''No-hair'' theorem for spontaneously broken Abelian models in static black holes

Eloy Ayón-Beato
Departamento de Física, CINVESTAV-IPN, Apartado Postal 14-740, C.P. 07000, México, D.F., Mexico
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The vanishing of the electromagnetic field, for purely electric configurations of spontaneously broken Abelian models, is established in the domain of outer communications of a static asymptotically flat black hole. The proof is gauge invariant, and is accomplished without any dependence on the model. In the particular case of the Abelian Higgs model, it is shown that the only solutions admitted for the scalar field become the vacuum expectation values of the self-interaction.

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I. INTRODUCTION

The classical and strongest version of the ''no-hair'' conjecture establishes that a stationary black hole is uniquely described by global charges, i.e., conserved charges associated with massless gauge fields, expressed by surface integrals at the spatial infinity i^o [1]. In particular, the conjecture excludes the existence of massive fields in the domain of outer communications $\langle\langle \mathcal{J} \rangle\rangle$ of a stationary black hole. This fact rests on the idea that in the black hole transition to stationarity ''everything that can be radiated away will be radiated away'' (see Ref. [2]), so, the only classical degrees of freedom of a stationary black hole are those corresponding to nonradiative multipole moments; massive fields are automatically excluded because all their multipoles are radiative [1].

The absence of massive ''hair'' was shown early in the Bekenstein pioneering works for massive scalar fields, Proca-massive spin-1 fields, and massive spin-2 fields [3–5]. An alternative demonstration for Proca fields can be found in Ref. [6]. The ''no-hair'' theorem for massive vector fields is a useful tool for excluding the existence of new black hole solutions for very complicated theories as metric-affine gravity, where a relevant sector of this theory reduces to an effective Einstein-Proca system [7].

It is well known that fields acquire mass not only kinematically, as in the previous cases, but also through a dynamical mechanism of spontaneous symmetry breaking. This is the case of spontaneously broken Abelian models describing a charged scalar field with a self-interaction having nonzero vacuum expectation values, and minimally coupled to a massless Abelian gauge field. The ''no-hair'' conjecture for this model has been previously articulated as follows [8]: any stationary black hole solution, such that all gauge-invariant observables are nonsingular, must have a vanishing electromagnetic field, in the domain of outer communications $\langle\langle \mathcal{J} \rangle\rangle$ of the black hole. The simplest of this systems is the Abelian Higgs model (Mexican-hat self-interaction) for which a ''no-hair'' theorem was shown in Ref. [9], proving the vanishing of the gauge field for spherically symmetric static black holes. This proof has been considered unsatisfactory [10] because it is based on an inconsistent gauge choice. Improved versions have been recently given [11–13], without the original restrictions criticized in Ref. [10].

The subject of this paper is twofold, first, to relax the spherically symmetric assumption in the previously quoted contributions, by working with general static asymptotically flat systems, and second, to extent the ''no-hair'' theorem to more general Abelian models than the Higgs model, i.e., for general spontaneously broken self-interactions. Emphasis is given on asymptotically flat black holes only, this way we exclude from consideration black holes pierced by a cosmic string [14] —with the corresponding nontrivial behavior of the Abelian field— as it has been previously pointed by Bekenstein [13,15], these last configurations are not asymptotically flat since they present the angular deficit inherent to the presence of topological defects. The basic difference between these configurations is that for the string-pierced black holes the scalar field satisfy boundary conditions at infinity in accordance with the existence of a topological defect, i.e., the scalar field is confined to the vacuum only in a circle at infinity, which implies the developing of a cosmic string at the interior of the circle, whereas for asymptotically flat black holes the scalar field approaches the vacuum in all directions at infinity.

For a static black hole, the Killing field $\boldsymbol{k}$ coincides with the null generator of the event horizon $\mathcal{H}^+$ and is timelike and hypersurface orthogonal in all the domain of outer communications $\langle\langle \mathcal{J} \rangle\rangle$. This allow us to choose, by simply connectedness of $\langle\langle \mathcal{J} \rangle\rangle$ [16], a global coordinate system (t,x^i), $i=1,2,3$, in all $\langle\langle \mathcal{J} \rangle\rangle$ [17], such that $\boldsymbol{k}=\partial/\partial t$ and the metric reads

$$\boldsymbol{g}=-V\boldsymbol{dt}^2+\gamma_{ij}\boldsymbol{dx}^i\boldsymbol{dx}^j, \tag{1}$$

where V and $\boldsymbol{\gamma}$ are t independent, $\boldsymbol{\gamma}$ is positive definite in all $\langle\langle \mathcal{J} \rangle\rangle$, and V is positive in all $\langle\langle \mathcal{J} \rangle\rangle$ and vanishes in $\mathcal{H}^+$. From Eq. (1) it can be noted that staticity implies the existence of a time-reversal isometry $t\mapsto -t$.

In Sec. II the vanishing of the electromagnetic field in the domain of outer communications $\langle\langle \mathcal{J} \rangle\rangle$ of a static asymptotically flat black hole is demonstrated for purely electric configurations of a generic spontaneously broken Abelian model. At the end of Sec. II the conditions for establishing a ''no-hair'' theorem for purely magnetic configurations are also analyzed. Section III is devoted to show, in the particular case of the Abelian Higgs model, that the charged scalar field is confined to its vacuum in $\langle\langle \mathcal{J} \rangle\rangle$. Conclusions are given in Sec. IV.

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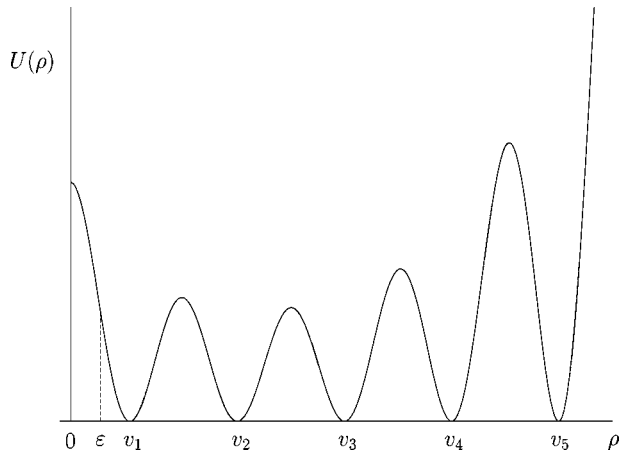

FIG. 1. Example of a spontaneously broken potential with five types of nonvanishing vacuum expectation values. The positive real number ε is such that $0<\varepsilon\leqslant v_1$, and it will be used to show that ρ is a nonvanishing function at the horizon.

II. ''NO-HAIR'' THEOREM FOR THE ABELIAN GAUGE FIELD

The action describing the coupling to gravity of the relevant models to be considered is

$$\mathcal{S}=\frac{1}{2}\int\left(\frac{1}{\kappa}R-\frac{1}{8\pi}F_{\alpha\beta}F^{\alpha\beta}-(D_\alpha\Phi)^\dagger D^\alpha\Phi - U(\Phi^\dagger\Phi)\right)dv, \tag{2}$$

where R is the scalar curvature, $F_{\alpha\beta}\equiv 2\nabla_{[\alpha}A_{\beta]}$ is the field strength of the Abelian gauge field A_α, $D_\alpha\equiv\nabla_\alpha - ieA_\alpha$ is the gauge covariant derivative, and $U(\Phi^\dagger\Phi)$ is a nonnegative self-interaction with nonvanishing vacuum expectation values, as for instance, in the Higgs model where $U_{\mathrm{H}}=(\lambda/2)(|\Phi|^2-v^2)^2$; here $(\cdots)^\dagger$ denotes complex conjugation. Parametrizing the complex scalar field by $\Phi=\rho\ \exp i\theta$ the Lagrangian becomes

$$L=\frac{1}{2\kappa}R-\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta}-\frac{1}{2}\nabla_\alpha\rho\nabla^\alpha\rho-\frac{1}{2e^2\rho^2}J_\alpha J^\alpha - \frac{1}{2}U(\rho), \tag{3}$$

with $J_\alpha\equiv e\rho^2(\nabla_\alpha\theta-eA_\alpha)$. The potential $U(\rho)$ is a nonnegative function achieving its minima at nonzero values v_a, see Fig. 1, and it is assumed that ρ asymptotically approaches to any one of this values. The Abelian symmetry of the models is expressed by the invariance of the Lagrangian (3) under the gauge transformations $\theta\mapsto\theta+e\Lambda, A_\alpha\mapsto A_\alpha+\nabla_\alpha\Lambda$. From Lagrangian (3), the Einstein-Maxwell-Scalar equations for the involved fields are established

$$\frac{1}{\kappa}R_{\mu\nu}=\frac{1}{4\pi}F_\mu{}^\alpha F_{\nu\alpha}+\nabla_\mu\rho\nabla_\nu\rho+\frac{1}{e^2\rho^2}J_\mu J_\nu - \frac{1}{2}g_{\mu\nu}\left(\frac{F_{\alpha\beta}F^{\alpha\beta}}{8\pi}-U(\rho)\right), \tag{4}$$

$$\nabla_\beta F^{\alpha\beta}=4\pi J^\alpha, \tag{5}$$

$$\Box\rho=\frac{1}{2}U'(\rho)+\frac{1}{e^2\rho^3}J_\alpha J^\alpha, \tag{6}$$

where $U'(\rho)\equiv dU(\rho)/d\rho$.

We would like to emphasize that the Reissner-Nordström black hole is not a solution of the above equations; the system we are dealing with is an Abelian Higgs model, i.e., a charged $(e\neq 0)$ scalar field minimally coupled to an Abelian gauge field, and with a self-interaction having nonvanishing vacuum expectation values. The coupling of this system to gravity Eqs. (4)-(6), does not reduce in no one case to the Einstein-Maxwell system, and therefore, it does not contain the Reissner-Nordström black hole as a solution. This becomes apparent from the Lagrangian (2): for constant values of the charged scalar field, $\Phi=\mathrm{const}$, a mass term, $e^2|\mathrm{const}|^2A_\mu A^\mu$, is present, which converts the Abelian gauge field in a massive Proca-like spin-1 field, for which there exist no static black hole solutions except the Schwarzschild one, as it was pointed out in the Introduction [4,6]. For a zero value of the scalar field, the mass term vanishes, but, an effective cosmological constant, $\Lambda_{\mathrm{eff}}=\kappa U(0)/2$, arises, this is due to the spontaneously broken behavior of the self-interaction, which requires $U(0)\neq 0$. In this case, we lost asymptotic flatness and, consequently, the Reissner-Nordström black hole cannot be a solution of the resulting system. Other is the situation when there is no spontaneously symmetry breaking, i.e., $U(0)=0$, in this case the model reduces to the Einstein-Maxwell system for vanishing scalar field and the existence of the Reissner-Nordström black hole is assured, but this is not the case we will deal with in the paper.

We shall assume that the gauge field shares the same symmetries of the metric, namely, it is stationary, $\pounds_{\boldsymbol{\xi}}\boldsymbol{F}=0$. Consequently with a (metric-) static configuration (1), we will also assume the existence of electromagnetic staticity, i.e., the Maxwell field $F^{\alpha\beta}$ and the Maxwell equations (5) are invariant under time-reversal transformations. The time-reversal invariance of Maxwell equations (5) requires that, in the coordinates chosen in Eq. (1), J^t and F^{ti} remain unchanged while J^i and F^{ij} change sign, or the opposite scheme, i.e., J^t and F^{ti} change sign as long as J^i and F^{ij} remain unchanged under time reversal [4]. However, this isometry should not change gauge-invariant observables, therefore J^i and F^{ij} must vanish in the first case, while J^t and F^{ti} vanish in the second one. Hence, staticity on the metric and material sources implies the existence of two nonoverlapping cases: a purely electric case (I) and a purely magnetic case (II).

Now we are ready to prove the ''no-hair'' statement for the gauge field, i.e., for spontaneously broken Abelian model the electromagnetic field vanishes in the domain of outer communications $\langle\langle \mathcal{J} \rangle\rangle$ of a static asymptotically flat black hole. Let $\mathcal{V} \subset \langle\langle \mathcal{J} \rangle\rangle$ be the open region bounded by the spacelike hypersurface Σ, the spacelike hypersurface Σ', and pertinent portions of the horizon $\mathcal{H}^+$, and the spatial infinity i^o. The spacelike hypersurface Σ' is obtained by shifting each point of Σ a unit parametric value along the integral curves of the Killing field $\boldsymbol{k}$. Multiplying the Maxwell equations (5) by J_α/ρ^2 and integrating by parts over $\mathcal{V}$, after applying the Gauss theorem, and using that $J_\alpha/\rho^2 = 2e(\nabla_\alpha\theta - eA_\alpha)$, one obtains

$$\left[\int_{\Sigma'} - \int_{\Sigma} + \int_{\mathcal{H}^+\cap\overline{\mathcal{V}}} + \int_{i^o\cap\overline{\mathcal{V}}}\right]\frac{1}{\rho^2}J_\alpha F^{\alpha\beta}d\Sigma_\beta = \int_{\mathcal{V}}\left(\frac{e^2}{2}F_{\alpha\beta}F^{\alpha\beta} + \frac{4\pi}{\rho^2}J_\alpha J^\alpha\right)dv. \tag{7}$$

The boundary integral over Σ' cancels out that one over Σ, since Σ' and Σ are isometric hypersurfaces. At spatial infinity i^o the scalar field modulus ρ approaches to one of the nonvanishing values v_a, minimizing the potential function $U(\rho)$, which implies [see Lagrangian (3)] that the gauge field behaves as an effective massive field at spatial infinity i^o, due to the spontaneous breaking of the gauge symmetry at this region. The usual Yukawa fall-off of massive fields at infinity cause the boundary integral over $i^o\cap\overline{\mathcal{V}}$ vanishes [9,11]. For the remaining boundary integral at the portion of the horizon $\mathcal{H}^+\cap\overline{\mathcal{V}}$ we make use of the standard measure on this region $d\Sigma_\beta = 2n_{[\beta}l_{\mu]}l^\mu d\sigma$ [18,19], where $\boldsymbol{l}$ is the null generator of the horizon, $\boldsymbol{n}$ is the other future-directed null vector ($n_\mu l^\mu = -1$), orthogonal to the spacelike cross sections of the horizon, and $d\sigma$ is the surface element —the standard measure follows from choosing the natural volume three-form at the horizon, i.e., $\boldsymbol{\eta}_3 = {}^*(\boldsymbol{n}\wedge\boldsymbol{l})\wedge\boldsymbol{l}$. Using the quoted measure the integrand over the horizon can be rewritten as

$$\frac{1}{\rho^2}J_\alpha F^{\alpha\beta}d\Sigma_\beta = \left(\frac{J_\alpha F^{\alpha\beta}l_\beta}{\rho^2} + \frac{J_\alpha F^{\alpha\beta}n_\beta}{\rho^2}(l_\mu l^\mu)\right)d\sigma. \tag{8}$$

In order to demonstrate that the last integrand vanishes it is sufficient to prove that the quantities appearing at the right hand side of Eq. (8) are such that: $J_\alpha F^{\alpha\beta}l_\beta/\rho^2$ vanishes and $J_\alpha F^{\alpha\beta}n_\beta/\rho^2$ remains bounded at the horizon. We shall establish the behavior of these quantities at the horizon by studying some invariants constructed from the curvature. By using Einstein equations (4), we obtain the following two equivalent expressions:

$$\begin{aligned}\frac{2}{\kappa^2}R_{\mu\nu}R^{\mu\nu} &= \frac{G^2}{2\pi} + \left(\frac{2J^\mu\nabla_\mu\rho}{e\rho}\right)^2 + \left(\frac{F}{2\pi} - \frac{J_\mu J^\mu}{e^2\rho^2}\right)^2 \\ &+ \left(\frac{F}{2\pi} + \nabla_\mu\rho\nabla^\mu\rho\right)^2 + \left(U(\rho) + \frac{J_\mu J^\mu}{e^2\rho^2}\right)^2 \\ &+ [U(\rho) + \nabla_\mu\rho\nabla^\mu\rho]^2 + \frac{1}{\pi e^2\rho^2}J^\mu F_\mu{}^\alpha J^\nu F_{\nu\alpha} \\ &+ \frac{1}{\pi}\nabla^\mu\rho\,{}^*F_\mu{}^\alpha\nabla^\nu\rho\,{}^*F_{\nu\alpha}, \end{aligned} \tag{9}$$

$$\begin{aligned} &= \frac{G^2}{2\pi} + \left(\frac{2J^\mu\nabla_\mu\rho}{e\rho}\right)^2 + \left(\frac{F}{2\pi} - \frac{J_\mu J^\mu}{e^2\rho^2}\right)^2 \\ &+ \left(\frac{F}{2\pi} - \nabla_\mu\rho\nabla^\mu\rho\right)^2 + \left(U(\rho) + \frac{J_\mu J^\mu}{e^2\rho^2}\right)^2 \\ &+ [U(\rho) + \nabla_\mu\rho\nabla^\mu\rho]^2 + \frac{1}{\pi e^2\rho^2}J^\mu F_\mu{}^\alpha J^\nu F_{\nu\alpha} \\ &+ \frac{1}{\pi}\nabla^\mu\rho F_\mu{}^\alpha\nabla^\nu\rho F_{\nu\alpha}, \end{aligned} \tag{10}$$

where $F \equiv F_{\alpha\beta}F^{\alpha\beta}/4$, $G \equiv {}^*F_{\alpha\beta}F^{\alpha\beta}/4$, and ${}^*F_{\alpha\beta}$ stands for the Hodge dual (${}^*F_{\alpha\beta} = \eta_{\mu\nu\alpha\beta}F^{\mu\nu}/2$). It is important to note that the previous Eqs. only differ in the sign inside the fourth term, and in the fact that the last term is written in each case with ${}^*F_{\alpha\beta}$ and $F_{\alpha\beta}$, respectively.

Since the horizon is a smooth surface, the left hand side of the above Eqs. is bounded on it. For the purely electric case (I), the last two terms in the right-hand side of Eq. (9) are nonnegative, the remaining terms are perfect square, and consequently each term in the right-hand side of Eq. (9) is bounded at the horizon. In particular, the bounded behavior of the sixth term involving the quantities $U(\rho)$ and $\nabla_\mu\rho\nabla^\mu\rho$ implies, from the non-negativeness of these quantities, that they are also bounded at the horizon. It follows from the bounded behavior of the perfect-square terms where $U(\rho)$ and $\nabla_\mu\rho\nabla^\mu\rho$ are combined with the quantities $J_\mu J^\mu/e^2\rho^2$ and F, respectively, that the last mentioned quantities are also bounded at the horizon. Thus, any quantity appearing in the right-hand side of Eq. (9) is bounded at the horizon, in particular $U(\rho)$, F and $J_\mu J^\mu/\rho^2$. The same conclusions can be achieved, along the same lines of reasoning, for the purely magnetic case (II), but this time using the right-hand side of Eq. (10). Other invariants can be built from the Ricci curvature (4) by means of $\boldsymbol{l}$ and $\boldsymbol{n}$, which are well-defined smooth vector fields on the horizon. The first invariant reads

$$\frac{1}{\kappa}R_{\mu\nu}n^\mu n^\nu = \frac{1}{4\pi}I_\mu I^\mu + (n^\mu\nabla_\mu\rho)^2 + \frac{1}{e^2\rho^2}(J_\mu n^\mu)^2 - \frac{n_\mu n^\mu}{2}\left(\frac{F}{2\pi} - U(\rho)\right), \tag{11}$$

where $I^{\mu}\equiv F^{\mu\nu}n_{\nu}$. The last term above vanishes because the bounded behavior of both the invariant F and the potential $U(\rho)$. Since $\boldsymbol{I}$ is orthogonal to the null vector $\boldsymbol{n}$, it must be spacelike or null ($I_{\mu}I^{\mu}\geqslant 0$), therefore each one of the remaining terms on the right hand side of Eq. (11) must be bounded. The next invariant to be considered, which vanishes at the horizon due to the Raychaudhuri equation for the null generator [20], reads

$$0=\frac{1}{\kappa}R_{\mu\nu}l^{\mu}l^{\nu}=\frac{1}{4\pi}E_{\mu}E^{\mu}+(l^{\mu}\nabla_{\mu}\rho)^2+\frac{1}{e^2\rho^2}(J_{\mu}l^{\mu})^2 - \frac{l_{\mu}l^{\mu}}{2}\left(\frac{F}{2\pi}-U(\rho)\right), \tag{12}$$

where $E^{\mu}\equiv F^{\mu\nu}l_{\nu}$ is the electric field at the horizon. Once again the bounded behavior of the invariant F and the potential $U(\rho)$ can be used to achieve the vanishing of the last term of Eq. (12). Since $\boldsymbol{E}$ is orthogonal to the null generator $\boldsymbol{l}$, it must be spacelike or null ($E_{\mu}E^{\mu}\geqslant 0$), consequently each term on the right-hand side of Eq. (12) vanishes independently, which implies that $J_{\mu}l^{\mu}/\rho=0$ and that $\boldsymbol{E}$ is proportional to the null generator $\boldsymbol{l}$ at the horizon, i.e., $\boldsymbol{E}=-(E_{\alpha}n^{\alpha})\,\boldsymbol{l}$. The vanishing of $l^{\mu}\nabla_{\mu}\rho$, only reproduces the fact that $\boldsymbol{l}$ coincides with the Killing field at the horizon. The last invariant to be studied gives the following relation:

$$\frac{1}{\kappa}R_{\mu\nu}l^{\mu}n^{\nu}-\frac{F}{4\pi}+\frac{U(\rho)}{2}=\frac{1}{4\pi}(E_{\mu}n^{\mu})^2+(l^{\mu}\nabla_{\mu}\rho)(n^{\nu}\nabla_{\nu}\rho) + \left(\frac{J_{\mu}l^{\mu}}{e\rho}\right)\left(\frac{J_{\nu}n^{\nu}}{e\rho}\right), \tag{13}$$

where it has been used that $\boldsymbol{E}=-(E_{\alpha}n^{\alpha})\,\boldsymbol{l}$. Because $n^{\nu}\nabla_{\nu}\rho$ and $J_{\nu}n^{\nu}/\rho$ are bounded at the horizon, and $J_{\mu}l^{\mu}/\rho=0=l^{\mu}\nabla_{\mu}\rho$, the last two terms in the right-hand side of Eq. (13) vanish, thus $E_{\mu}n^{\mu}$ is bounded at the horizon as consequence of the bounded behavior of the left-hand side of Eq. (13).

Summarizing, the study of the quoted invariants at the horizon leads to the following conclusions: $E_{\mu}n^{\mu}$, $J_{\mu}n^{\mu}/\rho$, $n^{\mu}\nabla_{\mu}\rho$, $J_{\mu}J^{\mu}/\rho^2$, and $I_{\mu}I^{\mu}$ are bounded at the horizon, while $J_{\mu}l^{\mu}/\rho=0$ and $\boldsymbol{E}=-(E_{\alpha}n^{\alpha})\,\boldsymbol{l}$ in the same region. Now we are in position to make a more detailed analysis of the sufficient conditions for the vanishing of the integrand (8) over the horizon, i.e., $J_{\alpha}F^{\alpha\beta}l_{\beta}/\rho^2$ vanishes and $J_{\alpha}F^{\alpha\beta}n_{\beta}/\rho^2$ remains bounded at the horizon. Using the definition $E^{\mu}\equiv F^{\mu\nu}l_{\nu}$ and that $\boldsymbol{E}=-(E_{\alpha}n^{\alpha})\,\boldsymbol{l}$, we obtain for the first quantity at the horizon

$$\frac{J_{\alpha}F^{\alpha\beta}l_{\beta}}{\rho^2}=-\frac{1}{\rho}(E_{\mu}n^{\mu})\frac{J_{\nu}l^{\nu}}{\rho}. \tag{14}$$

Since $E_{\mu}n^{\mu}$ is bounded and $J_{\nu}l^{\nu}/\rho$ vanishes at the horizon, it follows that the last expression vanishes at the horizon if the scalar field modulus ρ does not vanishes in this region. We shall show at the end of this section that ρ is a nonvanishing function at the horizon and in all the domain of outer communications $\langle\langle\mathcal{J}\rangle\rangle$.

For the second quantity we note that $\boldsymbol{J}$ and $\boldsymbol{I}$ are orthogonal to the null vectors $\boldsymbol{l}$ and $\boldsymbol{n}$, respectively. Therefore, $\boldsymbol{J}$ must be spacelike or proportional to $\boldsymbol{l}$, and $\boldsymbol{I}$ must be spacelike or proportional to $\boldsymbol{n}$. Using a null tetrad basis, constructed with $\boldsymbol{l}$, $\boldsymbol{n}$, and a pair of linearly independent spacelike vectors, these last ones being tangent to the spacelike cross sections of the horizon, the $\boldsymbol{J}$ and $\boldsymbol{I}$ vectors can be written as

$$\boldsymbol{J}=-(J_{\alpha}n^{\alpha})\boldsymbol{l}+\boldsymbol{J}^{\perp}, \tag{15}$$

$$\boldsymbol{I}=-(I_{\alpha}l^{\alpha})\boldsymbol{n}+\boldsymbol{I}^{\perp}, \tag{16}$$

where $\boldsymbol{J}^{\perp}$ and $\boldsymbol{I}^{\perp}$ are the projections, orthogonal to $\boldsymbol{l}$ and $\boldsymbol{n}$, on the spacelike cross sections of the horizon. Using Eqs. (15) and (16) it is clear that $J_{\mu}J^{\mu}=J^{\perp}_{\mu}J^{\perp\mu}$, and $I_{\mu}I^{\mu}=I^{\perp}_{\mu}I^{\perp\mu}$, i.e., the contribution to these bounded magnitudes comes only from the spacelike sector orthogonal to $\boldsymbol{l}$ and $\boldsymbol{n}$. With the help of Eqs. (15) and (16) the other quantity appearing in the integrand (8) can be written as

$$\frac{J_{\alpha}F^{\alpha\beta}n_{\beta}}{\rho^2}=\frac{J_{\alpha}I^{\alpha}}{\rho^2}=\frac{1}{\rho}\left\{(E_{\alpha}n^{\alpha})\frac{J_{\beta}n^{\beta}}{\rho}+\frac{J^{\perp}_{\alpha}I^{\perp\alpha}}{\rho}\right\}, \tag{17}$$

where the identity $I_{\alpha}l^{\alpha}=-E_{\alpha}n^{\alpha}$ has been used. The first term inside the braces in Eq. (17) is bounded because $E_{\alpha}n^{\alpha}$ and $J_{\beta}n^{\beta}/\rho$ are bounded. To the second term the Schwarz inequality applies, since $\boldsymbol{J}^{\perp}$ and $\boldsymbol{I}^{\perp}$ belong to a spacelike subspace. Thus, $(J^{\perp}_{\alpha}I^{\perp\alpha}/\rho)^2\leqslant(J^{\perp}_{\mu}J^{\perp\mu}/\rho^2)(I^{\perp}_{\nu}I^{\perp\nu})=(J_{\mu}J^{\mu}/\rho^2)(I_{\nu}I^{\nu})$ and since $J_{\mu}J^{\mu}/\rho^2$ and $I_{\nu}I^{\nu}$ are bounded at the horizon, the second term inside the braces of Eq. (17) is also bounded. Since the term enclosed by the braces in Eq. (17) is bounded, it follows that the bounded behavior at the horizon of the whole expression depends again in the nonvanishing property of the scalar field modulus ρ in this region.

The analysis of the sufficient conditions for the vanishing of the integrand (8) over the horizon shows that, the quantity (14) vanishes at the horizon and the quantity (17) remains bounded in this region if the scalar field modulus ρ is a nonvanishing function at the horizon. All the conclusions achieved up to now, can be applied to both cases the purely electric (I) and the purely magnetic (II) ones. To finish the demonstration of the vanishing of the integrand (8) over the horizon, it remains only to show that ρ is a nonvanishing function in this region. We are able, by using the function f_{ε} below, to complete the demonstration for the purely electric case (I). Unfortunately, the purely magnetic case (II) escapes to be treated along a similar way and it remains still as an open problem; we believe that the ''no-hair'' conjecture applies also to this case.

We proceed now to show that for the purely electric case (I) of spontaneously broken models ($v_a\neq 0$) ρ is a strictly positive function in all the domain of outer communications, $\langle\langle\mathcal{J}\rangle\rangle$, of a static asymptotically flat black hole. In fact, let $\varepsilon>0$ be any positive real number such that $0<\varepsilon\leqslant v_1$ (see Fig. 1), where $v_1\neq 0$ is the least value minimizing the potential function $U(\rho)$, then we shall show that $\rho\geqslant\varepsilon>0$ in all of $\langle\langle\mathcal{J}\rangle\rangle$. This result implies, by continuity of ρ, that $\rho\geqslant\varepsilon>0$ also at the horizon. In order to arrive at this conclusion,

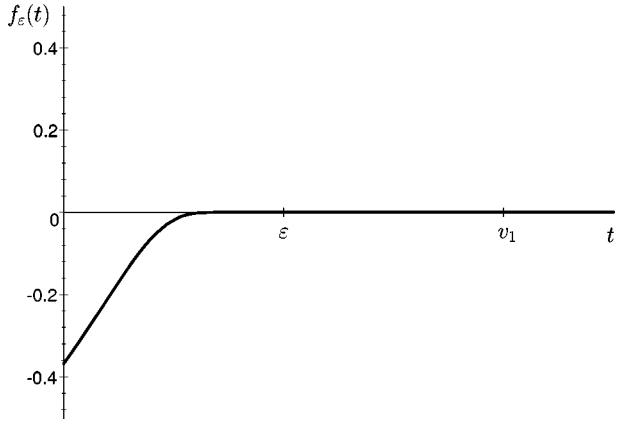

FIG. 2. The graph of the auxiliar function $f_\varepsilon(t)$.

the equation of motion (6) for ρ will be used. Let $f_\varepsilon \in C^\infty(\mathbb{R})$ be the real function defined by

$$f_\varepsilon(t)=\begin{cases} -\exp[-1/(\varepsilon-t)^2], & t<\varepsilon, \\ 0, & t\geqslant\varepsilon. \end{cases} \tag{18}$$

Such function satisfies the following conditions, see Fig. 2:

$$f_\varepsilon(v_a)=0, \qquad -1\leqslant f_\varepsilon(t)\leqslant 0, \qquad f'_\varepsilon(t)\geqslant 0, \tag{19}$$

where v_a is the value for which $U(\rho)$ achieves its ath minimum.

Multiplying Eq. (6) by $f_\varepsilon\circ\rho$ and integrating by parts over $\mathcal{V}$, after applying the Gauss theorem, one arrives at

$$\int_{\partial\mathcal{V}} f_\varepsilon(\rho)\nabla^\mu\rho\, d\Sigma_\mu=\int_{\mathcal{V}}\left(f'_\varepsilon(\rho)\nabla_\mu\rho\nabla^\mu\rho+\frac{1}{2}f_\varepsilon(\rho)U'(\rho) + \frac{f_\varepsilon(\rho)}{4e\rho^3}J_\mu J^\mu\right) dv. \tag{20}$$

We would like to point out that the term $1/\rho^3$ in the integrand above is well behaved in the domain of outer communications $\langle\langle\mathcal{J}\rangle\rangle$. This rests in the following: the integral identity (20) is obtained from the equation of motion (6). In order for this equation to be satisfied in the domain of outer communications $\langle\langle\mathcal{J}\rangle\rangle$, the function ρ must be $C^2(\langle\langle\mathcal{J}\rangle\rangle)$, i.e., twice differentiable in this region. On the other hand, most of the physically relevant potentials are smooth functions, in fact, the mayor part of them are polynomial. In this sense, the fulfillment of Eq. (6) implies, by the well-behaved nature of both its left-hand side and the term involving the derivative of the potential, that the remaining term in this Eq., going as $1/\rho^3$, is also well behaved in the domain of outer communications $\langle\langle\mathcal{J}\rangle\rangle$.

In $\partial\mathcal{V}$ the boundary integrals over Σ' and Σ cancel out again in the left-hand side of Eq. (20). The boundary integral over $i^o\cap\bar{\mathcal{V}}$ vanishes, because ρ takes asymptotically some of the values v_a minimizing the potential function $U(\rho)$, then by the conditions (19) the integrand vanishes there. The same happens to the integral over $\mathcal{H}^+\cap\bar{\mathcal{V}}$; using the natural measure at the horizon the integrand can be written as

$$f_\varepsilon(\rho)\nabla^\mu\rho\, d\Sigma_\mu=f_\varepsilon(\rho)[l_\mu\nabla^\mu\rho+(l_\nu l^\nu)n_\mu\nabla^\mu\rho]\, d\sigma, \tag{21}$$

where the vanishing of $l_\mu\nabla^\mu\rho$ and the bounded behavior of $n_\mu\nabla^\mu\rho$ at the horizon, together with the null character of $\boldsymbol{l}$ and the bounded behavior of $f_\varepsilon(\rho)$ (19) imply the vanishing of the whole integrand at the horizon. Since there are no contributions at the left-hand side of Eq. (20), the volume integral vanishes and we have for the purely electric case (I)

$$\int_{\mathcal{V}}\left(f'_\varepsilon(\rho)\gamma_{ij}\nabla^i\rho\nabla^j\rho+\frac{1}{2}f_\varepsilon(\rho)U'(\rho)-V\frac{f_\varepsilon(\rho)}{4e\rho^3}(J^t)^2\right) dv = 0, \tag{22}$$

where the coordinates from Eq. (1) has been used. From the properties of f_ε, U, V, and $\boldsymbol{\gamma}$ it follows that each term in the integrand above is non-negative, so, Eq. (22) is fulfilled only if each of them vanishes identically in $\mathcal{V}$. In particular, $f_\varepsilon(\rho)U'(\rho)|_{\mathcal{V}}=0$, this condition can be satisfied if $f_\varepsilon(\rho)|_{\mathcal{V}}=0$ which implies, from the definition of f_ε (18), that $\rho|_{\mathcal{V}}\geqslant\varepsilon>0$. Conversely, let us now suppose that $f_\varepsilon(\rho)|_p\neq 0$ for some $p\in\mathcal{V}$ this requires, from the quoted condition, that $U'(\rho)|_p=0$ and, from the definition of f_ε (18), that $0\leqslant\rho|_p<\varepsilon$, but the only extreme of $U(\rho)$ in this interval is at $\rho=0$ (see Fig. 1), hence, $f_\varepsilon(\rho)|_p\neq 0\Rightarrow\rho|_p=0$. The function ρ cannot vanish in all of $\mathcal{V}$ because it asymptotically approaches one of the values v_a for which $U(\rho)$ achieves its minima. Thus, by the connectedness of $\mathcal{V}$ and the continuity of the function ρ, $\rho(\mathcal{V})$ is an interval in $\mathbb{R}^+$ containing the points $\{0,v_a\}$, which implies that the inverse image of the open interval $]0,\varepsilon[\subset\rho(\mathcal{V})$ under the function ρ is a nonempty open subset of $\mathcal{V}$; it is clear that on this subset both $f_\varepsilon(\rho)$ and $U'(\rho)$ are nonvanishing functions (see Figs. 1 and 2). Summarizing, the assumption $f_\varepsilon(\rho)|_p\neq 0$ for some $p\in\mathcal{V}$, implies the existence of a nonempty open subset of $\mathcal{V}$ for which the condition $f_\varepsilon(\rho)U'(\rho)|_{\mathcal{V}}=0$ is violated. So, this contradiction implies the vanishing of $f_\varepsilon(\rho)$ in all of $\mathcal{V}$, which requires, by the definition of f_ε (18), that $\rho|_{\mathcal{V}}\geqslant\varepsilon>0$, result which can be extended to all of $\langle\langle\mathcal{J}\rangle\rangle$. This result finally implies, by the continuity of the function ρ that $\rho|_{\mathcal{H}^+}\geqslant\varepsilon>0$.

With the nonvanishing of ρ at the horizon we have that Eq. (14) vanishes and Eq. (17) remains bounded in this region, which implies, together with the null character of $\boldsymbol{l}$ at the horizon, the vanishing of the whole integrand (8) over the horizon. With no contribution from boundary integrals in Eq. (7), the volume integral for the purely electric case (I) is written, using the coordinates from Eq. (1), as

$$\int_{\mathcal{V}}-V\left(e^2\gamma_{ij}F^{ti}F^{tj}+\frac{4\pi}{\rho^2}(J^t)^2\right) dv=0. \tag{23}$$

The nonpositiveness of the above integrand, which is minus the sum of squared terms, implies that the integral is vanishing only if F^{ti} and J^t vanish everywhere in $\mathcal{V}$, and hence in all of $\langle\langle\mathcal{J}\rangle\rangle$.

Finally, we would like to explain why our proof on the nonvanishing of ρ fails in the purely magnetic case (II). This is due to the fact that the last term in the volume integral (22) must be replaced, in the purely magnetic case (II), by the nonpositive quantity $f_\varepsilon(\rho)\gamma_{ij}J^iJ^j/4e\rho^3$ (see Fig. 2); since the first two terms are again non-negative the integrand have no definite sign and it is impossible to deduce the vanishing of it from the vanishing of the integral. So, the nonvanishing of ρ for the purely magnetic case (II) must be justified using a different approach. We are looking for a shortcut to solve this impasse, since we believe that the ''no-hair'' conjecture applies also to this case. For any successful justification of the condition $\rho|_{\mathcal{H}^+}\neq 0$, the rest of the proof follows in this way: the nonvanishing of ρ implies again the vanishing of the integrand (8) over the horizon, having no contribution from boundary integrals in Eq. (7), the volume integral for the purely magnetic case (II) can be written, using the coordinates from Eq. (1), as

$$\int_{\mathcal{V}}\left(\frac{e^2}{2}\gamma_{ij}\gamma_{kl}F^{ik}F^{jl}+\frac{4\pi}{\rho^2}\gamma_{ij}J^iJ^j\right)dv=0, \tag{24}$$

where again the non-negativeness of the integrand implies that the vanishing of the integral is satisfied only if F^{ik} and J^i vanish everywhere in $\mathcal{V}$, and hence in all of $\langle\langle\mathcal{J}\rangle\rangle$.

III. ''NO-HAIR'' THEOREM FOR THE SCALAR FIELD IN THE ABELIAN HIGGS MODEL

It is reasonable to expect, from the ''no-hair'' conjecture, that the only possible solutions for a scalar model in the domain of outer communications $\langle\langle\mathcal{J}\rangle\rangle$ of a stationary asymptotically flat black hole become the vacuum expectation values of the self-interaction. In the models considered in this paper this implies the uniqueness of the scalar states $\Phi_a=v_a\exp i\theta$, where $v_a\neq 0$ are the values minimizing the potential function $U(\rho)$. We now concentrate our attention in the Abelian Higgs model, for which $U(\rho)$ has a single minimum at v, and we shall show the truthfulness of the last statement for the purely electric case (I), without any dependence on the specific choice of the potential. The result is obtained by applying the same procedure used above for the Eq. (6), with the function $f_\varepsilon(\rho)$ replaced this time by the function $\tanh(\rho-v)$, and taking into account that $J_\mu=0$, arriving now at

$$\int_{\mathcal{V}}[\operatorname{sech}^2(\rho-v)\ \gamma_{ij}\nabla^i\rho\nabla^j\rho+\tanh(\rho-v)U'(\rho)]dv=0, \tag{25}$$

where the boundary integral vanishes by the same arguments yielding to the vanishing of the boundary integral in Eq. (20). Since $U(\rho)$ has a single minimum at v, again the integrand at the left-hand side of Eq. (25) is non-negative, so the integral vanishes only if $\rho=v$ in all of $\mathcal{V}$, and hence in all of $\langle\langle\mathcal{J}\rangle\rangle$. We believe that this result can be extended to more general Abelian models.

IV. CONCLUSIONS

The ''no-hair'' theorem for purely electric configurations of spontaneously broken Abelian models has been extended to general static asymptotically flat black holes. The theorem is gauge invariant, and is established for any model with nonvanishing vacuum expectation values. It is shown that the gauge field vanishes outside the black hole. This vanishing is physically due to the effective behavior of the gauge field as a massive field by the spontaneous symmetry breaking. For the particular case of the Abelian Higgs model—Mexican-hat potential—it is additionally shown that the scalar field is confined to the vacuum in all the black hole exterior, which implies a zero contribution to the right-hand side of the Einstein equations (4), and that the only black hole admitted is the Schwarzschild solution. We discuss the main conditions to establish the theorem for purely magnetic configurations, but the problem remains still open; we believe that the ''no-hair'' conjecture applies also to this case.

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