eventually almost hit the strongest-field region of spacetime. This greatly improves the effectiveness of the factor-of-2 grid refinements, but this method was not required for the results presented here.⁴

By moving our grid points along null geodesics, the physical domain of dependence is automatically contained in the numerical domain of dependence, so our time step is not restricted by the usual Courant-Friedrichs-Lewy (CFL) stability limit [29,30]. However, in order to control time resolution we require (following Refs. [10,11]) that

$$(V/r) \Delta u \leq C \Delta r \quad (\text{A1})$$

everywhere in the grid, where C is a constant which we typically take to be on the order of unity. The time step Δu is thus limited such that grid points fall inwards by no more than $C/2$ grid point spacings per time step. Most of our results reported here were obtained with $C = 1.5$. [Note that for a null-cone evolution similar to ours, but with grid points at constant r (Ref. [15]), there is a CFL stability limit, which is in fact just Eq. (A1) with $C = 2$.]

For $r \Theta_+$ sufficiently small, a large value of V/r decreases the time step Δu as follows: From Eq. (2.6) it is clear that for small $r \Theta_+$ the function β —which is monotonically increasing with r —becomes large (it blows up at an apparent horizon). Furthermore, by Eq. (2.15) we get $V/r = e^{2\beta} (1 - 2m/r)$. Outside of the outermost local maximum of $2m/r$ both $e^{2\beta}$ and $1 - 2m/r$ are monotonically increasing with r , and thus so is V/r . If the outer boundary of the grid is taken sufficiently far out, this is therefore the location of the maximum of V/r and thus of the most stringent slowdown condition. If $\Delta u < 10^{-15}$ (i.e. close to machine precision), the evolution is terminated.

For the remainder of this appendix, we adopt the usual

⁴By fine-tuning r_{outer} in this way, we have also observed DSS in the massless scalar (Klein-Gordon) field, with up to 5 echoes visible (in the sense of Fig. 1). This provides a strong additional test of our numerical scheme, since the dynamic range of the DSS is much larger in the Klein-Gordon case: $\Delta/2 \approx 1.73$ there [as defined by Eq. (3.3)], much larger than the values we find for the σ field.

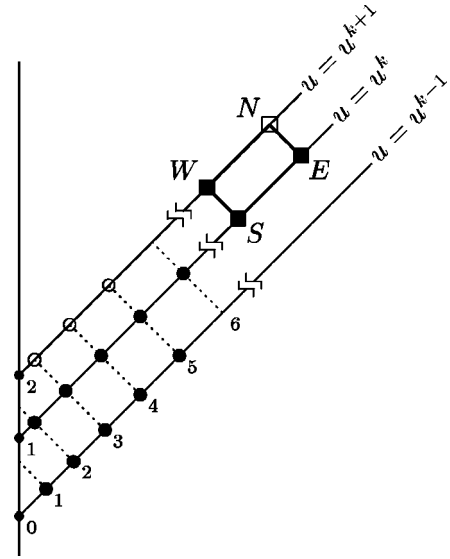


FIG. 7. This figure shows our finite differencing grid. Individual grid points are labeled with integers 0–6, and their ingoing-null-geodesic trajectories are shown as dotted lines. Grid points at the origin are marked with small points. Grid points used in the least-squares fitting procedure (cf. Appendix A 2) are marked with large solid circles, while the grid points where the field variables are calculated from the Taylor series are marked with large open circles. Grid points where ψ is already known in the diamond-integral scheme (cf. Appendix A 3) are marked with solid squares, while the grid point where ψ is computed in this scheme is marked with an open square.

notation where superscripts denote “temporal” (u) levels. Figure 7 shows the typical organization of our grid. All discretizations in time (u) and space (r) use nonuniform grid spacings to allow for the free fall of the grid points and the adaptive time stepping (A1). Our numerical scheme uses the geometry fields β , V/r and $(V/r)'$, and the rescaled matter field $\psi \equiv r\phi$.

Assuming that these fields are known at all grid points on the $u = u^k$ and $u = u^{k-1}$ slices, we determine the fields on the $u = u^{k+1}$ slice as follows:

- (i) For the innermost 3 non-origin grid points in the $u = u^{k+1}$ slice, we use a Taylor series expansion as described in Appendix A 2.
- (ii) We then sweep outwards over the remaining spatial grids of the $u = u^{k+1}$ slice as discussed in Appendix A 3.

2. Taylor expansions near the symmetry axis

The coupled Einstein-matter equations and regularity determine the generic behavior of ψ near the origin as

$$\psi(u + \Delta u, r) = c_1 r^2 + c_2 r^3 + c_2 r^2 \Delta u + O(\Delta u^4 + r^4). \quad (\text{A2})$$

Substitution of this series expansion into the hypersurface equations (2.12) yields corresponding series expansions for the geometry fields β and V/r .

To determine the geometry and matter fields near the origin on the $u=u^{k+1}$ slice, we first least-squares fit the functional form (A2) to the numerically computed ψ values at the 5 innermost non-origin points of the $u=u^k$ and $u=u^{k-1}$ time levels (these points are marked by large solid circles in Fig. 7). This determines the coefficients c_1 and c_2 .

For each of the 3 innermost non-origin grid points on the $u=u^{k+1}$ slice (these points are marked by open circles in Fig. 7), we first integrate the ingoing null geodesic equation (2.5) from $u=u^k$ to $u=u^{k+1}$, as described below. Then, using the coefficients c_1 and c_2 , we determine ψ at this grid point from the series expansion (A2). Finally, we compute β , V/r , and $(V/r)'$ from their corresponding series expansions.

3. Integration schemes

In order to integrate out from the Taylor series region to the outer boundary, our general strategy at each grid point is as follows:

(i) We first determine the grid point's r coordinate on the $u=u^{k+1}$ slice by integrating the ingoing null geodesic equation (2.5) from $u=u^k$ to $u=u^{k+1}$.

(ii) We then determine ψ at this grid point using a ‘‘diamond integral’’ scheme of Gómez and Winicour [15,16,14].

(iii) We compute the geometry fields by integrating the hypersurface equations one grid point outwards on the $u=u^{k+1}$ slice.

For the hypersurface equations (2.12) and the geodesic equation Eq. (2.5) we use a second order iterated Runge-Kutta scheme [adapted from Sec. 5.2.1, Eq. (5.6), of Ref. [31]]. For a generic ordinary differential equation (ODE) system $dy/dx=f(x,y)$ the scheme is as follows:

$$y_{\text{pred}}^{k+1} = y^k + \Delta x f(x^k, y^k) \quad (\text{A3a})$$

$$y^{k+1} = y^k + \frac{1}{2} \Delta x [f(x^k, y^k) + f(x^{k+1}, y_{\text{pred}}^{k+1})]. \quad (\text{A3b})$$

While this allows straightforward integration of the hypersurface equations (2.12), the geodesic equation (2.5) needs special care: The corrector (A3b) requires evaluating the right-hand-side function f at the x^{k+1} time level. For the geodesic equation this requires knowing the field V/r on the $u=u^{k+1}$ slice, which is not yet computed at the time the geodesic integration is done. We thus linearly extrapolate the needed V/r value from V/r and $(V/r)'$ values one spatial grid point inwards on the same ($u=u^{n+1}$) slice.

The matter field equation is integrated using a ‘‘diamond integral’’ scheme of Gómez and Winicour [15,16,14]. The basic idea is to integrate the nonlinear wave equation (2.10) over the null parallelogram Σ spanned by the 4 grid points N , S , W , and E in Fig. 7. This allows the nonlinear wave equation (2.10) to be written as

$$\begin{aligned} \psi(N) &= \psi(W) + \psi(E) - \psi(S) \\ &- \frac{1}{2} \int_{\Sigma} \left[\left(\frac{V}{r} \right)' \frac{\Psi}{r} + e^{2\beta} \sin \left(2 \frac{\psi}{r} \right) \right] du dr. \end{aligned} \quad (\text{A4})$$

We evaluate the integral numerically by approximating the integrand as constant over the null parallelogram Σ , with a value which is the average of its values at the grid points W and E . This gives second order overall accuracy for ψ .

4. Diagnostics

Within a single evolution, we use several diagnostics to assess the accuracy of our numerical computations. We numerically check the satisfaction of the subsidiary and redundant Einstein equations (2.13) and (2.14). We also compare the two ‘‘different’’ forms of the mass function, m_{MS} and m_{ρ} : These are in fact identical by virtue of the Einstein equations, but they are computed in very different ways [via Eqs. (2.15) and (2.16) respectively], and numerically they will generally differ by a small amount due to finite differencing errors. This difference is a useful diagnostic of the code's accuracy. To this end, we define

$$\delta m(u, r) = \frac{m_{\text{MS}} - m_{\rho}}{m_{\text{total, init}}} \quad (\text{A5})$$

where $m_{\text{total, init}} \equiv m_{\text{MS}}(u=0, r_{\text{max}})$ is the total mass of our initial slice. δm is then a dimensionless diagnostic of how well our field variables approximate the Einstein equations; we must have $|\delta m| \ll 1$ everywhere in the grid at all times for our results to be trustworthy.

APPENDIX B: CONVERGENCE TESTS

We use convergence tests of the type popularized by Choptuik [32–34] both to better understand the performance of our numerical algorithms, and to quantitatively assess the accuracy of our numerical results. In particular, it is only through such convergence tests that we can be confident our conclusions reflect properties of the continuum Einstein-matter equations, rather than numerical artifacts.

As an example of the convergence properties of our computational scheme, we discuss a series of near-critical $\eta = 0.5$ evolutions. We begin by considering the effects of varying grid resolutions (specified by the number of grid points N) on the critical parameter p^* . Figure 8(a) shows these effects for the supercritical mass-scaling law. Notice that the dominant effect is to simply shift each entire critical curve to a slightly different $p^*[N]$. Table II shows these $p^*[N]$ values, and their convergence to a continuum limit (which we denote by $p^*[\infty]$) as the grid resolution is increased. Notice that the ratios of the successive differences $(p^*[2N] - p^*[N]) / (p^*[4N] - p^*[2N])$ are very nearly equal to 4; i.e. the p^* values show second order convergence to $p^*[\infty]$.

Besides shifting the effective p^* , what other effects does varying the grid resolution have on the critical behavior? Figure 8(b) shows the same data as Fig. 8(a), but plotted using the usual logarithmic mass-scaling-law coordinates, and with each grid resolution's data plotted using that resolution's own $p^*[N]$ value. It is clear that the different resolutions all yield the same mass scaling law.

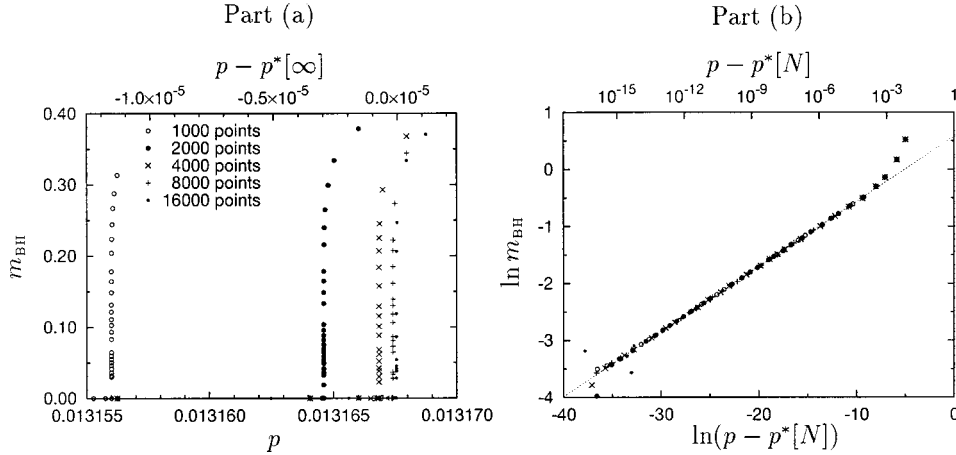


FIG. 8. This figure shows the convergence of the supercritical mass-scaling law with increasing finite difference grid resolution, for $\eta = 0.5$ evolutions. Part (a) shows the mass scaling behavior for 5 different grid resolutions, plotted on *linear* scales in both p (here the ‘‘amplitude’’ A) and m_{BH} . Notice how the main effect of changes in the grid resolution is to simply shift the entire critical curve to a slightly different p^* . The actual p^* values are given in Table II, and all these evolutions used the same initial data parameters as given in Table I. Part (b) shows the same data plotted on logarithmic scales, with each resolution’s p values being taken relative to that resolution’s own p^* value. Notice that all the resolutions satisfy the same scaling law, even down to $p - p^*[N]$ far smaller than the resolution shifts shown in part (a); this is discussed further in the text.

(In order to get the same mass scaling law at different resolutions, it is essential here to use each resolution’s own $p^*[N]$ value, since Fig. 8(b) shows the mass scaling law continuing down to $p - p^*[N]$ values some 10 orders of magnitude smaller than the typical $p^*[N]$ shifts from one resolution to another. Equivalently, if we did *not* use each resolution’s own $p^*[N]$ value in Fig. 8(b), then the mass scaling law would fail to hold below $p - p^*[N] \sim 10^{-5}$ (the typical $p^*[N]$ shifts seen in Fig. 8(a)), whereas by using each resolution’s own $p^*[N]$ value, it actually continues down to $p - p^*[N] \sim 10^{-15}$.)

We now consider convergence behavior within a single evolution, or more precisely between the 3 evolutions whose $\max 2m/r$ time developments are shown in Fig. 9(a):

(1) The first evolution uses 8000 grid points, with $p = p^*[8000] + 10^{-12}$, so this evolution is just slightly supercritical, by about 1 part in 10^{10} . This can be seen in the

TABLE II. This table shows the convergence of p^* with the finite difference grid resolution N , for the Gaussian-like data plotted in Fig. 8. These evolutions used the same initial data parameters as given in Table I. The first two columns give the p^* values for the various resolutions N . The third column gives the differences $\delta p^* \equiv p^*[2N] - p^*[N]$ between consecutive p^* values as the resolution is doubled, and the last column gives the ratios of consecutive differences. The values in the last column are very nearly equal to 4, showing second order convergence of p^* .

N	$p^*[N]$	δp^*	ratio
1000	0.013 156 008	} 8.61×10^{-6}	} 3.87
2000	0.013 164 618		
4000	0.013 166 841	} 5.64×10^{-7}	} 3.94
8000	0.013 167 405		
16 000	0.013 167 548	} 1.42×10^{-7}	} 3.97

$\max 2m/r$ plot: $\max 2m/r$ first oscillates a number of times, then eventually rises to 1.

(2) The second evolution uses 16 000 grid points, with the same p as evolution (1). Because of the shift in the effective p^* with N , this evolution is now subcritical, in fact subcritical by a relatively large amount: $\max 2m/r$ oscillates only about half as many times as in evolution (1), then eventually decays to zero.

(3) The third evolution also uses 16 000 grid points, but this time p is adjusted to compensate for the shift in the effective p^* with N : we take $p = p^*[16\,000] + 10^{-12}$. By construction, this evolution is supercritical again, by the same amount as evolution (1); in fact its $\max 2m/r$ plot is almost identical to that of evolution (1).

We use δm as a diagnostic of our code’s numerical accuracy for these evolutions. Figure 9(b) shows the convergence of δm to zero for evolutions (1) and (2). These evolutions eventually yield very different spacetimes (one forming a black hole, the other not), but here we consider $u = \text{const}$ slices at an early enough time, $u = 13.08$ [shown by the left vertical dashed line in Fig. 9(a)] that the evolutions have not drifted very far apart yet. From Fig. 9(b) it is clear that δm is almost precisely a factor of 4 smaller at the higher resolution than at the lower one; i.e. δm shows second order convergence to zero, as expected from the construction of our finite differencing schemes. Notice also that this convergence is *uniform*, which is a considerably stronger numerical-fidelity requirement than requiring only pointwise or gridwise-norm convergence.

Now consider a convergence test between evolutions (1) and (3). Because evolution (3) adjusts p to compensate for the shift in the effective p^* with N , these two evolutions have very similar behavior, so we can consider much later

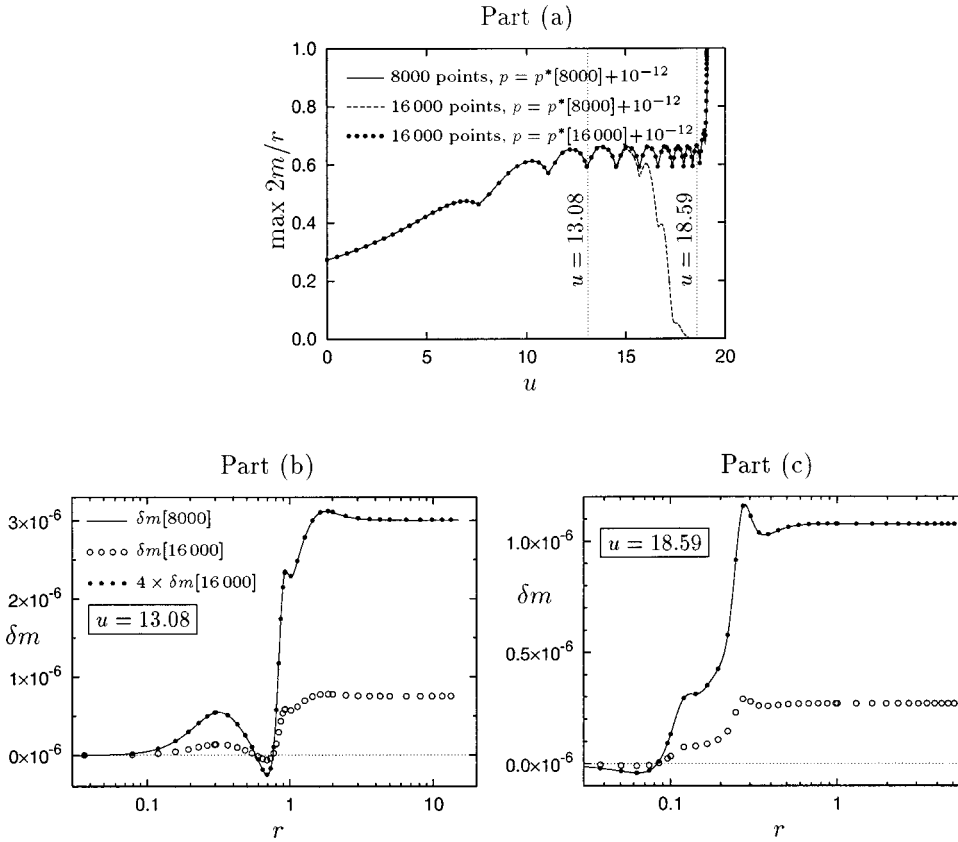


FIG. 9. This figure shows the convergence of δm to zero with increasing grid resolution, for near-critical $\eta=0.5$ evolutions. Part (a) shows the time development of $\max 2m/r$ for each of 3 evolutions described in the text. Part (b) shows the convergence of δm between evolutions (1) and (2), at a relatively early time. Part (c) shows the convergence of δm between evolutions (1) and (3), at a relatively late time. (Note that in all cases, the marked points are spaced for ease of reading, and represent only a small subset of the time steps or spatial grid points.)

u =constant slices and still obtain good convergence. For example, Fig. 9(c) shows the convergence of δm at the relatively late time $u = 18.59$ [shown by the right vertical dashed line in Fig. 9(a)]. The convergence is (again) very accurately second order. In other words, once we compensate for the

shift in the effective p^* with N , we have excellent—and uniformly pointwise—convergence even for evolutions that are *very* close to critical ($p - p^*[N]$ here is about 5 orders of magnitude smaller than the $p^*[N]$ shifts between the two resolutions), and hence *very* sensitive to small perturbations.

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