Sphalerons, merons, and unstable branes in AdS space

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We construct unstable classical solutions of Yang-Mills theories and their dual unstable states of type IIB on anti–de Sitter (AdS_5) space. An example is the unstable D0-brane of type IIB located at the center of AdS space. This has a field theory dual which is a sphaleron in gauge theories on $S^3 \times \mathbb{R}$. We argue that the two are dual because both are sphalerons associated with the topology of the instanton or D-instanton. This agreement provides a non-supersymmetric test of the AdS-conformal field theory duality. As an illustration, many aspects of Sen's hypothesis regarding the unstable branes can be seen easily in the weakly coupled dual field theory description. In Euclidean AdS space the D0-branes are dual to gauge theory merons. This implies that the two ends of a D0-brane world-line carry half the charge of a D-instanton. Other examples involve unstable strings in the Coulomb phase.

I. INTRODUCTION

Like any other strong-weak duality which cannot be proven directly, the anti–de Sitter–conformal-field-theory (AdS/CFT) duality [1] was tested using Bogomol'nyi-Prasad-Sommerfeld (BPS) configurations. Such configurations are protected by supersymmetry and can be traced while interpolating from weak to strong coupling. Non-BPS configurations are not protected and in general any result obtained using the duality is considered to be a prediction rather than a test.

In this paper we study some non-BPS states of gauge theories at weak and strong coupling. The configurations we discuss are unstable classical solutions which sit at the top of non-contractible loops in configuration space (sphalerons) [2–5].

Let us remind the reader what a sphaleron is. Say there exists a one parameter family of field configurations that form a non-contractible loop. One should think of all homotopically equivalent loops and find the point with maximal energy along each loop. Now consider the minimum of all those energies; since the loops are not contractible, that energy has to be greater than zero, and the corresponding field configuration is a saddle point—the sphaleron. In practice, once one understands the topology, it is usually easy to find the loop going through the sphaleron. A schematic picture is given in Fig. 1.

If there is a d-dimensional topologically charged object in the theory, then in general there would be a $(d+1)$-dimensional sphaleron. A simple example is a theory which has an instanton. Then consider the one parameter family of static field configurations where the extra parameter replaces the Euclidean time. This family of field configurations has the same topological charge as the instanton.

By varying the parameter, one starts and ends at the vacuum, and at the middle point there will be an unstable solution to the equations of motion. It sits at the top of a non-contractible loop in the space of field configurations. This is the sphaleron.

It was recently argued by Harvey, Hořava, and Kraus [6] that unstable D-branes of string theory [7,8] are sphalerons. For example, the type IIB D0-brane can decay to the vacuum, but its existence is dictated by the same topology as the D-instanton, whose charge is classified by K theory [9]. One can construct a one parameter family of static configurations whose topology is that of the D-instanton. The D0-brane sits at the top of the loop.

This will serve as our first example. We consider the configuration of a D0-brane at the center of AdS space. This is a massive, non-BPS object in the large N and large coupling classical limit of the theory. In global AdS space
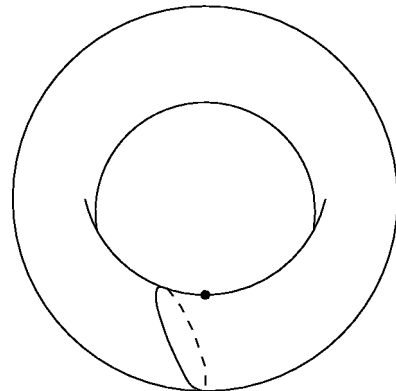

FIG. 1. The existence of a non-contractible loop, as illustrated in the picture, proves that there is an unstable saddle point. It is marked by the point in the picture.

geometry, where the topology of the boundary is $S^3 \times \mathbb{R}$, this is a static, spherically symmetric, configuration.

A similar configuration exists at weak 't Hooft coupling. It is explained in detail in Sec. II, let us just say now that it is a ''half pure gauge'' configuration. If one considers the $SU(2)$ instanton [10], this is the configuration half-way through the tunneling process, which is at the top of the potential. That is why it is a solution of the equations of motion with one unstable mode. This gauge theory sphaleron has many properties similar to the D0-brane in AdS space. It is static, spherically symmetric and has a single tachyonic mode. We will argue that it is dual to the D0-brane in AdS space. We also find duals of the configuration with k coincident D0-branes, which have k^2 unstable modes, in string theory and in the gauge theory.

It is rather perplexing at first that we are able to find a dual description for a non-BPS object. But there is, in fact, a good reason for that. The D0-brane sits in the middle of a non-contractible loop with the same topology as the D-instanton, while the gauge theory solution is at the middle of a loop with the topology of the gauge theory instanton which is dual to the D-instantons.

Put differently, the instanton describes a tunneling process under a potential barrier, and the sphaleron sits at the top of the potential. The mass of the sphaleron is the maximum height of the potential. In the dual theory, the D-instanton also describes a tunneling event, and the sphaleron is again at the top of the potential barrier. The mass of the D0-brane is the height of the potential. Since the Yang-Mills (YM) instanton and D-instanton are dual, they describe the same tunneling process in the dual pictures. The shape of the potential is altered by quantum corrections, but there is always an unstable point in the middle.

It is very simple to calculate the potential through which the instanton tunnels, it is given by a quartic of the field. The potential of string theory is much more complicated, understanding this potential is crucial to proving the brane anti-brane annihilation procedure, which is in the heart of Sen's construction, and the classification of D-brane charges by K theory. This issue was addressed recently by using level truncation in string field theory [11] with impressive results. Our dual description fits neatly with Sen's conjecture.

One should contrast this with other strong-weak dualities. It is more typical for the topological excitations of one theory to become the elementary excitations of the dual theory. For example, the kinks of the sine-Gordon model become the fermions in the dual Thirring model. The same is true in the S-duality of $\mathcal{N}=4$ Yang-Mills (and type IIB), where the topologically charged monopole goes over to the W-boson which is the elementary excitation. Here we find that one topologically charged object goes to another topologically charged object, and therefore there are sphalerons associated with those topologies. Roughly speaking, the AdS-CFT duality is special since it is a strong-weak duality with respect to the 't Hooft coupling, while the solitons' masses are of the order of $1/g_{YM}^2$.

These ''half pure gauge'' configurations were considered in the past on $\mathbb{R}^4$. They are singular at the origin and at infinity, but the singularities can be smoothed out. Those objects were named merons [12]. The singularity at the origin and at infinity are replaced with half an instanton, interpolating between the vacuum and the meron.

This has an exact analog in Euclidean AdS space, where a D0-brane appearing out of the vacuum, propagating and annihilating is dual to the meron. The D0-brane follows a geodesic in AdS space, and its action depends logarithmically on the separation of the two end points. The same logarithmic behavior (up to a coefficient which depends on the 't Hooft coupling) shows up on the gauge theory side. Because of the entropy of those configurations, they might dominate the path integral for large g_{YM}.

We will also argue that each of the two end points of the D0-brane carries half a unit of D-instanton charge. The D0-brane serves as a flux tube carrying half a unit of flux from one end to the other, thus preserving the Dirac quantization condition of D-instanton charge. A similar story applies to higher dimensional branes, so the unstable D-branes can be regarded as D-merons. Unlike AdS space, where the action of the D0-brane is logarithmic, in flat space it is linear, therefore it would not be dynamically favorable for D-branes to break by this mechanism.

The paper is organized as follows. We describe the details of the sphaleron on $S^3 \times \mathbb{R}$ and the D0-brane in Lorentzian AdS space in Sec. II. In Sec. III we describe the meron configurations. We review the old construction in the gauge theory, and then we describe its dual. We interpret the unstable branes as D-merons in Sec. IV. In Sec. V we consider another example of a duality between unstable classical solutions. We show that gauge theories in the Coulomb phase admit unstable string solutions which do not carry gauge invariant magnetic or electric fluxes. We describe the AdS dual of this solution. The unstable string can also serve as a meron, and we explain how a monopole can be separated into two halves as long as they are connected by one of those strings.

II. SPHALERON PARTICLE

In this section we consider sphaleron particles in four dimensional $U(N)$ Yang-Mills theory, and their AdS duals. Since Yang-Mills theory is a conformal theory there are no static finite energy (stable or unstable) solutions on $\mathbb{R}^4$ simply because there is no scale to fix the mass of the solution. However, there is a sphaleron particle if we consider the gauge theory on $S^3 \times \mathbb{R}$. In that case the size of the sphere, R, is the only scale in the theory and so the mass of any static solution is $\sim 1/R$.

We consider first the perturbative YM description, and then the AdS dual. While the duality is true only for the theory with the $\mathcal{N}=4$ matter content, in perturbation theory the particle exists already in the pure gauge theory.

A. Gauge theory description

The topology that supports a stable particle in four dimensions is the map from the S^2 at spatial infinity to the fields. For $U(N)$ pure gauge theory the only relevant topology is

$\pi_2(U(N))=0$. Hence this theory does not admit any topologically charged stable particles (on either $\mathbb{R}^4$ or $\mathbb{R}\times S^3$). However, since

$$\pi_{2l+1}(U(N))=\mathbb{Z}, \quad \text{for } l<N, \tag{1}$$

there are unstable solutions to YM theory. These solutions, which we describe below, sit at the top of a non-contractible S^{2l-1} in configuration space.

We start by considering the simplest case of $l=1$. In that case we have a non-contractible loop in the configuration space of $SU(2)$ gauge theory which we embed in $SU(N)$. The topology of the non-contractible loop is the same as the instanton topology. It is useful to recall the instanton solution, given by the ansatz

$$A_\mu = -if(r)\partial_\mu UU^\dagger,$$

$$U = \frac{x^\mu \sigma_\mu}{r} = \frac{x_0 + ix_i \sigma_i}{r}, \quad r^2 = x_0^2 + x_i^2, \tag{2}$$

where σ_i are the Pauli matrices and x_0, x_i the four Euclidean directions. The Yang-Mills action now yields

$$S = \frac{1}{4g_{YM}^2}\int_0^\infty dr\, 96\pi^2\left(\frac{r}{2}f'^2 + \frac{2}{r}f^2(1-f)^2\right). \tag{3}$$

The equations of motions have three constant solutions $f=0$, $f=1$ and $f=1/2$. $f=0,1$ are stable solutions which correspond to two vacua. The instanton solution, $f(r)=r^2/(a^2+r^2)$, interpolates between $f=0$ at the origin and $f=1$ at infinity. The configuration with $f=1/2$ is an unstable solution, it solves the second-order equation of motion, but unlike the two vacua and the instanton solution, does not solve the first-order BPS equation.

On $\mathbb{R}^4$ we see from Eq. (2) that $f=1/2$ is a non-static singular solution. It was first discussed in [13] and was studied further in [12]. Those are the merons which we will discuss in the next section. On $S^3\times\mathbb{R}$ however, the solution is static, regular and completely delocalized[1] on S^3. To see this, note that the conformal transformation that takes $\mathbb{R}^4$ with metric $ds^2 = dr^2 + r^2 d\Omega_3^2$ to $S^3\times\mathbb{R}$ with metric $ds^2 = dt^2 + R^2 d\Omega_3^2$ is

$$r = \exp(t/R). \tag{4}$$

Therefore the action of the sphaleron on $S^3\times\mathbb{R}$ is

$$S = \int_0^\infty dr\, \frac{3\pi^2}{g_{YM}^2 r} = \frac{3\pi^2}{g_{YM}^2 R}\int_{-\infty}^\infty dt. \tag{5}$$

We see that the action does not depend on t and that the sphaleron mass is

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[1] Since the solution is smeared over the entire S^3, it could be considered a tachyonic vacuum, rather than an unstable particle. Since the space is compact, it is hard to distinguish between the two notions.

$$M_{Sp} = \frac{3\pi^2}{g_{YM}^2 R}. \tag{6}$$

A non-contractible loop of static field configurations going between the two vacua and through the sphaleron is given by Eq. (2) with

$$f(r) = \alpha, \quad 0\leq\alpha\leq 1. \tag{7}$$

Equation (2) implies that for constant f we get $A_r=0$ (on $\mathbb{R}^4$) and hence $A_t=0$ (on $S^3\times\mathbb{R}$) and that A_θ does not depend on t. Therefore, $F_{t\theta}=0$ (where θ represents the S^3 coordinates). This has important implications for the non-contractible loop. First, the field configurations along the entire non-contractible loop (7) do not depend on t, and can be described in terms of the three dimensional theory on S^3. Second, even though the conformal map with Lorentzian signature (see e.g. [14]) is different from the Euclidean conformal map (4), the Wick rotation to the Lorentzian signature (on $S^3\times\mathbb{R}$) is trivial along the entire non-contractible loop. This is not the case for the instanton solution, which depends on r. Finally,

$$\text{Tr}\, F\tilde{F} = 0, \quad \text{while } \text{Tr}\, F^2 = \frac{6}{R^4}\neq 0. \tag{8}$$

These features will prove to be important for the dual description, as we shall see in the next section.

Next we turn to the cases when $l>1$. In those cases the solution exists only for $SU(N)$ with $N>2$. Finding all sphaleron solutions for $SU(N)$ gauge theory is beyond the scope of the paper. However, there is a very simple construction which yields sphalerons related to arbitrarily high homotopy groups. Those are dual to the coincident D0-branes in AdS space.

We can generalize the spherically symmetric ansatz (2) to larger gauge groups by replacing the Pauli matrices and the identity by

$$A_\mu = -if(r)\partial_\mu UU^\dagger, \quad U = \frac{x^\mu \gamma_\mu}{r}, \tag{9}$$

where the γ's satisfy the algebra $\gamma_\mu \gamma_\nu^\dagger + \gamma_\nu \gamma_\mu^\dagger = 2\delta_{\mu\nu}$. We use the simple choice

$$\gamma_\mu = \sigma_\mu \otimes I_k = \begin{pmatrix} \sigma_\mu & 0 & \cdots & 0 \\ 0 & \sigma_\mu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_\mu \end{pmatrix}, \tag{10}$$

where I_k is the identity matrix of rank k. It is easy to see that this is still a solution of the equations of motion if $f=1/2$. The action simply scales as the rank, $2k$, of the matrices γ_μ. Therefore the mass of the k sphaleron is

$$M_k = kM_{Sp}. \tag{11}$$

This sphaleron solution has k^2 unstable modes, which correspond to each of the 2×2 entries in the matrix in Eq. (10). The number of unstable modes alone does not fix the topology of the non-contractible loops associated with the sphaleron. For example, the fact that we have k^2 unstable modes does not mean that the sphaleron sits at the top of S^{k^2}. This would be inconsistent with $\pi_{k^2}(U(N))=0$ for even k. In fact the topology is exactly that of $U(k)$. The sphaleron sits at the point $-I_k$ in the group, which is opposite to the identity[2]. The k^2 unstable modes are the tangent vectors in the algebra of $U(k)$. Since the group $U(k)$ has non-contractible S^{2l-1} for all $0<l\leq k$, there are such loops going through the sphaleron. So we can choose to classify the tangent directions by those spheres. All together there are indeed $1+3+\cdots+2k-1=k^2$ unstable directions. The sphaleron sits, therefore, at the top of $S^1, S^3, \ldots, S^{2k-1}$. In the next section we shall see that this fits neatly with the results of [6].

Let us show this explicitly for $k=2$. Consider

$$A_\mu = -i\partial_\mu U U^\dagger \otimes H, \tag{12}$$

where U is of rank two, as defined in Eq. (2), and H is any 2×2 Hermitian matrix. We can parametrize

$$H = \frac{1}{2}(1+\alpha)I_2 + \frac{1}{2}\beta_i\sigma_i. \tag{13}$$

The sphaleron is at $\alpha=\beta_i=0$, which has $H=\frac{1}{2}I_2$. Two vacua are given by $\alpha=\pm1$, $\beta_i=0$, so that $H=0,1$. There is another family of vacua, at $\alpha=0$, $|\beta|=1$, those are parametrized by an S^2, the direction of β_i. Those vacua give H with one eigenvalue equal to zero and the other equal to one.

Identifying the two vacua at the end of the interval $-1\leq\alpha\leq1$ gives the non-contractible S^1. The parameters β_i (with $|\beta|<1$) take values in the ball B^3. Identifying all the boundary points gives a non-contractible S^3.

The parameter α in Eq. (7) gives a one-dimensional family of configurations in $SU(2)$. In the previous paragraphs α and β_i gave a one and a three dimensional family of configurations in $SU(4)$. Those are actually related to the non-trivial π_3 of $SU(2)$ and to the non-trivial π_3 and π_5 of $SU(4)$. This is true in general. To see this we have to include the spatial manifold S^3.

The parameters α, β_i and the higher dimensional ones live in B^{2l-1}. At every point there is a static field configuration on S^3. So we have an S^3 for every point in B^{2l-1}. At the boundary of the ball the field configuration is the vacuum, which is trivial on the S^3, so we can take the sphere to shrink to a point. This fibration of S^3 over B^{2l-1} gives S^{2l+2}. Now recall the well known fact that if the gauge group has a non-trivial π_{2l+1} then there is a non-trivial gauge bundle over S^{2l+2} (the map from S^{2l+1} to the group is the transition function on the equator of S^{2l+2}).
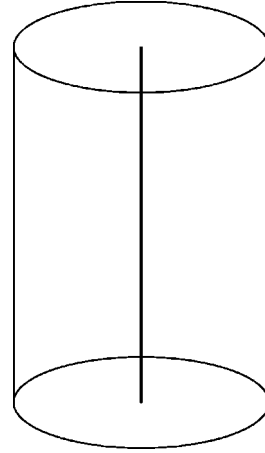
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[2]We described the sphaleron as the point in the algebra of $U(k)$ with $f=\frac{1}{2}I_k$, in the group that corresponds to $\exp(2\pi if)=-I_k$.

FIG. 2. An unstable D0-brane in the center of AdS$_5$ space. The vertical direction is time, and the radial direction is the radial coordinate of AdS space. The boundary of global AdS$_5$ space is $S^3\times\mathbb{R}$.

In the simplest case, adding the parameter α to S^3 allows us to build an S^4, on which there are configurations with the topology of the instanton.

B. Supergravity side—unstable D0-branes in $\mathrm{AdS}_5\times S^5$

The AdS-CFT duality is a strong or weak duality and as such it takes classical configuration of one description into a quantum excitation of the other description. Therefore, it is very hard to trace a generic (non-BPS) classical solution of weakly coupled SYM to the AdS description. A sphaleron is a non-supersymmetric solution sitting at the top of a non-contractible loop in the *classical* configuration space. Therefore, it is natural to suspect that the quantum corrections will blur the non-contractible loop, and, that by the time the 't Hooft coupling is large there will be no trace of the non-contractible loop and the sphaleron.

However, as we saw, the non-contractible loop associated with the sphaleron of the previous subsection is described by the topology of the instanton. The dual of the instanton is a D-instanton in AdS space, which carries a charge in K-theory. So we should look for a non-contractible loop with the topology of the D-instanton. Such non-contractible loops in flat space-time were constructed in [6]. There it was argued that the sphaleron at the top of the loop is the type IIB D0-brane. We claim, therefore, that the dual of the solution of the previous section are the unstable D0-branes located at the origin of AdS space. This is illustrated in Fig. 2.

Let us mention a few properties of the unstable D0-branes and how they fit into the claim that they are dual to the field theory sphalerons.

A D0-brane (or k coincident D0-branes) which are located at the center of AdS space are static objects with respect to the global time. Therefore they correspond to static objects in the gauge theory. The center of AdS space corresponds to the extreme infrared of the gauge theory, so the energy is uniformly distributed over S^3.

From the closed string theory point of view the low energy supergravity fields which are excited by the D0-branes

are the NS-NS graviton and dilaton. The RR-fields are not excited. That would correspond, using the dictionary of [15,16], to $\mathrm{Tr}F\widetilde{F}=0$ and to $\mathrm{Tr}F^2\neq0$, in agreement with the field theory results (8). Note that the mass of D0-branes (and $\mathrm{Tr}\,F^2$) do receive quantum corrections for they are not protected by supersymmetry,[3]

$$M_{D0}=\sqrt{\frac{2}{g_s\sqrt{\alpha'}}}=\frac{4\sqrt{2}\,\pi\lambda^{1/4}}{g_{YM}^2R}. \qquad (14)$$

In [6] it was shown that the type IIB D0-branes are sphalerons of string theory. That is, in flat space-time they sit at the top of a non-contractible loop in the configuration space of string theory. Since for large 't Hooft coupling the ''center'' of AdS space can be approximated by flat space-time, one can simply embed the construction of [6] in AdS space. There is also a global way to construct the D0-branes in AdS space. Starting with a system of D1-brane–anti-D1-brane stretching all the way to the boundary of AdS space, just like in flat space-time, this system contains a complex tachyon mode which can support an unstable D0-brane.

It was further argued in [6] that k coincident D0-branes, which have k^2 tachyonic modes correspond to sphalerons at the top of $S^1,S^3,\ldots,S^{2k-1}$ in $U(k)$. This is exactly what we found from the field theory side. It is worth while to note that in both descriptions the mass is proportional to k.

The NS sector of the excitations living on the D0-branes contains a real scalar tachyonic mode. According to Sen's conjecture at the bottom of the tachyon potential the negative energy cancels the tension of the brane and we are left with the vacuum. This was tested, to a good accuracy, via the level truncation method in string field theory [11,18,19]. On the field theory side, we see that indeed the bottom of the potential [$f=0$, 1 in Eq. (3)] is the vacuum. While calculating the tachyon potential in string theory is complicated, in the field theory it is just a quartic (3).

Since the tachyon is real, the potential can support a stable lower dimensional brane, D-instantons in our case. Again, the energy of such a configuration was calculated in string field theory with impressive agreement with expectations [18]. On the field theory side the instanton indeed interpolates between the two minima of the potential.

Of all the instanton solutions on $\mathbb{R}^4$, the one of radius R centered around the origin is special when translating to $S^3\times\mathbb{R}$. It goes over to a spherically symmetric solution on S^3. In that theory, this instanton can be described as a quantum mechanical tunneling process between the two minima of the quartic potential in Eq. (3). The gauge theory sphaleron sits at the middle of the potential. The width of the potential is R and the height, which is the mass of the sphaleron, is proportional to $1/g_{YM}^2R$. The action of the instanton is the area under the potential. In string theory the same is true, only that R is replaced by l_s. The height of the potential is

$\lambda^{1/4}/g_{YM}^2R=1/g_{YM}^2l_s$, and the width is of order l_s. Since the action of the D-instanton is the same as the gauge theory instanton, the area is the same, but the shape is altered.

We see therefore, that indeed the field theory sphaleron is dual to the unstable D0-branes in AdS space. It is important to emphasize [6] that the D0-branes are *not* sphalerons of the low energy supergravity. That is, there is no supergravity solution associated with the non-BPS D-branes which sits at the top of a non-contractible loop of field configurations of the classical supergravity. The unstable branes are sphalerons of the full string theory including all the quantum corrections to the sigma model. Since the full string theory on AdS space contains all the information about the dual super YM (SYM) theory it is not surprising that *in principle* the field theory sphalerons can be described by string theory on AdS space. What is remarkable is that the description is so simple.

A natural question that arises is whether the dual weakly coupled description sheds new light on the diagonal U(1) problem associated with the unstable D0-branes. Unfortunately, even though we can trace the D0-branes to the weakly coupled region, we cannot trace the gauge theory living on them to the weakly coupled description. Thus, as far as we can tell, the dual description does not lead to any new insight on the U(1) problem. It is worth mentioning that this problem of tracing the gauge theory living on the brane to the weakly coupled description is not special to D0-branes. For example, we know that the dual of a D1-brane stretching all the way to the boundary is the BPS monopole. But in weakly coupled field theory there are no fields living on the monopole, while there is a $1+1$ gauge theory living on D1-branes in AdS space. The reason is that the size of the D1-brane is larger than the string scale only for large 't Hooft coupling and so for small coupling the excitations which were supposed to live on the monopole cannot be separated from the other excitations.

It is interesting to note that when we have k D0-branes the full topology of the non-contractible loop, $U(k)$, with its non-contractible $S^1,S^3,\ldots,S^{2k-1}$, can be interpolated from the weakly to the strongly coupled region. The S^1 is ''protected'' by the instanton which is BPS. It should be interesting to understand why the other spheres are ''protected'' as well.

We would like to end this section with a comment on finite N. Our construction of the field theory solution which is dual to k coincident D0-branes is valid for $k\leq N/2$. Equation (1) implies that a dual solution should be found at up to $k=N$. Presumably, a more complicated ansatz will indeed yield the right solution. It should be interesting to see if the mass is still linear with k. Another question is what happens when $k>N$. In the field theory side we get out of the stable regime. Is there any stringy exclusion principle associated with that? Recall that the global construction of k D0-branes in AdS space involves k D1-branes and anti-D1-branes stretched all the way to the boundary (this is a simple generalization of the discussion in [6]). Now, when $k=N$ the D1-branes can end on a NS-brane which wraps S^5 [20,21]. So it seems that the existence of a baryon vertex in AdS

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[3]The origin of the $\sqrt{2}$ is the fact that the open strings living on an unstable brane carry two Chan-Paton factors I and σ_1 [17].

space is the underlying mechanism which bounds the number of coincident D0-branes in AdS space to N. Clearly, it would be helpful to understand this better.

III. MERONS IN GAUGE THEORIES AND IN AdS

In Sec. II A we studied the field configuration of "half pure gauge" on $S^3 \times \mathbb{R}$, and interpreted it as a sphaleron. As we mentioned, those same configurations can be considered in the Euclidean theory on $\mathbb{R}^4$, they are still classical solutions, but there is a singularity at the origin and at infinity. By smoothing out the singularities one gets a configuration that solves the equations of motion almost everywhere and has finite action. Those are the merons [12].

We give a brief review of the merons in gauge theories and then will find analogous configurations in string theory on AdS space.

A. Short review of merons

Let us write again the instanton ansatz (2)

$$A_\mu = -if(r)\partial_\mu UU^\dagger,$$

$$U = \frac{x^\mu \sigma_\mu}{r} = \frac{x_0 + ix_i\sigma_i}{r}, \quad r^2 = x_0^2 + x_i^2, \quad (15)$$

where $f = 0,1$ are vacuum solutions, and $f = \frac{1}{2}$, the meron, is an unstable solution which is singular at $r = 0, \infty$. The action (5) is logarithmically divergent

$$S = \frac{3\pi^2}{g_{YM}^2} \int_0^\infty \frac{dr}{r}. \quad (16)$$

To regularize this divergence consider the following configuration:

$$f(r) = \begin{cases} \dfrac{r^2}{r^2 + R_1^2}, & r < R_1, \\[2mm] \dfrac{1}{2}, & R_1 < r < R_2, \\[2mm] \dfrac{r^2}{r^2 + R_2^2}, & R_2 < r. \end{cases} \quad (17)$$

This is the meron for $R_1 < r < R_2$, glued to half an instanton at the origin and half at infinity. This carries the same topological charge as the instanton, but it is broken in two parts. If one takes $R_1 = R_2$, the instanton solution is recovered. For $R_1 \neq R_2$ this is a solution of the equations of motion everywhere but at the spheres which separate the three regions.

This is illustrated in Fig. 3(a). Region I and III are the half instantons near the origin and infinity. Region II is the meron which connects the two. The action can be easily calculated, and is equal to

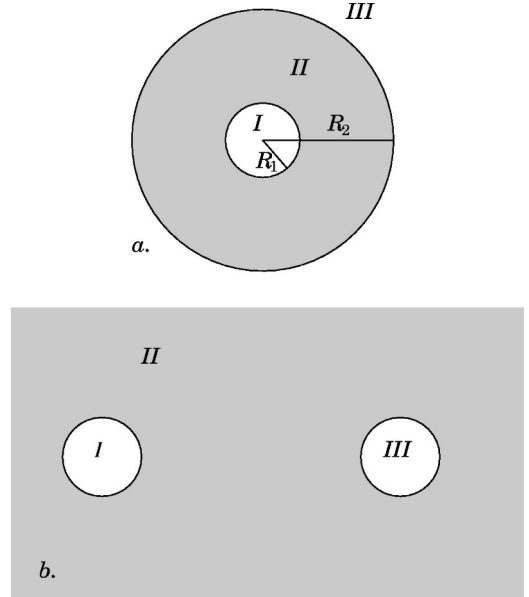$$S = \frac{8\pi^2}{g_{YM}^2} + \frac{3\pi^2}{g_{YM}^2} \ln \frac{R_2}{R_1}. \quad (18)$$

FIG. 3. (a) The meron configuration. Region I is half an instanton, region II is the meron with exactly half a pure gauge transformation, and region III is another half instanton. By a large conformal transformation that takes the point at infinity to finite distance and region III to a finite sphere this can be mapped to the two meron configuration (b).

Since classical YM theory is conformally invariant, we can use a large gauge transformation to map region III to a sphere at finite distance. The new configuration is shown in Fig. 3(b). Regions I and III each carry half the topological charge of the instanton, so at infinity this configuration is pure gauge.

One can, of course, replace the meron with an anti-meron, where instead of half an instanton there is half an anti-instanton. The meron anti-meron pair will have zero topological charge and two anti-merons -1 topological charge. The interaction between a meron and anti-meron is the same as that between two merons.

The action of a meron grows with the distance. Thus a first guess is that the contribution of merons to the partition function is negligible. However, the action grows only logarithmically so it can be compensated by a large entropic factor.[4] The entropy contribution to the partition function goes like L^4, hence the partition function associated with a meron is

$$Z \sim L^4 \exp\left(-\frac{1}{g_{YM}^2} \ln L\right) = L^{(4 - 1/g_{YM}^2)}. \quad (19)$$

This suggests a phase transition at $g_{YM}^2 \geq \frac{1}{4}$, wherein the meron charges that made up the instanton dipole are liber-

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[4]In thermodynamics this is, of course, common. At finite temperature one has to minimize the free energy, $F = E - ST$ rather than the energy. Thus a phase transition between minimizing E and maximizing the entropy can take place. Here the coupling constant plays the role of the temperature.

ated. In the non-supersymmetric theories it was suggested that the appearance of this new phase at large coupling, or large scale size, is closely related to confinement, where the merons play the role of the three dimensional instantons in Polyakov's mechanism for confinement [22]. However, the full story is much more complicated for one has to consider a gas of merons and their interactions. This, as well as the fact that the coupling runs, made it very hard to estimate the relevance of merons to confinement.

Even though the coupling does not run for $\mathcal{N}=4$, the main problem of understanding the interactions among the merons is still very complicated. In fact, in the $\mathcal{N}=4$ theory, because of the fermions and scalars and the fact that a meron breaks all supersymmetry, it is probably even more complicated. We however cannot resist the temptation of speculating that meron physics might be a clue for understanding $\mathcal{N}=4$ theory at the self-dual point ($g_{YM}^2=4\pi$).

B. Merons in AdS space

We would now like to describe merons in the strong coupling limit of the field theory, using string theory on AdS space. We saw in Sec. II that the sphaleron solution of the gauge theory on $S^3\times\mathbb{R}$ is described in the dual theory by an unstable D0-brane. Since the meron is the same field configuration as the sphaleron, only on $\mathbb{R}^4$, it is also described by a D0-brane in Euclidean AdS space. Here we use the metric

$$\frac{ds^2}{\alpha'}=\frac{\sqrt{\lambda}}{U^2}dU^2+\frac{U^2}{\sqrt{\lambda}}dx^2. \qquad (20)$$

Consider a D0-brane which is created at some point U_1, propagates till U_2 (and the same point in $\mathbb{R}^4$) and annihilates. This is the AdS dual of the configuration (17) which was illustrated in Fig. 3(a). By the UV-IR relation, for $U_1>U_2$, the internal circle has a radius $R_1=\sqrt{\lambda}/U_1$ and the external circle $R_2=\sqrt{\lambda}/U_2$.

The action of this configuration is

$$S=S_{D(-1)}+T_{D0}\int ds=\frac{2\pi}{g_s}+\frac{\sqrt{2}\lambda^{1/4}}{g_s}\ln\frac{U_1}{U_2}, \qquad (21)$$

where the first term $2\pi/g_s=8\pi^2/g_{YM}^2$ is equal to the instanton action and is related to the creation of the brane and its annihilation, like in the gauge theory. This contribution will be justified in the next section. Comparing this to the gauge theory result (18), the constant part of the action is unchanged, but the coefficient of the log is renormalized by a factor proportional to $\lambda^{1/4}$, like the sphaleron mass (14). Again, one should not be surprised, since this is a non-BPS configuration.

Just as was explained in the previous section a conformal transformation will take this geodesic into a D0-brane which is created and annihilated at the same value of U, but at a distance L on $\mathbb{R}^4$, this is the AdS dual of the configuration in Fig. 3(b). The size of the two half instantons is simply $R=\sqrt{\lambda}/U$. Those two configurations are shown in Fig. 4. It is not surprising, therefore, that the corresponding action is
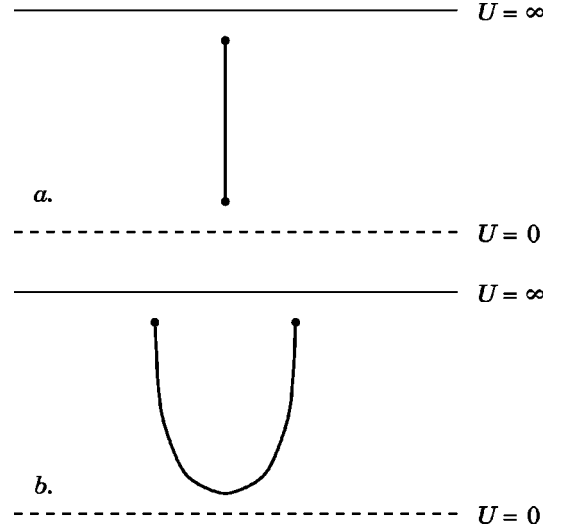

FIG. 4. Two examples of unstable D0-branes created and annihilated in Euclidean AdS space. The boundary of AdS spacetime is marked by the solid line at $U=\infty$. Between the creation and annihilation point the particle travels along a geodesic.

$$S=\frac{8\pi^2}{g_{YM}^2}+\frac{4\pi\sqrt{2}\lambda^{1/4}}{g_{YM}^2}\ln(L/R). \qquad (22)$$

The fact that the logarithmic term is now proportional to $\lambda^{1/4}/g_{YM}^2$, rather than just $1/g_{YM}^2$ as in the weakly coupled theory seems to imply that the entropy contribution cannot compete with the energy in strong coupling. That is,

$$Z\sim L^4\exp\left(-\frac{\lambda^{1/4}}{g_s}\ln(L)\right). \qquad (23)$$

So a phase transition at $g_s\sim1$ is very unlikely for large λ.

IV. UNSTABLE BRANES AS D-MERONS

In the previous section we studied D0-branes in Euclidean AdS space. Since they are unstable they can appear out of the vacuum, propagate some distance and disappear again. This was dual to the meron in the gauge theory which connects two regions where there are half instantons. Since the AdS dual of the instanton is the D-instanton, it is natural to suspect that at each end of the D0-brane sits half a D-instanton.

We reached that conclusion by studying D0-branes in AdS space, but this is true in any string theory background, and the argument does not have to rely on the AdS-CFT correspondence. After all, the D0-brane is a sphaleron at the top of a non-contractible loop with the same topology of the D-instanton. Therefore the entire event of a D0-brane creation, propagation and annihilation can carry a unit of D-instanton charge. In fact, it can carry 1, 0, or -1 units of D-instanton charge.

The creation or annihilation of a D0-brane is an event that carries half (or minus a half) of D-instanton charge. This

FIG. 5. An unstable D2 brane of type IIB can end on two half D1-branes.

might seem to contradict the charge quantization condition. The product of the charge of a single D7-brane and the charge of a single D-instanton is 2π, so how can a D-instanton break in two? The answer is that the two halves of the D-instanton are connected by a D0-brane, which must carry half a unit of D-instanton flux.

This is analogous to a bar magnet, or a solenoid in electro-magnetism. Outside the magnet the magnetic field looks like that of two separated, oppositely charged, monopoles. But the monopole charge need not satisfy the Dirac quantization condition, as the magnet (or solenoid), carries the flux from one to the other.

It is amusing to push this analogy further. Just as the magnetic field in a magnet is created by the angular momentum of the electric charges, the D0-brane can be regarded as a very thin solenoid in which a current of D7-brane charge produces a dual flux, connecting the one-half D(-1) charges. It would be interesting to pursue this analogy even further.

Since the unstable D0-branes connect pairs of 1/2 D-instantons, they could be called D-merons.

Thus far we considered only D0-branes, but the same is true for higher dimensional branes as well. A D1-brane can break into two halves with an unstable D2-brane in the middle. That is the same as saying that the boundary of a Euclidean D2-brane could carry half-D1-brane charge. Likewise in type IIA, a D0-brane can break in two with an unstable D1-brane in the middle, and so on. A D2-brane ending on two half D1-branes is shown in Fig. 5.

In AdS space the action of the D0-brane is logarithmic, however in flat space it will be linear. Therefore half D-instantons are clearly confined in flat space. The same is true for the higher dimensional half-branes.

V. UNSTABLE STRINGS IN THE COULOMB PHASE

In previous sections we discussed how the existence of the instanton implies that there is a point like sphaleron solution. By the same logic, the 't Hooft–Polyakov monopole implies the existence of a string-like sphaleron solution in gauge theories in the Coulomb phase. We discuss the field theory construction of the string and its supergravity dual.

A. Field theory description

We first study the unstable string in the $SU(2)$ gauge theory broken to $U(1)$ by an adjoint Higgs. The details of the construction, the relevant non-contractible loop in con-
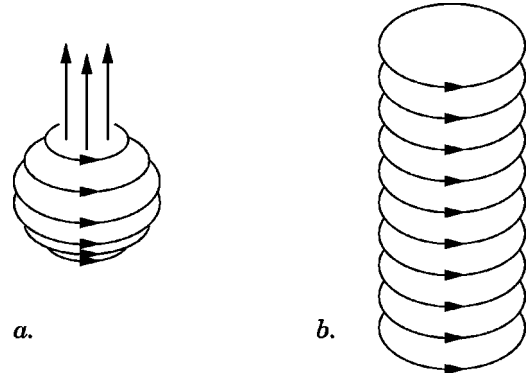

FIG. 6. (a) The 't Hooft–Polyakov monopole. (b) The sphaleron string is very similar to cutting the monopole in the middle and smearing it in the x_3 direction. The width of the string is of order $1/W$, where there is a non-trivial $SU(2)$ flux.

figuration space and the unstable string sitting at the top of the loop can be found in [23,24]. Those papers considered the theory in three dimensions, where the monopole is an instanton and the sphaleron is a particle. We are interested in uplifting this to four dimensions. We shall not review the explicit construction but rather deduce the relevant properties from general arguments.

The monopole solution [25] yields a radial $U(1)$ magnetic field,[5]

$$F_{ij} = -\frac{1}{er^3}\epsilon_{ijk}x_k,. \qquad (24)$$

To construct the non-contractible loop associated with this solution we have to consider configurations which are invariant under translation in one direction, say x_3. Then we replace the coordinate with a parameter in configuration space $x_3 \rightarrow \tan\alpha$. This is pictured in Fig. 6. Note that to get configurations which are independent of the x_3 coordinate one has to perform an α dependent gauge transformation. This does not change the topology of the loop, but it does change the action. Therefore one cannot simply replace x_3 with $\tan\alpha$ in the solution.

After the gauge transformation, the sphaleron string is given by

$$A_a = f(x)\epsilon_{ab}x_b\sigma_3, \quad \Phi = g(x)x_a\sigma_a, \qquad (25)$$

with $a,b = 1,2$. For more details see [23,24].

For $\alpha = 0$ we see [from Fig. 6, Eq. (24) or [23,24]] that there is a solution localized in the x^1, x^2 plane with no magnetic flux in the plane. Thus we have an unstable string solution (stretched along the x_3 direction). The string does not carry gauge invariant $U(1)$ flux, but it does carry $SU(2)$ magnetic flux in the x^3 direction. Dimensional analysis implies that the tension of such a string is

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[5]We remind the reader that the $U(1)$ components of the $SU(2)$ are defined with respect to the Higgs field, $F_{\mu\nu} = F_{\mu\nu}^a W^a$.

$$T \sim \frac{W^2}{g_{YM}^2}, \qquad (26)$$

where W is the Higgs expectation value. For $\alpha \neq 0$ there is a $U(1)$ magnetic field and the full non-contractible loop $-\pi/2 \leqslant \alpha \leqslant \pi/2$ describes a transition which changes the total magnetic flux of the vacuum by one unit. Note that in the Coulomb phase this does not cost any energy as the flux expands to infinity and we are still in the vacuum.

Put differently, as one starts from the vacuum, $\alpha = -\pi/2$ and goes around the non-contractible loop through the sphaleron, $\alpha = 0$ back to the vacuum $\alpha = \pi/2$, one unit of magnetic flux is added in the x_3 direction. Thus the non-contractible loop goes between vacua with different Chern numbers.

B. Supergravity description

The AdS-CFT correspondence is not useful to describe $SU(2)$ broken to $U(1)$. Instead, we describe $SU(2N)$ gauge symmetry broken to $[U(N) \times U(N)]/U(1)$ by the Higgs mechanism. The relevant supergravity background is [1]

$$\frac{ds^2}{\alpha'} = \frac{1}{\sqrt{4\pi gN\left(\dfrac{1}{\vec{U}^4} + \dfrac{1}{|\vec{U}-\vec{W}|^4}\right)}} dx_{\parallel}^2$$
$$+ \sqrt{4\pi gN\left(\frac{1}{\vec{U}^4} + \frac{1}{|\vec{U}-\vec{W}|^4}\right)} d\vec{U}^2, \qquad (27)$$

where $\vec{W}$ is the vector that represents the Higgs expectation value.

Since the dual of the monopole is a D1-brane in the U direction and since the sphaleron associated with the D1-brane charge is the unstable D2-brane [6] it is natural to suspect that the dual of the unstable string is a D2-brane along the x_0, x_3 and U directions. However, unlike in $\mathbb{R}^{10}$, where the boundary conditions are set at infinity, there is nothing holding the D2-brane to the horizon. One can easily see that such a D2-brane will not solve the equations of motion with free boundary conditions. Therefore, the unstable D2-brane cannot be the dual of the unstable gauge theory string.

To resolve this puzzle we should find another object. From the discussion in Sec. IV, the D2-brane can carry half a unit of D1-brane charge at each end. Another configuration with the same charge is a D1-brane (in the x_0, x_3 directions). To preserve the symmetry between $\vec{U}=0$ and $\vec{U}=\vec{W}$, the D1-brane should sit precisely at the center $\vec{U}=\vec{W}/2$. This is shown in Fig. 7.[6]

Indeed, suppose that we place a D1-brane along the x_3 direction at some value of $\vec{U}$ (we could compactify the x^3

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[6]Other strings in this geometry were considered in [26].

FIG. 7. A D1-brane (solid line) right in between two AdS-like regions in the double-centered AdS geometry.

direction to get a finite mass object). The field theory tension of such a string is calculated with respect to the field theory coordinates and is therefore

$$T_{D1} = \frac{\sqrt{g_{00}g_{11}}}{2\pi\alpha' g_s}. \qquad (28)$$

From Eq. (27) we see that the tension vanishes on the branes ($\vec{U}=0$ and $\vec{U}=\vec{W}$) and that the string would like to fall towards one of the branes.

There is one exception, the string located precisely in the middle $\vec{U}=\vec{W}/2$. It solves the classical equation of motion, however it is unstable. Any perturbation along such a string will eventually lead to either $\vec{U}=0$ or $\vec{U}=\vec{W}$. This is the instability of the string in the AdS description.

The tension of such a D1-brane is

$$T \sim \frac{W^2}{g_{YM}^2\sqrt{\lambda}}. \qquad (29)$$

Again, we see that because this is not a BPS configuration, the tension is not protected as one interpolates from the weakly coupled region (26).

Such a D1 string carries magnetic flux in the diagonal $U(1)$ (which decouples from the bulk degrees of freedom), but not in the relative $U(1)$. If it falls towards one of the collections of branes, a flux is turned on in the relative gauge group. We see that if we start with a string at $\vec{U}=0$ and move it to $\vec{U}=\vec{W}$ we go back to the vacuum, but we changed the flux in the relative gauge group by one. This is the topological structure of the non-contractible loop and the configuration at the middle is a sphaleron.

One can, of course, consider the configuration with a fundamental string along the x_0, x_3 direction. Such a string carries an electric flux and has the same instability. However, on the field theory side there is no dual electric unstable string. This is an example of the case where a sphaleron of the strongly coupled theory does not have a weakly coupled analog. The reason is that the BPS configuration which is supposed to guarantee its existence is the W-boson. But unlike the monopole, the W-boson is an elementary excitation in the weakly coupled theory, and not a classical solution,
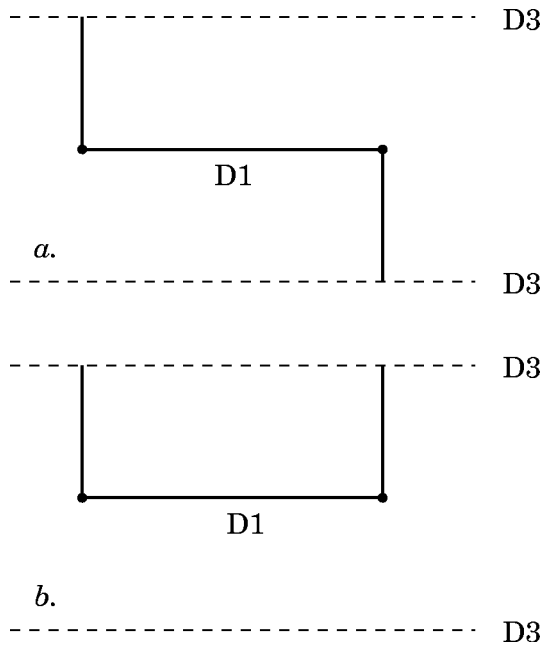
FIG. 8. A D-string stretched between one horizon and the center point is half a monopole. This configuration cannot exist on its own, but the unstable string can connect two half-monopoles (a) or a half monopole and a half anti-monopole (b).

and there is no related topological charge. This is related to the fact that the fundamental string does not carry a charge in K-theory.

C. 1/2 monopole configuration

In Secs. III and IV we showed that the unstable D0-brane is a meron connecting two half D-instantons. In this subsection we generalize the construction of merons to the 't Hooft–Polyakov monopole. Consider a D1-brane in the double center of the AdS solution (27) which follows one of the trajectories indicated in Fig. 8. Such a brane will solve the equations of motion everywhere along the trajectory except for the two turning points. From the field theory side this corresponds to a monopole broken into two half monopoles.
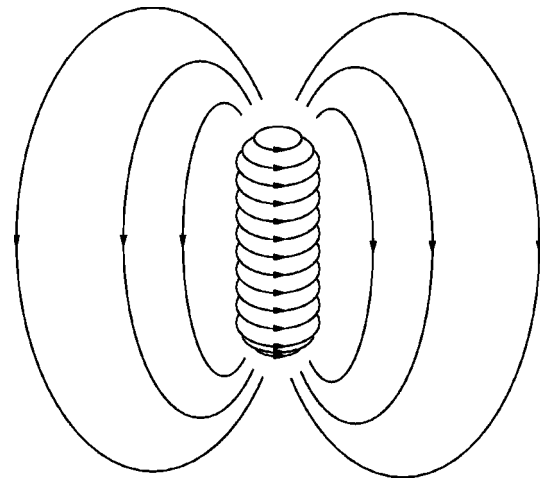

FIG. 9. The unstable string can end on half a monopole. Here we draw the string with a half monopole at one end and half an anti-monopole at the other. From far away it looks like a $U(1)$ dipole, but near the core, at distances of order $1/W$, a non-Abelian flux is carried by the string.

Notice that, unlike the meron case, the energy of the configuration is linear with the distance between the 1/2 monopoles and hence it will not contribute to the partition function for any value of the coupling constant.

Half a monopole seems to contradict the Dirac quantization condition. Again there is a magnet connecting the two half monopoles. One might wonder how this works, since we argued that this string does not carry any $U(1)$ flux. The resolution of the puzzle is simple. Recall that the thickness of the string is $\sim 1/W$, so the string is in the region of unbroken gauge symmetry. The flux is carried, therefore, in $SU(2)$. See Fig. 9.

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