q-deformed conformal quantum mechanics

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We construct a q-deformed version of the conformal quantum mechanics model of de Alfaro, Fubini, and Furlan for which the deformation parameter is complex and the unitary time evolution of the system is preserved. We also study the differential calculus on the q-deformed quantum phase space associated with such a system.

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I. INTRODUCTION

It has long been suggested that the description of spacetime based on the usual notion of geometry may not be valid at the Planck scale, and perhaps the spacetime becomes noncommutative or may show a non-Archimedean structure at such a small length scale. It has therefore been believed that the noncommutative description of spacetime might be relevant to a quantum theory of gravity. It is a generic property of a noncommutative space that the notion of a point has no meaning but the lattice structure of spacetime emerges due to the uncertainty in the measurement of the particle position in space. So in some cases such a lattice structure of spacetime at the Planck scale eliminates the ultraviolet divergence problem. Recently, there has been active investigation of noncommutative theories after it was found that noncommutative spacetime emerges naturally in M theory compactified in the presence of constant background three-form field [1] and in the world volume theory of D-branes with nonzero constant Neveu-Schwarz (NS) B field [2,3].

It is the purpose of this paper to study the noncommutative generalization of the conformal quantum mechanics of de Alfaro, Fubini, and Furlan [4]. In Ref. [5], it is observed that such a conformal quantum mechanics model can be realized as a special limit of the mechanics of a massive charged point particle in the near-horizon background of the extremal Reissner-Nordström black hole, indicating the possible relevance of the conformal quantum mechanics model to the quantum theory of black holes. Furthermore, the fact that the $SO(1,2)\cong SU(1,1)$ isometry symmetry of the $AdS_2\times S^n$ near-horizon geometry of the extremal Reissner-Nordström black hole coincides with the $SL(2,\mathbf{R})\cong SU(1,1)$ symmetry of the conformal quantum mechanics indicates that conformal quantum mechanics may have some relevance to the poorly understood AdS_2 one-dimensional conformal field theory (CFT_1) duality.

In the case of one spatial dimension, it is unclear what is meant by noncommutative space (in this paper we assume that the time coordinate is a commutative variable), since a coordinate always commutes with itself. A possible way of introducing noncommutativity for such a case is by following Manin's proposal [6–8], which is based upon a differential calculus on a quantum plane [9], that the Heisenberg algebra can be modified through q deformation of phase space. Manin's proposal was first applied to some nonrelativistic dynamical systems in Refs. [10,11]. Following this line of approach, we shall construct a q-deformed version of the conformal quantum mechanics model of Ref. [4] and study its properties.

The paper is organized as follows. In Sec. II, we review the relevant aspect of conformal quantum mechanics. In Sec. III, we discuss the q-deformed Heisenberg algebra with a complex deformation parameter and with unitary time evolution of the system, and apply this to construct a q-deformed version of conformal quantum mechanics. In Sec. IV, we study the dynamics of the system and from this we construct a differential calculus on q-deformed phase space.

II. CONFORMAL QUANTUM MECHANICS

In the following, we briefly summarize the conformal quantum mechanics studied in Refs. [4,12,13]. The Lagrangian density for the system is given by

$$\mathcal{L}=\frac{1}{2}m\dot{x}^2-\frac{g}{2x^2}. \tag{1}$$

The action is invariant under the following $SL(2,\mathbf{R})$ conformal algebra, spanned by the Hamiltonian $\mathcal{H}$, the dilation generator $\mathcal{D}$, and the special conformal generator $\mathcal{K}$:

$$[\mathcal{D},\mathcal{H}]=2i\mathcal{H},\quad [\mathcal{D},\mathcal{K}]=-2i\mathcal{K},\quad [\mathcal{H},\mathcal{K}]=-i\mathcal{D}. \tag{2}$$

Here, the $SL(2,\mathbf{R})$ generators are explicitly given by

$$\mathcal{H}=\frac{p^2}{2m}+\frac{g}{2x^2},\quad \mathcal{D}=\frac{1}{2}(px+xp),\quad \mathcal{K}=\frac{1}{2}mx^2. \tag{3}$$

The problem with the above Hamiltonian $\mathcal{H}$ is that its eigenspectrum is continuous and bounded from below but without an end point or ground state, and its eigenstates are not normalizable. Such problem was circumvented [4] by redefining the Hamiltonian as a linear combination of the above $SL(2,\mathbf{R})$ generators with a suitable condition on the coefficients. Particularly, the following choice is found to be convenient [4]:

$$L_0=\frac{1}{2}\left(a\mathcal{H}+\frac{1}{a}\mathcal{K}\right), \tag{4}$$

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where the introduction of the constant a leads to a breakdown of scale invariance. Then, the potential term in L_0 has a minimum and the energy eigenstates become discrete and normalizable. Furthermore, along with the linear combinations

$$L_{\pm 1}=\frac{1}{2}\left(a\mathcal{H}-\frac{1}{a}\mathcal{K}\mp i\mathcal{D}\right), \tag{5}$$

L_0 satisfies the following $SL(2,\mathbf{R})$ algebra in the Virasoro form:

$$[L_1,L_{-1}]=2L_0, \quad [L_0,L_{\pm 1}]=\mp L_{\pm 1}. \tag{6}$$

III. *q* DEFORMATION OF CONFORMAL QUANTUM MECHANICS

For ordinary commutative quantum mechanics in two spacetime dimensions, the Heisenberg algebra of observables can be defined as the quotient

$$H(I,x,p)=C[I,x,p]/J(I,x,p), \tag{7}$$

where $C[I,x,p]$ is a unital associative algebra freely generated by the identity I, the position operator x, and the canonical momentum operator p, and $J(I,x,p)$ is a two-sided ideal in $C[I,x,p]$ generated by the following relation corresponding to the Heisenberg rule:

$$xp-px=iI, \tag{8}$$

where we are using units in which $\hbar=1$. The operators x and p are assumed to have the following property under the antilinear anti-involution operation in $C[I,x,p]$:

$$x^*=x, \quad p^*=p. \tag{9}$$

The formalism of Manin's quantum space [6–8] can be applied to the above Heisenberg algebra by making use of the q-deformed differential calculus developed in Ref. [9]. Namely, one can deform the above Heisenberg algebra by deforming the usual Heisenberg rule (8) as follows:

$$xp-qpx=iI, \tag{10}$$

where the deformation parameter q can be either complex or real. First, if q is a complex number, the consistency of relation (10) along with the Hermiticity condition (9) on x and p requires that $|q|=1$. According to Ref. [10], which first studied q-deformed classical and quantum mechanics (with a complex deformation parameter q) of a particle in one-dimensional space and whose work was later generalized to the relativistic case in Ref. [14], the parameters of the dynamics such as the inertial mass m of the particle do not commute with the generators x and p of the algebra and there is no unitary time evolution of the system at the quantum level. Later, it was found [15–17] that to achieve unitary noncommutative q dynamics on the quantum level, i.e., for the Heisenberg equation of motion $\dot{\Omega}=(i/\hbar)[\mathcal{H},\Omega]+\partial_t\Omega$ to be satisfied after the q deformation, one has to introduce additional generators into the algebra. Second, if q is a real number, x and p cannot both be Hermitian, as can be seen by applying the involution operation to Eq. (10). So one has to assume that only one of p and x is Hermitian and the involution of the other is a separate operator [11]. One can alternatively describe the q-deformed Heisenberg algebra with a real q by redefining the generators, say, in terms of the above generators p, x, and x^* so that new momentum and position operators can be both Hermitian, as was done in Ref. [11]. In such a case, an additional generator (expressed in terms of p, x and x^*), which approaches I as $q\to 1$, other than the Hermitian position and momentum operators, is introduced into the algebra. Such an alternative algebra can be obtained [18] also by making use of the Leibniz rule $\partial_{\mathsf{x}}\mathsf{x}=1+q\mathsf{x}\partial_{\mathsf{x}}$ for the differential calculus in the one-dimensional q-deformed Euclidean space $\mathbf{R}^1_q$. In the present paper, we shall apply the first approach for studying the q-deformed generalization of the conformal quantum mechanics of Refs. [4,12,13].

The q-deformed Heisenberg algebra of observables with a complex deformation parameter is given by the following quotient:

$$H=A[I,x,p,K,\Lambda]/J(I,x,p,K,\Lambda). \tag{11}$$

In the case of a particle under the influence of a nontrivial potential V, with the assumption of the proper limit of no q deformation, the two-sided ideal J is defined by the following q deformed Heisenberg relations or the Bethe ansatz reordering rules:

$$xp=q^2px+iq\Lambda^2, \quad x\Lambda=\xi\Lambda x, \quad p\Lambda=\xi^{-1}\Lambda p,$$
$$xK=\xi^{-2}Kx, \quad pK=Kp, \quad \Lambda K=\xi^{-1}K\Lambda. \tag{12}$$

Here, the generators K and Λ are assumed to be invertible and time independent, and one can consistently (with the above q-deformed Heisenberg relations) impose the following reality conditions on the generators under the involution operation:

$$x^*=x, \quad p^*=p, \quad K^*=K, \quad \Lambda^*=\Lambda, \tag{13}$$

along with $|q|=1=|\xi|$.

In the q-deformed quantum phase space described above, the Hamiltonian (3) of the conformal quantum mechanics of Ref. [4] is deformed in the following way:

$$\mathcal{H}_\xi=p^2K^2+\frac{mg}{\xi^4}x^{-2}K^2\Lambda^4, \tag{14}$$

where we obtained this form of the Hamiltonian from the requirement of the consistency of the Hamiltonian form of the Heisenberg equations with the q-deformed Heisenberg relations (12), namely, the requirement of the unitary time evolution of the system. To further impose the naturalness condition that the velocity $\dot{x}$ be linear in the momentum p in the Heisenberg equation of motion $\dot{x}=i[\mathcal{H}_\xi,x]$, one has to further let $\xi=q$, which we assume from now on. Note that in the limit of no q deformation K and Λ belong to the center of the algebra. The requirement of irreducibility of the representation level implies that they should be proportional to the identity I when $\xi=q=1$. We choose $K=(1/\sqrt{2m})I$ and $\Lambda=I$ when $\xi=q=1$ so that the Hamiltonian (14) reduces to the form (3) in the limit of no q deformation.

The dilation generator $\mathcal{D}$ and the special conformal generator $\mathcal{K}$ in Eq. (3) of the $SL(2,\mathbf{R})$ algebra can be q deformed in such a way that the commutation relations (2) of the $SL(2,\mathbf{R})$ algebra continue to be satisfied after the q deformation. Such q-deformed $SL(2,\mathbf{R})$ generators are given by

$$\mathcal{H}_q=p^2K^2+\frac{mg}{q^4}x^{-2}K^2\Lambda^4,$$

$$\mathcal{D}_q=\frac{1}{2}(qpx+q^{-1}xp)\Lambda^{-2},$$

$$\mathcal{K}_q=\frac{1}{4q^4}x^2K^{-2}\Lambda^{-4}. \tag{15}$$

By using the q-deformed Heisenberg relations (12) with $\xi=q$, one can show that these q-deformed generators satisfy the following commutation relations:

$$[\mathcal{D}_q,\mathcal{H}_q]=2i\mathcal{H}_q,\quad [\mathcal{D}_q,\mathcal{K}_q]=-2i\mathcal{K}_q,$$

$$[\mathcal{H}_q,\mathcal{K}_q]=-i\mathcal{D}_q. \tag{16}$$

Just as in the case of undeformed conformal quantum mechanics, one can redefine the generators through linear combinations so that the resulting new generators satisfy the $SL(2,\mathbf{R})$ algebra in the Virasoro form. The q-deformed forms of the $SL(2,\mathbf{R})$ generators (4) and (5) are given by

$$L_0^q=\frac{1}{2}\left(a\mathcal{H}_q+\frac{1}{a}\mathcal{K}_q\right),$$

$$L_{\pm1}^q=\frac{1}{2}\left(a\mathcal{H}_q-\frac{1}{a}\mathcal{K}_q\mp i\mathcal{D}_q\right), \tag{17}$$

where $\mathcal{H}_q$, $\mathcal{D}_q$, and $\mathcal{K}_q$ are defined in Eqs. (15). It is straightforward to show that the q-deformed generators (17) still satisfy Eqs. (6). It might be possible to construct a q-deformed version of the de Alfaro–Fubini– and Furlan conformal quantum mechanics in such a way that the generators instead satisfy the q-deformed commutation relations and thereby the $SL(2,\mathbf{R})$ algebra (6) is deformed to the quantum $SL_q(2,\mathbf{R})$ algebra. The q-deformed version of the symmetry group, the so-called quantum group, of a dynamical system was originally studied in Refs. [19,20] within the context of the quantum noncommutative harmonic oscillator with q-deformed creation and annihilation operators.

IV. DIFFERENTIAL CALCULUS ON THE q-DEFORMED PHASE SPACE

Note that the above q-deformed algebra (11) generated by I, x, p, K, and Λ, satisfying the commutation relations (12), is the zero-form sector of the q-deformed quantum de Rham complex generated by these generators and their differentials. The quantum de Rham complex contains information about not only the algebra of observables but also the dynamics of the theory. Namely, by relating the velocity vector $(\dot{x},\dot{p})$ for a particle in quantum phase space to the one-forms dx and dp as $dx=\dot{x}dt$ and $dp=\dot{p}dt$, one can learn about the dynamics of a particle moving on quantum phase space from the commutation relations among the generators of the algebra and their differentials. Here, the overdot denotes the derivative with respect to the time coordinate t, which we assume to be a commuting parameter. According to Ref. [21], there are three families[1] of possible differential calculi associated with the Manin's plane defined by commutation relations among the generators. In this section, we construct the q-deformed quantum de Rham complex directly from the Heisenberg equations of motion, instead of applying the result of Refs. [9,21]. In the following, we restore Planck's constant $\hbar$ in the equations just for the purpose of making it easy to see various limits. In particular, in the $q\to1$ and $\hbar\to0$ limit (i.e., the limit of undeformed classical theory), the formulas obtained in the following reduce to those of the usual commutative classical geometry.

The Heisenberg equations associated with the q-deformed Hamiltonian (14) with $\xi=q$ have the following form:[2]

$$\dot{x}=\frac{i}{\hbar}[\mathcal{H}_q,x]=2qpK^2\Lambda^2,$$

$$\dot{p}=\frac{i}{\hbar}[\mathcal{H}_q,p]=\frac{2mg}{q^4}x^{-3}K^2\Lambda^6. \tag{19}$$

One can also show by using Eq. (12) that $\dot{K}=(i/\hbar)[\mathcal{H}_q,K]=0$ and $\dot{\Lambda}=(i/\hbar)[\mathcal{H}_q,\Lambda]=0$. The Aref'eva-Volovich limit [10] is achieved by further letting $\Lambda=I$. In this case, the system does not evolve unitary with time, as can be seen from the fact that the reordering rules (12), which are derived from the condition of unitary time evolution, cannot be consistent when $\Lambda=I$. As expected, in the limit of no q deformation, the above Heisenberg equations of motion reduce to the Heisenberg equations associated with the Hamiltonian given by Eq. (3).

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[1]In the case of the relativistic motion of a particle in two-dimensional noncommutative Minkowski spacetime, one of the families is excluded and the remaining two coincide [14] under the condition of a reasonable description of the particle dynamics.

[2]For the more general $\xi\neq q$ case, the Heisenberg equations take the following form:

$$\dot{x}=\left[\frac{i}{\hbar}(\xi^4-q^4)p^2x+q(q^2+\xi^2)p\Lambda^2\right]K^2,$$

$$\dot{p}=-\frac{img}{\hbar q^4}\left[\left(\frac{q}{\xi}\right)^4-1\right]px^{-2}K^2\Lambda^4+\frac{mg}{q^4}\left[\left(\frac{q}{\xi}\right)^2+1\right]x^{-3}K^2\Lambda^6. \tag{18}$$

As mentioned in the previous section, the velocity $\dot{x}$ becomes linear in the momentum p when $\xi=q$.

From the above Heisenberg equations, one can express the differentials of the generators as follows:

$$dx=\dot{x}dt=2qpK^2\Lambda^2dt,\quad dp=\dot{p}dt=\frac{2mg}{q^4}x^{-3}K^2\Lambda^6dt,$$

$$dK=\dot{K}dt=0,\qquad d\Lambda=\dot{\Lambda}dt=0. \tag{20}$$

By using the fact that we defined the time coordinate t to be a commuting parameter, one can derive the commutation relations among the generators and their differentials. The following commutation relations can be obtained by making use of the relations (20) and the q-deformed Heisenberg relations (12) with $\xi=q$:

$$\begin{aligned}
x\,dx&=dx(x+i\hbar q^{-1}p^{-1}\Lambda^2),\\
p\,dx&=q^{-2}dx\,p,\\
K\,dx&=q^2dx\,K,\\
\Lambda\,dx&=q^{-1}dx\,\Lambda,\\
x\,dp&=q^2dp\,x,\\
p\,dp&=dp\,(p+3i\hbar qx^{-1}\Lambda^2),\\
K\,dp&=dp\,K,\\
\Lambda\,dp&=qdp\,\Lambda.
\end{aligned}\tag{21}$$

The first and the sixth relations in Eq. (21) can be rewritten in more symmetric forms as follows:

$$px\,dx=q^{-4}dx\,xp,\quad x^{-3}K^2\Lambda^4p\,dp=q^{-4}dp\,px^{-3}K^2\Lambda^4. \tag{22}$$

By assuming the usual Leibniz rule and nilpotency condition for the external differential operator d, one obtains the following product rules for the differentials:

$$\begin{aligned}
(dx)^2&=\frac{i\hbar}{2}q^{-7}p^{-2}\,dxdp,\\
(dp)^2&=\frac{3}{2}i\hbar qdxdp\,(\partial_xx^{-1})\Lambda^2,\\
dxdp&=-q^2dpdx,
\end{aligned}\tag{23}$$

where ∂_x denotes the generalized q- and $\hbar$-deformed partial derivative with respect to x, which we define in the following. We see that the first commutation relation in Eq. (21) is similar to that of the universal calculus over a lattice, except that the 'lattice spacing' is an operator. In fact, the q deformation leads to a deformation of continuous phase space to a lattice structure, where the Hilbert space of representations of the q-deformed system has a discrete spectrum, putting physics on a q lattice.

We further enlarge the algebra by defining the derivatives ∂_x and ∂_p on the q-deformed phase space in the following way:

$$d=dx\,\partial_x+dp\,\partial_p, \tag{24}$$

along with the assumption of the usual Leibniz rule and the nilpotency condition as above. Note that we have seen in the above that $dK=0=d\Lambda$. The following commutation relations between the partial derivatives and the generators can be obtained by applying the Leibniz rule:

$$\begin{aligned}
\partial_xx&=1+(x+i\hbar q^{-1}p^{-1}\Lambda^2)\partial_x,\\
\partial_xp&=q^{-2}p\partial_x,\\
\partial_xK&=q^2K\partial_x,\\
\partial_x\Lambda&=q^{-1}\Lambda\partial_x,\\
\partial_px&=q^2x\partial_p,\\
\partial_pp&=1+(p+3i\hbar qx^{-1}\Lambda^2)\partial_p,\\
\partial_pK&=K\partial_p,\\
\partial_p\Lambda&=q\Lambda\partial_p,
\end{aligned}\tag{25}$$

The commutation relations between the partial derivatives and the differentials can be obtained by demanding their consistency with the q-deformed Heisenberg rules and the product rules obtained in the above. We have not yet been successful in obtaining the commutation relations for the general $\hbar\neq0$ case. In the q-deformed classical phase space (the $\hbar=0$ case), the commutation relations are given by

$$\begin{aligned}
\partial_xdx&=dx\,\partial_x,\quad \partial_xdp=q^{-2}dp\,\partial_x,\\
\partial_pdx&=q^2dx\,\partial_p,\quad \partial_pdp=dp\,\partial_p.
\end{aligned}\tag{26}$$

In the $q\rightarrow1$ and $\hbar\rightarrow0$ limit (i.e., the limit of undeformed classical theory), the formulas obtained in the above reduce to those of the usual commutative classical geometry. Particularly interesting limits are the $q\rightarrow1$ limit and the $\hbar\rightarrow0$ limit, which, respectively, correspond to the $\hbar$ deformation (or quantization) and the q deformation of the differential calculus on the "classical" commutative phase space. Note that the q deformation and the $\hbar$ deformation generally do not commute with one another; i.e., the so-called Faddeev's rectangle is not always commutative. In this paper, we consider the case of the q deformation of commutative-quantized (or $\hbar$-deformed) classical conformal mechanics. Had we first q deformed the commutative classical conformal quantum mechanics and then quantized it, we might have obtained a different noncommutative theory.

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