

Noncommutative monopole at the second order in θ

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We study the noncommutative U(2) monopole solution to second order in the noncommutativity parameter θ^{ij} . We solve the BPS equation in noncommutative super Yang-Mills theory to $\mathcal{O}(\theta^2)$, transform the solution to the commutative description by the Seiberg-Witten (SW) map, and evaluate the eigenvalues of the scalar field. We find that, by tuning the free parameters in the SW map, we can make the scalar eigenvalues precisely reproduce the configuration of a tilted D-string suspended between two parallel D3-branes. This gives an example of how the ambiguities inevitable in the higher order SW map are fixed by physical requirements.

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I. INTRODUCTION

An explicit relation between the noncommutative fields and the commutative ones has been presented in Ref. [1], called “the Seiberg-Witten (SW) map.” The noncommutative Dirac-Born-Infeld (DBI) theory and the ordinary one appear as low-energy effective theories of D-branes in a constant Neveu-Schwarz–Neveu-Schwarz (NSNS) B field. They differ by the choice of regularization for the worldsheet theory: the Pauli-Villars regularization for the commutative description and the point-splitting regularization for the noncommutative one. This means that these two descriptions are connected by some field redefinition and this is the SW map.

The relation between the commutative and noncommutative DBI theories has been examined in various aspects. Now in particular, let us concentrate on the Bogomol’nyi-Prasad-Sommerfield (BPS) solutions and compare them in both the descriptions. The reason is that the BPS solutions are considered as powerful tools beyond the perturbative understanding. Noncommutative BPS monopoles describe, by the brane interpretation of [2], the configurations of tilted D-strings ending on parallel D3-branes in a constant NSNS B -field [3] and have been investigated in various papers [4–8]. In Ref. [4], noncommutative U(2) monopole was considered at the first order in the noncommutativity parameter θ^{ij} . The analysis using the noncommutative eigenvalue equation for the scalar field successfully reproduced the tilted D-string picture. In Ref. [6], the similar analysis was carried out for the string junction and the anticipated result was obtained. Study of the noncommutative monopoles using the SW map was carried out in Refs. [7,8] (see also Ref. [9]). There, the noncommutative BPS solutions were transformed into the commutative description via the SW map, and then the brane interpretation was done for the eigenvalues of the mapped scalar field to give the expected tilted D-string picture.

The purpose of this paper is to extend the analysis of the noncommutative U(2) monopole using the SW map to second order in θ . The motivating fact is as follows: the SW map possesses some ambiguities in higher orders in θ [10].

This map is derived from the requirement of the gauge equivalence of the two descriptions. Since this is a very weak requirement, arbitrary parameters appear in the map. There are two types of ambiguities in it. One is in the form of the gauge transformation and has no physical effect. However, the other type of ambiguity consists of gauge covariant quantities and can cause physical differences.

We apply the SW map to the noncommutative monopole solution at the second order in θ and examine the effects of the ambiguities. Concretely, we compare the eigenvalues of the scalar field obtained by the SW map with that in the commutative Yang-Mills theory in a background magnetic field. Note that the ambiguities in the SW map can change the scalar eigenvalues (which are gauge invariant quantities) and hence change their brane interpretation. It is found that we can make these two eigenvalues coincide with each other by tuning the free parameters in the SW map. This gives an example of how the ambiguities in the SW map are fixed in concrete physical situations.

The rest of this paper is organized as follows. In Sec. II, we solve the noncommutative version of the BPS equation to second order in θ . In Sec. III, we apply the SW map to the solution and evaluate the eigenvalues of the scalar field in the commutative description. In Sec. IV, we compare the scalar eigenvalues of Sec. III with those in the commutative Yang-Mills theory in a constant magnetic field, and examine the effect of the ambiguities in the SW map. In Sec. IV, we summarize the paper and give some discussions. The SW map to second order in the change of θ is presented in the Appendix.

II. NONCOMMUTATIVE BPS MONOPOLE SOLUTION AT θ^2

We shall consider the $\mathcal{N}=4$ U(2) noncommutative super Yang-Mills theory in 1+3 dimensions with the metric $G_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, and construct the BPS monopole solution to second order in the noncommutativity parameter. The BPS equation reads

$$\hat{D}_i \hat{\Phi} + \frac{1}{2} \epsilon_{ijk} \hat{F}_{jk} = 0, \quad (2.1)$$

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where the quantities with a hat denote those in the noncommutative description. In particular, we have

$$\hat{F}_{ij} \equiv \partial_i \hat{A}_j - \partial_j \hat{A}_i - i \hat{A}_i^* \hat{A}_j + i \hat{A}_j^* \hat{A}_i, \quad (2.2)$$

$$\hat{D}_i \hat{\Phi} \equiv \partial_i \hat{\Phi} - i \hat{A}_i^* \hat{\Phi} + i \hat{\Phi}^* \hat{A}_i, \quad (2.3)$$

where the $*$ product is defined by

$$\begin{aligned} (f^*g)(x) &\equiv f(x) \exp\left(\frac{i}{2} \theta^{ij} \vec{\partial}_i \vec{\partial}_j\right) g(x) \\ &= f(x)g(x) + \frac{i}{2} \theta^{ij} \partial_i f(x) \partial_j g(x) \\ &\quad - \frac{1}{8} \theta^{ij} \theta^{kl} \partial_i \partial_k f(x) \partial_j \partial_l g(x) + \mathcal{O}(\theta^3). \end{aligned} \quad (2.4)$$

In order to solve the BPS equation (2.1), we expand the fields in powers of θ :

$$\begin{aligned} \hat{\Phi} &\equiv (\hat{\Phi}^{a(0)} + \hat{\Phi}^{a(1)} + \hat{\Phi}^{a(2)}) \frac{1}{2} \sigma_a + (\hat{\Phi}^{0(1)} + \hat{\Phi}^{0(2)}) \frac{1}{2} \mathbb{1}, \\ \hat{A}_i &\equiv (\hat{A}_i^{a(0)} + \hat{A}_i^{a(1)} + \hat{A}_i^{a(2)}) \frac{1}{2} \sigma_a + (\hat{A}_i^{0(1)} + \hat{A}_i^{0(2)}) \frac{1}{2} \mathbb{1}, \end{aligned} \quad (2.5)$$

where the superscript (n) denotes the order of θ . As the solution at $\mathcal{O}(\theta^0)$, we adopt the BPS monopole [11,12] with vanishing U(1) components $\hat{\Phi}^{0(0)} = \hat{A}_i^{0(0)} = 0$:

$$\hat{\Phi}^{a(0)} = \frac{\hat{x}_a}{r} H(\xi), \quad \hat{A}_i^{a(0)} = \epsilon_{aij} \frac{\hat{x}_j}{r} (1 - K(\xi)), \quad (2.6)$$

with $\hat{x}_i \equiv x_i/r$ and $\xi \equiv Cr$ (C is the parameter characterizing the mass of the monopole). The functions H and K are defined by

$$H(\xi) = \frac{\xi}{\tanh \xi} - 1, \quad K(\xi) = \frac{\xi}{\sinh \xi}. \quad (2.7)$$

These two functions behave asymptotically as

$$H(\xi) = \xi - 1 + \mathcal{O}(e^{-\xi}), \quad K(\xi) = 0 + \mathcal{O}(e^{-\xi}). \quad (2.8)$$

Note that the solution (2.6) has an invariance under the rotation by the diagonal subgroup $\text{SO}(3)$ of $\text{SO}(3)_{\text{space}} \otimes \text{SO}(3)_{\text{gauge}}$.

The solution at $\mathcal{O}(\theta)$ was constructed in Refs. [4,5] and it is given by

$$\begin{aligned} \hat{A}_i^{a(1)} &= 0, \quad \hat{A}_i^{0(1)} = \theta^{ij} \hat{x}_j \frac{1}{4r^3} (1 - K)(1 - K + 2H), \\ \hat{\Phi}^{a(1)} &= 0, \quad \hat{\Phi}^{0(1)} = 0. \end{aligned} \quad (2.9)$$

This solution is invariant under the generalized rotation, namely, the simultaneous rotation of the diagonal $\text{SO}(3)$ and the indices of the noncommutativity parameter θ_{ij} . Note that the noncommutativity has no influence on the scalar solution at $\mathcal{O}(\theta)$.

Now let us consider the components at the second order in θ in the expansion (2.5). The $\mathcal{O}(\theta^2)$ part of the BPS equation reads

$$\partial_i \hat{\Phi}^{0(2)} + \epsilon_{ijk} \partial_j \hat{A}_k^{0(2)} = 0, \quad (2.10)$$

$$\begin{aligned} \partial_i \hat{\Phi}^{a(2)} + \epsilon_{ijk} \partial_j \hat{A}_k^{a(2)} + \epsilon_{abc} (\hat{A}_i^{b(0)} \hat{\Phi}^{c(2)} \\ - \hat{\Phi}^{b(0)} \hat{A}_i^{c(2)} + \epsilon_{ijk} \hat{A}_j^{b(0)} \hat{A}_k^{c(2)}) \\ = \frac{1}{2} \theta^{kl} \partial_k \hat{\Phi}^{a(0)} \partial_l \hat{A}_i^{0(1)} - \frac{1}{2} \epsilon_{ijk} \theta^{lm} \partial_l \hat{A}_j^{a(0)} \partial_m \hat{A}_k^{0(1)} \\ + \frac{1}{8} \epsilon_{abc} \theta^{lm} \theta^{pq} \left(\partial_l \partial_p \hat{A}_i^{b(0)} \partial_m \partial_q \hat{\Phi}^{c(0)} \right. \\ \left. + \frac{1}{2} \epsilon_{ijk} \partial_l \partial_p \hat{A}_j^{b(0)} \partial_m \partial_q \hat{A}_k^{c(0)} \right), \end{aligned} \quad (2.11)$$

where the first equation (2.10) is the U(1) part of Eq. (2.1), while the second equation (2.11) is the SU(2) part. The U(1) part has no regular solutions and we shall concentrate on the SU(2) part (2.11).

In order to solve Eq. (2.11), we adopt the generalized rotational invariance used in the construction of the $\mathcal{O}(\theta)$ part (2.9), and expand $\hat{\Phi}^{a(2)}$ and $\hat{A}_i^{a(2)}$ as

$$\begin{aligned} \hat{\Phi}^{a(2)} &= \frac{1}{r^5} [\phi_1(\xi) (\theta \hat{x}) \theta_a + \phi_2(\xi) \theta^2 \hat{x}_a + \phi_3(\xi) (\theta \hat{x})^2 \hat{x}_a], \\ \hat{A}_i^{a(2)} &= \frac{1}{r^5} [a_1(\xi) \theta^2 \epsilon_{aij} \hat{x}_j + a_2(\xi) (\theta \hat{x}) \epsilon_{aij} \theta_j \\ &\quad + a_3(\xi) \epsilon_{ajk} \theta_i \theta_j \hat{x}_k + a_4(\xi) (\theta \hat{x})^2 \epsilon_{aij} \hat{x}_j \\ &\quad + a_5(\xi) (\theta \hat{x}) \epsilon_{ajk} \hat{x}_i \theta_j \hat{x}_k], \end{aligned} \quad (2.12)$$

where we have used $\theta_i \equiv (1/2) \epsilon_{ijk} \theta^{jk}$, $\theta^2 \equiv \theta_i \theta_i$ and $(\theta \hat{x}) \equiv \theta_i \hat{x}_i$. One can check that this is the most general expression satisfying the generalized rotational invariance, by using the identities

$$\epsilon_{aij} x^2 = (\epsilon_{kij} x_a + \epsilon_{akj} x_i + \epsilon_{aik} x_j) x_k, \quad (2.13)$$

and the same one with x_i replaced by θ_i . Putting the expansion (2.12) into Eq. (2.11), we obtain the following linear differential equations with inhomogeneous terms for the unknown functions $\phi_k(\xi)$ and $a_k(\xi)$:

$$\begin{aligned}
\mathcal{D}(-a_1 - a_3) + \phi_2 + 4a_1 - a_2 + 5a_3 + a_5 - (1 - K)\phi_2 - Ha_1 &= I_1, \\
\mathcal{D}(\phi_2 + a_1 + a_3) - 6\phi_2 - 6a_1 - 6a_3 - a_5 + (1 - K)(\phi_2 + 2a_1 + a_3) + Ha_1 &= I_2, \\
\mathcal{D}a_3 + \phi_1 + a_2 - 5a_3 - a_5 - Ha_3 &= I_3, \\
\mathcal{D}(a_2 - a_3) + 2\phi_3 - 6a_2 + 6a_3 + 2a_5 + (1 - K)(\phi_1 + a_2 - a_3 - a_5) + Ha_3 &= I_4, \\
\mathcal{D}(\phi_1 - a_3) - 6\phi_1 + 6a_3 + 2a_4 + a_5 + (1 - K)a_2 + H(a_2 - a_5) &= I_5, \\
\mathcal{D}(-a_2 + a_3 - a_4) + \phi_3 + 6a_2 - 6a_3 + 4a_4 - 2a_5 - (1 - K)(\phi_1 + \phi_3) - H(a_2 + a_4) &= I_6, \\
\mathcal{D}(\phi_3 + a_4) - 8\phi_3 - 8a_4 + (1 - K)(\phi_3 + 2a_4 + a_5) + H(a_4 + a_5) &= I_7,
\end{aligned} \tag{2.14}$$

where the differential operator $\mathcal{D} \equiv \xi(d/d\xi)$ has been introduced. The seven equations (2.14) correspond to seven independent structures of Eq. (2.11): $\theta^2 \delta_{ai}$, $\theta^2 \hat{x}_a \hat{x}_i$, $\theta_a \theta_i$, $(\theta \hat{x}) \hat{x}_a \theta_i$, $(\theta \hat{x}) \theta_a \hat{x}_i$, $(\theta \hat{x})^2 \delta_{ai}$, and $(\theta \hat{x})^2 \hat{x}_a \hat{x}_i$, respectively. The functions I_k ($k=1, \dots, 7$) are the inhomogeneous terms which are polynomials of H and K :

$$\begin{aligned}
I_1(\xi) &= -\frac{13}{8} + \frac{3H}{8} - \frac{H^2}{2} + \frac{9K}{4} + \frac{5HK}{4} + \frac{3H^2K}{8} \\
&+ \frac{H^3K}{8} + \frac{K^2}{8} - \frac{5HK^2}{8} - \frac{H^2K^2}{4} - \frac{5K^3}{8} \\
&- HK^3 - \frac{3H^2K^3}{8} - \frac{K^5}{8},
\end{aligned}$$

$$\begin{aligned}
I_2(\xi) &= \frac{3}{8} + \frac{7H}{8} + \frac{H^2}{2} - \frac{9K}{8} - \frac{3HK}{2} - \frac{3H^2K}{8} - \frac{H^3K}{8} \\
&+ K^2 + \frac{HK^2}{8} - \frac{H^2K^2}{2} - \frac{H^3K^2}{4} + \frac{3HK^3}{4} \\
&+ \frac{3H^2K^3}{8} - \frac{3K^4}{8} - \frac{HK^4}{4} + \frac{K^5}{8},
\end{aligned}$$

$$\begin{aligned}
I_3(\xi) &= \frac{5}{8} + \frac{9H}{8} + \frac{H^2}{2} - \frac{17K}{8} - 3HK - \frac{5H^2K}{4} \\
&- \frac{H^3K}{4} + \frac{21K^2}{8} + \frac{21HK^2}{8} + \frac{3H^2K^2}{4} - \frac{11K^3}{8} \\
&- \frac{3HK^3}{4} + \frac{K^4}{4},
\end{aligned}$$

$$\begin{aligned}
I_4(\xi) &= -\frac{11}{8} - \frac{17H}{8} - \frac{H^2}{2} + \frac{17K}{4} \\
&+ 5HK + \frac{3H^2K}{2} + \frac{H^3K}{4} - \frac{9K^2}{2} - \frac{29HK^2}{8} \\
&- H^2K^2 + \frac{7K^3}{4} + \frac{3HK^3}{4} - \frac{K^4}{8},
\end{aligned}$$

$$\begin{aligned}
I_5(\xi) &= -1 - \frac{3H}{2} - \frac{3H^2}{4} + \frac{13K}{4} + \frac{31HK}{8} + \frac{3H^2K}{2} \\
&+ \frac{H^3K}{4} - \frac{15K^2}{4} - \frac{13HK^2}{4} - \frac{3H^2K^2}{4} + \frac{7K^3}{4} \\
&+ \frac{7HK^3}{8} - \frac{K^4}{4},
\end{aligned}$$

$$\begin{aligned}
I_6(\xi) &= 2 - H + \frac{3H^2}{4} - \frac{19K}{8} - \frac{3HK}{4} - \frac{3H^2K}{8} - \frac{H^3K}{8} \\
&- K^2 + \frac{HK^2}{2} + \frac{H^2K^2}{4} + \frac{5K^3}{4} + \frac{5HK^3}{4} \\
&+ \frac{3H^2K^3}{8} + \frac{K^5}{8},
\end{aligned}$$

$$\begin{aligned}
I_7(\xi) &= \frac{7}{4} + \frac{7H}{4} - \frac{43K}{8} - \frac{39HK}{8} - \frac{11H^2K}{8} - \frac{H^3K}{8} \\
&+ \frac{23K^2}{4} + \frac{19HK^2}{4} + \frac{7H^2K^2}{4} + \frac{H^3K^2}{4} - \frac{5K^3}{2} \\
&- \frac{15HK^3}{8} - \frac{3H^2K^3}{8} + \frac{K^4}{2} + \frac{HK^4}{4} - \frac{K^5}{8}. \tag{2.15}
\end{aligned}$$

We can solve the differential equations (2.14) by the same polynomial assumption as used in the construction of the noncommutative 1/4 BPS solution [6], that is, we assume that the functions $\mathcal{F} = \phi_k$ and a_k are given as polynomials of H and K :

$$\mathcal{F} = \sum_{n=0}^{n_{\max}} \sum_{m=0}^{m_{\max}} \mathcal{F}_{nm} H^n K^m, \quad (2.16)$$

with suitably large n_{\max} and m_{\max} . This assumption is owing to the property of H and K ,

$$\begin{aligned} \mathcal{D}K &= -HK, \\ \mathcal{D}H &= 1 + H - K^2, \end{aligned} \quad (2.17)$$

which implies that the operation of \mathcal{D} on a polynomial of H and K just reproduces another polynomial of them. With the assumption (2.16), the differential equations (2.14) are reduced to a set of linear algebraic equations for the coefficients \mathcal{F}_{nm} , which can be solved straightforwardly. Note that we originally had seven differential equations (2.14) for eight unknown functions, so the solution contains one undetermined function. This is the gauge freedom which preserves the generalized rotational invariant form (2.12), and the corresponding gauge transformation function is

$$\lambda^a = \epsilon_{aij} \theta_i x_j (\theta \hat{x}) \frac{1}{r^4} \lambda(\xi). \quad (2.18)$$

Using this freedom to choose $a_5(\xi) = 0$, the solution to the $\mathcal{O}(\theta^2)$ part (2.11) of the noncommutative BPS equation is given as follows:

$$\begin{aligned} \phi_1(\xi) &= -\frac{1}{4}H + \frac{1}{4}H^2 - \frac{1}{8}H^3 + \frac{1}{4}HK^2, \\ \phi_2(\xi) &= \frac{1}{8} - \frac{3}{8}H + \frac{1}{8}H^2 - \frac{1}{4}K^2 + \frac{3}{8}HK^2 + \frac{1}{8}H^2K^2 \\ &\quad + \frac{1}{8}K^4, \\ \phi_3(\xi) &= -\frac{1}{8} + \frac{7}{8}H - \frac{5}{8}H^2 + \frac{1}{8}H^3 + \frac{1}{4}K^2 - \frac{7}{8}HK^2 \\ &\quad - \frac{1}{8}H^2K^2 - \frac{1}{8}K^4, \\ a_1(\xi) &= -\frac{1}{8} + \frac{1}{2}H - \frac{1}{8}K - \frac{1}{2}HK - \frac{1}{4}H^2K + \frac{5}{8}K^2 \\ &\quad + \frac{1}{4}HK^2 - \frac{3}{8}K^3 - \frac{1}{4}HK^3, \end{aligned}$$

$$\begin{aligned} a_2(\xi) &= \frac{1}{4} + \frac{1}{2}H - \frac{3}{8}H^2 - \frac{3}{4}K - \frac{1}{2}HK + \frac{1}{8}H^2K \\ &\quad + \frac{3}{4}K^2 - \frac{1}{4}K^3, \end{aligned}$$

$$\begin{aligned} a_3(\xi) &= -\frac{1}{8} - \frac{1}{4}H - \frac{1}{8}H^2 + \frac{3}{8}K + \frac{1}{2}HK + \frac{1}{8}H^2K \\ &\quad - \frac{3}{8}K^2 - \frac{1}{4}HK^2 + \frac{1}{8}K^3, \end{aligned}$$

$$\begin{aligned} a_4(\xi) &= -\frac{1}{4} - \frac{3}{2}H + \frac{1}{2}H^2 + \frac{5}{4}K + \frac{3}{2}HK + \frac{1}{4}H^2K \\ &\quad - \frac{7}{4}K^2 - \frac{1}{4}HK^2 + \frac{3}{4}K^3 + \frac{1}{4}HK^3. \end{aligned} \quad (2.19)$$

III. SW MAP AND THE EIGENVALUES OF THE SCALAR FIELD

Having obtained the classical solution to the noncommutative BPS equation to $\mathcal{O}(\theta^2)$, our next task is to transform it into the commutative description via the SW map to get the eigenvalues of the scalar field [7]. For this purpose, we first have to establish the SW map to second order in the change $\delta\theta$ of the noncommutativity parameter.

It was pointed out in Ref. [10] that the SW map has inherent ambiguities. There are two types of ambiguities in it. One is of the form identifiable as gauge transformations. The other type of ambiguity consists of gauge covariant quantities. The latter can cause physical differences and must be fixed by some physical requirements. This type of ambiguities comes from the path dependence of the map in the θ space. In other words, even if we perform the map to go round in the θ space, we do not come back to the original configuration. This means that the SW map at $\mathcal{O}(\delta\theta^2)$ and higher has such a type of ambiguities.

The SW map for the gauge field to $\mathcal{O}(\delta\theta^2)$, including the ambiguities, is presented in the Appendix. Here, we need the SW map for the scalar field from a noncommutative space with small θ to the commutative space. This map is obtained by performing the dimensional reduction of the map for the gauge field and taking

$$\delta\theta^{ij} = -\theta^{ij}. \quad (3.1)$$

Then, the scalar field Φ in the commutative description is expressed in terms of $\hat{\Phi}$ and \hat{A}_i in the noncommutative description as

$$\Phi = \hat{\Phi} + \Delta\hat{\Phi}^{(1)} + \Delta\hat{\Phi}^{(2)}, \quad (3.2)$$

with $\Delta\hat{\Phi}^{(1)}$ and $\Delta\hat{\Phi}^{(2)}$ given by

$$\Delta\hat{\Phi}^{(1)} = -\frac{1}{4}\delta\theta^{kl}\{A_k, (\partial_l + D_l)\Phi\} - i\alpha\delta\theta^{kl}[\Phi, F_{kl}] - 2\beta\delta\theta^{kl}[\Phi, A_k A_l], \quad (3.3)$$

$$\begin{aligned} \Delta\hat{\Phi}^{(2)} &= \frac{1}{4}\text{Im}(\overline{\partial A \partial \Phi}) + \frac{1}{4}\text{Re}(\overline{A \partial A \partial \Phi}) - \frac{1}{4}\text{Re}(\overline{\partial A A \partial \Phi}) \\ &\quad - \frac{1}{4}\text{Re}(\overline{A \partial D \Phi A}) + \frac{1}{4}\text{Re}(\overline{A A \partial D \Phi}) + \frac{1}{4}\text{Re}(\overline{A F D \Phi}) \\ &\quad + \frac{1}{4}\text{Re}(\overline{A D \Phi F}) - \frac{1}{8}\text{Re}(\overline{\partial A \partial A \Phi}) + \frac{1}{8}\text{Re}(\overline{\partial A \Phi \partial A}) \\ &\quad - \frac{1}{8}\text{Re}(\overline{A A A \Phi}) + \frac{1}{4}\text{Re}(\overline{A A A \Phi A}) - \frac{1}{8}\text{Re}(\overline{A A \Phi A A}) \\ &\quad - \left(\frac{1}{16} + 8\alpha\beta + 4\beta^2\right)\text{Re}(\overline{A A}[\Phi, \overline{A A}]) + 8\alpha\beta\text{Im}(\overline{\partial A}[\Phi, \overline{A A}]) \\ &\quad - \gamma_1\text{Re}(\overline{F}[\Phi, \overline{F}]) - \gamma_2\text{Re}(\overline{F}[\Phi, \overline{F}]) + \gamma_3\text{Im}(\overline{D \Phi} D \overline{F}) \\ &\quad + \Delta\hat{\Phi}_{\text{metric}}^{(2)} + (\text{gauge-type ambiguities}). \end{aligned} \quad (3.4)$$

In Eqs. (3.3) and (3.4), all the fields are defined at θ and all the products are the $*$ products (we have omitted the hats on the fields and the $*$ for the products). We have used the following simplified notation:

$$\overline{AB} \equiv \delta\theta^{kl} A_k B_l, \quad \overline{AF_i} \equiv \delta\theta^{kl} A_k F_{li}, \quad \overline{F} \equiv \delta\theta^{kl} F_{kl}, \quad (3.5)$$

and

$$\text{Re } \mathcal{O} \equiv \frac{1}{2}(\mathcal{O} + \mathcal{O}^\dagger), \quad \text{Im } \mathcal{O} \equiv \frac{1}{2i}(\mathcal{O} - \mathcal{O}^\dagger). \quad (3.6)$$

Note that the contraction symbol has the property that $(\overline{AB})^\dagger = -\overline{B^\dagger A^\dagger}$ due to the anti-symmetry of $\delta\theta^{kl}$. We shall mention the $\Delta\hat{\Phi}_{\text{metric}}^{(2)}$ term in Eq. (3.4) soon below.

The SW map at $\mathcal{O}(\delta\theta)$ (3.3) contains two ambiguities parametrized by α and β . They are the gauge-type ambiguities. On the other hand, there exist three covariant-type ambiguities with coefficients γ_k ($k=1,2,3$) in the SW map at $\mathcal{O}(\delta\theta^2)$ (3.4). This type of ambiguities directly affects the eigenvalues of the scalar. Note that the gauge-type ambiguities at $\mathcal{O}(\delta\theta)$ have influence on the map at $\mathcal{O}(\delta\theta^2)$ and may possibly change the eigenvalues.

The term $\Delta\hat{\Phi}_{\text{metric}}^{(2)}$ in Eq. (3.4) represents the covariant-type ambiguities using the metric.¹ Note that, in all the terms written explicitly in Eq. (3.4), the upper indices of θ^{ij} are contracted with the lower ones of A_i and ∂_i without using the metric. On the other hand, the terms in $\Delta\hat{\Phi}_{\text{metric}}^{(2)}$ are constructed by using the metric. There are many terms belonging to $\Delta\hat{\Phi}_{\text{metric}}^{(2)}$. Examples are

¹SW map containing the metric was also considered in Ref. [13] in a different context.

$$\delta^{km}\delta^{ln}\text{Re}(F_{kl}[\Phi, F_{mn}])\theta^2,$$

$$\delta^{km}\delta^{ln}\text{Re}(D_k\Phi[\Phi, D_l\Phi])\theta_m\theta_n. \quad (3.7)$$

These are obtained by the dimensional reduction of the corresponding operators in the SW map for the gauge field given in Eq. (A8) of the Appendix.

Now we shall proceed to the evaluation of the eigenvalues of the scalar Φ (3.2) as a 2×2 matrix to $\mathcal{O}(\theta^2)$. For this purpose, we expand Φ in the commutative description in powers of θ :

$$\Phi = \Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)}, \quad (3.8)$$

where $\Phi^{(n)}$ is of order θ^n . Let us write explicitly the arguments of the SW map, Eqs. (3.3) and (3.4), as $\Delta\hat{\Phi}^{(n)}[\hat{\Phi}, \hat{A}_i, \theta]$ with the last argument representing the θ dependence only through the $*$ product (2.4). Then, using the noncommutative classical solution (2.5), we have

$$\Phi^{(0)} = \hat{\Phi}^{a(0)} \frac{1}{2} \sigma_a, \quad (3.9)$$

$$\Phi^{(1)} = \Delta\hat{\Phi}^{(1)}[\hat{\Phi}^{(0)}, \hat{A}_i^{(0)}, \theta=0], \quad (3.10)$$

$$\begin{aligned} \Phi^{(2)} &= \hat{\Phi}^{a(2)} \frac{1}{2} \sigma_a + \Delta\hat{\Phi}^{(1)}[\hat{\Phi}, \hat{A}_i, \theta]_{\theta^2} \\ &\quad + \Delta\hat{\Phi}^{(2)}[\hat{\Phi}^{(0)}, \hat{A}_i^{(0)}, \theta=0]. \end{aligned} \quad (3.11)$$

We shall add some explanations about Eqs. (3.9)–(3.11). First, we have set $\theta=0$ in Eq. (3.10) and the last term of Eq. (3.11). This implies that we take the commutative products among the fields. Next, the second term on the RHS of Eq. (3.11) means the sum of all the terms quadratic in θ in

$\Delta\hat{\Phi}^{(1)}[\hat{\Phi},\hat{A}_i,\theta]$. There are three sources of θ : $\delta\theta^{kl}=-\theta^{kl}$ in Eq. (3.3), θ in the $*$ product, and θ in the noncommutative classical solution (2.9).

Then, the two eigenvalues λ_{\pm} of the scalar Φ (3.8) are given using the well-known perturbation theory formula as

$$\lambda_{\pm}=\lambda_{\pm}^{(0)}+\lambda_{\pm}^{(1)}+\lambda_{\pm}^{(2)}, \quad (3.12)$$

with

$$\lambda_{\pm}^{(0)}=\pm\frac{H(\xi)}{2r}, \quad (3.13)$$

$$\lambda_{\pm}^{(1)}=\langle\pm|\Phi^{(1)}|\pm\rangle, \quad (3.14)$$

$$\lambda_{\pm}^{(2)}=\langle\pm|\Phi^{(2)}|\pm\rangle+\frac{\langle\pm|\Phi^{(1)}|\mp\rangle\langle\mp|\Phi^{(1)}|\pm\rangle}{\lambda_{\pm}^{(0)}-\lambda_{\mp}^{(0)}}, \quad (3.15)$$

where the kets $|\pm\rangle$ are the eigenvectors of $\Phi^{(0)}$ satisfying

$$\hat{x}\cdot\sigma|\pm\rangle=\pm|\pm\rangle. \quad (3.16)$$

Plugging the noncommutative classical solution obtained in Sec. II into Eqs. (3.14) and (3.15), we get after a tedious but straightforward calculation

$$\lambda_{+}^{(1)}=-\frac{1}{4r^3}H(1-K^2)(\theta\hat{x}), \quad (3.17)$$

$$\begin{aligned} \lambda_{+}^{(2)}&=\frac{1}{16r^5}\left(H^2-HK^2+\frac{1}{2}H^3K^2+HK^4\right)\theta^2 \\ &+\frac{1}{16r^5}\left(2H-3H^2-HK^2-\frac{1}{2}H^3K^2-HK^4\right)(\theta\hat{x})^2 \\ &+c_1f_1(x,\theta)+c_2f_2(x,\theta) \\ &+\langle+|\Delta\hat{\Phi}^{(2)}[\hat{\Phi}^{(0)},\hat{A}_i^{(0)},\theta=0]_{\text{metric}}|+\rangle, \end{aligned} \quad (3.18)$$

with

$$c_1=-\frac{1}{2}\gamma_1+\gamma_2-\frac{1}{2}\gamma_3+\alpha^2, \quad c_2=\frac{1}{2}\gamma_3, \quad (3.19)$$

and

$$\begin{aligned} f_1(x,\theta)&=\frac{1}{r^5}H^3K^2(\theta^2-(\theta\hat{x})^2), \\ f_2(x,\theta)&=\frac{1}{r^5}(HK^2-H^2K^2-HK^4)(\theta^2-3(\theta\hat{x})^2). \end{aligned} \quad (3.20)$$

The other eigenvalue λ_{-} is given by $\lambda_{-}^{(1)}=\lambda_{+}^{(1)}$ and $\lambda_{-}^{(2)}=-\lambda_{+}^{(2)}$. The first order eigenvalue (3.17) is already obtained in Ref. [7]. The origin of the α^2 term in c_1 Eq. (3.19) is the last term of Eq. (3.15). There are no other contributions to

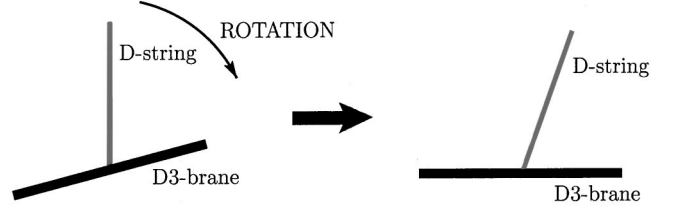


FIG. 1. When the gauge group is U(1), the target space rotation precisely connects the tilted D3-brane picture (left) with the tilted D-string picture (right).

$\lambda_{+}^{(2)}$ from the last term of Eq. (3.15) since we have $\Phi^{(1)}|_{\alpha=\beta=0}\propto 1$. The terms in the SW map (3.4) quadratic in α and β do not contribute to $\lambda_{+}^{(2)}$ owing to the property $[\Phi^{(0)},\overline{A^{(0)}}A^{(0)}]=0$ for the zeroth order solution. All the constituents of $\lambda_{+}^{(2)}$ (3.18), which are polynomials of H and K divided by r^5 , vanish at the origin $r=0$.

IV. TILTED D-STRING PICTURE

We would like to compare the scalar eigenvalues obtained in the previous section with those which are obtained by different ways and are expected to describe the same physical situation of the tilted D-string between two parallel D3-branes. In the U(1) case, there are three ways giving the same result [7,8]; the SW map of the noncommutative BPS solution, the nonlinear BPS solution in the commutative space, and the target space rotation of the linear BPS solution in the commutative space. In particular, the linear BPS solution (under a constant magnetic field) gives the tilted D3-brane picture, which is related to the tilted D-string picture by the target space rotation (see Fig. 1). In the non-Abelian case, the nonlinearly realized supertransformation of the DBI theory is not well understood. Therefore, we shall take the target space rotation of the linear BPS solution in the commutative space as the object to be compared with the eigenvalues of Sec. III.

Let us consider the U(2) super Yang-Mills theory in the commutative space with a constant U(1) magnetic field B_i . The BPS equation of this system, which we regard as describing the tilted D3-brane in a constant NSNS B -field $B_{ij}=\epsilon_{ijk}B_k$, is

$$D_i\Phi+\frac{1}{2}\epsilon_{ijk}\left(F_{jk}+B_{jk}\frac{1}{2}\mathbb{1}\right)=0. \quad (4.1)$$

For our present purpose of comparing with the previous section, we should in fact consider the Yang-Mills theory in the commutative space with the metric g_{ij} related to the metric $G_{ij}=\delta_{ij}$ in the noncommutative theory of Sec. II by [1]

$$G^{ij}+\frac{\theta^{ij}}{2\pi\alpha'}=\left(\frac{1}{g+2\pi\alpha'B}\right)^{ij}. \quad (4.2)$$

However, the desired scalar eigenvalue is obtained by considering the BPS equation (4.1) with $g_{ij}=\delta_{ij}$ and coordinate transforming back to the original g_{ij} afterwards [8].

The U(1) part of this equation

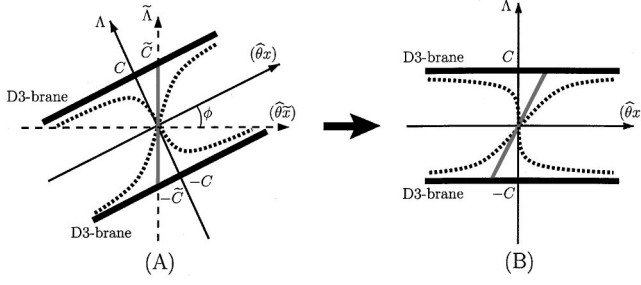


FIG. 2. The tilted D3-brane picture (A) in the U(2) case is expected to be related to the tilted D-string picture (B) by a target space rotation. The dotted curves represent the scalar eigenvalues. We have omitted $2\pi\alpha'$ which should multiply Λ , $\tilde{\Lambda}$, C , and \tilde{C} in the two figures.

$$\partial_i \Phi^0 + \frac{1}{2} \epsilon_{ijk} B_{jk} = 0, \quad (4.3)$$

is easily solved to give

$$\Phi^0 = -\frac{1}{2} \epsilon_{ijk} B_{jk} x^i = \frac{1}{(2\pi\alpha')^2} (\theta x), \quad (4.4)$$

where the relation $2\pi\alpha' B_i = -\theta_i/2\pi\alpha' + \mathcal{O}(\theta^3)$ has been used. As a solution to the non-Abelian part, we adopt the ordinary BPS monopole solution (2.6). We shall attach tilde to the space coordinates in the present system for distinguishing them from those in the rotated system to be discussed below. Then, the (larger) eigenvalue of the scalar field is

$$\tilde{\Lambda} = \frac{1}{2r} H(\tilde{\xi}) + \frac{1}{(2\pi\alpha')^2} (\theta \tilde{x}), \quad (4.5)$$

with $\tilde{\xi} \equiv \tilde{C} \tilde{r}$ and $\tilde{r} \equiv \tilde{x}_i \tilde{x}_i$ (\tilde{C} is the mass scale of the present monopole).

Now let us carry out the target space rotation and turn to the tilted D-string picture (see Fig. 2):

$$\begin{pmatrix} 2\pi\alpha' \Lambda \\ (\hat{\theta} x) \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} 2\pi\alpha' \tilde{\Lambda} \\ (\hat{\theta} \tilde{x}) \end{pmatrix}, \quad (4.6)$$

where $\hat{\theta}_i$ is the unit vector $\hat{\theta}_i \equiv \theta_i/|\theta|$, and the rotation angle ϕ is given as $\tan \phi = |\theta|/2\pi\alpha'$. The components perpendicular to θ_i are common between x_i and \tilde{x}_i . Expressing the new eigenvalue Λ in the tilted D-string picture as a function of the coordinate x^i , we have

$$\begin{aligned} \Lambda = & \frac{H}{2r} - \frac{1}{4r^3} H(1-K^2)(\theta \hat{x}) + \frac{1}{16r^5} (H^2 - H^2 K^2) \theta^2 \\ & + \frac{1}{16r^5} (2H - 3H^2 - 4HK^2 + 3H^2 K^2 + 2H^3 K^2 + 2HK^4) \\ & \times (\theta \hat{x})^2 - \frac{1}{(2\pi\alpha')^2} \frac{1}{4r} \{H\theta^2 + (1-K^2)(\theta \hat{x})^2\}, \quad (4.7) \end{aligned}$$

where the arguments of H and K are $\tilde{C}r$ with $r^2 = x^i x^i$. Now we make the coordinate transformation in Eq. (4.7) from the metric δ_{ij} to $g_{ij} = \delta_{ij} - (\theta^2 \delta_{ij} - \theta_i \theta_j)/(2\pi\alpha')^2$ corresponding to the open string metric $G_{ij} = \delta_{ij}$ adopted in Sec. II. This is accomplished by the replacement of r with

$$(g_{ij} x^i x^j)^{1/2} = \left(1 - \frac{1}{(2\pi\alpha')^2} \frac{1}{2} [\theta^2 - (\theta \hat{x})^2] \right) r. \quad (4.8)$$

Then, the eigenvalue Λ in the new coordinate system is given by Eq. (4.7) with the last $1/(2\pi\alpha')^2$ term omitted and the arguments of H and K replaced by Cr [see Ref. [8] for the U(1) case]. Here, C is the D3-brane separation, $C \equiv \tilde{C} \cos \phi$ (see Fig. 2). We can show that this eigenvalue is exactly the same as $\Lambda(x)$ obtained by solving

$$\Lambda(x) = \frac{1}{2r} H(|x_i - \Lambda(x) \theta_i|), \quad (4.9)$$

which implies the tilted D-string picture in Fig. 2(B). Namely, for a given value of Λ , the corresponding x_i lies on a sphere with its center at $x_i = \Lambda \theta_i$ (see Refs. [4,6,7]).

Having finished the preparation of obtaining the eigenvalue from the target space rotation of the linear BPS solution, let us proceed to the comparison between this eigenvalue Λ (4.7) [without the $1/(2\pi\alpha')^2$ term] and the eigenvalue (3.17) and (3.18) obtained from the noncommutative monopole via the SW map. First, the $\mathcal{O}(\theta)$ terms agree between them as was already shown in Ref. [7]. Second, the $\mathcal{O}(\theta^2)$ parts coincide perfectly in the asymptotic region $r \rightarrow \infty$ where we can drop the exponentially decaying terms [see Eq. (2.8)]. Note in particular that all the ambiguity terms in Eq. (3.18) disappear in the asymptotic region. [The last term of Eq. (3.18) using the metric is also exponentially decaying as $r \rightarrow \infty$.]

Let us compare the $\mathcal{O}(\theta^2)$ terms in the two eigenvalues for a general x_i not restricted to the asymptotic region. Since the SW map is defined by the gauge equivalence relation independent of the metric, we shall consider first the simpler case of Eq. (3.18) without the last term $\langle +|\Delta \hat{\Phi}_{\text{metric}}^{(2)}|+ \rangle$ using the metric. In this case, by taking $c_2 = 1/16$, we can make Eq. (3.18) agree with the $\mathcal{O}(\theta^2)$ part of Eq. (4.7) except only the $H^3 K^2$ terms. However, for the complete agreement between the two eigenvalues, the introduction of the metric term $\langle +|\Delta \hat{\Phi}_{\text{metric}}^{(2)}|+ \rangle$ is inevitable.

As we mentioned in Sec. III, there are many contributions to the covariant-type ambiguity using the metric. A complete analysis shows that the term $\langle +|\Delta \hat{\Phi}_{\text{metric}}^{(2)}|+ \rangle$ is a sum of three functions, f_1 and f_2 of Eq. (3.20) and a new one

$$f_3(x, \theta) = \frac{1}{r^5} H^3 K^2 \theta^2, \quad (4.10)$$

each multiplied by an arbitrary coefficient. In fact, we have $\langle +|\delta^{km} \delta^{ln} \text{Re}(F_{kl}[\Phi, F_{mn}]) \theta^2|+ \rangle = -f_3(x, \theta)$. Then, expressing the RHS of $\lambda_+^{(2)}$ (3.18) as the sum of its first two terms and $c_1 f_1 + c_2 f_2 + c_3 f_3$ with the redefined c_1 and c_2 ,

the complete agreement between $\lambda_+^{(2)}$ and the $\mathcal{O}(\theta^2)$ term of Λ (4.7) is achieved by taking the three parameters as $c_1 = -5/32$, $c_2 = 1/16$, and $c_3 = 1/8$. This is the unique choice for the coefficients c_k . Note that this agreement is a nontrivial one since we have to tune eight coefficients by using only three free parameters. Of course, the three coefficients c_k do not completely fix the ambiguity in the SW map since there are many contributions to c_k if we allow the $\Delta\hat{\Phi}_{\text{metric}}^{(2)}$ term using the metric. The use of the metric in the SW map seems not so unnatural if we recall that the noncommutative classical solution [see Eqs. (2.9) and (2.12)] as well as the BPS equation (2.1) already contains the metric.

V. SUMMARY AND DISCUSSIONS

In this paper, we considered the noncommutative monopole solutions at the second order in θ . We solved the noncommutative version of the BPS equation to $\mathcal{O}(\theta^2)$, mapped the solution to the commutative side, and obtained the eigenvalues of the resulting scalar field. We saw that the ambiguities in the SW map have explicit influence on the scalar eigenvalues. We made the brane interpretation to the scalar eigenvalues and examined whether they can reproduce the configuration of a tilted D-string suspended between two parallel D3-branes. In the asymptotic region, the effect of the ambiguities in the SW map disappears and at the same time the scalar eigenvalue precisely gives the expected D-string picture. Without the restriction to the asymptotic region, we found that we can tune the free parameters in the SW map so that the scalar eigenvalues reproduce the desired configuration. It is necessary to introduce the covariant-type ambiguity term using the metric. The number of free parameters c_k in the eigenvalues is just enough to adjust them to the expected ones.

We would like to make a few comments. Our first comment is on the covariant type ambiguity in the SW map. In this paper we have constructed the SW map first in the pure Yang-Mills system without the scalar field and then obtained the map for the scalar by the dimensional reduction of the map for the gauge field. This is natural if we recall the origin of the present super Yang-Mills theory via the dimensional reduction. However, if we forget this origin, there are other covariant-type ambiguities treating the scalar field Φ as a gauge covariant quantity from the start. For example, as an ambiguity for the scalar field at $\mathcal{O}(\theta)$, we have $\delta\theta^{ij}\{\Phi, F_{ij}\}$. However, this term gives the same contribution (with an arbitrary coefficient) to the $\mathcal{O}(\theta)$ eigenvalue as the existing one (3.17), and hence even the tilt angle at $\mathcal{O}(\theta)$ becomes a free parameter.

Next we shall comment on the noncommutative eigenvalue equation for the scalar field proposed and examined in Refs. [4,6]. At $\mathcal{O}(\theta)$, the eigenvalues of the noncommutative eigenvalue problem for the scalar gave the same asymptotic behavior as those obtained via the SW map [7]. We have carried out the analysis of the noncommutative eigenvalue equation for the classical solution at $\mathcal{O}(\theta^2)$ given in Sec. II. However, the resulting eigenvalues do not agree with those from the SW map even in the asymptotic region. Therefore,

the noncommutative eigenvalue equation seems to work well only at the first order in θ , though it is still an interesting subject to understand why it gives a good result at this order.

Finally, we would like to emphasize the usefulness of the analysis using the BPS solutions. The BPS solutions are expected to remain intact even if we include the α' corrections. Thus, the BPS solutions would be helpful for giving a support for the equivalence between the noncommutative description and the commutative one independently of the α' expansion. It is a very interesting subject to pursue the method which enables us to examine this equivalence to all orders in θ .

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APPENDIX: SEIBERG-WITTEN MAP TO $\mathcal{O}(\delta\theta^2)$

In this appendix, we present the SW map for the gauge field to second order in the change $\delta\theta$ of the noncommutativity parameter θ . The SW map is derived from the gauge equivalence relation [1]

$$A_i(\hat{A}) + \delta_\lambda A_i(\hat{A}) = A_i(\hat{A} + \hat{\delta}_\lambda \hat{A}), \quad (\text{A1})$$

where the quantities with a hat are defined at θ and those without hat at $\theta + \delta\theta$, a nearby point of θ . This unconventional meaning of hat is for the convenience of the use in Sec. III.

We expand A_i and λ in powers of $\delta\theta$:

$$\begin{aligned} A_i &= \hat{A}_i + \Delta\hat{A}_i^{(1)} + \Delta\hat{A}_i^{(2)} + \mathcal{O}(\delta\theta^3), \\ \lambda &= \hat{\lambda} + \Delta\hat{\lambda}^{(1)} + \Delta\hat{\lambda}^{(2)} + \mathcal{O}(\delta\theta^3). \end{aligned} \quad (\text{A2})$$

Substituting them into Eq. (A1), the first order part is solved in the most general form as [10]

$$\begin{aligned} \Delta\hat{A}_i^{(1)} &= -\frac{1}{4} \delta\theta^{kl} \{\hat{A}_k, \partial_l \hat{A}_i + \hat{F}_{li}\} + \alpha \delta\theta^{kl} \hat{D}_i \hat{F}_{kl} \\ &\quad - i\beta \delta\theta^{kl} \hat{D}_i [\hat{A}_k, \hat{A}_l], \end{aligned} \quad (\text{A3})$$

$$\Delta\hat{\lambda}^{(1)} = -\frac{1}{4} \delta\theta^{kl} \{\hat{A}_k, \partial_l \hat{\lambda}\} - 2i\beta \delta\theta^{kl} [\hat{A}_k, \partial_l \hat{\lambda}], \quad (\text{A4})$$

where α and β are arbitrary real coefficients. Note that these two ambiguity terms are both gauge-type ones. Next, we shall solve the second order part of Eq. (A1),

$$\begin{aligned} \hat{\delta}_\lambda \Delta \hat{A}_i^{(2)} + i[\Delta \hat{A}_i^{(2)}, \hat{\lambda}] - \hat{D}_i \Delta \hat{\lambda}^{(2)} = & \frac{i}{8} \delta \theta^{kl} \delta \theta^{mn} [\partial_k \partial_m \hat{A}_i, \partial_l \partial_n \hat{\lambda}] + \frac{1}{2} \delta \theta^{kl} \{\partial_k \hat{A}, \partial_l \Delta \hat{\lambda}^{(1)}\} \\ & + \frac{1}{2} \delta \theta^{kl} \{\partial_k \Delta \hat{A}_i^{(1)}, \partial_l \hat{\lambda}\} - i[\Delta \hat{A}_i^{(1)}, \Delta \hat{\lambda}^{(1)}], \end{aligned} \quad (\text{A5})$$

to obtain $\Delta \hat{A}_i^{(2)}$ and $\Delta \hat{\lambda}^{(2)}$. We solved Eq. (A5) by assuming the most general forms for $\Delta \hat{A}_i^{(2)}$ and $\Delta \hat{\lambda}^{(2)}$. The result is as follows:²

$$\begin{aligned} \Delta \hat{A}_i^{(2)} = & \frac{1}{4} \text{Im}(\overline{\partial \hat{A}} \partial \partial A_i) - \frac{1}{8} \text{Im}(\overline{\partial \hat{A}} \partial \partial_i A) + \frac{1}{4} \text{Re}(\overline{\hat{A}} \partial \hat{A} \partial A_i) - \frac{1}{4} \text{Re}(\overline{\partial \hat{A}} \hat{A} \partial A_i) \\ & - \frac{1}{4} \text{Re}(\overline{\hat{A}} \partial F_i \hat{A}) + \frac{1}{4} \text{Re}(\overline{\hat{A} \hat{A}} \partial F_i) + \frac{1}{4} \text{Re}(\overline{\hat{A} F} F_i) + \frac{1}{4} \text{Re}(\overline{\hat{A} F_i} F) \\ & - \frac{1}{8} \text{Re}(\overline{\partial \hat{A}} \partial \hat{A} A_i) + \frac{1}{8} \text{Re}(\overline{\partial \hat{A}} A_i \partial \hat{A}) - \frac{1}{8} \text{Im}(\overline{\hat{A} \hat{A}} \partial_i \hat{A}) + \frac{1}{8} \text{Im}(\overline{\hat{A} \hat{A}} \partial_i \hat{A} A) \\ & - \frac{1}{8} \text{Re}(\overline{\hat{A} \hat{A} \hat{A}} A_i) + \frac{1}{4} \text{Re}(\overline{\hat{A} \hat{A} \hat{A}} A_i A) - \frac{1}{8} \text{Re}(\overline{\hat{A} \hat{A} A_i} A) \\ & + \left(\frac{1}{16} + 8\alpha\beta + 4\beta^2\right) \text{Im}(\overline{\hat{A} \hat{A}} D_i(\overline{\hat{A} \hat{A}})) + 8\alpha\beta \text{Re}(\overline{\partial \hat{A}} D_i(\overline{\hat{A} \hat{A}})) \\ & + \gamma_1 \text{Im}(\overline{F D_i F}) + \gamma_2 \text{Im}(\overline{F D_i F}) + \gamma_3 \text{Im}(\overline{F_i D F}) \\ & + \Delta \hat{A}_{i \text{metric}}^{(2)} + (\text{gauge-type ambiguities}), \end{aligned} \quad (\text{A6})$$

$$\begin{aligned} \Delta \hat{\lambda}^{(2)} = & \frac{1}{8} \text{Im}(\overline{\partial \hat{A}} \partial \partial \lambda) - \frac{1}{8} \text{Re}(\overline{\partial \hat{A}} \hat{A} \partial \lambda) - \frac{1}{8} \text{Re}(\overline{\partial \hat{A}} \partial \lambda \hat{A}) + \frac{1}{4} \text{Re}(\overline{\hat{A}} \partial \hat{A} \partial \lambda) \\ & - \frac{1}{8} \text{Im}(\overline{\hat{A} \hat{A}} \partial \lambda) + \frac{1}{8} \text{Im}(\overline{\hat{A} \hat{A}} \partial \lambda \hat{A}) \\ & + \left(\frac{1}{16} + 8\alpha\beta + 4\beta^2\right) \text{Im}(\overline{\hat{A} \hat{A}} (\overline{\hat{A}} \partial \lambda + \overline{\partial \lambda} \hat{A})) + 8\alpha\beta \text{Re}(\overline{\partial \hat{A}} (\overline{\hat{A}} \partial \lambda + \overline{\partial \lambda} \hat{A})) \\ & + (\text{gauge-type ambiguities}), \end{aligned} \quad (\text{A7})$$

where the meanings of the contraction, $\text{Re } \mathcal{O}$ and $\text{Im } \mathcal{O}$ are as given by Eqs. (3.5) and (3.6). We have omitted hats on the RHS of Eqs. (A6) and (A7). The ambiguities of the SW map at $\mathcal{O}(\theta^2)$ are the homogeneous solutions to Eq. (A5). The terms in Eq. (A6) multiplied by γ_k ($k=1,2,3$) are the covariant-type ambiguities which cannot be identified as gauge transformation. All other covariant-type terms are reduced to the three γ_k terms owing to the Bianchi identity. The term $\Delta \hat{A}_{i \text{metric}}^{(2)}$ denotes the covariant-type ambiguity using the metric G_{ij} . There are many operators belonging to this type; for example,

$$G^{km} G^{ln} \text{Im}(F_{kl} D_i F_{mn}) \theta^2, \quad G^{kp} G^{mq} G^{ln} \text{Im}(F_{kl} D_i F_{mn}) \theta_p \theta_q. \quad (\text{A8})$$

The SW map for the scalar field Φ used in Sec. III is obtained from Eq. (A6) by the dimensional reduction using $A_\Phi = \Phi$, $F_{i\Phi} = D_i \Phi$ and $D_\Phi \mathcal{O} = -i[\Phi, \mathcal{O}]$. The second quantity in Eq. (3.7) is obtained from that in Eq. (A8) by setting $i = \Phi$ and taking the $G^{\Phi\Phi}$ part.

²The terms in $\Delta \hat{\Phi}^{(2)}$ of the form $\text{Re } \mathcal{O}$ ($\text{Im } \mathcal{O}$) with \mathcal{O} containing odd (even) number of derivatives do not contribute to the scalar eigenvalue formula (3.15). Therefore, in Eq. (A6) we have omitted such terms, which would appear as the covariant-type ambiguity terms and the terms quadratic in α and β .

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