Quark hadron continuity in QCD with one flavor

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We study QCD with one flavor at finite baryon density. In the limit of very high baryon density the system is expected to be a color superconductor. In the case of one flavor, the order parameter is in a $\bar{3}$ of color and has a total angular momentum of 1. We show that, in weak coupling perturbation theory, the energetically preferred phase exhibits ''color-spin locking''; i.e., the color and spin direction of the condensate are aligned. We discuss the properties of this phase and argue that it shares important features of the hadronic phase at low density. In particular, we find an unbroken rotational symmetry, spin-3/2 quasiparticles, and an unusual mechanism for quark-antiquark condensation. Our results are relevant to three flavor QCD in the regime where the strange quark mass is bigger than the critical value for color-flavor locking. We find that the gaps in this case are on the order of 1 MeV.

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I. INTRODUCTION

The behavior of hadronic matter in the regime of very high baryon density but small temperature has attracted a lot of interest recently. It was realized a long time ago that asymptotic freedom combined with the presence of a Fermi surface implies that high density quark matter is a superconductor [1–4]. More recently, it was pointed out that the corresponding gaps can be quite large, on the order of $\Delta \simeq 100$ MeV [5,6]. It was also realized that the phase structure is quite rich, and that matter at very high density exhibits a wealth of nonperturbative phenomena, such as a mass gap and chiral symmetry breaking, in a regime where the coupling is weak and systematic calculations are possible [7,8].

The structure of the superconducting state depends sensitively on the number of quark flavors and their masses [9–12]. For two light flavors the dominant order parameter pairs up and down quarks in a color antisymmetric wave function. The condensate is a flavor singlet and breaks the gauge symmetry $SU(3) \rightarrow SU(2)$. The up and down quarks that are singlets under the residual gauge symmetry remain gapless. In the case of three light flavors the pair condensate involves the coupling of color and flavor degrees of freedom, color-flavor locking [7,13]. Both the color and flavor symmetries are broken, but a vectorlike combination of the two remains a symmetry. This implies, among other things, that all gluons acquire a mass and that chiral symmetry is broken. In addition to that, the spectrum of low-lying states bears an uncanny resemblance to what is expected, on phenomenological grounds, for three flavor QCD at low baryon density. This has led us to the conjecture that the hyperon matter phase at low density might be continuously connected to quark matter at high density, without any phase transition [8].

If the strange quark mass is included, the structure of the ground state depends on the relative magnitude of the gap, the strange quark mass, and the Fermi momentum [9,10]. If $\Delta > m_s^2/(2p_F)$ all flavors participate and the system exhibits color-flavor locking. If the gap is smaller then pairing only takes place in the up-down sector. In this case, we expect the strange quarks to form an independent superfluid [4,10]. It is this state we wish to study in more detail in this work.

At moderate densities, instantons play an important role in determining the pairing gap for light quarks [5,6,14,15]. But instantons do not contribute to the scattering amplitude for two strange quarks. In the following, we will therefore focus on perturbative interactions [3,4,16–20]. This has the added advantage that our results are rigorous in the limit of very large chemical potential. If two strange quarks are in a color anti-symmetric wave function their combined spin and spatial wave function cannot be antisymmetric. This means that pairing between strange quarks has to involve total angular momentum one or greater. We will see that the perturbative one gluon exchange interaction is attractive for color anti-symmetric Cooper pairs in both the spin 0 and spin 1 states.

Quark superfluids with total angular momentum one have been studied using renormalization group methods [21–24], perturbative QCD [4,18,25,26], and Nambu–Jona-Lasinio models [26]. In the present work we wish to present a detailed investigation of the phase structure and of the symmetries. For this purpose, we will not only consider the realistic case of three flavor matter with $m_s > \sqrt{2p_F\Delta}$, but also the academic situation of one flavor QCD at large chemical potential. In particular, we shall argue that one flavor QCD provides a new and interesting realization of the concept of quark-hadron continuity.

The paper is organized as follows. In Sec. II we study the phase structure of one flavor QCD in the nonrelativistic limit. We show that the stable phase exhibits color-spin-locking, and discuss the symmetries of this phase in Sec. III. In Secs. IV and V we study the phase structure in the ultrarelativistic limit. We conclude in Sec. VI.

II. PHASE STRUCTURE IN THE NONRELATIVISTIC LIMIT

In practice we are mostly interested in the phase structure of superfluid strange quark matter. In this case, we expect that $p_F > m_s$ and the strange quark mass can be treated as a perturbation. Nevertheless, it is also interesting to consider the opposite limit $m > p_F$. The nonrelativistic limit simplifies the calculation and we shall study it first. In the nonrelativistic limit, the QCD Lagrangian simplifies as

$$\mathcal{L} = \psi^\dagger \left(p_0 + \mu - \frac{p^2}{2M} + gA_0 \right) \psi - \frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu}. \qquad (1)$$

We are interested in the behavior of one-flavor matter with $M > p_F > \Lambda_{QCD}$. This means that the density is sufficiently large to justify the perturbative treatment, but not so large as to invalidate the nonrelativistic limit. The dominant interaction between quarks is given by the Coulomb force. This interaction is attractive between quarks in a color antisymmetric $\bar{3}$ state. In a Fermi liquid, this attractive interaction will lead to an instability. We shall assume that this instability is resolved by the formation of a condensate of quark pairs in a color $\bar{3}$ state.

The Pauli principle requires that the wave function of two identical quarks is antisymmetric. In QCD with only one flavor this implies that color $\bar{3}$ pairs cannot condense in a total angular momentum zero state. The obvious alternative is to consider order parameters with total angular momentum one. In the nonrelativistic limit there are only two possibilities:

$$(\phi_{s=1})^a_i = \psi i \sigma_2 \sigma_i \lambda^a_A \psi, \quad (\phi_{l=1})^a_i = \psi i \sigma_2 \hat{q}_i \lambda^a_A \psi. \qquad (2)$$

Here, we have introduced a vector notation for the antisymmetric Gell-Mann matrices $\vec{\lambda}_A = (\lambda_2, \lambda_5, \lambda_7)$. The two order parameters are independent, because in the nonrelativistic limit spin and orbital angular momentum are separately conserved.

In order to derive a gap equation for the order parameters defined in Eq. (2) we follow the usual Nambu-Gorkov procedure and introduce a bispinor $\Psi = (\psi, \psi_c)$ with $\psi_c = i\sigma_2 \psi^\dagger$. In this basis, the fermionic part of the action becomes

$$S^{-1}(q) = \begin{pmatrix} q_0 - \omega_q & \Delta \\ \Delta & q_0 + \omega_q \end{pmatrix}, \qquad (3)$$

where we have defined $\omega_q = q^2/(2M) - \mu$. The interaction vertex is a diagonal matrix $\Gamma^a = \mathrm{diag}[\lambda^a/2, (\lambda^a)^T/2]$. The Nambu-Gorkov matrix (3) is easily inverted to give the normal and anomalous components of the quark propagator. The anomalous propagator is

$$S_{21}(q) = \frac{\Delta}{q_0^2 - \omega_q^2 - \Delta^2}, \qquad (4)$$

where we have to keep in mind that Δ is a color-spin matrix. The gap equation now follows from the Dyson-Schwinger equation

$$\Sigma(k) = -ig^2 \int \frac{d^4q}{(2\pi)^4} \Gamma^a S(q) \Gamma^b D^{ab}(q-k), \qquad (5)$$

where $\Sigma(k) = -[S^{-1}(k) - S_0^{-1}(k)]$ is the proper self energy and $D^{ab}(q-k)$ is the Coulomb gluon propagator.

Before we proceed, we have to specify the color-spin structure of the gap matrix. The gap is a 3×3 matrix which transforms as $\Delta \to U\Delta R$ under gauge transformations $U \in SU(3)$ and rotations $R \in SO(3)$. This is similar to the situation in liquid ^{3}He [27] and in the color superconducting phase of $N_f = 3$ QCD [7]. In liquid ^{3}He the order parameter describes the coupling of the nuclear spin to the orbital angular momentum of the pair. In the case of three flavor QCD the order parameter specifies the coupling between the color and flavor wave functions of the pair. While the order parameters in the three cases look similar, the symmetries involved are not the same. Liquid ^{3}He is characterized by a global $SO(3)\times SO(3)\times U(1)$ symmetry, high density QCD with one flavor by $SU(3)_c \times SO(3)\times U(1)$, and $N_f = 3$ QCD by $SU(3)_c \times SU(3)_L \times SU(3)_R \times U(1)$. The fact that the symmetry groups are not the same implies, among other things, that the number of independent components of the order parameter is not the same. The same statement applies to the number of independent structures in the Landau-Ginzburg functional. We find that the number of independent quartic terms in the Landau-Ginzburg functional is 5 in the case of ^{3}He, 3 in the case of one flavor QCD, and 2 in the case of three flavor QCD.

In any case, the preferred order parameter is governed by dynamics, not symmetry, and has to determined for each system separately. In the following, we shall consider the following order parameters:[1]

$$\begin{aligned} &\Delta^a_i = \Delta\, \delta^a_i, \quad B \text{ phase (CSL)}, \\ &\Delta^a_i = \Delta\, \delta^{a3} \delta_{i3}, \quad \text{polar phase (2SC)}, \\ &\Delta^a_i = \Delta\, \delta^{a3} (\delta_{i1} + i\,\delta_{i2}), \quad A \text{ phase}, \\ &\Delta^a_i = \Delta (\delta^{a1} \delta_{i1} + \delta^{a2} \delta_{i2}), \quad \text{planar phase}, \end{aligned} \qquad (6)$$

which correspond to the ''inert'' phases of liquid ^{3}He. These phases are characterized by having the largest residual symmetry groups, and by their stability under small perturbations that are consistent with the residual symmetry. There are three additional inert phases in liquid ^{3}He, the A_1, B_1, and β phases [27], but they do not lead to a gap on the Fermi surface in the case of $N_f = 1$ QCD.

In the following, we shall outline the calculation in the case of the B phase of the $s=1$ order parameter (2). We shall refer to this phase as the color-spin-locked state, in analogy with the color-flavor-locked phase of $N_f = 3$ QCD. The determination of the gap in the other phases proceeds along

[1] In the context of color-flavor locking, it is sometimes argued that one can make use of color and flavor symmetries in order to restrict the possible order parameters to diagonal matrices. This is not necessarily correct, because there is no reason to exclude non-Hermitian order parameters. An example is the A phase of liquid ^{3}He, which is one of the stable phases of ^{3}He at zero magnetic field.

similar lines and we will briefly summarize the results below. In order to determine the anomalous propagator we have to diagonalize the color-spin structure of the gap matrix. In the case of color-spin locking (CSL) this can be achieved using the observation that $(\vec{\sigma}\cdot\vec{\lambda}_A)^2 = 2-(\vec{\sigma}\cdot\vec{\lambda}_A)$. This implies that the eigenvalues of $(\vec{\sigma}\cdot\vec{\lambda}_A)$ are 1 and -2. It is useful to introduce the corresponding projection operators

$$P_{1/2} = \tfrac{1}{3}(1-\vec{\sigma}\cdot\vec{\lambda}_A), \tag{7}$$

$$P_{3/2} = \tfrac{2}{3}(1+\tfrac{1}{2}\vec{\sigma}\cdot\vec{\lambda}_A). \tag{8}$$

As we will explain in the next section, the subscript g denotes the grand spin of the eigenstate. In particular, we have $\mathrm{tr}(P_g) = (2g+1)$, so the degeneracies of the eigenvalues 1 and -2 are 4 and 2, respectively. It is now straightforward to determine the anomalous quark propagator. We find

$$S_{21}(q) = \frac{\Delta P_{3/2}}{q_0^2-\omega_q^2-\Delta^2} + \frac{-2\Delta P_{1/2}}{q_0^2-\omega_q^2-(2\Delta)^2}. \tag{9}$$

Using the explicit form of the projection operators, we can calculate the color factors for the two terms in the propagator. We find

$$\left(\frac{\lambda^a}{2}\right) P_{1/2} \left(\frac{\lambda^a}{2}\right)^T = -\frac{1}{3}P_{1/2} + \frac{1}{3}P_{3/2}, \tag{10}$$

$$\left(\frac{\lambda^a}{2}\right) P_{3/2} \left(\frac{\lambda^a}{2}\right)^T = \frac{2}{3}P_{1/2}. \tag{11}$$

The gap matrix is proportional to $P_{3/2}-2P_{1/2}$. For the gap equation to close, the right hand side of Eq. (5) also has to be proportional to the same combination of projectors. In the weak coupling limit $\Delta \ll \mu$ we can neglect the difference of the gaps in the denominator of Eq. (9) and the gap equation indeed closes. At stronger coupling, the $g=1/2$ and $g=3/2$ gaps are independent, and there is a small admixture of a spin singlet, color symmetric gap. This is analogous to the situation in the color-flavor locked phase. Putting everything together, we find the following gap equation:

$$\Delta(k) = \frac{2g^2}{9}\int \frac{d^4q}{(2\pi)^4}\left\{\frac{\Delta(q)}{q_0^2-\omega_q^2-\Delta^2(q)} + \frac{2\Delta(q)}{q_0^2-\omega_q^2-[2\Delta(q)]^2}\right\} D(q-k). \tag{12}$$

We take $D(p)$ to be a screened Coulomb propagator $D(p) = 1/(\vec{p}^2+m_D^2)$ where $m_D^2 = g^2\mu p_F/(2\pi^2)$ is the Debye screening mass. In the weak coupling limit the gap equation is dominated by momenta in the vicinity of the Fermi surface. In this case, there is no dependence on k and we can approximate the gap by a constant, $\Delta(k) \simeq \Delta$. We get

$$\Delta = \frac{2}{3}\,\frac{g^2}{8\pi^2}\log\left(\frac{4p_F^2}{m_D^2}\right)\frac{1}{3}\int_0^\Lambda d\epsilon\left\{\frac{\Delta}{\sqrt{\epsilon^2+\Delta^2}} + \frac{2\Delta}{\sqrt{\epsilon^2+(2\Delta)^2}}\right\}, \tag{13}$$

where we have introduced a cutoff Λ. This equation is easily solved

$$\Delta_{csl} = 2^{-2/3}(2\Lambda)\exp(-1/G), \quad G = \frac{2}{3}\,\frac{g^2}{8\pi^2}\log\left(\frac{4p_F^2}{m_D^2}\right). \tag{14}$$

The factor $2^{-2/3}$ is due to the fact that the $g=\frac{1}{2}$ and $g=\frac{3}{2}$ gaps are not equal. There is a similar factor $2^{-1/3}$ in the case of the color-flavor-locked phase in $N_f = 3$ QCD [11]. We can now repeat this calculation for the other phases. The main ingredient is the spectrum of gap matrix. We find

$$\begin{aligned}
&\text{polar phase} && \Delta^\dagger\Delta = (\sigma_3\lambda_A^3)^2, \\
& && \lambda^2 = \{1\ (d=4), 0\ (d=2)\}, \\
&\text{planar phase} && \Delta^\dagger\Delta = (\sigma_1\lambda_A^1+\sigma_2\lambda_A^2)^2, \\
& && \lambda^2 = \{2\ (d=4), 0\ (d=2)\}, \\
&A\ \text{phase} && \Delta^\dagger\Delta = (\sigma^+\lambda_A^3)(\sigma^-\lambda_A^3), \\
& && \lambda^2 = \{1\ (d=2), 0\ (d=4)\},
\end{aligned} \tag{15}$$

where d indicates the degeneracy of the eigenvalue. We observe that the color-spin-locked phase is the only phase in which all excitations are gapped. The gaps in the other phases are easily calculated. We find

$$\Delta(\text{polar}) = \Delta(\text{planar}) = \Delta(A\ \text{phase}) = 2^{2/3}\Delta(\text{CSL}), \tag{16}$$

which is a simple consequence of the fact that in the A, polar, and planar phases all nonzero gaps are equal. Even though the gap in the CSL phase is smaller than the gap in the other phases, the critical temperature is not. The gap in the CSL phase is suppressed because the $g=\frac{1}{2},\frac{3}{2}$ gaps are not equal. But T_c is determined by the solution of the finite temperature gap equation in the limit $\Delta \to 0$, and does not depend on the spectrum of the gap matrix.

So far, we have only discussed the solution of the gap equation in the different phases. The stable phase is determined by the condition that the thermodynamic potential is minimal. The grand potential can be calculated from

$$\Omega = \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\{\mathrm{tr}\log[S_0^{-1}(q)S(q)] - \mathrm{tr}[S(q)\Sigma(q)]\}. \tag{17}$$

The traces can be calculated using the representation of the propagator in terms of projection operators (9). In the color-spin-locked phase, we find

$$\Omega = -\frac{\mu p_F}{2\pi^2} 12\Delta_{csl}^2 \log\left(\frac{2\mu}{\Delta_{csl}}\right). \tag{18}$$

In the polar, planar and A-phase the gap is bigger, but the number of condensed states is smaller. We find

$$\Omega(\text{polar}) = \Omega(A\ \text{phase}) = \frac{2^{4/3}}{3}\Omega(\text{CSL}), \tag{19}$$

and $\Omega(\text{planar}) = \frac{1}{2}\Omega(\text{polar})$. This result shows that the color-spin-locked phase is indeed favored, but only by a very small amount, $3\times 2^{-4/3} \simeq 1.2$.

Finally, we study the phases of the $l=1$ condensate $\langle \psi\sigma_2 \hat{q}_i \lambda_A^a \psi\rangle$. The gap depends on the matrix $[\vec{d}(\hat{q})\cdot\vec{\lambda}_A]$ where $d^a = \Delta_i^a \hat{q}_i$. The eigenvalues of $(\vec{d}\cdot\vec{\lambda}_A)$ can be found from the fact that $(\hat{d}\cdot\vec{\lambda}_A)^3 = (\hat{d}\cdot\vec{\lambda}_A)$. This means that $\lambda = \{|\vec{d}|, -|\vec{d}|, 0\}$, so that there are always two gapless excitations, independent of the structure of $\vec{d}$. In the B phase $\vec{d} = \hat{q}$, and the gap is isotropic. There are four gapped and two gapless modes. Even though the gap function is isotropic, the gap equation contains extra factors of $\cos(\theta)$ and the gap is reduced compared to the result in the polar phase of the $s=1$ order parameter. We will study these suppression factors in more detail in Sec. IV. In the other phases the gap is no longer isotropic. In the polar phase, for example, we have $\vec{d} = \hat{e}_3$ and the gap behaves as $\Delta(\hat{q}) \sim \cos(\theta)$. Because the gap is not isotropic, both the gap and the condensation energy are suppressed with respect to the B phase.

III. SYMMETRIES OF THE COLOR-SPIN-LOCKED PHASE

The results of the previous section show that, in weak coupling and in the limit that the quark mass is large, high density QCD with one quark flavor exhibits color-spin locking. In this section we wish to discuss some of the properties of the color-spin-locked phase. We will also contrast these properties with our expectations for the behavior of one flavor QCD at low density.

In the A phase the $SO(3)\times SU(3)\times U(1)$ symmetry is broken to $U(1)\times SU(2)$. Here, U(1) is the residual rotational symmetry and $SU(2)$ is the unbroken part of the gauge group. In the color-spin-locked phase the original rotational and gauge symmetries are completely broken, but there is a new SO(3) invariance which is generated by a combination of the original SO(3) generators and the SU(3) color generators.[2] Consider the ''grand spin'' generators

$$\vec{G} = \frac{\vec{\sigma}}{2} + \vec{\lambda}_A. \tag{20}$$

We can verify that the operators satisfy SO(3) commutation relations

$$[G_i, G_j] = i\epsilon_{ijk} G_k, \tag{21}$$

and commute with the gap matrix in the color-spin-locked phase

$$[\vec{G}, (\vec{\sigma}\cdot\vec{\lambda}_A)] = 0. \tag{22}$$

This means that the color-spin-locked state is invariant with respect to rotations generated by $\vec{G}$. We can combine grand spin $\vec{G}$ and orbital angular momentum $\vec{L} = \vec{r}\times\vec{p}$ to obtain the conserved total angular momentum generator $\vec{J} = \vec{L} + \vec{G}$. Away from the nonrelativistic limit only $\vec{J}$ is conserved, not $\vec{L}$ and $\vec{G}$ separately. We will discuss this issue further in Sec. V.

Excitations in the color-spin-locked phase are characterized by their grand spin quantum numbers g, g_3. The quantum numbers of quasiparticles can be determined using

$$\vec{G}^2 = \tfrac{11}{4} + (\vec{\sigma}\cdot\vec{\lambda}_A). \tag{23}$$

We can now verify that the projectors P_g defined in Eqs. (7),(8) satisfy $\vec{G}^2 P_g = g(g+1)P_g$. In the color-spin-locked phase there is a $g=\frac{3}{2}$ quartet of quasiparticles with gap Δ, and a $g=\frac{1}{2}$ doublet with gap 2Δ. All gluons acquire a mass via the Higgs mechanism. In the color-spin-locked phase the octet of gauge bosons splits into a $g=1$ triplet and a $g=2$ quintet. If we couple orbital angular momentum to grand spin we find one $j=0$ state, two $j=1$ and $j=2$ states, as well as one $j=3$ state.

In the color-spin-locked phase the U(1) of baryon number is spontaneously broken. As a consequence, the system exhibits superfluidity and the spectrum contains a massless phonon. The order parameter is charged, and the photon acquires a mass by the Higgs mechanism. This also implies that QCD with three flavors exhibits the Meissner effect if the strange quark mass is larger than the critical mass for color-flavor-locking. Both the color-flavor-locked phase and the phase with pairing in the up-down sector only do not exhibit the Meissner effect.

We would now like to compare these results with our expectations for the behavior of QCD with one flavor at low baryon density. QCD with one flavor has a $U(1)_A$ chiral symmetry which is broken by the anomaly. This means that there is no spontaneous symmetry breaking, and that there are no Goldstone bosons. $N_f=1$ QCD has a large mass gap in all channels. The lowest dimension operator with baryon number $B=1$ is a spin 3/2 current

$$\eta_\mu = \epsilon^{abc}(q^a C\gamma_\mu q^b)q^c. \tag{24}$$

This suggests that the lightest baryon has spin 3/2. This agrees with the expectation from the quark model. In order to construct a color singlet state in which three quarks of the

[2]Our discussion here applies to the idealized case of one flavor QCD. In QCD with three flavors but separate pairing in the up-down and strange quark sectors the color SU(3) is broken to SU(2) by the primary condensate. This means that the color-spin-locked phase of the strange superfluid in $N_f=3$ QCD cannot have exact rotational invariance.

same flavor occupy an s state the total spin has to be 3/2. Finally, this expectation is also borne out by phenomenology. In QCD, the lightest $I=3/2$ baryon has spin 3/2. The splitting between $s=1/2$ and $s=3/2$ baryons with $I=3/2$ is about 400 MeV. The situation in the meson sector is less clear. For light quarks, phenomenology suggests that both the scalar (a_0) and the pseudoscalar (η') are heavy, and the lightest state is a vector (ω). For heavy quarks, on the other hand, the lightest quark-anti-quark bound state is a pseudoscalar, and the first excited state a vector.

Let us now turn to the effects of a nonzero chemical potential. Since the theory has a mass gap, there has to be a critical chemical potential $\mu_c \simeq M_B/3$ below which the baryon density is zero. Depending on whether nuclear matter is self-bound, this transition is continuous or not. We have no information on the interaction between two spin-3/2 baryons. If the lightest meson is a scalar, it is natural to assume that the s-wave scattering length is attractive. Even if this is not the case at very small density, repulsive interactions may get screened as the density is increased. In this case, we expect one flavor nuclear matter to be a superconductor. A natural order parameter for s-wave superconductivity is

$$\phi = \langle \eta_\mu C \gamma_5 \eta_\mu \rangle. \tag{25}$$

This order parameter breaks the U(1) of baryon number, and will lead to the appearance of a massless phonon. It also breaks the U(1) of electromagnetism, and gives a mass to the photon.

If the density is very large we expect baryons to dissolve, and it becomes natural to describe the system in terms of quarks. As we have seen, one-gluon exchange causes an instability near the Fermi surface and the quark liquid is a color superconductor. In the color-spin-locked phase rotational symmetry is unbroken, and the only global symmetries that get broken are the U(1) of baryon number and of electromagnetism. The gauge symmetry is completely broken and all colored excitations have a mass. In this sense, the system remains confined. This means that in terms of symmetries, the high density phase cannot be distinguished from the low density phase. What is even more surprising is that the spectrum of fermions is very similar to the low energy phase. We saw that the spectrum contains spin-3/2 and spin-1/2 multiplets, where the spin-3/2 quasiparticles are lower in energy. This is the expected behavior in one flavor QCD at low density. These observations suggest that the high density phase might be continuously connected to the low density phase, similar to what we suggested in the case of three flavor QCD [8].

IV. THE POLAR PHASE IN THE ULTRA-RELATIVISTIC LIMIT

We would now like to consider the opposite limit of massless, ultrarelativistic quarks. In this section we shall study the polar phase [18,25]. More complicated phases will be considered in the following section. The main new ingredient in the relativistic limit is that the interaction preserves the chirality of the quarks. It is therefore useful to employ a chiral representation. In terms of left and right handed spinors $\psi_{L,R}$ the action becomes

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(\not{q} + \not{\mu} + g\not{A})\psi \\ &= \psi_R^\dagger(q\cdot\sigma + \mu + gA\cdot\sigma)\psi_R + \psi_L^\dagger(q\cdot\bar{\sigma} + \mu + gA\cdot\bar{\sigma})\psi_L,\end{aligned} \tag{26}$$

where we have introduced $\sigma_\mu = (1,\vec{\sigma})$ and $\bar{\sigma}_\mu = (1,-\vec{\sigma})$. There are two types of order parameters with total angular momentum one, depending on whether the condensate couples quarks of the same or opposite chirality. We begin with order parameters that connect quarks of the same chirality. We have

$$\psi C\gamma_5 \hat{q}\lambda_2\psi = \psi_R i\sigma_2 \hat{q}\lambda_2\psi_R - \psi_L i\sigma_2\hat{q}\lambda_2\psi_L, \tag{27}$$

$$\psi C\vec{\alpha}\lambda_2\psi = -\psi_R i\sigma_2\vec{\sigma}\lambda_2\psi_R + \psi_L i\sigma_2\hat{\sigma}\lambda_2\psi_L, \tag{28}$$

where $\vec{\alpha} = \gamma_0\vec{\gamma}$ and we have selected a particular direction in color space. There are two additional order parameters with the opposite parity, $C\gamma_5 \to C$ and $C\vec{\alpha} \to C\vec{\Sigma}$. In this case, the relative sign between the RR and LL terms is flipped. As usual, perturbative interactions do not distinguish between order parameters of different parity. In the weak coupling limit, only states with the same chirality and helicity contribute. In order to make this manifest, we introduce the helicity projectors $H_\pm = \frac{1}{2}(1 \pm \vec{\sigma}\cdot\hat{q})$. Using the fact that

$$\psi_R \sigma_2 H_+ \vec{\sigma} H_+ \psi_R = \psi_R \sigma_2 \hat{q} H_+ \psi_R \tag{29}$$

we see that, in the weak coupling limit, the second order parameter (28) is not independent of the first (27).

In order to derive the gap equation we again consider the Dyson-Schwinger equation for the fermion self-energy in the Nambu-Gorkov representation. We concentrate on right handed quarks and introduce the bispinor $\Psi = (\psi_R, \psi_{c,R})$ with $\psi_{c,R} = -i\sigma_2\psi_R^\dagger$. The inverse fermion propagator takes the form

$$S^{-1}(q) = \begin{pmatrix} q\cdot\sigma + \mu & \vec{\Delta}\cdot\hat{q}H_+ \\ \vec{\Delta}^*\cdot\hat{q}H_+ & q\cdot\bar{\sigma} - \mu \end{pmatrix}. \tag{30}$$

The normal and anomalous components of the Nambu-Gorkov propagator are determined by the inverse of the matrix (30). The anomalous propagator is given by

$$S_{21} = -\frac{1}{q_-\cdot\bar{\sigma}}\Delta\frac{1}{q_+\cdot\sigma - \Delta^\dagger(q_-\cdot\bar{\sigma})^{-1}\Delta} \tag{31}$$

with $q_\pm = (q_0 \pm \mu, \vec{q})$ and $\Delta = \vec{\Delta}\cdot\hat{q}H_+$. Except for the angular dependence of the gap parameter this propagator has the same form as the propagator in the spin 0 case. The corresponding gap equation has been discussed many times in the literature [16–19]. Here, we simply quote the result

$$\Delta(k_0)=\frac{g^2}{12\pi^2}\int dq_0\int d\cos(\theta)\frac{\cos(\theta)\Delta(q_0)}{\sqrt{q_0^2+\cos(\theta)^2\Delta(q_0)^2}}\times\left\{\frac{\frac{1}{2}[1+\cos(\theta)]}{1-\cos(\theta)+F^2/(2\mu^2)}+\frac{\frac{1}{2}[3-\cos(\theta)]}{1-\cos(\theta)+G^2/(2\mu^2)}\right\}. \quad (32)$$

G and F are the magnetic and electric components of the gluon self-energy. For $q_0\ll\vec{q}\ll\mu$ we have $F^2=m_D^2$ and $G^2=(\pi/4)m_D^2x$ with $x=|q_0-k_0|/|\vec{q}-\vec{k}|\simeq|q_0-k_0|/\{\sqrt{2}\mu[1-\cos(\theta)]\}^{1/2}$.

The gap equation is dominated by collinear scattering with $\cos(\theta)\simeq1$. To leading order, we can solve the gap equation by setting $\cos(\theta)=1$ in the numerator. We also drop the angular dependence in the denominator of the anomalous quark propagator. In this limit, the gap is the same as for the spin zero case

$$\Delta_0=512\pi^4(2/N_f)^{5/2}\mu g^{-5}\exp\left(-\frac{3\pi^2}{\sqrt{2}g}\right), \quad (33)$$

where N_f is the number of flavors that are active in determining the screening mass. Here, we only show the contribution of electric and magnetic gluon exchanges to the pre-exponent. Additional contributions from the fermion self-energy were found in Ref. [25]. We can calculate corrections to the leading order result $\Delta_{l=1}=\Delta_{l=0}$ by expanding the numerator around $\cos(\theta)=1$. The correction term has no collinear singularity and the gluon self-energy terms F and G can be neglected. We find $\Delta_{l=1}=\exp(3c_1)\Delta_{l=0}$ with

$$c_1=\frac{1}{2}\int\frac{d\cos(\theta)}{1-\cos(\theta)}\left\{\cos(\theta)\left(\frac{3}{2}-\frac{1}{2}\cos(\theta)\right)+\cos(\theta)\left(\frac{1}{2}+\frac{1}{2}\cos(\theta)\right)-2\right\}=-2. \quad (34)$$

This implies that $\Delta_{l=1}=\exp(-6)\Delta_{l=0}\simeq0.004\Delta_{l=0}$ [25], which shows that the angular momentum $l=1$ gap is strongly suppressed with respect to the s-wave gap. While the natural scale of the s-wave gap is $\Delta=100$ MeV, the p-wave gap is expected to be less than 1 MeV.

We now come to superfluid order parameters that couple quarks of opposite chirality. In weak coupling, the only option is

$$\psi C\vec{\gamma}\lambda_2\psi=-\psi_L i\sigma_2\vec{\sigma}\lambda_2\psi_R-\psi_R i\sigma_2\vec{\sigma}\lambda_2\psi_L. \quad (35)$$

In this case, the parity of the order parameter is fixed. The order parameter $\psi C\gamma_5\vec{\gamma}\psi$ has the opposite symmetry under the exchange of the two fermion fields and cannot be color antisymmetric.

We can derive the gap equation for $\psi C\vec{\Delta}\cdot\vec{\gamma}\lambda_2\psi$ following the same steps as in the spin zero case. We introduce the bispinor $\Psi=(\psi_R,\psi_{c,L})$ with $\psi_{c,L}=i\sigma_2\psi_L^\dagger$. The inverse fermion propagator takes the form

$$S^{-1}(q)=\begin{pmatrix} q\cdot\sigma+\mu & H_+\vec{\Delta}\cdot\vec{\sigma}H_- \\ H_-\vec{\Delta}\cdot\vec{\sigma}H_+ & q\cdot\sigma-\mu\end{pmatrix}. \quad (36)$$

The anomalous quark propagator is determined by the inverse of this matrix. The gap equation is

$$\Delta(k_0)(\hat{\Delta}_\perp^k)^2=\frac{g^2}{12\pi^2}\int dq_0\int d\cos(\theta)\times\frac{\Delta(q_0)}{\sqrt{q_0^2+\Delta(q_0)^2(\hat{\Delta}_\perp^q)^2}}\frac{1}{2}\{[1+(\hat{k}\cdot\hat{\Delta})(\hat{q}\cdot\hat{\Delta})]\times(1+\hat{k}\cdot\hat{q})-(\hat{k}\cdot\hat{\Delta}+\hat{q}\cdot\hat{\Delta})^2-i(\hat{k}\cdot\hat{\Delta}+\hat{q}\cdot\hat{\Delta})\hat{k}(\hat{\Delta}\times\hat{q})\}\times\left\{\frac{1}{1-\cos(\theta)+F^2/(2\mu^2)}+\frac{1}{1-\cos(\theta)+G^2/(2\mu^2)}\right\}. \quad (37)$$

with $(\hat{\Delta}_\perp^k)^2=1-(\hat{k}\cdot\hat{\Delta})^2$. We observe that in the absence of screening and damping, $F=G=0$, electric and magnetic gluon exchanges contribute equally to the spin 1 gap. We also note that the gap has a node if the direction of the order parameter is aligned with the pair momentum. This is due to the fact that the condensate connects quarks of opposite helicity. In order to produce a gap for a quark moving with momentum $\vec{p}$, the condensate has to flip the helicity of the quark. But this cannot happen if the spin of the condensate is parallel to $\vec{p}$. We can see this explicitly from the fact that

$$\bar{\Delta}\Delta=H_-\vec{\Delta}\cdot\vec{\sigma}H_+\vec{\Delta}\cdot\vec{\sigma}H_-=H_-[(\vec{\Delta}\cdot\vec{\sigma})^2-(\vec{\Delta}\cdot\hat{q})^2]H_-. \quad (38)$$

If $\Delta\sim\hat{q}$ this expression vanishes, so there is no gap if $\hat{q}$ is parallel to $\hat{\Delta}$.

To leading order, we can solve the gap equation by evaluating the angular factors for $\hat{k}\cdot\hat{\Delta}=0$ and $\hat{k}\cdot\hat{q}=1$. In this case, the complicated matrix element in Eq. (37) reduces to the expression for the spin zero case [18]. The leading correction to this result can be determined as in Eq. (34). We find

$$c_{s=1}=\frac{1}{2}\int\frac{d\cos(\theta)}{1-\cos(\theta)}\left\{\cos(\theta)+\frac{1}{2}\cos(\theta)^2-\frac{3}{2}\right\}=-\frac{3}{2}. \quad (39)$$

This implies $\Delta^{LR}=\exp(-9/2)\Delta_0\simeq0.01\Delta_0$, which is bigger than the LL,RR gap by a factor ~4.5.

Finally, we have to consider the possibility that pairing takes place both between quarks of the same and of opposite chirality. In QCD with one flavor the chiral $U(1)_A$ symmetry is anomalous, so there is no symmetry that prevents the two condensates from mixing. But even if the chemical potential is infinitely large, and the effects of the anomaly can be neglected, simultaneous pairing may still take place. As we shall see, by combining the two order parameters we can obtain a gap with enhanced symmetries, and simultaneous pairing could be energetically favored. In the following we will consider a gap matrix of the form

$$\Delta = \Lambda_+ (\vec{\Delta}_1 \cdot \hat{p} + \vec{\Delta}_2 \cdot \vec{\gamma}) \Lambda_+ \lambda_2, \tag{40}$$

where $\Lambda_\pm = \frac{1}{2}(1 \pm \vec{\alpha} \cdot \hat{p})$ projects on positive (negative) energy states. This is equivalent to projecting on states with equal chirality and helicity, $\Lambda_+ = P_R H_+ + P_L H_-$. The derivation of the gap equation proceeds along the same lines as before, only that now we have to include both chiralities. The propagator takes the form

$$S_{21} = -\frac{1}{\not{q} - \not{\mu}} \Delta \frac{1}{(\not{q} + \not{\mu}) + \bar{\Delta}(\not{q} - \not{\mu})^{-1} \Delta}. \tag{41}$$

The quadratic form in the denominator determines the structure of the gap. We find

$$\begin{aligned} \bar{\Delta}\tilde{\Delta} &\equiv \bar{\Delta}(\not{q} - \not{\mu})^{-1} \Delta (\not{q} - \not{\mu}) \\ &= \{(\vec{\Delta}_1 \cdot \hat{q})^2 + [\vec{\Delta}_2^2 - (\vec{\Delta}_2 \cdot \hat{q})^2]\} \Lambda_-, \end{aligned} \tag{42}$$

where we have assumed that there is no relative phase between $\vec{\Delta}_1$ and $\vec{\Delta}_2$. If this is not the case, Eq. (42) contains interference terms $\sim \mathrm{Im}(\Delta_1^* \Delta_2)$ with a more complicated chiral structure. There is no general reason why such terms should be absent. Indeed, we shall find that interference between the LL and LR terms is important in the color-spin-locked phase. Here, we neglect interference effects for simplicity. Equation (42) shows that the case $\vec{\Delta}_1 = \vec{\Delta}_2 \equiv \vec{\Delta}$ is special, because the gap function is completely isotropic. Nevertheless, rotational invariance is still broken. The gap is straightforward to determine. Since the two structures in Eq. (40) have different chirality, the one-gluon exchange matrix elements decouple. As a result, the gap equation is just a linear combination of the gap equations in the LL and LR case. To leading order, the gap is again identical to the spin zero gap. Taking into account the angular dependence of the matrix elements, the gap is suppressed by $\Delta^{LL+LR} = \exp(-5)\Delta_0$. This result simply corresponds to an average of the correction terms in the LL and LR channel. Because the gap is isotropic, the condensation energy is increased by a factor 3/2 over pure LR pairing. This is not sufficient, however, to overcome the bigger angular suppression factor. We conclude that in the limit of massless quarks, pairing in the polar phase is dominated by the LR and RL channels.[3]

[3] To leading order in the coupling, condensates with an arbitrary linear combination of LL and LR diquarks are degenerate. The degeneracy is lifted once higher order corrections are included. At present, however, it is not clear whether the next-to-leading order calculation performed in the present work is complete.

V. COLOR-SPIN-LOCKING IN THE ULTRARELATIVISTIC LIMIT

In the polar phase the color and spin orientation of the condensate are completely uncorrelated. In this section, we shall deal with the more complicated cases in which the two are entangled. In practice, we will only discuss the color-spin-locked phase. The calculation in the A and planar phases is very similar, and we have verified that, in weak coupling, they do no compete with the color-spin-locked phase. As in the previous section, we have to deal with three different cases, pairing between quarks of the same chirality, pairing between different chiralities, or a combination of the two.

We begin with the simplest case, which is pairing between quarks of the same chirality. The gap matrix has the form $\Delta = (\hat{q} \cdot \vec{\lambda}_A)\Lambda_+$. As in Sec. II, this matrix can be diagonalized using the relation $\Delta^3 = \Delta$. This gives the eigenvalues ± 1 and 0. Projectors on the nonzero eigenvalues are given by $P_{\pm 1} = \frac{1}{2}(\pm 1 + \Delta)\Delta$. We observe that, just as in the nonrelativistic limit, the B phase of the RR (or LL) order parameter is not fully gapped. The value of the gap is easy to determine. Using the projectors introduced above we observe that the gap equation reduces to the one we found in the polar phase, Eq. (32). This implies $\Delta_{csl}^{LL} = \Delta_{\mathrm{pol}}^{LL} = \exp(-6)\Delta_0$. Even though the gap in the B phase is equal to the one in the polar phase, the condensation energy is not. Because the gap in the B phase is isotropic, the condensation energy is three times larger and the B phase is energetically favored.

The B phase of the $(LR+RL)$ order parameter is more complicated. As in Sec. IV we concentrate on the LR sector and consider the gap matrix

$$\Delta = H_-(\vec{\sigma} \cdot \vec{\lambda}_A) H_+. \tag{43}$$

The physical gaps are determined by the eigenvalues of

$$\bar{\Delta}\Delta = H_+[(\vec{\sigma} \cdot \vec{\lambda}_A)^2 - (\hat{q} \cdot \vec{\lambda}_A)^2] H_+. \tag{44}$$

From $(\bar{\Delta}\Delta)^2 = 2(\bar{\Delta}\Delta)$ it follows that $\lambda = 2,0$. The corresponding projectors are $P_2 = \frac{1}{2}\bar{\Delta}\Delta$ and $P_0 = 1 - \frac{1}{2}\bar{\Delta}\Delta$. This shows that the eigenvalue $\lambda = 2$ has degeneracy 2, and the eigenvalue $\lambda = 0$ has degeneracy 1. Of course, the actual degeneracies are 4 and 2, because of the trivial degeneracy $LR \to RL$. In the nonrelativistic limit $\psi C \vec{\gamma} \cdot \vec{\lambda}_A \psi$ reduces to $\psi \sigma_2 \vec{\sigma} \cdot \vec{\lambda}_A \psi$. But while the nonrelativistic order parameter $\psi \sigma_2 \vec{\sigma} \cdot \vec{\lambda}_A \psi$ leads to a fully gapped state, we find that in the ultrarelativistic limit two modes remain gapless. This is related to the result that in the polar phase the $LR+RL$ gap has a node on the Fermi surface, see Eq. (38).

This also means that, unlike the nonrelativistic case (12) the gap equation is not modified as compared to the polar phase. We find $\Delta_{csl}^{LR} = \Delta_{\mathrm{pol}}^{LR} = \exp(-9/2)\Delta_0$ where Δ_0 is the spin zero gap. On the other hand, because the gap is isotropic, the condensation energy is increased compared to the polar phase by a factor 3/2.

Finally, we have to consider the possibility that pairing takes place in both the LL,RR and LR,RL channels. In particular, we would like to consider the gap matrix

$$\Delta=\Lambda_+(\hat{q}+\vec{\gamma})\cdot\vec{\lambda}_A\Lambda_+ . \tag{45}$$

We note that this order parameter has positive parity, and that the parity is fixed, even if only perturbative interactions are taken into account. The physical gaps are determined by [cf. Eq. (42)]

$$\overline{\Delta}\tilde{\Delta}=\Lambda_-[(\vec{\alpha}\cdot\vec{\lambda}_A)^2+i\vec{\gamma}\cdot(\hat{q}\times\vec{\lambda}_A)]\Lambda_- . \tag{46}$$

We note that, similar to the polar case, the structure $(\hat{q}\cdot\vec{\lambda}_A)$ has disappeared, but this time an interference term is present. We can now follow the standard procedure and determine the characteristic equation for $\overline{\Delta}\tilde{\Delta}$. We find $(\overline{\Delta}\tilde{\Delta})^3-5(\overline{\Delta}\tilde{\Delta})^2+4(\overline{\Delta}\tilde{\Delta})=0$ which leads to the eigenvalues $\lambda=0,1,4$. The corresponding projectors are

$$P_0=\Lambda_+ ,\quad P_{3/2}=\tfrac{4}{3}\,(1-\tfrac{1}{4}\,(\overline{\Delta}\tilde{\Delta}))\Lambda_- ,$$

$$P_{1/2}=-\tfrac{1}{3}\,[1-(\overline{\Delta}\tilde{\Delta})]\Lambda_- , \tag{47}$$

where we follow the notation used in Eq. (7),(8). We note that in the weak coupling limit, all particles are gapped. The eigenvalue $\lambda=1$ has multiplicity 4, while $\lambda=4$ has multiplicity 2. This implies that the spectrum is identical to the one we found in the nonrelativistic color-spin-locked phase. As a consequence, the structure of the gap equation is also very similar to Eq. (12). We find

$$\Delta(k_0)=g^2\int\frac{d^4q}{(2\pi)^4}\left\{\frac{\Delta(q_0)M^{\mu\nu}_{3/2}}{q_0^2+\omega_q^2+[\Delta(q_0)]^2}+\frac{\Delta(q_0)M^{\mu\nu}_{1/2}}{q_0^2+\omega_q^2+[2\Delta(q_0)]^2}\right\}D_{\mu\nu}(q-k), \tag{48}$$

with $\omega_q^2=(q-\mu)^2$ and the matrix elements

$$M^{\mu\nu}_g=\frac{1}{12}\mathrm{tr}\left[\gamma_\mu\left(\frac{\lambda^a}{2}\right)\Lambda^q_-\vec{\lambda}_A\cdot(\hat{q}-\vec{\gamma})\Lambda^q_-P_g\gamma_\nu\times\left(\frac{\lambda^a}{2}\right)^T\Lambda^k_+\vec{\lambda}_A\cdot(\hat{k}-\vec{\gamma})\Lambda^k_+\right], \tag{49}$$

where $\Lambda^q_\pm=\frac{1}{2}(1\pm\vec{\alpha}\cdot\hat{q})$, $\Lambda^k_\pm=\frac{1}{2}(1\pm\vec{\alpha}\cdot\hat{k})$, and P_g are the projectors defined in Eq. (47). The structure of these matrix elements is quite complicated, but the traces simplify in the weak coupling limit. In this case, we can evaluate the matrix elements in the forward direction $\vec{q}\simeq\vec{k}$ and find

$$\Delta(k_0)=\frac{g^2}{12\pi^2}\int dq_0\int d\cos(\theta)\left\{\frac{1}{3}\frac{\Delta(q_0)}{\sqrt{q_0^2+(\Delta(q_0))^2}}+\frac{2}{3}\frac{\Delta(q_0)}{\sqrt{q_0^2+(2\Delta(q_0))^2}}\right\}\left\{\frac{1}{1-\cos(\theta)+F^2/(2\mu^2)}+\frac{1}{1-\cos(\theta)+G^2/(2\mu^2)}\right\}. \tag{50}$$

In leading order we can neglect the difference between the gaps and find $\Delta_{csl}^{LL+LR}=\Delta_0$. Taking the difference into account we get $\Delta_{csl}^{LL+LR}=2^{-2/3}\Delta_0$, as in the nonrelativistic case. Again, the condensation energy is bigger as compared to the polar state, even though the gap is smaller. We can also determine the corrections which come from nonforward scattering. The calculation is identical to the one in the polar phase and we find $\Delta_{csl}^{LL+LR}=2^{-2/3}\exp(-5)\Delta_0$.

The color-spin-locked state (46) has a number of interesting properties. First we note that, as in the nonrelativistic limit, rotational invariance is unbroken, and the low-lying fermions are organized into a spin-3/2 and a spin-1/2 multiplet. What is new in the relativistic case is the fact that both the $U(1)_V$ and $U(1)_A$ symmetries are broken. The $U(1)_A$ symmetry is also broken in the polar phase, but there is an important difference here. Without the chirally odd interference term in Eq. (46) the diquark condensate does not induce a quark-antiquark condensate $\langle\overline{\psi}_L\psi_R\rangle$. In the color-spin-locked state, there is a nonvanishing condensate

$$\Sigma_{csl}=\langle\overline{\psi}\vec{\alpha}\cdot(\hat{q}\times\vec{\lambda}_A)\psi\rangle. \tag{51}$$

In the weak coupling limit we find $\Sigma_{csl}=-(1/6\pi^2)\mu^3\log(2)$. The condensate is a scalar under rotations generated by the grand angular momentum operator. This means that is has the same symmetries as the quark condensate $\langle\overline{\psi}\psi\rangle$. Once higher order perturbative corrections are included, we expect the primary condensate Σ_{csl} to induce a non-zero $\langle\overline{\psi}\psi\rangle$ as well. As a result, there will be a nonzero quark condensate even at very large density, where instantons are exponentially suppressed.

We saw that the gap in the state (45), which has an equal mixture of LL and LR components, is given by $\Delta_{csl}^{LL+LR}=2^{-2/3}\exp(-5)\Delta_0$. This is suppressed with respect to the gap in the color-spin-locked phase of the pure LR order parameter $\Delta_{csl}^{LR}=\exp(-4.5)\Delta_0$, and the larger number of condensed species is insufficient to overcome this suppression. On the other hand, we saw that the spectrum in the $LL+LR$ state corresponds exactly to the spectrum in the nonrelativistic limit. This suggests that, if the baryon density is increased in $N_f=1$ QCD with massive quarks, the order parameter evolves from the fully gapped $LL+LR$ state to the partially gapped LR state. In order to see whether this can happen continuously we would like to study the spectrum for a general linear combination of the LL and LR order parameters

$$\Delta=\Lambda_+[\cos(\beta)\hat{q}+\sin(\beta)\vec{\gamma}]\cdot\vec{\lambda}_A\Lambda_+ . \tag{52}$$

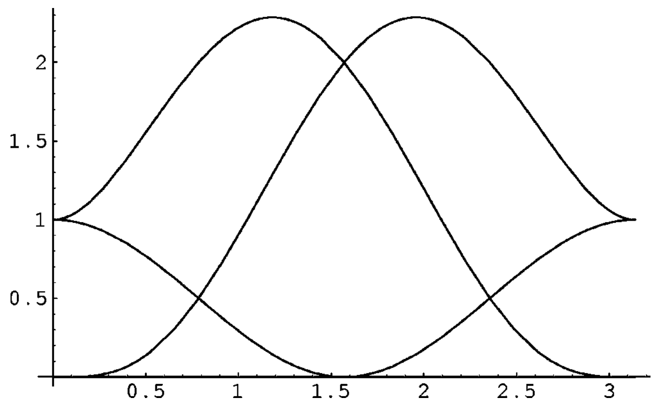

FIG. 1. Quasiparticle spectrum in the color-spin-locked phase as a function of the mixing angle β between the LL and LR condensates, see Eq. (53).

For $\beta=\pi/2$ this corresponds to the pure LR order parameter (43), and for $\beta=\pi/4$ we get an equal mixture of LL and LR as in Eq. (45). For an arbitrary value of β the spectrum of $\bar{\Delta}\tilde{\Delta}$ is given by

$$\lambda_1=\cos(\beta)^2,$$

$$\lambda_{2,3}=\frac{1}{4}\left[5\pm\sqrt{2}\cos(\beta)\sqrt{9-7\cos(2\beta)}-3\cos(2\beta)\right], \quad (53)$$

where all eigenvalues are doubly degenerate. We show the spectrum as a function of β in Fig. 1. We note that the spectrum is fully gapped for all values of β except for $\beta=0,\pi/2$. There are three values of β, $\beta=0,\pi/4,\pi/2$, for which two pairs of eigenvalues meet, and the degeneracy of the spectrum is enhanced. These correspond to the cases we already discussed in detail, pure LL and LR pairing, and equal LL and LR pairing. To leading order in the coupling, the gap and the condensation energy are independent of the mixing angle β. Taking subleading corrections into account, we found that in the nonrelativistic limit the state corresponding to $\beta=\pi/4$ is favored. In the ultrarelativistic limit, the energetically preferred state has $\beta=\pi/2$. We therefore conjecture that as a function of p_F/m the order parameter evolves from $\beta=\pi/4$ to $\beta=\pi/2$.

VI. CONCLUSIONS

In summary, we have studied QCD with one flavor at high baryon density. Our results are relevant to QCD with three flavors in the case when the strange quark mass is bigger than the critical value for color-flavor locking [9,10]. They also apply to the situation in two flavor QCD when the difference between the chemical potentials for up and down quarks is bigger than the gap [28,29]. In both cases there is no pairing between quarks of different flavors, and the possible phases are identical to those in one flavor QCD. We should note, however, that even if the pair condensate involves only a single flavor, there will still be some dependence on the number flavors. This dependence arises from the N_f dependence of the screening mass, and from higher order corrections that may couple condensates of different flavors.

In QCD with one flavor, as in QCD with two or more flavors, the Fermi surface is unstable with respect to the formation of color antisymmetric Cooper pairs. However, because the wave function of the pair cannot be antisymmetric in flavor, the Cooper pairs have to have angular momentum one or greater. As a result, the magnitude of the gap is suppressed with respect to the spin zero gap in two flavor QCD. Using weak coupling perturbation theory, we find $\Delta<\exp(-4.5)\Delta_0$ where Δ_0 is the spin zero gap. If the typical magnitude of the gap in two flavor QCD is 100 MeV, we find that the one flavor gap is $\Delta\sim 1$ MeV.

In one flavor QCD the order parameter is a spin-color matrix, and interesting phases can arise because of the possibility that color and spin degrees of freedom become entangled. The situation is superficially similar to the phase structure of liquid ^{3}He [27] and high density QCD with three flavors [11,30,31], but the dynamics and the symmetries involved are different. Nevertheless, as in BCS (or Eliashberg) studies of $N_f=3$ QCD or liquid ^{3}He, we find that, in weak coupling perturbation theory, the B phase is energetically favored. In analogy with the color-flavor-locked phase of $N_f=3$ QCD we refer to this phase as color-spin locked.

The situation is particularly simple in the nonrelativistic limit. In this case, there is a unique ground state, and the spectrum in the color-spin-locked phase is fully gapped. In the ultrarelativistic limit the situation is more complicated. The order parameter exhibits color-spin locking, but to leading order in the coupling constant there is a continuous family of states which differ by the mixing angle between the $\psi_R\psi_R-\psi_L\psi_L$ and $\psi_R\psi_L+\psi_L\psi_R$ components of the order parameter. Except at special points, the spectrum is again fully gapped.

In the color-spin-locked phase the original rotational symmetry is broken, but there is an unbroken SO(3) symmetry which is generated by a combination of the original angular momentum and color generators. The only non-anomalous symmetry which is broken in the color-spin-locked phase is the U(1) of baryon number. This means that the global symmetries of the color-spin-locked phase agree with what we expect, on phenomenological grounds, for one flavor QCD at low density. We also found that the color-spin-locked phase has certain other features that are characteristic of $N_f=1$ QCD. In particular, we saw that the color-spin-locked phase supports low energy spin-3/2 quasiparticles, and that there is a mechanism for generating quark-antiquark condensates. These observations lead us to conjecture that in one flavor QCD the low and high density phases are continuously connected. In the case of one flavor QCD this suggestion is less radical than in the case of three flavors. In particular, it is known that for sufficiently small values of the quark mass there is no phase transition along the finite temperature axis [32]. In this case, we expect the only phase transition in the T-μ plane to be the nuclear onset transition.

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[1] S. C. Frautschi, in *Proceedings of the Workshop on Hadronic Matter at Extreme Energy Density*, edited by N. Cabibbo (Erice, Italy, 1978).
[2] B. C. Barrois, Nucl. Phys. **B129**, 390 (1977).
[3] F. Barrois, Ph.D. thesis, Caltech, UMI Microfiche No. 79-04847-mc.
[4] D. Bailin and A. Love, Phys. Rep. **107**, 325 (1984).
[5] M. Alford, K. Rajagopal, and F. Wilczek, Phys. Lett. B **422**, 247 (1998).
[6] R. Rapp, T. Schäfer, E. V. Shuryak, and M. Velkovsky, Phys. Rev. Lett. **81**, 53 (1998).
[7] M. Alford, K. Rajagopal, and F. Wilczek, Nucl. Phys. **B537**, 443 (1999).
[8] T. Schäfer and F. Wilczek, Phys. Rev. Lett. **82**, 3956 (1999).
[9] M. Alford, J. Berges, and K. Rajagopal, Nucl. Phys. **B558**, 219 (1999).
[10] T. Schäfer and F. Wilczek, Phys. Rev. D **60**, 074014 (1999).
[11] T. Schäfer, Nucl. Phys. **B575**, 269 (2000).
[12] R. D. Pisarski, Phys. Rev. C (to be published), nucl-th/9912070.
[13] M. Srednicki and L. Susskind, Nucl. Phys. **B187**, 93 (1981).
[14] G. W. Carter and D. Diakonov, Phys. Rev. D **60**, 016004 (1999).
[15] R. Rapp, T. Schäfer, E. V. Shuryak, and M. Velkovsky, Ann. Phys. (N.Y.) **280**, 35 (2000).
[16] D. T. Son, Phys. Rev. D **59**, 094019 (1999).
[17] T. Schäfer and F. Wilczek, Phys. Rev. D **60**, 114033 (1999).
[18] R. D. Pisarski and D. H. Rischke, Phys. Rev. D **61**, 074017 (2000).
[19] D. K. Hong, V. A. Miransky, I. A. Shovkovy, and L. C. Wijewardhana, Phys. Rev. D **61**, 056001 (2000).
[20] W. E. Brown, J. T. Liu, and H. Ren, Phys. Rev. D **61**, 114012 (2000).
[21] N. Evans, S. D. Hsu, and M. Schwetz, Nucl. Phys. **B551**, 275 (1999).
[22] N. Evans, S. D. Hsu, and M. Schwetz, Phys. Lett. B **449**, 281 (1999).
[23] T. Schäfer and F. Wilczek, Phys. Lett. B **450**, 325 (1999).
[24] S. D. Hsu and M. Schwetz, Nucl. Phys. **B572**, 211 (2000).
[25] W. E. Brown, J. T. Liu, and H. Ren, Phys. Rev. D (to be published), hep-ph/9912409.
[26] J. Hosek, hep-ph/9812516; M. Iwasaki and T. Iwado, Phys. Lett. B **350**, 163 (1995).
[27] D. Vollhardt and P. Wölfe, *The Superfluid Phases of Helium 3* (Taylor and Francis, New York, 1990).
[28] P. Bedaque, hep-ph/9910247.
[29] D. T. Son and M. A. Stephanov, hep-ph/0005225.
[30] I. A. Shovkovy and L. C. Wijewardhana, Phys. Lett. B **470**, 189 (1999).
[31] N. Evans, J. Hormuzdiar, S. D. Hsu, and M. Schwetz, Nucl. Phys. **B581**, 391 (2000).
[32] C. Alexandrou, A. Borici, A. Feo, P. de Forcrand, A. Galli, F. Jegerlehner, and T. Takaishi, Phys. Rev. D **60**, 034504 (1999).