Construction of a minimal Higgs $SO(10)$ supersymmetric grand unified model

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A full account is given of the procedure used by the authors to construct an $SO(10)$ supersymmetric grand unified model of the fermion mass matrices. Various features of the model which gives remarkably accurate results for the quark and lepton masses and mixings were presented earlier in separate publications. The construction of the matrices is first discussed in the framework of effective operators, from which one naturally obtains the maximal $\nu_\mu - \nu_\tau$ mixing, while the small angle or maximal mixing solutions for the solar neutrinos depend upon the nature of the Majorana matrix. A set of Higgs and fermion superfields is then introduced from which the Higgs and Yukawa superpotentials uniquely give the structure of the mass matrices previously obtained. The right-handed Majorana matrix arises from one Higgs field coupling to several pairs of super-heavy conjugate neutrino singlets. For the simple version considered, 10 input parameters accurately yield the 20 masses and mixings of the quarks and leptons, and the 3 masses of the right-handed neutrinos.

I. INTRODUCTION

In several recent papers [1–4] we have developed a highly predictive model of quark and lepton masses based on the grand unified group $SO(10)$. This model grew out of our attempt [1] to construct a realistic grand unified theory (GUT) in which $SO(10)$ was broken down to the standard model in the simplest possible, or ''minimal'' way [5]. In this model there emerged a new mechanism based on certain well-known features of $SU(5)$ for explaining the large mixing between the mu and tau neutrinos that is seen at Super-Kamiokande [6]. In [1,2] we gave the structure of the quark and lepton mass matrices for the second and third families, treating the first family as massless. In [3], it was shown how to extend the model to include the first family, which leads to several interesting predictions. In [4], it was observed that the mixing of the electron neutrino very naturally falls either within the range $0.004 \leq \sin^2 2\theta_{e\mu} \leq 0.008$, corresponding to the small angle Mikheyev-Smirnov-Wolfenstein (MSW) solution [7] of the solar neutrino problem, or very near to the value $\sin^2 \theta_{e\mu} = 1$, corresponding to what is called ''bimaximal mixing.''

In this paper we present the model in fuller detail, especially in regard to neutrino phenomenology, and to the structure of the Higgs sector, Yukawa interactions, and flavor symmetries that underlie the quark and lepton mass matrices.

The paper is organized as follows. In Sec. II we discuss in general terms, that is apart from a particular model, our mechanism for explaining the large mixing of ν_μ and ν_τ. In Sec. III, we explain what we mean by a ''minimal'' $SO(10)$ breaking scheme, and show how such minimal breaking and the requirements of simplicity lead one naturally to a certain form for the mass terms of the heavier two families of quarks

and leptons. We then observe that this form realizes the general mechanism for large $\nu_\mu - \nu_\tau$ mixing described in the previous section. It is important to emphasize that this mechanism emerged not from an attempt to explain neutrino phenomenology, but from other considerations entirely, in particular the attempt to simplify the Higgs structure of $SO(10)$. It is most interesting that the same mechanism has also independently been found by other groups attempting to make sense of neutrino phenomenology. In Sec. IV, it will be explained how this model is best extended to the first family of quarks and leptons, and how this gives rise to several distinctive predictions. Accurate analytic expressions for the predictions at the GUT scale will be presented. In Sec. V, the neutrino sector will be examined in detail. It will be seen how either the small-angle MSW solution of the solar neutrino problem or bimaximal mixing can result with equal simplicity. Finally, in Sec. VI, a concrete model, including all the details of flavor symmetries and of the Higgs and Yukawa superpotentials, will be presented, showing that the basic scheme is technically natural.

II. MECHANISM FOR LARGE $\nu_\mu - \nu_\tau$ MIXING

Before explaining our mechanism, it will be helpful to explain why the observed large mixing of ν_μ, presumably with ν_τ, has been a theoretical puzzle. The basic reason is simple: the mixing that is seen between the quarks of the second and third families is described by a small mixing angle, namely $V_{cb} \cong 0.04$, and therefore it was expected that the mixing between the second and third family of leptons would also be small.

The grounds for this expectation were twofold. First, there is the empirical fact that the masses of the quarks and leptons exhibit roughly similar ''hierarchical'' patterns, and therefore it was natural to assume that their mixing angles would be similar also. Second, the most promising theoretical approaches to understanding the pattern of quark and lepton

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masses, namely grand unification and flavor symmetry, tend to treat quarks and leptons in similar ways. For instance, small quark mixing angles might suggest an underlying fundamental ''family symmetry'' that is weakly broken, in which case the lepton mixings might be expected also to be small. And in grand unification based on $SO(10)$ there is a close connection between the quark and lepton mass matrices.

There are actually two puzzles associated with the mixing of the second and third families: First, why is the lepton mixing $|U_{\mu3}| \sim 0.7$ so large? And, second, why is the quark mixing $V_{cb} \cong 0.04$ so small? What we mean by saying that these are distinct puzzles is that they are both unexpected within the most commonly assumed framework for explaining quark and lepton masses, the Weinberg-Wilczek-Zee-Fritzsch (WWZF) idea [8].

The WWZF idea was that the Cabibbo angle could be understood if the mass matrices of the first and second families of quarks had the following form:

$$
L_{mass} = (\overline{u_R}, \overline{c_R}) \begin{pmatrix} 0 & b \\ b & a \end{pmatrix} \begin{pmatrix} u_L \\ c_L \end{pmatrix} + (\overline{d_R}, \overline{s_R}) \begin{pmatrix} 0 & b' \\ b' & a' \end{pmatrix} \begin{pmatrix} d_L \\ s_L \end{pmatrix}.
$$

$$(1)$$

This gives $|m_d/m_s| \cong |b'/a'|^2$, $|m_u/m_c| \cong |b/a|^2$, and $V_{us} \cong b'/a' - b/a$, and thus the famous relation

$$
V_{us} \cong \sqrt{m_d/m_s} - e^{i\alpha} \sqrt{m_u/m_c}.
$$

$$(2)$$

Since $|V_{us}| \cong 0.22$, $\sqrt{m_d/m_s} \cong 0.22$, and $\sqrt{m_u/m_c} \cong 0.07$, this relation is satisfied for $\alpha \sim \pm \pi/2$. If we apply the same idea to the leptons of the first two families we get

$$
U_{e2} \cong \sqrt{m_e/m_\mu} - e^{i\beta} \sqrt{m_{\nu_1}/m_{\nu_2}}.
$$

$$(3)$$

The second term on the right is not known, but if it is assumed to be small one has the rough prediction that $U_{e2} \sim \sqrt{m_e/m_\mu} \cong 0.07$. This could be consistent with the small angle MSW solution of the solar neutrino problem, which requires that $U_{e2} \sim 0.04$. Thus the WWZF idea appears to work well where it was originally applied, namely to the first and second families.

Fritzsch [9] later extended this idea to explain the mixing of the third family. If a WWZF form is assumed to hold for the second and third family, i.e., if one takes $(u,c) \rightarrow (c,t)$ and $(d,s) \rightarrow (s,b)$ in Eq. (1), one obtains

$$
V_{cb} \cong \sqrt{m_s/m_b} - e^{i\gamma} \sqrt{m_c/m_t}
$$

$$(4)$$

and

$$
U_{\mu3} \cong \sqrt{m_\mu/m_\tau} - e^{i\delta} \sqrt{m_{\nu_2}/m_{\nu_3}}.
$$

$$(5)$$

Since $\sqrt{m_s/m_b} \cong 0.14$, and $\sqrt{m_c/m_t} \cong 0.04$, one sees that the observed value of $V_{cb} \cong 0.04$ is too small by a factor of three or so. Assuming that the neutrino mass ratio in Eq. (5) is small, and given that $\sqrt{m_\mu/m_\tau} \cong 0.24$, one sees that the nearly maximal value of $U_{\mu3} \sim 1/\sqrt{2} \cong 0.7$ that is observed is too large by a factor of three or so.

These Eqs. (2)–(5) are based on the assumption of a hierarchical and symmetric form for the mass matrices. A key feature in our mechanism for understanding the large mixing of the tau neutrino is that it involves highly asymmetric mass matrices. As we shall see, the assumption of asymmetric mass matrices naturally explains why $U_{\mu3}$ is larger than the Fritzsch value and V_{cb} is smaller than the Fritzsch value by approximately the same factor.

Consider, a toy model with $SU(5)$ symmetry, which has a set of Yukawa terms of the following form: $\lambda_{33}(\overline{5}_3 10_3)\overline{5}_H + \lambda_{23}(\overline{5}_2 10_3)\overline{5}_H + \lambda_{32}(\overline{5}_3 10_2)\overline{5}_H$, with $\lambda_{32} \ll \lambda_{23} \sim \lambda_{33}$ and the subscript H denoting a Higgs representation. These terms yield the following mass matrices for the second and third families of down quarks and charged leptons:

$$
(\overline{d_{2R}}, \overline{d_{3R}}) \begin{pmatrix} 0 & \sigma \\ \epsilon & 1 \end{pmatrix} \begin{pmatrix} d_{2L} \\ d_{3L} \end{pmatrix} M_D + (\overline{l_{2R}}, \overline{l_{3R}}) \begin{pmatrix} 0 & \epsilon \\ \sigma & 1 \end{pmatrix} \begin{pmatrix} l_{2L} \\ l_{3L} \end{pmatrix} M_D,
$$

$$(6)$$

with $\epsilon \ll \sigma \sim 1$. Here we have labeled the fermions with a family index, instead of the names s, b, μ, and τ, since the mass matrices in this case are far from diagonal. A crucial point to notice is that the matrix for the leptons, which we will denote by L, is the transpose of the matrix for the down quarks, which we will denote by D. This is a feature of minimal $SU(5)$. It arises from the fact that the $\overline{5}$ representation of fermions contains the left-handed leptons, l_L, and the charge conjugate of the right-handed down-quarks, d_R, while the **10** representation of fermions contains the charge conjugate of the right-handed leptons, l_R, and the left-handed down-quarks, d_L. Thus, $SU(5)$ relates D to L, but only up to a left-right transposition: $D = L^T$.

The transposition feature of $SU(5)$ unification appearing in Eq. (6) results in the large element, σ, of L producing an O(1) mixing of l_{2L} with l_{3L} for the leptons, while in D for the quarks it produces a large mixing of the right-handed fields d_{2R} and d_{3R}. The mismatch between the large $l_{2L} - l_{3L}$ mixing and the $\nu_{2L} - \nu_{3L}$ mixing, which is small (as will soon be seen), leads to a large $U_{\mu3}$ mixing element. But the right-handed mixings of the quarks are not observable through standard model physics. What matters is the left-handed mixing of d_{2L} with d_{3L}, which contributes to V_{cb}, and is controlled by the small parameter ϵ.

The common statement that grand unification relates quark and lepton mixing angles, and thus V_{cb} to $U_{\mu3}$, is very misleading. What is really true in general is that grand unification relates the mixing of quarks of one handedness to the mixing of leptons of the other handedness. Thus V_{cb} and $U_{\mu3}$ need not be directly related to each other. Of course, if the mass matrices are symmetric, as has almost always been assumed, the left-handed and right-handed mixings are the same, and hence V_{cb} is directly related to $U_{\mu3}$. The most natural interpretation, then, of the experimental discovery that $|U_{\mu3}| \gg |V_{cb}|$ is that the mass matrices are highly asymmetric. This is the essential point first made in [10].

Not only does a highly asymmetric, or, as we will call it, ''lopsided,'' form of the mass matrices explain the difference between the size of $U_{\mu3}$ and V_{cb}, but it also explains the

fact, noted above, that $U_{\mu 3}$ is larger than the Fritzsch value and V_{cb} is smaller than the Fritzsch value by about the same factor. The point is that the product of the two off-diagonal elements, ϵ and σ, is controlled by the fermion mass ratio. As is evident from Eq. (6), $m_s/m_b \cong \epsilon\sigma/(1+\sigma^2) \sim \epsilon\sigma$. That means that the Fritzsch prediction for the mixing of d_L and s_L, which is $\sqrt{m_s/m_b}$, goes approximately as $\sqrt{\epsilon\sigma}$. That shows that the Fritzsch prediction for the mixing angles is roughly the geometric mean between the true value of $U_{\mu 3}$ $\sim \sigma$ and the true value of $V_{cb} \sim \epsilon$. In other words, in our hypothesis of lopsided mass matrices, the surprising largeness of $U_{\mu 3}$ and the surprising smallness of V_{cb} are two sides of the same coin.

Another important feature of this mechanism should be emphasized. Almost all published explanations of the largeness of the $\nu_\mu - \nu_\tau$ mixing trace it to some special feature or form of the neutrino mass matrix. Perhaps this is due to the purely linguistic fact that we talk about ''neutrino mixing angles.'' But they could just as well be called the ''charged-lepton mixing angles.'' They are really the angles expressing the mismatch between the neutrino mass eigenstates and the charged lepton mass eigenstates, just as the Cabibbo-Kobayashi-Maskawa (CKM) angles are the mismatch between the up and down quark eigenstates. In our mechanism, the large value of $U_{\mu 3}$ is traceable to a peculiarity of the charged lepton mass matrix L, namely, having a large off-diagonal entry σ. As we shall see in the next section, having such a large entry helps to explain several other features of the quark and lepton mass spectrum.

To sum up, the mechanism for explaining large $\nu_\mu - \nu_\tau$ mixing proposed in [1–3] has three salient features: (1) the largeness of this mixing is due to the charged lepton mass matrix, which is (2) highly asymmetric, and which is (3) related to the transpose of the down quark mass matrix by $SU(5)$.

In the next section we will see how a model with precisely these features arises very naturally in $SO(10)$ from very different considerations.

III. FERMION MASS MATRICES IN MINIMAL SCHEMES OF $SO(10)$ BREAKING

The model that we shall examine in this paper emerged originally from our attempt to construct a realistic model based on $SO(10)$ [11] in which $SO(10)$ is broken to the standard model group, $G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$ in the simplest possible way. We shall therefore start by explaining what we mean by minimal $SO(10)$ breaking.

Since $SO(10)$ is a rank 5 group, it requires for its breakdown to G_{SM} at least two Higgs fields. One Higgs field is needed to break the rank of the group to 4, but this generally leaves an unbroken $SU(5)$. The second Higgs field is needed to break $SU(5)$ down to G_{SM}. The two breakings can occur in either sequence depending upon which Higgs field has the larger vacuum expectation value (VEV) and effects the first breaking.

Whatever Higgs field gives superlarge mass to the right-handed neutrinos, as required for the standard seesaw explanation of the lightness of the left-handed neutrinos, will also break $SO(10)$ down to $SU(5)$, and thus the rank to 4. There are two simple choices for this Higgs field: either an anti-symmetric five-index tensor $\overline{\bf 126}$ or a spinor $\overline{\bf 16}$. In either case, one also expects a Higgs field in the conjugate representation, **126** or **16**, to go along with it. A nice feature of the $\overline{\bf 126}$ is that this tensorial representation leaves unbroken a Z_2 subgroup of the center of $SO(10)$ that acts as an automatic matter parity, whereas if a spinor Higgs field is introduced, then matter parity is not automatic. On the other hand, to introduce $\mathbf{126} + \overline{\bf 126}$ is to introduce quite large representations that tend to make the unified gauge coupling go non-perturbative below the Planck scale, and that may be hard to obtain from superstring theory. In any event, it would seem that the use of a spinor-antispinor pair, $\mathbf{16} + \overline{\bf 16}$, is more economical. Thus we assume that the rank of $SO(10)$ is broken at the unification scale and the right-handed neutrinos get mass from one such spinor-antispinor pair of Higgs fields.

To break the group the rest of the way to G_{SM} requires the existence of Higgs fields in the adjoint representation **45** and/or in the symmetric two-index tensor representation **54**. Most published realistic $SO(10)$ models have several of both kinds of multiplets. However, it has been shown that it is possible to break $SO(10)$ to G_{SM} with only a *single* adjoint Higgs and no larger representations [5].

This, then, is what we call the ''minimal breaking scheme for $SO(10)$'': *The breaking of $SO(10)$ to G_{SM} is accomplished by the expectation values of a set of Higgs fields consisting of* $\mathbf{45}_H + \mathbf{16}_H + \overline{\bf 16}_H$, *with the model containing no multiplets larger than the single* **45**. There, of course, have to be other Higgs fields to break the $SU(2)_L \times U(1)_Y$ group of the electroweak interactions.

This minimality assumption is restrictive enough that it is possible to say in which direction the expectation values of these fields point. This can be done by considering the problem of doublet-triplet splitting, whereby the colored partners of the weak-doublet Higgs fields of the standard model become superheavy while the weak-scale masses of the doublets themselves are preserved. In $SO(10)$ the only known way of doing this in a technically natural manner is the Dimopoulos-Wilczek or ''missing VEV'' mechanism [12]. The idea is that if an adjoint Higgs field that has an expectation value proportional to the $SO(10)$ generator $B-L$ couples to Higgs fields in the vector representation, it will make their color-triplet components heavy (since they have $B-L = \pm 2/3$) while leaving their weak-doublet components massless (since they have $B-L = 0$). The needed coupling is simply of the form $\mathbf{10}_{1H} \mathbf{45}_H \mathbf{10}_{2H}$. Of course, the expectation value of the adjoint, by virtue of the definition of the adjoint representation, is necessarily a linear combination of generators of the group. $B-L$ is one of the $SO(10)$ generators that is picked out by simple forms of the Higgs superpotential for the adjoint multiplet. It should be noted that there is another version of the missing VEV mechanism that works in $SO(10)$, in which the VEV of the adjoint is proportional to the generator I_{3R} of the $SU(2)_R$ subgroup of $SO(10)$ [13]. However, that version is significantly more complicated. Therefore, simplicity dictates the choice that

$$\langle \mathbf{45}_H \rangle \propto B - L. \qquad (7)$$

Since the assumption of a minimal $SO(10)$ breaking scheme included the supposition that only *one* adjoint exists in the model, no adjoint exists except the one that points in the $B-L$ direction. As we shall see, this puts an important limitation on the possibilities for constructing realistic mass matrices for the quarks and leptons. The assumption of a minimal $SO(10)$ breaking scheme thus acts as an important guide in searching for good models.

The simplest possible terms that would give mass to quarks and leptons in $SO(10)$ would be $\lambda_{ij}\mathbf{16}_i\mathbf{16}_j\mathbf{10}_H$, where the subscripts i and j are family indices. This would lead to four proportional Dirac mass matrices for the up-quarks (U), down quarks (D), charged leptons (L), and neutrinos (N). In fact one would have $D=L\propto U=N$. Moreover, all these matrices would be symmetric, which is why one can write $D=L$ instead of $D=L^T$ as in minimal $SU(5)$. Some of the predictions that follow from these relations are good, notably the famous prediction $m_b^0=m_\tau^0$, where the superscript zero stands for quantities evaluated at the unification scale M_G. However, $D=L$ also predicts that $m_s^0=m_\mu^0$ and $m_d^0=m_e^0$. Empirically, one finds instead that $m_s^0\simeq\frac{1}{3}m_\mu^0$ and $m_d^0\simeq 3m_e^0$. These factors of three are called the Georgi-Jarlskog factors [14]. The simplest possible $SO(10)$ Yukawa terms also predict that all the CKM angles vanish, since $U\propto D$. While not exactly true, this is at least a good zeroth order relation, since the CKM angles are all small compared to unity. By contrast, in $SU(5)$ the matrices D and U are not related by the unified symmetry and so the CKM angles are unconstrained. The smallness of the CKM angles can be regarded, therefore, as evidence for $SO(10)$. On the other hand, the proportionality of D and U in $SO(10)$ also predicts that $m_c^0/m_t^0=m_s^0/m_b^0$, which fails badly by over an order of magnitude.

What one can conclude is that a way of going beyond the simplest possible $SO(10)$ Yukawa scheme must be found which preserves some of its predictions while breaking others. One way to do this involves using larger representations to break the electroweak interactions. For instance, in the original Georgi-Jarlskog model, a $\overline{\mathbf{45}}$ multiplet of $SU(5)$ [not to be confused with the adjoint of $SO(10)$] participates in breaking $SU(2)_L\times U(1)_Y$. In the context of $SO(10)$, this $\overline{\mathbf{45}}$ is contained in a $\overline{\mathbf{126}}$, which is inconsistent with our minimality assumptions. More economical is to assume that the Higgs fields that break $SO(10)$ at the unification scale, i.e., the $\mathbf{45}_H+\mathbf{16}_H+\overline{\mathbf{16}}_H$, couple to quarks and leptons and thus introduce the effects of that $SO(10)$ breaking into the quark and lepton mass relations. This is the assumption we make.

To describe the third family it is simplest to assume the minimal Yukawa term $\mathbf{16}_3\mathbf{16}_3\mathbf{10}_H$ as pictured in Fig. 1(a). By itself, this would make all the mass matrices have the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (8)$$
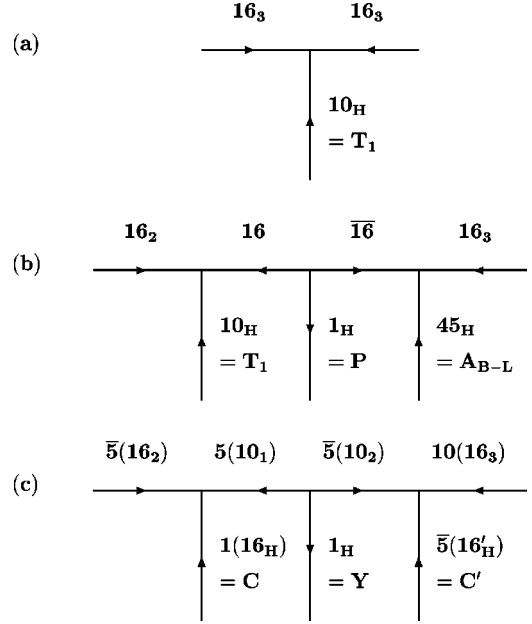

FIG. 1. Diagrams that generate the elements in the quark and lepton mass matrices shown in Eqs. (10) with the Higgs labeling corresponding to that appearing in Table I of Sec. VI. (a) The 33 elements denoted ''1.'' (b) The 23 and 32 elements denoted ''ϵ.'' Note that because of the appearance of the VEV of the adjoint Higgs field $\mathbf{45}_H\equiv A$, they are proportional to the $SO(10)$ generator $B-L$. (c) The asymmetric elements denoted ''σ'' arise from this diagram. That they do not contribute to the up quark masses, and contribute asymmetrically to the down quark and charged lepton mass matrices are consequences of the fact that the $SO(10)$ $\mathbf{10}$'s, i.e., $\mathbf{10}_1$ and $\mathbf{10}_2$, contain $\overline{\mathbf{5}}$ but not $\mathbf{10}$ of $SU(5)$.

That would give the following predictions, all of which are at least good zeroth approximations to reality: $m_b^0=m_\tau^0$, $V_{cb}=0$, and $m_1/m_3=m_2/m_3=0$, where m_i is a mass of a fermion of the ith family. Note that these are just the ''good'' $SO(10)$ predictions mentioned above.

The second family presents more of a challenge. The main issue is how to get the Georgi-Jarlskog factor of 3 between m_μ^0 and m_s^0. Breaking of $SU(5)$ must be involved, since the bad relation $m_s^0=m_\mu^0$ arises already at the $SU(5)$ level. The only field that breaks $SU(5)$ in the framework of minimal $SO(10)$ breaking is the adjoint, $\mathbf{45}_H$. Since $\langle\mathbf{45}_H\rangle\propto B-L$, and the $B-L$ of leptons is -3 times that of quarks, this field has the possibility of giving the needed Georgi-Jarlskog factor. Thus one must seek an effective Yukawa term that involves the $\mathbf{45}_H$. The simplest such term [15], in the sense of the term of lowest dimension, is of the form $(\mathbf{16}_i\mathbf{16}_j)\mathbf{10}_H\mathbf{45}_H/M_G$. Moreover, this term can arise in a simple way by the integration out of a $\mathbf{16}+\overline{\mathbf{16}}$ family plus antifamily at the unification scale, as shown in Fig. 1(b).

There are actually two ways to contract the $SO(10)$ indices of such a term: the product $(\mathbf{16}_i\mathbf{16}_j)$ can be contracted symmetrically or antisymmetrically. It is easy to show that if $\langle\mathbf{45}\rangle\propto B-L$, only the antisymmetric contraction contributes to the quark and lepton mass matrices. [The reason is simple. If the VEV of the adjoint is proportional to a generator Q, then the symmetric-antisymmetric contractions give contri-

butions to fermion masses that go as $Q(f) \pm Q(\bar{f})$. Since $B-L$ of an antifermion is minus that of the fermion, the contribution cancels for the symmetric contraction.] Thus, one need only consider the flavor-antisymmetric term, which means only $ij=23$ and not $ij=22$, or 33. Consequently, the only operator of interest is $(\mathbf{16_2 16_3})\mathbf{10}_H\mathbf{45}_H$ which, together with the operator $\mathbf{16_3 16_3 10}_H$, gives

$$
U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon/3 \\ 0 & -\epsilon/3 & 1 \end{pmatrix} M_U, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon/3 \\ 0 & -\epsilon/3 & 1 \end{pmatrix} M_D,
$$

$$
N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\epsilon \\ 0 & \epsilon & 1 \end{pmatrix} M_U, \quad L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\epsilon \\ 0 & \epsilon & 1 \end{pmatrix} M_D. \tag{9}
$$

The desired factor of 3 has been achieved between leptons and quarks, due to the generator $B-L$ to which the adjoint VEV is proportional. One also can see that the ϵ entries are flavor antisymmetric for reasons already explained. As they stand, these forms of the matrices are inadequate to explain even the features of the second and third families of fermions. There are three inadequacies. (1) The factor of 3 comes in squared between the mass of the leptons and quarks of the second family. The reason is that, for ϵ small due to the mass hierarchy between families, the second eigenvalue of L is given by the seesaw formula $m_\mu^0 \cong \epsilon^2 M_D$, while the second eigenvalue of D is given by $m_s^0 \cong (\epsilon/3)^2 M_D$. (2) The matrices D and U are still exactly proportional. This is a consequence of the fact that the generator $B-L$ does not distinguish up and down quarks. Therefore, the CKM angle V_{cb} still exactly vanishes. (3) Because D and U are exactly proportional, one still has the bad prediction $m_c^0/m_t^0 = m_s^0/m_b^0$.

It is clear that the breaking of $SO(10)$ due to the adjoint cannot cure all of these problems, since $B-L$ does not distinguish D from U. Thus the breaking of $SO(10)$ done by $\mathbf{16}_H + \overline{\mathbf{16}}_H$ must come into play. As we shall now show, a single simple operator exists, which involves one of these spinor Higgs and cures at one stroke all three of the problems we have identified.

The lowest dimension effective Yukawa operators that involve the spinor Higgs fields are quartic in spinors. Consider, therefore, operators of the form $\mathbf{16}_i \mathbf{16}_j \mathbf{16}'_H \mathbf{16}_H / M_G$. The $\mathbf{16}_H$ is the spinor Higgs field that breaks $SO(10)$ at M_G down to $SU(5)$. The $\mathbf{16}'_H$ is a spinor Higgs field that has a weak-scale VEV that breaks $SU(2)_L \times U(1)_Y$. In principle, these two spinors could be the same field. However, if they were, it would mean that they had to be contracted symmetrically by Bose statistics, which in turn would mean that $\mathbf{16}_i$ and $\mathbf{16}_j$ would also have to be contracted symmetrically. A careful examination shows that the resulting flavor-symmetric contributions to the mass matrices do not lead to realistic forms, though it is possible to achieve realistic mass matrices by adding yet another Yukawa operator, as in the interesting model of Babu, Pati, and Wilczek [16]. Therefore $\mathbf{16}'_H$ must be a distinct field. As will be seen later, introduc-

ing this $\mathbf{16}'_H$ involves no loss of economy, since it allows a very elegant explanation of the largeness of the ratio m_t/m_b without making $\tan \beta$ large.

There are still several operators of this type to be considered: the family indices can take the values $ij=33,22,23$, or 32, and there are three ways to contract the four spinors to make an $SO(10)$ singlet. Here again, one must examine the various cases to see which gives the most realistic mass matrices. As it turns out, there is one operator that is much superior to the others, in the sense that it much more cleanly and simply fits the data. It is of the form $[\mathbf{16_2 16}_H][\mathbf{16_3 16}'_H]$, where $[\ldots]$ means that the spinors inside are contracted into a $\mathbf{10}$. This can arise very simply by integrating out a $\mathbf{10}$ of fermions, as shown in Fig. 1(c).

Let us write the resulting mass operator in $SU(5)$ language. Denote by $\mathbf{p(q)}$ a $\mathbf{p}$ multiplet of $SU(5)$ that is contained in a $\mathbf{q}$ multiplet of $SO(10)$. The VEV of $\mathbf{16}_H$ lies, of course, in the $\mathbf{1(16)}$ direction, while the VEV of $\mathbf{16}'_H$ that breaks the weak interactions lies in the $\overline{\mathbf{5}}(\mathbf{16})$ direction. Therefore, the resulting mass term is of the form $[\overline{\mathbf{5}}(\mathbf{16_2})\mathbf{1}(\mathbf{16}_H)][\mathbf{10}(\mathbf{16_3})\overline{\mathbf{5}}(\mathbf{16}'_H)]$, which in $SU(5)$ terms gives effectively the operator $(\overline{\mathbf{5}}_2 \mathbf{10_3})\overline{\mathbf{5}}_H$. Note that this has the same form as the $SU(5)$ operator discussed in the last section, which gave the σ entries in Eq. (6). The result, then, of including this operator [2] is to make the mass matrices take the form

$$
U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon/3 \\ 0 & -\epsilon/3 & 1 \end{pmatrix} M_U,
$$

$$
D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma+\epsilon/3 \\ 0 & -\epsilon/3 & 1 \end{pmatrix} M_D, \tag{10}
$$

$$
N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\epsilon \\ 0 & \epsilon & 1 \end{pmatrix} M_U, \quad L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\epsilon \\ 0 & \sigma+\epsilon & 1 \end{pmatrix} M_D.
$$

The new term has given the entries we call σ. Note that these lopsided entries appear only in D and L. The reason is simply that the $\mathbf{16}'_H$ contains a $\overline{\mathbf{5}}$ of $SU(5)$ but no $\mathbf{5}$.

It is easy to see that the new term with $\sigma \gg \epsilon$ cures at once all three of the problems we identified with the forms given in Eq. (9): (1) Instead of $m_\mu^0 \cong \epsilon^2 M_D$ and $m_s^0 \cong (\epsilon/3)^2 M_D$, one has approximately that $m_\mu^0 \propto (\epsilon)(\sigma+\epsilon) \cong \epsilon\sigma$ and $m_s^0 \propto (\epsilon/3)(\sigma+\epsilon/3) \cong \epsilon\sigma/3$. More exact expressions will be given later. Thus the desired Georgi-Jarlskog factor of 1/3 is obtained, instead of 1/9. The σ entry has dominated over one of the factors of $\epsilon/3$ and thus prevented the factor of 1/3 from coming in squared.

(2) The σ entry comes into D but not U, and thus breaks the proportionality of the two matrices. As a result, V_{cb} no longer vanishes, but is given approximately by

$(\epsilon/3)[\sigma^2/(\sigma^2+1)]$. Note that this is of the same order in ϵ as $m_s/m_b \cong (\epsilon/3)[\sigma/(\sigma^2+1)]$, rather than $\sqrt{m_s/m_b}$ as is the case with Fritzsch forms, and accords much better with the actual experimental values.

(3) The fact that σ breaks the proportionality of U and D also means that the bad relation $m_c^0/m_t^0 = m_s^0/m_b^0$ is broken. Specifically, m_s^0/m_b^0 is of order ϵ, while m_c^0/m_t^0 is still of order ϵ^2 and therefore much smaller. This also accords well with the experimental numbers. In fact, as we shall see, if one uses V_{cb} and m_μ/m_τ to fix the parameters σ and ϵ, one finds that $m_c(1 \text{ GeV})$ is predicted to be in agreement with the experimentally determined value of 1.27 ± 0.1 GeV. It should also be noted that the prediction $m_b^0 = m_\tau^0$ is only very slightly affected by the addition of the σ term, both m_b^0 and m_τ^0 being given to leading order in ϵ by $\sqrt{\sigma^2+1} M_D$.

The economy of the above mass matrix forms is seen in the fact that five quantities (V_{cb}, m_μ/m_τ, m_s/m_b, m_c/m_t, and m_τ/m_b) are successfully fit with only the two parameters σ and ϵ. No other published form succeeds in accurately reproducing the masses and mixing of the the heavier two families with so few parameters. The predictions and fits will be discussed in detail in Sec. IV.

We see that the matrices in Eq. (10) were arrived at by a process of reasoning that had nothing to do with the question of neutrino mass but rather with an attempt to get realistic masses and mixings for the quarks and charged leptons using as simple a Higgs sector as possible in $SO(10)$. But what has emerged is a structure with precisely the three critical features identified in the last section as giving a simple explanation of the large mixing of ν_μ and ν_τ. In fact, a fit of V_{cb} and m_μ/m_τ gives $\sigma \cong 1.8$ and $\epsilon \cong 0.14$. Consequently, as can be seen directly from Eq. (10), the angle $\theta_{\mu\tau} = \sin^{-1} U_{\mu3} = \tan^{-1}\sigma - O(\epsilon) = 60° - O(8°)$. This is quite consistent with what is observed. We will look more carefully at these predictions later.

To summarize, with two parameters, ϵ and σ, four mass ratios and two mixing angles are satisfactorily accounted for, if we include $U_{\mu3}$. No greater economy could be hoped for in explaining the spectrum of the heavy two families. Moreover, as we shall see in the next section, the forms in Eq. (10) can be extended to include the first family with equal economy: the introduction of two new parameters (one of which is complex) will nicely account for seven quantities pertaining to the first family.

Before explaining how the model is extended to the first family we will expand on a couple of points made earlier. First, we said that the introduction of the $\mathbf{16}_H'$ allows a simple explanation [1,16] of why $m_t \gg m_b$ that does not require a $\tan\beta \gg 1$. The point is that the Higgs doublet of the minimal supersymmetric standard model (MSSM) that is often called H_U is purely contained in the $\mathbf{10}_H$ that couples to $\mathbf{16}_3\mathbf{16}_3$ and gives rise to the ''1'' entry in the mass matrices of Eq. (10). However, the Higgs doublet of the MSSM that is called H_D does not come purely from $\mathbf{10}_H$. Rather it is a mixture of doublets in $\mathbf{10}_H$ and $\mathbf{16}_H'$, since they both contain $\bar{\mathbf{5}}$'s of $SU(5)$. Thus we may write

$$H_U = H(\mathbf{10}_H),$$

$$H_D = \bar{H}(\mathbf{10}_H)\cos\gamma + \bar{H}(\mathbf{16}_H')\sin\gamma, \quad (11)$$

where γ is some mixing angle that depends on the parameters of the Higgs sector. Since the 33 elements of the mass matrices all arise purely from the coupling of the $\mathbf{10}_H$, the parameters we called M_U and M_D in Eq. (10) are given by

$$M_U = \lambda_{33}\langle H_U\rangle,$$

$$M_D = \lambda_{33}\langle H_D\rangle\cos\gamma, \quad (12)$$

leading to the ratio $M_U/M_D = \langle H_U\rangle/(\langle H_D\rangle\cos\gamma) = \tan\beta/\cos\gamma$, where $\tan\beta$ is defined to be the ratio of the Higgs VEV $\langle H_U\rangle$ giving mass to the top quark to the Higgs VEV $\langle H_D\rangle$ giving mass to the bottom quark. Hence

$$m_t^0/m_b^0 \cong (\sigma^2+1)^{-1/2}(\tan\beta/\cos\gamma). \quad (13)$$

The point is simply that the large ratio of the top to bottom masses could be the result of $\cos\gamma$ being small rather than $\tan\beta$ being large. In fact, since we do not know anything *a priori* about the angle γ, we cannot say whether $\tan\beta$ is large or small. It should be noted that if one assumes that H_D lies mostly in the $\mathbf{16}_H'$ (so that $\cos\gamma \ll 1$), it would explain why the parameter σ is large since it comes from a coupling to $\mathbf{16}_H'$, and also explain why $\tan\beta$ might be small.

A second point we wish to underline here has to do with the reasonableness of asymmetric mass matrices. In many models it is assumed that all the mass matrices are symmetric. However, this is not something that is called for by the group theory of grand unification. It is true that with the minimal Yukawa terms $SU(5)$ gives a symmetric U. But even with minimal Yukawa terms $SU(5)$ does not predict any symmetry of the D and L matrices. And in $SO(10)$, as we have seen, once one introduces the effects of $SO(10)$ breaking into the Yukawa sector, as one virtually must, one easily obtains effective Yukawa terms that are asymmetric. Figure 1(c) shows that very simple diagrams can give terms that are lopsided, in the sense that they contribute only above or below the diagonal. From the point of view of the fundamental grand unified theory, then, lopsided terms are as natural as symmetric ones. The preference for symmetric terms has been the result not of examining what kinds of terms are obtained in a simple way in unification, but rather from the desire to reduce the number of parameters at the level of mass matrices with the aim of making models which are highly predictive. However, putting oneself in the straightjacket of symmetric matrices makes it hard to get a good fit to all the quark and lepton masses and mixings. It turns out, as we have seen, and will see further below, that allowing asymmetric matrices makes possible a model which gives both a very good fit to the data and is actually much more predictive than most models which assume symmetric matrices.

IV. EXTENSION TO THE FIRST FAMILY

In arriving at the form of the mass matrices for the heavy two families we were limited in the choices that were possible by the assumption we made about the simplicity of the $SO(10)$-breaking sector. In extending to the first family we are not quite so limited. Nevertheless, the number of simple possibilities is not very large. There are several discrete choices: Should the contributions to the first row and column of the mass matrices be flavor symmetric like the 1's in Eq. (10), antisymmetric like the ϵ's, or lopsided like the σ's? Should they contribute to all the matrices equally like the 1's, to all the matrices but with non-trivial Clebsch factors like the ϵ's or only to D and L like the σ's? It is fairly easy to run through the various cases and see what kinds of relations among masses and mixings result. As it turns out, one of the simplest possibilities gives a remarkably good fit to the data. This uniquely simple choice [3] is the following:

$$
U = \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & \epsilon/3 \\ 0 & -\epsilon/3 & 1 \end{pmatrix} M_U,
$$

$$
D = \begin{pmatrix} \eta & \delta & \delta' \\ \delta & 0 & \sigma+\epsilon/3 \\ \delta' & -\epsilon/3 & 1 \end{pmatrix} M_D, \tag{14}
$$

$$
N = \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & -\epsilon \\ 0 & \epsilon & 1 \end{pmatrix} M_U, \quad L = \begin{pmatrix} \eta & \delta & \delta' \\ \delta & 0 & -\epsilon \\ \delta' & \sigma+\epsilon & 1 \end{pmatrix} M_D.
$$

We have already mentioned that fits give $\sigma \cong 1.8$ and $\epsilon \cong 0.14$. The new parameters δ and δ' both have magnitude of about 0.008. The parameter η is by far the smallest, being about 8×10^{-6}. The only role that η plays in the sector of quarks and charged leptons is in giving the up quark a mass, for it makes negligible contributions to the down quark and electron masses as determined from D and L, respectively. In Fig. 2(a) we have illustrated a higher-order diagram that can contribute to the parameter η. Since it is not excluded that the up quark is exactly massless, it is possible to set η to zero. In any event, one can see that $\eta \cong m_u^0/m_t^0$, which is by orders of magnitude smaller than any other interfamily ratio of masses in the standard model. It will, however, be of some significance for neutrino masses. If η vanishes, this model gives only the small-angle MSW solution to the solar neutrino problem. But even if η is as small as 8×10^{-6}, it allows either the small-angle MSW solution or bimaximal neutrino mixing to arise in a simple way.

Turning to the parameters δ and δ', we see that they appear symmetrically and only in D and L. Such terms are easily obtained in $SO(10)$ from simple diagrams such as that shown in Figs. 2(b) and 2(c). The effective operators arising from these diagrams are of the form $[16_1 16_j][16_H 16'_H]$ with $j=2,3$, where again the spinors in brackets are contracted symmetrically into a **10** of $SO(10)$ which is integrated out. Note, however, that the symmetric contributions δ and δ'
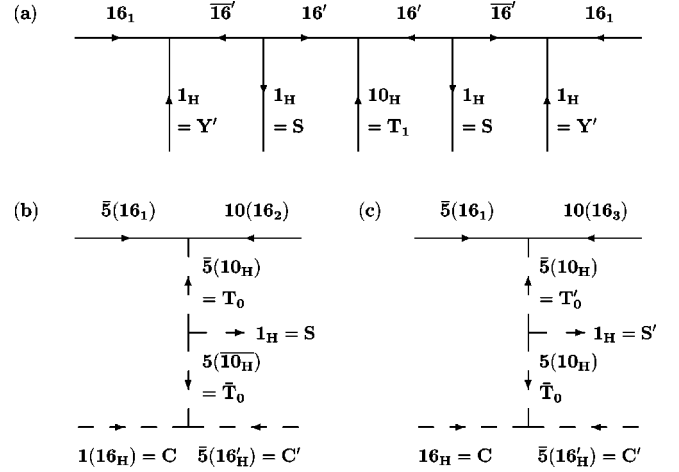

FIG. 2. Diagrams that generate the masses of the first family of quarks and leptons. See Eq. (14). (a) The 11 element called "η." (b) The 12 and 21 elements called "δ." Because the 10_H couples to the symmetric product of $16_1 16_2$, δ appears symmetrically in the mass matrices. (c) The 13 and 31 elements called "δ'," which also appear symmetrically.

from the two Higgs contraction $[1(16_H)\bar{5}(16'_H)]$ contributes only to D and L by virtue of their $SU(5)$ structure. Contrast these effective operators for $j=2,3$ with that occurring previously for the term σ arising from the diagram shown in Fig. 1(c).

The three new parameters we have introduced are, as we shall see, sufficient to account for everything about the first family. Before proceeding, however, we must be careful about complex phases. It is easy to show that if we allow all parameters of the model to be complex all but two phase angles can be rotated away from the mass matrices U, D, L and N, provided we now neglect the negligible η contributions to D and L. We will call these physical phases α and ϕ which appear as follows,

$$
U = \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & \epsilon/3 \\ 0 & -\epsilon/3 & 1 \end{pmatrix} M_U,
$$

$$
D = \begin{pmatrix} 0 & \delta & \delta' e^{i(\phi+\alpha)} \\ \delta & 0 & \sigma+\epsilon e^{i\alpha}/3 \\ \delta' e^{i\phi} & -\epsilon/3 & 1 \end{pmatrix} M_D,
$$

$$
N = \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & -\epsilon \\ 0 & \epsilon e^{i\alpha} & 1 \end{pmatrix} M_U, \tag{15}
$$

$$
L = \begin{pmatrix} 0 & \delta & \delta' e^{i\phi} \\ \delta & 0 & -\epsilon \\ \delta' e^{i(\phi+\alpha)} & \sigma+\epsilon e^{i\alpha} & 1 \end{pmatrix} M_D,
$$

where in these matrices and henceforth ϵ, σ, δ, δ' and η denote the magnitudes of these parameters and the phases are

written explicitly. The phase α only comes into the fits of masses at higher order in the small quantity $\epsilon/\sigma \cong 0.08$. Numerically, its effect is only a few percent; moreover, the fits (especially to m_c) prefer a value near zero. Therefore, we can ignore α and will do so from now on. That leaves only the phase ϕ. Its only significant effect, but a very important one, is to give the CP-violating phase angle δ_{CP}.

Instead of using the parameters δ, δ' and $e^{i\phi}$, it will be somewhat more convenient to use the parameters t_L, t_R, and $e^{i\theta}$, which are defined in terms of them as follows:

$$t_L e^{i\theta} \equiv \frac{\delta - \sigma \delta' e^{i\phi}}{\sigma \epsilon/3}, \tag{16}$$

and

$$t_R \equiv \frac{\delta \sqrt{\sigma^2+1}}{\sigma \epsilon/3}. \tag{17}$$

The significance of these parameters is that they are essentially the left-handed and right-handed Cabbibo angles. This can be seen by taking the forms for D and L given in Eq. (15) and diagonalizing the 2-3 block. When this is done the 1-2 blocks of these matrices take the form

$$D^{[12]} \propto \begin{pmatrix} 0 & t_R \\ t_L & 1 \end{pmatrix}, \quad L^{[12]} \propto \begin{pmatrix} 0 & t_L \\ t_R & 3 \end{pmatrix}. \tag{18}$$

In terms of the five dimensionless parameters ϵ, σ, t_L, t_R, and $e^{i\theta}$ with η set equal to zero, we now write down expressions for fourteen observable quantities: seven ratios of quark and lepton masses, three CKM angles and one phase, and three lepton mixing angles

$$m_b^0/m_\tau^0 \cong 1 - \frac{2}{3} \frac{\sigma}{\sigma^2+1} \epsilon, \tag{19}$$

$$m_c^0/m_t^0 \cong \frac{1}{9} \epsilon^2 \cdot \left[1 - \frac{2}{9} \epsilon^2 \right],$$

$$m_\mu^0/m_\tau^0 \cong \epsilon \frac{\sigma}{\sigma^2+1} \cdot \left[1 + \epsilon \frac{1-\sigma^2-\sigma\epsilon}{\sigma(\sigma^2+1)} + \frac{1}{18}(t_L^2+t_R^2) \right],$$

$$m_s^0/m_b^0 \cong \frac{1}{3} \epsilon \frac{\sigma}{\sigma^2+1} \cdot \left[1 + \frac{1}{3} \epsilon \frac{1-\sigma^2-\sigma\epsilon/3}{\sigma(\sigma^2+1)} + \frac{1}{2}(t_L^2+t_R^2) \right],$$

$$m_u^0/m_t^0 = 0,$$

$$m_e^0/m_\mu^0 \cong \frac{1}{9} t_L t_R \cdot \left[1 - \epsilon \frac{\sigma^2+2}{\sigma(\sigma^2+1)} \right.$$
$$\left. + \epsilon^2 \frac{\sigma^4+9\sigma^2/2+3}{\sigma^2(\sigma^2+1)^2} - \frac{1}{9}(t_L^2+t_R^2) \right],$$

$$m_d^0/m_s^0 \cong t_L t_R \cdot \left[1 - \frac{1}{3} \epsilon \frac{\sigma^2+2}{\sigma(\sigma^2+1)} \right.$$
$$\left. - (t_L^2+t_R^2) + (t_L^4+t_L^2 t_R^2+t_R^4) \right],$$

$$V_{cb} \cong \frac{1}{3} \epsilon \frac{\sigma^2}{\sigma^2+1} \cdot \left[1 + \frac{2}{3} \epsilon \frac{1}{\sigma(\sigma^2+1)} \right],$$

$$V_{us} \cong t_L \left[1 - \frac{1}{2} t_L^2 - t_R^2 + t_R^4 + \frac{5}{2} t_L^2 t_R^2 + \frac{3}{8} t_L^4 \right.$$
$$\left. - \frac{\epsilon}{3\sigma\sqrt{\sigma^2+1}} \frac{t_R}{t_L} e^{-i\theta} \right],$$

$$V_{ub} \cong \frac{1}{3} t_L \epsilon \frac{1}{\sigma^2+1} \left[\sqrt{\sigma^2+1} \frac{t_R}{t_L} e^{-i\theta} \left(1 - \frac{1}{3} \epsilon \frac{\sigma}{\sigma^2+1} \right) \right.$$
$$\left. - \left(1 - \frac{2}{3} \epsilon \frac{\sigma}{\sigma^2+1} \right) \right],$$

$$U_{\mu 3}^0 \equiv \sin \theta_{\mu\tau} \cong \frac{\sigma}{\sqrt{\sigma^2+1}} + O(\epsilon),$$

$$U_{e2}^0 \cong \cos \theta_{\mu\tau} \left(\frac{1}{3} t_R \right) \cdot \left[1 + \epsilon \left(\frac{\tan \theta_{\mu\tau}}{\sigma^2+1} - \frac{t_L}{t_R} e^{i\theta} \right. \right.$$
$$\left. \left. \times \frac{(1+\sigma \tan \theta_{\mu\tau})}{\sigma\sqrt{\sigma^2+1}} \right) - \frac{1}{18} t_R^2 - \frac{1}{9} t_L^2 \right],$$

$$U_{e3} \cong \tan \theta_{\mu\tau} U_{e2} \cdot \left[1 + \epsilon \frac{2}{\sin 2\theta_{\mu\tau}} \right.$$
$$\left. \times \left(\frac{t_L}{t_R} e^{i\theta} \frac{1}{\sqrt{\sigma^2+1}} - \frac{1}{\sigma^2+1} \right) \right].$$

These expansions have been carried to sufficiently high order in small quantities to be accurate to within 0.2% and are useful in doing the fits to the data. However, the leading terms in these expansions have much simpler forms and thus allow one to see more readily the relationships among various quantities in this model. We therefore write these simpler expressions for purposes of discussion:

$$m_b^0/m_\tau^0 \cong 1, \tag{20}$$

$$m_c^0/m_t^0 \cong \frac{1}{9} \epsilon^2,$$

$$m_\mu^0/m_\tau^0 \cong \epsilon \frac{\sigma}{\sigma^2+1},$$

$$m_s^0/m_b^0 \cong \frac{1}{3}\epsilon\frac{\sigma}{\sigma^2+1},$$

$$m_u^0/m_t^0 = 0,$$

$$m_e^0/m_\mu^0 \cong \frac{1}{9}t_L t_R,$$

$$m_d^0/m_s^0 \cong t_L t_R,$$

$$V_{cb}^0 \cong \frac{1}{3}\epsilon\frac{\sigma^2}{\sigma^2+1},$$

$$V_{us}^0 \cong t_L,$$

$$V_{ub}^0 \cong \frac{1}{3}t_L\epsilon\frac{1}{\sigma^2+1}\left(\sqrt{\sigma^2+1}\frac{t_R}{t_L}e^{-i\theta}-1\right),$$

$$U_{\mu3}^0 \equiv \sin\theta_{\mu\tau} \cong \frac{\sigma}{\sqrt{\sigma^2+1}}+O(\epsilon) \sim 0.7,$$

$$U_{e2}^0 \cong \cos\theta_{\mu\tau}\left(\frac{1}{3}t_R\right),$$

$$U_{e3} \cong \sin\theta_{\mu\tau}\left(\frac{1}{3}t_R\right).$$

It might at first seem surprising that without any information about the Majorana mass matrix M_R of the right-handed neutrinos we are able to write down predictions for the three neutrino mixing angles. However, if $\eta = 0$, as we are assuming at present, then the Dirac mass matrix of the neutrinos (N) has vanishing first row and column, and therefore, obviously, the same will be true of the mass matrix of the light neutrinos, which is given by the well-known ''see-saw'' formula $M_\nu = -N^T M_R^{-1} N$. This means that the two mixing elements of the electron neutrino, U_{e2} and U_{e3}, get no contribution from diagonalizing M_ν, but come entirely from diagonalizing L. Since L is a known matrix in our model, these two mixing elements are predicted. In the case of the mixing of the mu and tau neutrinos, $U_{\mu3}$ does receive a contribution from diagonalizing M_ν. However, as can be seen from the form of N this is an effect of $O(\epsilon)$. The contribution to $U_{\mu3}$ coming from diagonalizing L, on the other hand, is of order unity, since it arises from the large parameter σ. Thus $U_{\mu3}$ is predicted, although not precisely.

Since we have written fourteen quantities in terms of five parameters, there are altogether nine predictions of the model. Which quantities one takes as ''predicted'' depends on which quantities are used to determine the values of the parameters. We will use the lepton mass ratios and the angles V_{cb} and V_{us} for this purpose as they are the best measured. As one can see from the third and eighth of Eqs. (20), one can get the value σ from the ratio $3V_{cb}^0/(m_\mu^0/m_\tau^0)$. One finds (of course, taking the renormalization effects into account as was done in [2]) that numerically $\sigma \cong \sqrt{3}$. Substituting this

into the expression for m_μ^0/m_τ^0, one obtains that $\epsilon \cong 0.14$. This is the small parameter of the model that is responsible for the hierarchy between the second and third families, and is small enough that the expressions in Eqs. (20) are fairly accurate. One can use m_e/m_μ and V_{us} to determine t_L and t_R. A careful fit, described later, gives $t_L = 0.236$ and $t_R = 0.205$. That $t_L \cong t_R$ is easily understood from Eq. (18) and the well-known Weinberg-Wilczek-Zee-Fritzsch result [8] that the Cabbibo angle is well acounted for by symmetric mass matrices for the first two families; cf. Eq. (1). The near equality of t_L and t_R is also apparant from the seventh and ninth relations of Eqs. (20) and the fact that numerically $V_{us} \cong \sqrt{m_d/m_s}$. The phase factor $e^{i\theta}$ will be determined from the CP-violating phase δ_{CP}.

The nine predictions, then, are the following. To begin with, there are the three famous predictions, **(1)** $m_b^0/m_\tau^0 \cong 1$, **(2)** $m_s^0 \cong \frac{1}{3}m_\mu^0$, and **(3)** $m_d^0 \cong 3m_e^0$. The first is the ''good'' prediction of minimal $SU(5)$ unification, and the latter two are the Georgi-Jarlskog relations. These predictions are manifest from the first, third, fourth, sixth, and seventh of Eqs. (20). It is hardly surprising that the model gives these relations, since we were guided by them in constructing the model. The fourth prediction is **(4)** $m_u^0/m_t^0 = 0$. Even if the u quark is not exactly massless this relation is a very good approximation to reality. If one takes the favored value of $m_u \approx 4$ MeV, then, with reasonable assumptions about thresholds in doing the running up to the GUT scale, one obtains $\eta \approx m_u^0/m_t^0 \approx 8 \times 10^{-6}$. This is far smaller than any other interfamily ratio of masses. For instance, the comparable ratio for down quarks is $m_d^0/m_b^0 \cong 10^{-3}$, and for leptons is $m_e^0/m_\tau^0 \cong 3 \times 10^{-4}$. Like the previous three relations, $m_u \approx 0$ is a reflection of basic group-theoretical aspects of the model. It comes from the fact, explained above, that the δ and δ' entries only appear in D and L.

The remaining five predictions are not simple group-theoretical relations like the foregoing, but are non-trivial quantitative predictions. They are predictions for **(5)** m_c, **(6)** V_{ub}, **(7)** $U_{\mu3}$, **(8)** U_{e2}, and **(9)** U_{e3}.

The prediction for m_c is particularly interesting. We see immediately that, for reasons having to do with the group-theoretic structure of the model, the ratio m_c^0/m_t^0 is much less than the corresponding ratio m_s^0/m_b^0 for the down quarks because it is of higher order in the small parameter ϵ. This is a highly significant success, because the minimal Yukawa terms of $SO(10)$ notoriously give these ratios to be equal. Moreover, the success is not merely a qualitative one. When ϵ and σ are fit (using V_{cb} and m_μ/m_τ) and the renormalization effects are later taken into account, it is found that m_c comes out within about 5% of the experimentally preferred value, which is quite remarkable given the various experimental and theoretical uncertainties. This success is nontrivial, because the reasoning that led to the forms of the mass matrices did not depend upon the value of m_c, and hence it could have been expected that m_c would come out wrong by a large factor.

Another non-trivial quantitative hurdle for the model is the prediction for V_{ub}. The eighth, ninth, and tenth relations of Eqs. (20) give $V_{ub}^0 \cong V_{us}^0 V_{cb}^0 (1/\sigma^2)[\sqrt{\sigma^2+1}(t_R/t_L)e^{-i\theta}$
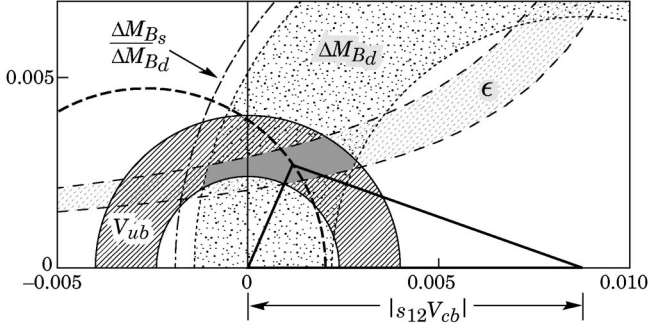
FIG. 3. The unitarity triangle for $V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$ is displayed along with the experimental constraints on $V_{ud}V_{ub}^*$, which is the upper vertex in the triangle. The constraints following from $|V_{ub}|$, B-mixing and ϵ extractions from experimental data are shown in the lightly shaded regions. The experimentally allowed region is indicated by the darkly shaded overlap. The model predicts that $V_{ud}V_{ub}^*$ will lie on the dashed circle; cf. Eq. (21). The particular point on this circle used to draw the triangle shown is obtained from the numbers given in Sec. VI; cf. Eq. (62).

-1]. If we use the facts that $\sigma \simeq \sqrt{3}$ and $t_L \simeq t_R$, this gives $V_{ub} \simeq V_{us}V_{cb}(\frac{2}{3}e^{-i\theta} - \frac{1}{3})$. A careful fit gives

$$V_{ub} = V_{us}V_{cb}(0.558 e^{-i\theta} - 0.315). \qquad (21)$$

In other words the model predicts that V_{ub} should lie on a certain circle in the complex plane. As can be seen from Fig. 3, the circle for $V_{ud}V_{ub}^*$ slices neatly through the middle of the presently allowed region. Again, this is a very significant success, since the reasoning that led to the forms in Eq. (15) was not based on the value of V_{ub}.

The prediction for the mixing of ν_μ and ν_τ has already been discussed. It is one of the key successes of this model that this mixing turns out to be nearly maximal. The fact that $\sigma \simeq \sqrt{3}$ tells us that the first term in the expression for $U_{\mu3}^0$ in Eq. (20) corresponds to an angle near $\pi/3$. As we shall see in the next section, the $O(\epsilon)$ corrections easily bring this down close to the maximal mixing value of $\pi/4$.

The prediction of this model for the mixing of ν_e and ν_μ with $\eta = 0$ is quite interesting. From the sixth relation of Eqs. (20) and the fact that $t_L \simeq t_R$, one sees that $\frac{1}{3}t_R \simeq \sqrt{m_e/m_\mu}$. Thus the model predicts that $U_{e2} \cong \cos\theta_{\mu\tau}\sqrt{m_e/m_\mu}$. The factor of $\cos\theta_{\mu\tau}$ is crucial [17] since without it one would have $\sin^2 2\theta_{\text{solar}} = 4|U_{e1}|^2|U_{e2}|^2 \approx 4(m_e/m_\mu) \cong 2 \times 10^{-2}$, which is about twice the value needed for the small-angle MSW solution to the solar neutrino problem. Since atmospheric neutrino data tells us that $\cos\theta_{\mu\tau} \simeq 1/\sqrt{2}$, the model gives just the correct value for the small-angle MSW solution.

In the future both V_{ub} and $\sin^2 2\theta_{\text{solar}}$ will be known better and will provide a sharp test of the model. The theoretical uncertainties in the predictions for V_{ub} and U_{e2} are estimated to be only a few percent.

In discussing the $\nu_e - \nu_\mu$ mixing above, we have assumed that $\eta = 0$. If η does not vanish, but is around 8×10^{-6}, corresponding to $m_u \approx 4.5$ MeV, then it turns out that both

the small-angle MSW solution we just discussed and bimaximal mixing are possible. This will be discussed in detail in the next section.

Finally, there is the prediction of $\nu_e - \nu_\tau$ mixing. One sees from Eq. (20) that there is a prediction that $U_{e3} \cong \tan\theta_{\mu\tau}U_{e2} \cong 0.05$. It is interesting that even for the bimaximal mixing case that will be discussed in the next section, the numerical value of U_{e3} is virtually unaffected. Thus this prediction is a "robust" one of this model.

V. NEUTRINO MIXING

A. Mixing of $\nu_\mu - \nu_\tau$

In analyzing the predictions of this model for $\nu_\mu - \nu_\tau$ mixing, we may make the approximation that $\eta = 0$. This means that N has vanishing first row and column. Therefore, in computing $M_\nu = -N^T M_R^{-1} N$ the first row and column of M_R^{-1} are irrelevant. Thus we may write M_R^{-1} as

$$M_R^{-1} = \begin{pmatrix} - & - & - \\ - & X & Y \\ - & Y & Z \end{pmatrix}, \qquad (22)$$

where X, Y, and Z are in general complex. There are consequently five real parameters (the over all phase does not matter) that come into the masses and mixing of ν_μ and ν_τ from M_R. As observed earlier, this does not prevent us from making a qualitative prediction for the mixing parameter $U_{\mu3}$, since the contribution to it from diagonalizing the mass matrix M_ν is only of order ϵ and $U_{\mu3}$ comes predominantly from diagonalizing the known matrix L. However, in order to see if more precise predictions can be obtained, we shall look at two simple special cases:

$$(I) \quad M_R = \begin{pmatrix} - & 0 & 0 \\ 0 & 0 & Be^{i\beta}\epsilon \\ 0 & Be^{i\beta}\epsilon & 1 \end{pmatrix} \Lambda_R e^{i\gamma}, \qquad (23)$$

and

$$(II) \quad M_R = \begin{pmatrix} - & 0 & 0 \\ 0 & Be^{i\beta}\epsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Lambda_R e^{i\gamma}. \qquad (24)$$

In these cases only three parameters in M_R, namely Λ_R, B and $e^{i\beta}$, contribute to the neutrino observables of the second and third families, since ϵ has appeared previously and is used as a natural scaling parameter. In the first case,

$$M_\nu^I = - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & \epsilon & 2 + B^{-1}e^{-i\beta} \end{pmatrix} \frac{M_U^2}{B\Lambda_R} e^{-i(\beta+\gamma)}. \quad (25)$$

The neutrino mixing matrix U, now known as the Maki-Nakagawa-Sakata (MNS) mixing matrix [18], is given by

$U = U_L^\dagger U_\nu$, where U_L is the unitary matrix that diagonalizes $L^\dagger L$, and U_ν is the unitary matrix that diagonalizes $M_\nu^\dagger M_\nu$. For case I, U_ν is given by

$$U_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23}^{\nu*} & \sin\theta_{23}^\nu \\ 0 & -\sin\theta_{23}^{\nu*} & \cos\theta_{23}^\nu \end{pmatrix}, \quad (26)$$

where $\tan 2\theta_{23}^\nu = 2\epsilon/K$, and $K \equiv 2 + B^{-1}e^{i\beta}$. The ratio of eigenvalues of M_ν gives $m_{\nu_2}/m_{\nu_3} \cong (\epsilon^2/|K|^2)(1 - \epsilon^2/|K|^2 + \cdots)$. One can choose $\cos\theta_{23}^\nu$ to be real, and one can write

$$\sin\theta_{23}^\nu \cong \sqrt{m_{\nu_2}/m_{\nu_3}} \, e^{-i\xi} \left[1 + \left(\frac{1}{2} - e^{-2i\xi} \right) \frac{m_{\nu_2}}{m_{\nu_3}} \right], \quad (27)$$

where $e^{i\xi}$ is the phase of K. One readily sees from the form of the charged-lepton mass matrix L in Eq. (15) that $\sin\theta_{23}^L \equiv (U_L)_{23}$ is given by $\tan 2\theta_{23}^L = -2(\sigma+\epsilon)/(\sigma^2 - 1 + 2\sigma\epsilon) = -2\sigma/(\sigma^2 - 1) + O(\epsilon)$. Since $\sigma \simeq \sqrt{3}$ it is evident that $\theta_{23}^L \simeq 60°$. Using the best fit values of σ and ϵ one finds, more precisely, that $\theta_{23}^L \cong 63°$.

Altogether, then, the mixing parameter of ν_μ and ν_τ is given by

$$\begin{aligned} U_{\mu 3} &\equiv \sin\theta_{\mu\tau} \\ &= -\sin\theta_{23}^L \cos\theta_{23}^\nu + \cos\theta_{23}^L \sin\theta_{23}^\nu \\ &\cong -0.898(1 - m_{\nu_2}/m_{\nu_3}) + 0.441\sqrt{m_{\nu_2}/m_{\nu_3}} \, e^{-i\xi}. \end{aligned} \quad (28)$$

If neutrino masses are hierarchical, and atmospheric neutrino oscillations are $\nu_\mu - \nu_\tau$ oscillations, then $m_{\nu_3} \simeq 0.06$ eV. And if one further assumes the small-angle MSW solution to the solar neutrino problem, then $m_{\nu_2} \simeq 0.003$ eV. Thus, $m_{\nu_2}/m_{\nu_3} \simeq 0.05$, within a factor of two or so. Taking it to have the value 0.05, and the phase ξ to vanish, Eq. (28) gives $U_{\mu 3} \cong -0.756$, and $\sin^2 2\theta_{\mu\tau} \cong 0.984$. With the same value of the neutrino mass ratio and ξ taken to be $\pi/4$, $\sin^2 2\theta_{\mu\tau} \cong 0.943$. We see that there is excellent agreement with the experimental limits from SuperKamiokande if the complex phase is not too large. But if $\xi = \pi/2$, with the same mass ratio, $\sin^2 2\theta_{\mu\tau} \cong 0.77$.

The value of $m_{\nu_2}/m_{\nu_3} = 0.05$ corresponds to $|K| = 0.63$. Since $B^{-1}e^{i\beta} = |K|e^{i\xi} - 2$, for $\xi = 0$ this gives $B = 0.73$, or $B\epsilon = 0.1$. In other words, no very great hierarchy is required in M_R.

Turning now to case II, we have that

$$M_\nu^{II} = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon^2 & \epsilon \\ 0 & \epsilon & 1 + B^{-1}e^{-i\beta} \end{pmatrix} \frac{M_U^2}{\Lambda_R} e^{-i\gamma}. \quad (29)$$

Consequently, for this case

$$\tan 2\theta_{23}^\nu \cong 2\epsilon/K', \quad (30)$$

where $K' \equiv 1 + B^{-1}e^{i\beta}$. The ratio of eigenvalues of M_ν gives $m_{\nu_2}/m_{\nu_3} \cong \epsilon^2 \sqrt{1 - |K'-1|^2}/|K'|^2$. If we take, $m_{\nu_2}/m_{\nu_3} = 0.05$, as before, and assume that K' is real, we have that $K' \cong 0.6$. This gives $\theta_{23}^\nu \cong 12.3°$, $\theta_{\mu\tau} \cong 50.5°$, and $\sin^2 2\theta_{\mu\tau} \cong 0.96$. Again, there is good agreement with the SuperKamiokande results. Moreover, since this value of K' corresponds to $B = -2.5$, or $B\epsilon^2 \cong -0.05$, we see that in this case also no great hierarchy is needed in M_R.

The foregoing discussion is all based on the assumption that the mixing with the first family is small, so that one has the small-angle MSW solution to the solar neutrino problem. This will certainly be the case if $\eta = 0$. As we will now see, if instead $\eta \cong 8 \times 10^{-6}$, as needed to have $m_u \cong 4.5$ MeV, either the small mixing of ν_e that we have been considering or large mixing of ν_e is possible, depending on the form of M_R.

B. Mixing of the first family

In the previous discussion, we set $\eta = 0$ in which case, no matter what the form of M_R, the matrix $M_\nu = -N^T M_R^{-1} N$ has vanishing first row and column, and the matrix U_ν that diagonalizes $M_\nu^\dagger M_\nu$ has the form of Eq. (26). It is easy to show that the matrix U_L which diagonalizes $L^\dagger L$ has the form

$$U_L \cong \begin{pmatrix} \cos\theta_{12}^L & -\sin\theta_{12}^L & 0 \\ \sin\theta_{12}^L & \cos\theta_{12}^L & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23}^L & -\sin\theta_{23}^L \\ 0 & \sin\theta_{23}^L & \cos\theta_{23}^L \end{pmatrix}, \quad (31)$$

where $\sin\theta_{12}^L \cong \frac{1}{3}t_R$, t_R is defined in Eq. (17), and θ_{23}^L is given after Eq. (27). Putting these together, one has that the total mixing matrix of the neutrinos, $U = U_L^\dagger U_\nu$, is

$$U = \begin{pmatrix} \cos\theta_{12}^L & -\sin\theta_{12}^L \cos\theta_{\mu\tau} & -\sin\theta_{12}^L \sin\theta_{\mu\tau} \\ \sin\theta_{12}^L & \cos\theta_{12}^L \cos\theta_{\mu\tau} & \cos\theta_{12}^L \sin\theta_{\mu\tau} \\ 0 & -\sin\theta_{\mu\tau} & \cos\theta_{\mu\tau} \end{pmatrix}, \quad (32)$$

where $\theta_{\mu\tau} = \theta_{23}^L - \theta_{23}^\nu$. This yields the results, already given in Eq. (20), for U_{e2} and U_{e3}.

Now we will consider what happens under what is presumably the more realistic assumption that $\eta \cong 8 \times 10^{-6}$.

With $\eta \neq 0$, there are two basic possibilities to consider. One possibility is that M_R has a form in which its 12, 21, 13, and 31 elements all vanish or are negligibly small. If such is the case, then the previous analysis applies, and the mixing of ν_e is due entirely to the matrix L. The only effect of the parameter η in the lepton sector is then to give ν_1 a mass of about 4×10^{-7} eV. The second possibility is that M_R does have significant 12, 21 and/or 13, 31 elements. If this is the case then a strikingly different situation can arise [4], namely "bimaximal" mixing [19], [20].

We will first illustrate what happens with a simple example. Consider the following form for M_R:

$$M_R = \begin{pmatrix} 0 & A\epsilon^3 & 0 \\ A\epsilon^3 & B\epsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Lambda_R. \tag{33}$$

We normalize A and B by powers of ϵ simply for later convenience. The mass matrix of light neutrinos resulting from this form is

$$M_\nu = -N^T M_R^{-1} N = \begin{pmatrix} \dfrac{\eta^2}{\epsilon^4} \dfrac{B}{A^2} & 0 & -\dfrac{\eta}{\epsilon^2} \dfrac{1}{A} \\ 0 & \epsilon^2 & \epsilon \\ -\dfrac{\eta}{\epsilon^2} \dfrac{1}{A} & \epsilon & 1 \end{pmatrix} \dfrac{M_U^2}{\Lambda_R}. \tag{34}$$

One sees that the 2-3 block has vanishing determinant, so that a rotation in the 2-3 plane by an angle $\theta_{23}^\nu \cong \epsilon$ brings M_ν to the form

$$M'_\nu \cong \begin{pmatrix} \dfrac{\eta^2}{\epsilon^4} \dfrac{B}{A^2} & \dfrac{\eta}{\epsilon} \dfrac{1}{A} & -\dfrac{\eta}{\epsilon^2} \dfrac{1}{A} \\ \dfrac{\eta}{\epsilon} \dfrac{1}{A} & 0 & 0 \\ -\dfrac{\eta}{\epsilon^2} \dfrac{1}{A} & 0 & 1 \end{pmatrix} \dfrac{M_U^2}{\Lambda_R}. \tag{35}$$

This can be put in a more transparent form by a rotation in the 1-3 plane by an angle $\theta_{13}^\nu \cong -\eta/(\epsilon^2 A)$. This angle is less than or of order 10^{-4} and thus negligible, practically speaking, so

$$M''_\nu \cong -\begin{pmatrix} \dfrac{\eta^2}{\epsilon^4} \dfrac{(B-1)}{A^2} & \dfrac{\eta}{\epsilon} \dfrac{1}{A} & 0 \\ \dfrac{\eta}{\epsilon} \dfrac{1}{A} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dfrac{M_U^2}{\Lambda_R}. \tag{36}$$

It is clear that the 11 element, being higher order in η, is likely to be much smaller than the 12 and 21 elements. The condition for this to be the case is that $A/(B-1) > \eta/\epsilon^3 \cong 2\times 10^{-3}$. If this very weak condition is satisfied, then the form of the matrix manifestly corresponds to the situation in which the ν_e and ν_μ together form a pseudo-Dirac pair. That in turn would mean that the mixing of these two neutrinos is very close to maximal.

One sees from Eq. (36) that $m_{\nu_3} = M_U^2/\Lambda_R$, and that the splitting between m_{ν_1} and m_{ν_2} is given by $\Delta m_{21}^2 \cong 2(\eta^3/\epsilon^5)[(B-1)/A^3](M_U^2/\Lambda_R)^2$. For the vacuum solution to the solar neutrino problem, one has $\Delta m_{21}^2 \approx 10^{-10}$ eV^2, so that $\Delta m_{21}^2/m_{\nu_3}^2 \approx 3\times 10^{-8} \cong 2(\eta^3/\epsilon^5)(B-1)/A^3$. This gives $A(B-1)^{1/3} \approx 0.06$. Thus no great hierarchy is required in M_R to get the vacuum oscillation solution. The

reason for this is that in this scheme the smallness of Δm_{21}^2 is due to the extreme smallness of the parameter η, which is equal to the ratio m_u/m_t.

It is easy to see from what has already been said that the matrix U_ν needed to diagonalize $M_\nu^\dagger M_\nu$ is of the form

$$U_\nu \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23}^\nu & \sin\theta_{23}^\nu \\ 0 & -\sin\theta_{23}^\nu & \cos\theta_{23}^\nu \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{37}$$

where we have neglected the tiny rotation θ_{13}^ν. The matrix U_L is already given in Eq. (31), so that the full neutrino mixing matrix can be written

$$U \cong U_{SMA} \cdot \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{38}$$

where U_{SMA} is given in Eq. (32), and is just the form that results in the small mixing angle (SMA) MSW case of this model. In other words, the net result of the large mxing of the first family produced by the A entries in Eq. (34) is simply to multiply the (SMA) MSW form of U on the right by a rotation of $\pi/4$ in the 1-2 plane. Consequently, the predictions for $U_{\mu3}$ and U_{e3} are essentially unaffected. However, U_{e2} becomes $1/\sqrt{2}$ instead of the value given in Eq. (20).

The interesting lesson is that ''bimaximal mixing'' is easy to achieve if the large mixing of ν_μ and ν_τ comes from the charged lepton sector, i.e., from diagonalizing L, while the large mixing of ν_e comes from diagonalizing M_ν.

The simple form given in Eq. (33) gives $\theta_{23}^\nu \cong \epsilon \cong 8°$, and thus $\theta_{\mu\tau} = \theta_{23}^L - \theta_{23}^\nu \cong 55°$, corresponding to $\sin^2 2\theta_{\mu\tau} = 0.88$. Somewhat larger values of $\sin^2 2\theta_{\mu\tau}$ can arise from a more general form M_R. Consider, for example,

$$M_R = \begin{pmatrix} 0 & A\epsilon^3 & C\epsilon^2 \\ A\epsilon^3 & B\epsilon^2 & 0 \\ C\epsilon^2 & 0 & 1 \end{pmatrix} \Lambda_R. \tag{39}$$

Then

$$M_\nu = -\begin{pmatrix} \dfrac{\eta^2}{\epsilon^4} B & \dfrac{\eta}{\epsilon} BC & \dfrac{\eta}{\epsilon}(BC-A) \\ \dfrac{\eta}{\epsilon} BC & \epsilon^2 A^2 & \epsilon A(A+C) \\ \dfrac{\eta}{\epsilon}(BC-A) & \epsilon A(A+C) & (A+C)^2 \end{pmatrix}$$

$$\times (A^2 + BC^2)^{-1} \dfrac{M_U^2}{\Lambda_R}. \tag{40}$$

Here, as in Eq. (34), the 2-3 block has vanishing determinant. The crucial difference is that the diagonalization of this matrix involves a rotation in the 2-3 plane by an angle θ_{23}^ν

TABLE I. Higgs superfields in the proposed model.

Higgs Fields Needed to Solve the 2-3 Problem:	
$\mathbf{45}_{B-L}$:	$A(0)^{+-}$
$\mathbf{16}$:	$C(\frac{3}{2})^{-+}$, $C'(\frac{3}{2}-p)^{++}$
$\overline{\mathbf{16}}$:	$\bar{C}(-\frac{3}{2})^{++}$, $\bar{C}'(-\frac{3}{2}-p)^{-+}$
$\mathbf{10}$:	$T_1(1)^{++}$, $T_2(-1)^{+-}$
$\mathbf{1}$:	$X(0)^{++}$, $P(p)^{+-}$, $Z_1(p)^{++}$, $Z_2(p)^{++}$
Additional Higgs Fields for the Mass Matrices:	
$\mathbf{10}$:	$T_0(1+p)^{+-}$, $T'_o(1+2p)^{+-}$,
	$\bar{T}_o(-3+p)^{-+}$, $\bar{T}'_o(-1-3p)^{-+}$
$\mathbf{1}$:	$Y(2)^{-+}$, $Y'(2)^{++}$, $S(2-2p)^{--}$, $S'(2-3p)^{--}$,
	$V_M(4+2p)^{++}$

$\cong \epsilon[A/(A+C)]$. With $C=-\frac{1}{2}A$, for instance, $\theta_{\mu\tau}$ comes out very close to $45°$. Otherwise, this case is quite similar to that of Eq. (33).

VI. DETAILS OF A SPECIFIC MODEL

In the previous sections we have presented the construction of our $SO(10)$ minimal Higgs model in the framework of effective $SO(10)$ and $SU(5)$ operators. We now show that one can construct a technically-natural realization of this scheme by introducing sets of Higgs superfields and matter superfields with a well-defined family symmetry. We first address the Higgs sector.

A. Higgs sector with $U(1)\times Z_2\times Z_2$ family symmetry

The doublet-triplet splitting problem in $SU(5)$, and therefore $SO(10)$, arises because the colored Higgs fields in the $\mathbf{5}-\bar{\mathbf{5}}$ pairs of each $\mathbf{10}_H$ must be made superheavy at the GUT scale, while just one pair of Higgs doublets should remain massless there and be free to develop VEV's at the electroweak scale. This problem has been addressed and solved in [5] by the introduction of just one $\mathbf{45}$ adjoint Higgs field with its VEV pointing in the $B-L$ direction, together with two pairs of $\mathbf{16}+\overline{\mathbf{16}}$ spinor Higgs fields, two $\mathbf{10}$ Higgs fields in the vector representation plus several Higgs singlets. We shall briefly summarize the solution, but first we note that it is necessary to introduce several more Higgs fields in the vector and singlet representations in order to generate the Yukawa structure for the fermion mass matrices presented earlier.

The authors of [5] found that the Higgs superpotential required to solve the doublet-triplet splitting problem could be neatly obtained from their list of Higgs fields by introducing a global family symmetry group of the type $U(1)\times Z_2\times Z_2$, which can arise in a natural fashion from string theory. With this in mind, we now list in Table I all the Higgs fields to be considered together with their family charge assignments. As noted in the table, in order to complete the construction of the Dirac mass matrices, four more vector Higgs fields and four additional Higgs singlets are needed, while one Higgs singlet is introduced to generate the right-handed Majorana neutrino mass matrix.

TABLE II. Matter superfields in the proposed model.

$\mathbf{16}_1(-\frac{1}{2}-2p)^{+-}$	$\mathbf{16}_2(-\frac{1}{2}+p)^{++}$	$\mathbf{16}_3(-\frac{1}{2})^{++}$
$\mathbf{16}(-\frac{1}{2}-p)^{-+}$	$\mathbf{16}'(-\frac{1}{2})^{-+}$	
$\overline{\mathbf{16}}(\frac{1}{2})^{+-}$	$\overline{\mathbf{16}}'(-\frac{3}{2}+2p)^{+-}$	
$\mathbf{10}_1(-1-p)^{-+}$	$\mathbf{10}_2(-1+p)^{++}$	
$\mathbf{1}_1(2+2p)^{+-}$	$\mathbf{1}_2(2-p)^{++}$	$\mathbf{1}_3(2)^{++}$
$\mathbf{1}_1^c(-2-2p)^{+-}$	$\mathbf{1}_2^c(-2)^{+-}$	$\mathbf{1}_3^c(-2-p)^{++}$

It is then possible to write down explicitly the full Higgs superpotential from the Higgs $SO(10)$ and family assignments, where we have written it as the sum of five terms:

$$W_{\mathrm{Higgs}}=W_A+W_{CA}+W_{2/3}+W_{H_D}+W_R$$

$$W_A=\mathrm{tr}\,A^4/M+M_A\mathrm{tr}\,A^2$$

$$W_{CA}=X(\bar{C}C)^2/M_C^2+F(X)+\bar{C}'(PA/M_1+Z_1)C$$
$$+\bar{C}(PA/M_2+Z_2)C' \qquad (41)$$

$$W_{2/3}=T_1AT_2+Y'T_2^2$$

$$W_{H_D}=T_1\bar{C}\bar{C}Y'/M+\bar{T}_0CC'+\bar{T}_0(T_0S+T'_0S')$$

$$W_R=\bar{T}_0\bar{T}'_0V_M .$$

The Higgs singlets are all assumed to develop VEV's at the GUT scale. We can then determine the fate of the other Higgs fields from the F-flat and D-flat conditions. In particular, W_A fixes $\langle A\rangle$ through the $F_A=0$ condition where one solution is $\langle A\rangle\propto B-L$, the Dimopoulos-Wilczek solution [12]. W_{CA} gives a GUT-scale VEV to $\bar{C}$ and C by the $F_X=0$ condition and also couples the adjoint A to the spinors C, $\bar{C}$, C' and $\bar{C}'$ without destabilizing the Dimopoulos-Wilczek solution or giving Goldstone modes, as shown in [5]. $W_{2/3}$ gives the doublet-triplet splitting by the Dimopoulos-Wilczek mechanism [12,5]. W_{H_D} mixes the $(1,2,-1/2)$ doublet in T_1 with those in C' (by $F_{\bar{C}}=0$), and in T_0 and T'_0 (by $F_{\bar{T}_0}=0$).

B. Yukawa sector

We now turn to the Yukawa sector and specify the matter fields and their $U(1)\times Z_2\times Z_2$ charge assignments which will complete the realization of the specific model in question. For this purpose, we require three spinor fields $\mathbf{16}_i$, one for each light family, two vector-like pairs of $\mathbf{16}-\overline{\mathbf{16}}$ spinors which can get superheavy, a pair of superheavy $\mathbf{10}$ fields in the vector representation, and three pairs of superheavy $\mathbf{1}-\mathbf{1}^c$ fermion singlets. The complete listing is given in Table II.

In terms of these fermion fields and the Higgs fields previously introduced, one can then spell out all the terms in the Yukawa superpotential which follow from their $SO(10)$ and $U(1)\times Z_2\times Z_2$ assignments:

$$
\begin{aligned}
W_{Yukawa} = {}& \mathbf{16}_3 \cdot \mathbf{16}_3 \cdot T_1 + \mathbf{16}_2 \cdot \mathbf{16} \cdot T_1 + \mathbf{16}' \cdot \mathbf{16}' \cdot T_1 \\
& + \mathbf{16}_3 \cdot \mathbf{16}_1 \cdot T_0' + \mathbf{16}_2 \cdot \mathbf{16}_1 \cdot T_0 + \mathbf{16}_3 \cdot \overline{\mathbf{16}} \cdot A \\
& + \mathbf{16}_1 \cdot \overline{\mathbf{16}}' \cdot Y' + \mathbf{16} \cdot \overline{\mathbf{16}} \cdot P + \mathbf{16}' \cdot \overline{\mathbf{16}}' \cdot S \\
& + \mathbf{16}_3 \cdot \mathbf{10}_2 \cdot C' + \mathbf{16}_2 \cdot \mathbf{10}_1 \cdot C + \mathbf{10}_1 \cdot \mathbf{10}_2 \cdot Y \\
& + \mathbf{16}_3 \cdot \mathbf{1}_3 \cdot \bar{C} + \mathbf{16}_2 \cdot \mathbf{1}_2 \cdot \bar{C} + \mathbf{16}_1 \cdot \mathbf{1}_1 \cdot \bar{C} + \mathbf{1}_3 \cdot \mathbf{1}_3^c \cdot Z \\
& + \mathbf{1}_2 \cdot \mathbf{1}_2^c \cdot P + \mathbf{1}_1 \cdot \mathbf{1}_1^c \cdot X + \mathbf{1}_3^c \cdot \mathbf{1}_3^c \cdot V_M + \mathbf{1}_1^c \cdot \mathbf{1}_2^c \cdot V_M ,
\end{aligned}
\tag{42}
$$

where the coupling parameters have been suppressed. To obtain the GUT scale structure for the fermion mass matrix elements, all but the three chiral spinor fields in the first line of Table II will be integrated out to yield Froggatt-Nielsen diagrams [21] of the type pictured earlier. Note that the right-handed Majorana matrix elements will all be generated through the Majorana couplings of the V_M Higgs field with conjugate singlet fermions given in the last two terms of Eq. (42).

An important question arises concerning proton instability due to R-parity violating terms in the low-energy effective theory. In the minimal $SO(10)$ model, ''matter'' fields (i.e. quark and lepton fields) are contained in spinorial representations of $SO(10)$, while Higgs fields are contained in tensorial representations. Consequently, there is an automatic Z_2 matter parity [contained in the Z_4 center of $SO(10)$] under which matter superfields are odd and Higgs superfields are even. In our model, there are some matter fields in tensorial representations and some Higgs fields in spinorial representations. Hence there is a danger that some R-parity violating terms that also violate B and L may exist in the low-energy theory. In fact, a careful study shows that the family symmetry specified above does allow some terms of this kind, the most dangerous of which comes from the higher-dimension operator $\mathbf{16}_2 \mathbf{16} \mathbf{10}_2 Y / M_{Pl}$, for example. Although suppressed by $1/M_{Pl}$ this term would lead to excessive proton decay. Therefore, in addition to family symmetry, one must impose by hand a Z_2 matter parity under which all the matter superfields are odd and all the Higgs superfields are even. This in no way complicates the model, and makes precise the distinction we have been drawing between matter and Higgs fields. With such a matter parity, it can be shown that proton stability is not a problem. The leading contributions to proton decay are then the usual dimension 5 operators that arise from colored-Higgsino exchange and that are generically present in $SO(10)$ models.

In order to present a clearer description of how the GUT scale mass matrices are determined from the Yukawa and Higgs superpotentials, we shall illustrate the procedure for the up quark mass matrix U. The three massless color-triplet quark states each with charge 2/3 are obtained as linear combinations of all such color and charge states within the fermion supermultiplets given in Eq. (41). In particular, the basis for the left-handed states u_L and left-handed conjugate states u_L^c can be ordered as follows:

$$
\mathcal{B}_{u_L} = \left\{ \left| 3,2,\frac{1}{6} \right\rangle_{10(16_1)} , \ \left| 3,2,\frac{1}{6} \right\rangle_{10(16_2)} , \ \left| 3,2,\frac{1}{6} \right\rangle_{10(16_3)} , \ \left| 3,2,\frac{1}{6} \right\rangle_{10(16)} , \ \left| 3,2,\frac{1}{6} \right\rangle_{10(16')} , \ \left| 3,1,\frac{2}{3} \right\rangle_{\overline{10}(\overline{16})} , \ \left| 3,1,\frac{2}{3} \right\rangle_{\overline{10}(\overline{16}')} \right\},
\tag{43}
$$

$$
\mathcal{B}_{u_L^c} = \left\{ \left| \bar{3},1,-\frac{2}{3} \right\rangle_{10(16_1)} , \ \left| \bar{3},1,-\frac{2}{3} \right\rangle_{10(16_2)} , \ \left| \bar{3},1,-\frac{2}{3} \right\rangle_{10(16_3)} , \ \left| \bar{3},1,-\frac{2}{3} \right\rangle_{10(16)} \left| \bar{3},1,-\frac{2}{3} \right\rangle_{10(16')} , \right.
$$

$$
\left. \left| \bar{3},2,-\frac{1}{6} \right\rangle_{\overline{10}(\overline{16})} , \ \left| \bar{3},2,-\frac{1}{6} \right\rangle_{\overline{10}(\overline{16}')} \right\},
\tag{44}
$$

where the states are labeled by their representations and hypercharge according to $|SU(3)_c, SU(2)_L, Y\rangle_{SU(5)(SO(10))}$.

We then form the Yukawa contribution $u_L^{cT} C^{-1} D_u u_L$ by using the above bases and the superpotentials to obtain the matrix

$$
D_u = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & y' \\
0 & 0 & 0 & t_2 & 0 & 0 & 0 \\
0 & 0 & t_3 & 0 & 0 & a & 0 \\
0 & t_2 & 0 & 0 & 0 & p & 0 \\
0 & 0 & 0 & 0 & t' & 0 & s'' \\
0 & 0 & a & p & 0 & 0 & 0 \\
y' & 0 & 0 & 0 & s'' & 0 & 0
\end{pmatrix}
$$

where we have introduced the following shorthand notation:

$$
t_3 = \lambda_{16_3 16_3 T_1} \langle T_1 \rangle, \qquad t_2 = \lambda_{16_2 16 T_1} \langle T_1 \rangle, \qquad t' = \lambda_{16' 16' T_1} \langle T_1 \rangle,
$$

$$
a = \lambda_{16_3 \overline{16} A} \langle A \rangle, \qquad p = \lambda_{16 \overline{16} P} \langle P \rangle,
\tag{45}
$$

$$
s'' = \lambda_{16' \overline{16}' S} \langle S \rangle, \qquad y' = \lambda_{16_1 \overline{16}' Y'} \langle Y' \rangle.
$$

We can then determine from this matrix the three pairs of zero-mass eigenstates at the GUT scale where the electroweak VEV of T_1 vanishes:

$$|u_{1L}\rangle = \left[|10(16_1)\rangle - \frac{y'}{s''}|10(16')\rangle \right] \Big/ \sqrt{1+y'^2/s''^2}$$

$$|u_{2L}\rangle = |10(16_2)\rangle$$

$$|u_{3L}\rangle = \left[|10(16_3)\rangle - \frac{a}{p}|10(16)\rangle \right] \Big/ \sqrt{1+a^2/p^2}$$

$$|u_{1L}^c\rangle = \left[|10(16_1)\rangle - \frac{y'}{s''}|10(16')\rangle \right] \Big/ \sqrt{1+y'^2/s''^2}$$

$$|u_{2L}^c\rangle = |10(16_2)\rangle$$

$$|u_{3L}^c\rangle = \left[|10(16_3)\rangle - \frac{a}{p}|10(16)\rangle \right] \Big/ \sqrt{1+a^2/p^2}$$

(46)

and where the states are now simply labeled by their $SU(5)$ and $SO(10)$ representations.

Finally, the Dirac matrix U for the three light quark states u, c, t is obtained by bracketing the electroweak contributions by the appropriate u_{iL}^c state on the left and the u_{jL} state on the right. The result obtained for U has exactly the form found earlier in Eq. (15) from the previous effective operator approach, with the identifications:

$$M_U = (t_3)_{5(10)}$$

$$\epsilon M_U = |3(a_q/p)(t_2)_{5(10)}|$$

(47)

$$\eta M_U = (y'/s'')^2(t')_{5(10)}.$$

Here the subscript on a_q signifies a factor of 1/3 arising from the $B-L$ VEV of the A in the adjoint representation, while the subscripts on the t terms specify the appropriate doublet VEV in the **10** for T_1. We have neglected the state normalization factors in Eq. (47) but will later argue that they can all be taken to be approximately unity.

The Dirac matrices, D, N, L are constructed in a similar fashion. In the case of D and L, the bases corresponding to Eqs. (43) and (44) are enlarged by two states lying in the **10**$_1$ and **10**$_2$ representations of $SO(10)$. For N, on the other hand, in addition to the two above states, one must add the singlet fermion contributions from the representations **1**$_k$, $k=1,2,3$ for $\mathcal{B}_{\nu_L}$ and **1**$_k^c$, $k=1,2,3$ for $\mathcal{B}_{\nu_L^c}$. We list the zero-mass state vectors for D, N and L in analogy to Eqs. (46):

$$|d_{1L}\rangle = \left[|10(16_1)\rangle - \frac{y'}{s''}|10(16')\rangle \right] \Big/ \sqrt{1+y'^2/s''^2}$$ (48)

$$|d_{2L}\rangle = |10(16_2)\rangle$$

$$|d_{3L}\rangle = \left[|10(16_3)\rangle - \frac{a}{p}|10(16)\rangle \right] \Big/ \sqrt{1+a^2/p^2}$$

$$|d_{1L}^c\rangle = \left[[|\bar5(16_1)\rangle - \frac{y'}{s''}|\bar5(16')\rangle \right] \Big/ \sqrt{1+y'^2/s''^2}$$

$$|d_{2L}^c\rangle = \left[|\bar5(16_2)\rangle - \frac{c}{y}|\bar5(10_2)\rangle \right] \Big/ \sqrt{1+c^2/y^2}$$

$$|d_{3L}^c\rangle = \left[|\bar5(16_3)\rangle - \frac{a}{p}|\bar5(16)\rangle \right] \Big/ \sqrt{1+a^2/p^2}$$

$$|n_{1L}\rangle = \left[|\bar5(16_1)\rangle - \frac{y'}{s''}|\bar5(16')\rangle \right] \Big/ \sqrt{1+y'^2/s''^2}$$

$$|n_{2L}\rangle = \left[|\bar5(16_2)\rangle - \frac{c}{y}|\bar5(10_2)\rangle \right] \Big/ \sqrt{1+c^2/y^2}$$

$$|n_{3L}\rangle = \left[|\bar5(16_3)\rangle - \frac{a}{p}|\bar5(16)\rangle \right] \Big/ \sqrt{1+a^2/p^2}$$

(49)

$$|n_{1L}^c\rangle = \left[|1(16_1)\rangle - \frac{y'}{s''}|1(16')\rangle \right.$$
$$\left. - \frac{\bar c_1}{x}|1_1^c\rangle \right] \Big/ \sqrt{1+y'^2/s''^2+\bar c_1^2/x^2}$$

$$|n_{2L}^c\rangle = \left[|1(16_2)\rangle - \frac{\bar c_2}{p_{22}}|1_2^c\rangle \right] \Big/ \sqrt{1+\bar c_2^2/p_{22}^2}$$

$$|n_{3L}^c\rangle = \left[|1(16_3)\rangle - \frac{a}{p}|1(16)\rangle \right.$$
$$\left. - \frac{\bar c_3}{z}|1_3^c\rangle \right] \Big/ \sqrt{1+a^2/p^2+\bar c_3^2/z^2}$$

$$|l_{1L}\rangle = \left[|\bar5(16_1)\rangle - \frac{y'}{s''}|\bar5(16')\rangle \right] \Big/ \sqrt{1+y'^2/s''^2}$$

$$|l_{2L}\rangle = \left[|\bar5(16_2)\rangle - \frac{c}{y}|\bar5(10_2)\rangle \right] \Big/ \sqrt{1+c^2/y^2}$$

$$|l_{3L}\rangle = \left[|\bar5(16_3)\rangle - \frac{a}{p}|\bar5(16)\rangle \right] \Big/ \sqrt{1+a^2/p^2}$$

(50)

$$|l_{1L}^c\rangle = \left[|10(16_1)\rangle - \frac{y'}{s''}|10(16')\rangle \right] \Big/ \sqrt{1+y'^2/s''^2}$$

$$|l_{2L}^c\rangle = |10(16_2)\rangle$$

$$|l_{3L}^c\rangle = \left[|10(16_3)\rangle - \frac{a}{p}|10(16)\rangle \right] \Big/ \sqrt{1+a^2/p^2}.$$

In the above we have introduced, in analogy with Eqs. (45), the additional shorthand notation:

$$c = \lambda_{16_2 10_1 C}\langle C \rangle, \quad \bar{c}_i = \lambda_{16_1 1_i \bar{c}}\langle \bar{C} \rangle, i = 1,2,3,$$

$$x = \lambda_{1_1 1_1^c X}\langle \bar{X} \rangle, \quad y = \lambda_{10_1 10_2 Y}\langle Y \rangle, \tag{51}$$

$$z = \lambda_{1_3 1_3^c Z}\langle Z \rangle, \quad p_{22} = \lambda_{1_2 1_2^c P}\langle P \rangle.$$

The Dirac matrices D, N and L are found by forming matrix elements of the electroweak symmetry breaking VEV's with the appropriate basis vectors. Again these matrices have exactly the structures given in Eqs. (15), provided the state normalization factors are approximated by unity, i.e., we assume that the zero-mass states have their large components in the chiral representations 16_1, 16_2 and 16_3 and that all the other components are small. We shall return to this point in the next section. In the meantime we make the identifications

$$M_D = (t_3)_{\bar{5}(10)}$$

$$\epsilon M_D = |3(a_q/p)(t_2)_{\bar{5}(10)}|$$

$$\sigma M_D = -(c/y)(c')_{\bar{5}(16)} \tag{52}$$

$$\Delta M_D = t_0 \bar{t}_0 / s$$

$$\delta' e^{i\phi} M_D = t_0' \bar{t}_0 / s'$$

in terms of the notation given in Eqs. (45) and (51) and the following:

$$t_0 = \lambda_{16_1 16_2 T_0}, \quad t_0' = \lambda_{16_1 16_3 T_0'}$$

$$\bar{t}_0 = \lambda_{CC'\bar{T}_0}\langle C \rangle\langle C' \rangle, \quad c' = \lambda_{16_3 10_2 C'}\langle C' \rangle, \tag{53}$$

$$s = \lambda_{T_0 \bar{T}_0 S}\langle S \rangle, \quad s' = \lambda_{T_0' \bar{T}_0 S'}\langle S' \rangle.$$

The phase ϕ appearing in the δ' term can be understood to arise from a phase in the VEV for S'. The structures of the Dirac matrix elements given in Eqs. (15), (47) and (52) can be understood in terms of the simple Froggatt-Nielsen diagrams of Fig. 1 and 2, with the Higgs fields labeled as in Table I.

Turning to the right-handed Majorana mass matrix, we use the zero mass left-handed conjugate states that were obtained implicitly above for the Dirac matrix N to form the basis for M_R. The right-handed Majorana matrix is then obtained by bracketing the Majorana Higgs V_M with the appropriate zero mass conjugate neutrino states in Eqs. (49). We obtain

$$M_R = \begin{pmatrix} 0 & A\epsilon^3 & 0 \\ A\epsilon^3 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\Lambda_R \tag{54}$$
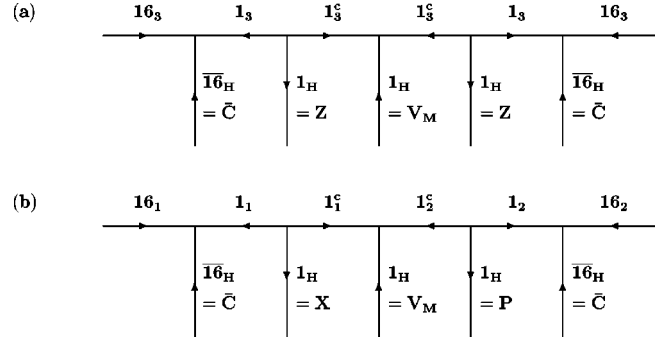
where

FIG. 4. Diagrams that generate the 33, 12, and 21 elements of the Majorana mass matrix M_R of the superheavy right-handed neutrinos.

$$M_3 = \Lambda_R = \lambda_{1_3^c 1_3^c V_M}\langle V_M\rangle(\bar{c}_3/z)^2,$$

$$\tag{55}$$

$$M_2 = -M_1 = A\epsilon^3 \Lambda_R = \lambda_{1_1^c 1_2^c V_M}\langle V_M\rangle(\bar{c}_1/x)(\bar{c}_2/p_{22}).$$

The lighter two right-handed Majorana masses are degenerate and have opposite CP-parity. Note that the whole right-handed Majorana matrix has been generated in this simple model by one Majorana VEV coupling superheavy conjugate fermion singlets as shown in Fig. 4.

We conclude this section with a summary of the GUT scale predictions derived from the Dirac and Majorana mass matrices with the particular parameters appropriate for the model in question. For convenience we give the whole set equations which are the counterpart of Eqs. (19):

$$m_t^0/m_b^0 \cong (\sigma^2+1)^{-1/2}M_U/M_D, \quad m_u^0/m_t^0 \cong \eta, \tag{56}$$

$$m_c^0/m_t^0 \cong \frac{1}{9}\epsilon^2 \cdot \left[1 - \frac{2}{9}\epsilon^2\right], \quad m_b^0/m_\tau^0 \cong 1 - \frac{2}{3}\frac{\sigma}{\sigma^2+1}\epsilon,$$

$$m_s^0/m_b^0 \cong \frac{1}{3}\epsilon\frac{\sigma}{\sigma^2+1} \cdot \left[1 + \frac{1}{3}\epsilon\frac{1-\sigma^2-\sigma\epsilon/3}{\sigma(\sigma^2+1)} + \frac{1}{2}(t_L^2+t_R^2)\right],$$

$$m_d^0/m_s^0 \cong t_L t_R \cdot \left[1 - \frac{1}{3}\epsilon\frac{\sigma^2+2}{\sigma(\sigma^2+1)} - (t_L^2+t_R^2)\right.$$

$$\left. + (t_L^4+t_L^2 t_R^2+t_R^4)\right],$$

$$m_\mu^0/m_\tau^0 \cong \epsilon\frac{\sigma}{\sigma^2+1} \cdot \left[1 + \epsilon\frac{1-\sigma^2-\sigma\epsilon}{\sigma(\sigma^2+1)} + \frac{1}{18}(t_L^2+t_R^2)\right],$$

$$m_e^0/m_\mu^0 \cong \frac{1}{9}t_L t_R \cdot \left[1 - \epsilon\frac{\sigma^2+2}{\sigma(\sigma^2+1)} + \epsilon^2\frac{\sigma^4+9\sigma^2/2+3}{\sigma^2(\sigma^2+1)^2}\right.$$

$$\left. - \frac{1}{9}(t_L^2+t_R^2)\right],$$

$$V_{cb}^0 \cong \frac{1}{3}\epsilon\frac{\sigma^2}{\sigma^2+1}\cdot\left[1+\frac{2}{3}\epsilon\frac{1}{\sigma(\sigma^2+1)}\right],$$

$$V_{us}^0 \cong t_L\left[1-\frac{1}{2}t_L^2-t_R^2+t_R^4+\frac{5}{2}t_L^2t_R^2+\frac{3}{8}t_L^4\right.$$

$$\left. -\frac{\epsilon}{3\sigma\sqrt{\sigma^2+1}}\frac{t_R}{t_L}e^{-i\theta}\right],$$

$$V_{ub}^0 \cong \frac{1}{3}t_L\epsilon\frac{1}{\sigma^2+1}\left[\sqrt{\sigma^2+1}\frac{t_R}{t_L}e^{-i\theta}\left(1-\frac{1}{3}\epsilon\frac{\sigma}{\sigma^2+1}\right)\right.$$

$$\left. -\left(1-\frac{2}{3}\epsilon\frac{\sigma}{\sigma^2+1}\right)\right],$$

$$m_2^0/m_3^0 \cong \left(\frac{\eta}{A\epsilon\sqrt{1+\epsilon^2}}\right)\left[1+\frac{\eta}{A\epsilon^3\sqrt{1+\epsilon^2}}\right],$$

$$m_1^0/m_3^0 \cong \left(\frac{\eta}{A\epsilon\sqrt{1+\epsilon^2}}\right)\left[1-\frac{\eta}{2A\epsilon^3\sqrt{1+\epsilon^2}}\right],$$

$$U_{\mu3}^0 \cong -\frac{1}{\sqrt{\sigma^2+1}}\left(\sigma-\epsilon\frac{\sigma^2}{\sigma^2+1}\right),$$

$$U_{e2}^0 \cong -\frac{1}{\sqrt{2}}\left[1-\frac{\epsilon}{3\sigma}t_Le^{i\theta}+\frac{1}{3\sqrt{\sigma^2+1}}(1+\epsilon\sigma)t_R\right],$$

$$U_{e3}^0 \cong \frac{1}{3\sqrt{\sigma^2+1}}(\sigma-\epsilon)t_R-\frac{\eta}{A\epsilon^2}.$$

To the quark equations we have added the ratio m_t^0/m_b^0 which involves the ratio of $\langle \mathbf{5}(T_1)\rangle$ to $\langle \overline{\mathbf{5}}(T_1)\rangle$, i.e., M_U/M_D, as well as giving the leptonic mass ratios and mixings specific to the model in question.

C. Numerical evaluation of matrix parameters

We have elaborated above how the simple explicit model proposed gives precisely the structure for the Dirac mass matrices that was obtained from the effective operator approach. We now show that the entries are also numerically in the range to fit the quark and lepton mass and mixing data.

In order to compare the GUT scale predictions in Eq. (56) with the low scale data, the GUT scale values are first evolved from $\Lambda_G=2\times10^{16}$ GeV down to the SUSY scale which is taken to be $\Lambda_{\mathrm{SUSY}}=m_t(m_t)=165$ GeV by use of the 2-loop MSSM beta functions. For this purpose, the mass ratios at the two scales are related by the $\eta_{i/j}$ running factors, while the quark mixing elements are scaled by the η_{ij} factor according to

$$\left(\frac{m_i}{m_j}\right)_{\mathrm{SUSY}}=\left(\frac{m_i^0}{m_j^0}\right)\Big/\eta_{i/j},$$

$$(V_{ij})_{\mathrm{SUSY}}=V_{ij}^0/\eta_{ij},\quad (ij)=(ub),\quad(cb),\quad(td),\quad(ts).\tag{57}$$

The remaining evolutions to the bottom and charm quark or tau lepton running mass scales, or to the 1 GeV scale in the case of the light quarks and leptons, is carried out with the 3-loop QCD and 1-loop QED renormalization group equations. Here the running factors are η_i with the mass ratios scaled according to

$$m_i(m_i)=(m_i)_{\mathrm{SUSY}}/\eta_i(m_t)\tag{58}$$

or similarly, with the running mass scale m_i replaced by 1 GeV. With $\tan\beta=5$ used for the numerical evaluations for reasons that will become apparently shortly, $\alpha_s(M_Z)=0.118$, $\alpha(M_Z)=1/127.9$ and $\sin^2\theta_W=0.2315$, the running factors are given by

$$\eta_{u/t}=\eta_{c/t}=0.6927,\quad \eta_{d/b}=\eta_{s/b}=0.8844$$

$$\eta_{\mu/\tau}=0.9988,\quad \eta_{b/t}=0.5094$$

$$\eta_{ub}=\eta_{cb}=\eta_{td}=\eta_{ts}=0.8853\tag{59}$$

$$\eta_u(m_t)=0.4235,\quad \eta_c(m_t)=0.4733,\quad \eta_t(m_t)=1.0000$$

$$\eta_d(m_t)=0.4262,\quad \eta_s(m_t)=0.4262,\quad \eta_t(m_t)=0.6540$$

$$\eta_e(m_t)=0.9816,\quad \eta_\mu(m_t)=0.4816,\quad \eta_\tau(m_t)=0.9836.$$

Finally, finite corrections must be applied to m_s, m_b and the evolved quark mixing matrix elements which arise from gluino and chargino loops. The correction factors are conventionally written as $(1+\Delta_s)$, $(1+\Delta_b)$ and $(1+\Delta_{cb})$ where we have used

$$\Delta_s=-0.20,\quad \Delta_b=-0.15,\quad \Delta_{cb}=-0.05\tag{60}$$

as explained below.

Using the quantities [22] $m_t(m_t) = 165$ GeV, $m_\tau = 1.777$ GeV, $m_\mu = 105.7$ MeV, $m_e = 0.511$ MeV, $m_u = 4.5$ MeV, $V_{us} = 0.220$, $V_{cb} = 0.0395$, and $\delta_{CP} = 64°$ to determine the input parameters, one obtains for them $M_U \simeq 113$ GeV, $M_D \simeq 1$ GeV, $\sigma = 1.780$, $\epsilon = 0.145$, $t_L = 0.236$, $t_R = 0.205$, $\theta = 34°$ (corresponding to $\delta = 0.0086$, $\delta' = 0.0079$, $\phi = 54°$), and $\eta = 8 \times 10^{-6}$. With these inputs the remaining quark masses and mixings are obtained, to be compared with the experimental values [22] in parentheses:

$$m_c(m_c) = 1.23 \text{ GeV} \quad (1.27 \pm 0.1 \text{ GeV})$$

$$m_b(m_b) = 4.25 \text{ GeV} \quad (4.26 \pm 0.11 \text{ GeV})$$

$$m_s(1 \text{ GeV}) = 148 \text{ MeV} \quad (175 \pm 50 \text{ MeV}) \tag{61}$$

$$m_d(1 \text{ GeV}) = 7.9 \text{ MeV} \quad (8.9 \pm 2.6 \text{ MeV})$$

$$|V_{ub}/V_{cb}| = 0.080 \quad (0.090 \pm 0.008)$$

where the finite SUSY loop corrections for m_b, m_s and V_{cb} have been rescaled to give $m_b(m_b) \simeq 4.25$ GeV for $\tan\beta = 5$. Had we chosen $\delta_{CP} = 70°$ as input, on the other hand, we would find instead $|V_{ub}/V_{cb}| = 0.085$. With the numerical values in Eqs. (61) we find for $\bar\rho$, $\bar\eta$ and the α, β and γ angles of the unitarity triangle pictured in Fig. 3

$$\bar\rho = 0.143, \quad \bar\eta = 0.305, \quad \alpha = 96°, \quad \beta = 20°, \quad \gamma = 64°. \tag{62}$$

The upper vertex of the triangle appears to be circled precisely in the allowed experimental region.

Additional predictions follow for the neutrino sector. The effective light neutrino mass matrix of Eq. (34) or (36) with $B = 0$ leads to bimaximal mixing with a large angle solution for atmospheric neutrino oscillations [6] and the ''just-so'' vacuum solution [19] involving two pseudo-Dirac neutrinos, if we set $\Lambda_R = 2.4 \times 10^{14}$ GeV and $A = 0.05$. We then find

$$m_3 = 54.3 \text{ meV}, \quad m_2 = 59.6 \text{ } \mu\text{eV}, \quad m_1 = 56.5 \text{ } \mu\text{eV}$$

$$M_3 = 2.4 \times 10^{14} \text{ GeV}, \quad M_2 = M_1 = 3.66 \times 10^{10} \text{ GeV}$$

$$U_{e2} = 0.733, \quad U_{e3} = 0.047, \quad U_{\mu3} = -0.818, \quad \delta'_{CP} = -0.2°$$

$$ \tag{63}$$

$$\Delta m_{23}^2 = 3.0 \times 10^{-3} \text{ eV}^2, \quad \sin^2 2\theta_{\text{atm}} = 4|U_{\mu3}|^2|U_{\tau3}|^2 = 0.89$$

$$\Delta m_{12}^2 = 3.6 \times 10^{-10} \text{ eV}^2, \quad \sin^2 2\theta_{\text{solar}} = 4|U_{e1}|^2|U_{e2}|^2 = 0.99$$

$$\Delta m_{13}^2 = 3.0 \times 10^{-3} \text{ eV}^2, \quad \sin^2 2\theta_{\text{reac}} = 4|U_{e3}|^2(1 - |U_{e3}|)^2 = 0.009.$$

The effective scale of the right-handed Majorana mass contribution occurs two orders of magnitude lower than the supersymmetric (SUSY) GUT scale of $\Lambda_G = 1.2 \times 10^{16}$ GeV. The effective two-component reactor mixing angle given above should be observable at a future neutrino factory, whereas the present limit from the CHOOZ experiment [23] is approximately 0.1 for the above Δm_{23}^2. In principle, the parameter A appearing in M_R can also be complex, but we find that in no case does the leptonic CP-violating phase, δ'_{CP} exceed 10° in magnitude. Hence the model predicts leptonic CP-violation will be unobservable.

The vacuum solar solution depends critically on the appearance of the parameter η in the matrix N, corresponding to the non-zero η entry in U which gives the up quark a mass at the GUT scale. Should we set $\eta = 0$, only the small-angle MSW solution [7] would be obtained for the solar neutrino oscillations. The bimaximal large mixing angle MSW solution requires a considerably larger hierarchy in M_R. But

such a large hierarchy may be desirable in order to obtain baryogenesis through the leptogenesis mechanism.

Finally we address the issue that the state normalization factors were all replaced by unity in Eqs. (47) and (52) for the various matrix parameters. This is a good approximation provided the three fermion spinor states $|16_1\rangle$, $|16_2\rangle$, $|16_3\rangle$ provide the dominant contributions to the zero-mass quark and lepton states at the GUT scale. In particular, the following ratios must be much less than unity:

$$(a/p)^2, \quad (y'/s'')^2, \quad (c/y)^2, \quad (\bar c_1/x)^2,$$

$$(\bar c_2/p_{22})^2, \quad (\bar c_3/z)^2 \ll 1. \tag{64}$$

Let us assume for simplicity that the electroweak couplings of $\langle T_1\rangle$ in t_3, t_2 and t' in Eq. (45) and of $\langle C'\rangle$ in c' of Eq. (53) are identical. Then with $\epsilon = |3(a_q/p)| = |a_l/p| = 0.14$, we find the condition $(a/p)^2 \simeq 0.02 \ll 1$ holds. To ob-

tain an up quark mass $m_u(1 \text{ GeV}) \simeq 4.5$ MeV, we need η $\cong (y'/s'')^2 \simeq 8 \times 10^{-6}$ at the GUT scale, which easily satisfies Eqs. (64).

Requiring that $(c/y)^2 \ll 1$ and with the result from Eqs. (52) that

$$\sigma \simeq \left| \frac{c}{y} \frac{\langle \bar{5}(C') \rangle}{\langle \bar{5}(T_1) \rangle} \right| \simeq 1.8 \qquad (65)$$

leads us to the results that

$$\tan \gamma \equiv \frac{\langle \bar{5}(C') \rangle}{\langle \bar{5}(T_1) \rangle} \gg \sigma$$

$$\tan \beta \simeq \sqrt{\sigma^2 + 1} \, m_t^0 (\cos \gamma) / m_b^0 \ll m_t^0 / m_b^0$$

$$(66)$$

in terms of the $T_1 - C'$ mixing angle, γ, in Eq. (11). With $c/y \cong 0.1$, for example, $\tan \gamma \simeq 18$ which implies $\tan \beta \simeq 6$, a very reasonable mid-range value allowed by experiment. For this reason, we have chosen to illustrate the numerical results above with $\tan \beta = 5$.

The remaining ratios in Eq. (64) can also be satisfied. For comparable $\bar{c}_i$'s, $A\epsilon^3 \sim 1.4 \times 10^{-4}$ obtained from Eq. (55) requires that $\langle Z \rangle / \sqrt{\langle X \rangle \langle P \rangle} \sim 0.01$. This ratio is consistent with the VEV's needed in the Higgs superpotential of Eq. (41) in order to solve the doublet-triplet splitting problem.

Turning now to the parameters δ and δ', we note that the near equality of their magnitudes leads to the ratio $\delta / |\delta'| \cong s / |s'| \simeq 1$. Moreover, if we assume $y \sim y'$, we obtain the estimate $\delta \sim c y' / (y s) \tan \gamma \sim 5 \times 10^{-3}$ with the numbers obtained earlier, whereas the actual value needed is $\delta \simeq 0.008$.

Thus we have found that not only are the desired forms of the Dirac (and Majorana) matrices generated by the model of this section, but that the numerical values required for the matrix parameters are also quite reasonable.

VII. SUMMARY

Both the largeness of the atmospheric neutrino mixing $U_{\mu 3}$ and the smallness of the quark mixing V_{cb} can be elegantly accounted for by the idea that the charged lepton mass matrix L is highly asymmetric or ''lopsided'' and that the down-quark mass matrix D is related to the transpose of L by an $SU(5)$ symmetry. This idea was discovered independently by several groups and has since been used in numerous models of fermion masses. Remarkably, exactly such mass matrices emerged in our work from quite other considerations than neutrino masses and mixings, specifically from an attempt to construct the simplest possible realistic $SO(10)$ model.

Advances have been made in recent years in simplifying the Higgs structure of SUSY $SO(10)$ models. If one assumes the minimal set of Higgs fields that can break $SO(10)$ down to the standard model group, the possibilities for Yukawa terms for the quarks and leptons become significantly restricted. It turns out that there is what seems to be a uniquely simple set of $SO(10)$ Yukawa terms that gives realistic masses and mixings. This set consists of only six effective Yukawa terms (five if $m_u = 0$) which satisfactorily fits all nine masses of the quarks and charged leptons as well as the four CKM parameters. In addition, large $\nu_\mu - \nu_\tau$ mixing emerges automatically. Moreover, in this uniquely simple model, the simplest possibilities for the Majorana mass matrix M_R of the right-handed neutrinos lead either to small angle MSW values for the solar neutrino mixing or to vacuum oscillation values. In this paper we have studied in detail the consequences of different forms of M_R for the neutrino mixing angles and mass ratios.

In the published literature no more predictive and economical a model for quark and lepton masses than the one discussed here exists that is also consistent with present knowledge. It is striking that in this model a single term and a single parameter (which we call σ) accounts for no less than four puzzling aspects of the light fermion spectrum: the largeness of $U_{\mu 3}$, the smallness of V_{cb}, the smallness of m_c / m_t compared to m_s / m_b, and the Georgi-Jarlskog factor of three between m_μ and m_s at the GUT scale. It should be emphasized that, while many satisfactory neutrino mixing ideas and also many interesting ideas for explaining the pattern of quark and charged lepton masses have been proposed, very few models exist which not only give a satisfactory account of neutrino phenomenology but are at the same time highly predictive.

We have shown that the model defined by the existence of these five (or six) effective Yukawa terms can be realized in a complete and specific renormalizable SUSY $SO(10)$ model that is technically natural. We have presented the details of such a model, including all the Higgs and quark and lepton superfields, the Abelian flavor symmetries, and the transformation properties of the fields under these symmetries. Finally, we have done a quantitative comparison of the predictions of the model to experiment.

In the future this model will be rigorously testable in several ways. The most important are (1) a relation between the real and imaginary parts of V_{ub} including a precise test of the angles of the unitarity triangle; (2) a prediction for U_{e2}, which in the small angle MSW case gives a sharp relation between the solar and atmospheric angles; and (3) a definite prediction for U_{e3}.

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