# Character expansions, Itzykson-Zuber integrals, and the QCD partition function

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A combinatorial formula to generate $U(N)$ character expansions is presented. It is shown that the resulting character expansion formulas greatly simplify a number of problems where integrals over the group manifolds need to be calculated. Several examples are given, including direct and very quick calculations of the Itzykson-Zuber integral and the finite volume effective partition function of QCD in the sector with a given topological charge.

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## I. INTRODUCTION

Expansion of a periodic function into its Fourier components is widely used in physics. Since sines and cosines can be considered as the characters of the $U(1)$ group, Fourier expansion is the simplest character expansion. In general expansion of an invariant function of a group into its characters (traces of the representation matrices) is not an easy task. Some time ago the author had given a combinatorial formula to write character expansions for the $U(N)$ group [1]. The purpose of this paper is first to extend this formula to more general situations than those covered in Ref. [1], and then to show that this formula can be profitably used to simplify a number of situations where integrals over the group manifolds need to be calculated.

The derivation of the general character expansion formula is given in the next section. In Sec. III we give some examples of character expansions obtained using our formula. The examples given there are meant to be illustrative of the technique, but not exhaustive of all the character expansions one can obtain. Some related determinantal identities are placed in the Appendices for easy reference. In Sec. IV we show that our techniques can be used to directly calculate the Itzykson-Zuber integral and its various extensions. In Sec. V we use our technique to calculate the finite volume effective partition function of QCD in the sector with a given topological charge. Even though the results presented in Secs. IV and V were previously obtained by other methods, it is shown that our method greatly simplifies the calculations. Finally in Sec. VI a brief discussion of the results, including extension of our results into the $U(N/M)$-type supergroups, and directions for future work concludes the paper. For continuity of the text several mathematical formulas, namely, a review of the properties of the symmetric functions and several determinantal identities are placed in two appendices.

## II. CHARACTER EXPANSION FORMULAS

First we review and extend the main result of Ref. [1]. Consider the representations of the $U(N)$ group labeled by a partition into $N$ parts $(n_1,n_2,\ldots,n_N)$ where $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_N$ (see for example Ref. [2]). The character (trace of the representation matrix) of the irreducible representation corresponding to the partition $(n_1,n_2,\ldots,n_N)$ of non-negative integers is given by Weyl's formula [2]

$$\chi_{(n_1,n_2,\ldots,n_N)}(U)=\frac{\det(t_i^{n_j+N-j})}{\Delta(t_1,\ldots,t_N)}, \tag{2.1}$$

where $t_i, i=1,\ldots,N$, are the eigenvalues of the group element $U$ in the fundamental representation and $\Delta(t_1,\ldots,t_N)$ is the Vandermonde determinant in the arguments $t_1,\ldots,t_N$:

$$\Delta(t_1,\ldots,t_N)=\det(t_i^{N-j}). \tag{2.2}$$

In these equations the arguments of the determinants indicate the $(ij)$-th element of the matrix the determinant of which is calculated. An alternative form for the character formula is given by

$$\chi_{(n_1,n_2,\ldots,n_N)}(U)=\det(h_{n_j+i-j}), \tag{2.3}$$

where $h_n$ is the complete symmetric function in the arguments $t_1,\ldots,t_N$ of degree $n$. (For a review of its properties see Appendix A.)

We now consider the power series expansion

$$G(x,t)=\sum_n A_n(x)t^n, \tag{2.4}$$

where the range of $n$ in the sum is not yet specified. $x$ stands for all the parameters needed to specify the coefficients $A_n$. We assume that this series is convergent for $|t|=1$. Given $N$ different $t$'s: $t_1,\ldots,t_N$, we next write down the equality using $N$ copies of Eq. (2.4)

$$\Delta(t_1,\ldots,t_N)\left(\prod_{i=1}^{N} G(x,t_i)\right)=\det\left[\sum_n A_n(x)t_i^{N+n-j}\right]. \tag{2.5}$$

Changing the variable in the sums to $p=n+N-j$, we can rewrite Eq. (2.5) as

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$$\Delta(t_1,\ldots,t_N)\left(\prod_{i=1}^{N} G(x,t_i)\right)=\det[f_j(t_i)], \tag{2.6}$$

where

$$f_j(z)=\sum_p A_{p+j-N}z^p. \tag{2.7}$$

The range of $n$ in Eq. (2.4), which is so far completely unrestricted, determines the range of $p$ in Eq. (2.7). Using the properties of determinants (see for example Ref. [3]) and Eq. (B2) of the Appendix B Eq. (2.6) can be written as

$$\Delta(t_1,\ldots,t_N)\left(\prod_{i=1}^{N} G(x,t_i)\right)$$

$$=\sum_{k_1>k_2>\cdots>k_N}\det(A_{k_j+i-N})\det(t_i^{k_j}). \tag{2.8}$$

In Eq. (2.8) the range (but not the ordering) of the variables $k_1,k_2,\ldots,k_N$ are still determined by the range of $n$ in Eq. (2.4). First we introduce

$$n_i=k_i-N+i. \tag{2.9}$$

The ordering of the variables indicated in the sum of Eq. (2.8) now becomes

$$n_1\geqslant n_2\geqslant\cdots\geqslant n_N. \tag{2.10}$$

This transformation is necessary since the partitions $(n_1,\ldots,n_N)$ that label the representations of $U(N)$ should satisfy the condition in Eq. (2.10), i.e., it is the $n_i$'s, not the $k_i$'s that label the irreducible representations. Next we want to show that because of the condition in Eq. (2.10) only one of the sums in Eq. (2.8) still spans the original range of $n$ in Eq. (2.4). To this end we introduce the non-negative quantities

$$m_j=n_j-n_{j+1},\quad j=1,\ldots,N-1, \tag{2.11}$$

one can write

$$n_i=m_i+m_{i+1}+\cdots+m_{N-1}+n_N. \tag{2.12}$$

As a result the right-hand side of Eq. (2.8) takes the form

$$\sum_{m_1=0}\sum_{m_2=0}\cdots\sum_{m_{N-1}=0}\sum_{n_N}\det(A_{n_j+i-j})\det(t_i^{n_j+N-j}). \tag{2.13}$$

The upper range of the $m_1$ through $m_{N-1}$ sums are still determined by the range of $n$ in Eq. (2.4), but these sums start with $m_i=0$. The entire range of the $n_N$ sum is still determined by the range of $n$ in Eq. (2.4). At this point we want to use Weyl's formula, Eq. (2.1), to write the last term in the right-hand side of Eq. (2.13) as the character. Since $n_N$ may take negative values, we cannot yet use Eq. (2.1), where all the $n_j$'s are non-negative. To achieve our goal we need to do yet another transformation of indices in Eq. (2.13) to those that take only non-negative values. This is achieved by introducing the quantities

$$l_i=\sum_{j=i}^{N-1} m_j=n_i-n_N,\quad i=1,\ldots,N-1, \tag{2.14a}$$

$$l_N=0. \tag{2.14b}$$

Then the second determinant on the right-hand side of Eq. (2.13) can be written as

$$\det(t_i^{n_j+N-j})=(t_1t_2\cdots t_N)^{n_N}\det(t_i^{l_j+N-j}). \tag{2.15}$$

Substituting Eq. (2.15) into Eq. (2.13) and then inserting the resulting expression into Eq. (2.8) one obtains

$$\Delta(t_1,\ldots,t_N)\left(\prod_{i=1}^{N} G(x,t_i)\right)$$

$$=\sum_{m_1=0}\sum_{m_2=0}\cdots\sum_{m_{N-1}=0}\sum_{n_N}\det(A_{n_j+i-j})$$

$$\times(t_1t_2\cdots t_N)^{n_N}\det(t_i^{l_j+N-j}). \tag{2.16}$$

We can now take $t_i$'s to be the eigenvalues of the fundamental representation of $U(N)$. Dividing both sides of Eq. (2.16) with the Vandermonde determinant and using Eq. (2.1) we obtain

$$\left(\prod_{i=1}^{N} G(x,t_i)\right)=\sum_{m_1=0}\sum_{m_2=0}\cdots\sum_{m_{N-1}=0}\sum_{n_N}\det(A_{n_j+i-j})$$

$$\times(\det U)^{n_N}\chi_{(l_1,l_2,\ldots,l_N)}(U). \tag{2.17}$$

This is the main result of this paper. In writing this equation we used the fact that the matrix $U$ can always be diagonalized by a unitary transformation which leaves the character invariant. Equation (2.17) is a generalization of the character expansion given in Ref. [1]. If the sum over $n$ in the expression Eq. (2.4) we started with is restricted to the non-negative values of $n$ (i.e., $A_n=0$ when $n<0$), then $n_N$ is non-negative and we can absorb the term $(\det U)^{n_N}$ into the character to obtain the result given in Ref. [1]:

$$\left(\prod_{i=1}^{N} G(x,t_i)\right)$$

$$=\sum_{n_1=0}\sum_{n_2=0}\cdots\sum_{n_N=0}\det(A_{n_j+i-j})\chi_{(n_1,n_2,\ldots,n_N)}(U). \tag{2.18}$$

Note that the summation in Eq. (2.18) is over all irreducible representations of $U(N)$, but in Eq. (2.17) is restricted to those representations where the number of boxes in the last row of the Young Tableau is zero *and* an additional summation over $n_N$, which, in general can take both positive and negative values. An application of Eq. (2.18) to the thermodynamics of two-dimensional QCD in the large-$N$ limit was given in Ref. [4]. In the next section we give some explicit examples of character expansions.

### III. EXAMPLES OF CHARACTER EXPANSIONS

For our first example we choose $G(x,t)=\exp(xt)$. Then $A_n = x^n/n!$ for $n\geqslant 0$ and $A_n=0$ for $n<0$. We can then use Eq. (2.18) to write

$$\exp(x\ \mathrm{Tr}\ U)=\sum_r \alpha_r(x)\chi_r(U), \tag{3.1}$$

where the sum is over all irreducible representations [$r$ stands for $(n_1,n_2,\ldots,n_N)$] and

$$\alpha_r(x)=\det\left(\frac{x^{n_j+i-j}}{(n_j+i-j)!}\right)=x^{n_1+n_2+\cdots+n_N}\times\begin{vmatrix} \frac{1}{n_1!} & \frac{1}{(n_2-1)!} & \frac{1}{(n_3-2)!} & \cdots \\ \frac{1}{(n_1+1)!} & \frac{1}{n_2!} & \frac{1}{(n_3-1)!} & \cdots \\ \frac{1}{(n_1+2)!} & \frac{1}{(n_2+1)!} & \frac{1}{n_3!} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}. \tag{3.2}$$

This particular character expansion can also be obtained by explicit integration over the group manifold [5]. It will be increasingly difficult to obtain more complicated character expansions by explicit integration. Using Eqs. (2.17) and (2.18) provides a much easier alternative to the explicit integration over the group manifold. The determinant in Eq. (3.2) can also be written in terms of the dimensions of the group representations

$$\alpha_{\{n_1,n_2,\ldots,n_N\}}=x^{n_1+n_2+\cdots+n_N}\left[\prod_{i=1}^{N}\frac{(N-i)!}{(N+n_i-i)!}\right]d_{\{n_1,n_2,\ldots,n_N\}}, \tag{3.3}$$

where $d_{\{n_1,n_2,\ldots,n_N\}}$ is the dimension of the representation corresponding to the partition $\{n_1,n_2,\ldots,n_N\}$. (The dimensions can be evaluated by calculating the character of the identity. See, e.g., Ref. [6] for explicit formulas.) A related character expansion can be obtained by noting

$$t^\nu e^{xt}=\sum_{m=\nu}^{\infty}\frac{x^{m-\nu}}{(m-\nu)!}t^m. \tag{3.4}$$

Using Eq. (2.18) we immediately get

$$(\det U)^\nu e^{x\ \mathrm{Tr}\ U}=\sum_r x^{n_1+\cdots+n_N-N\nu}\left(\det\frac{1}{(n_j-\nu+i-j)!}\right)\times\chi_{n_1,\ldots,n_N}(U) \tag{3.5}$$

and

$$(\det U)^{-\nu} e^{x\ \mathrm{Tr}\ U}=\sum_r x^{n_1+\cdots+n_N+N\nu}\left(\det\frac{1}{(n_j+\nu+i-j)!}\right)\times\chi_{n_1,\ldots,n_N}(U). \tag{3.6}$$

For our second example we pick $G(x,t)$ to be the generating function of the Hermite polynomials

$$G(x,t)=\exp(2tx-t^2)=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n. \tag{3.7}$$

The corresponding character expansion can again be found using Eq. (2.18):

$$\exp(a\ \mathrm{Tr}\ U-b\ \mathrm{Tr}\ U^2)=\sum_r b^{(n_1+n_2+\cdots+n_N)/2}\det\left(\frac{H_{n_j+i-j}(a/2\sqrt{b})}{(n_j+i-j)!}\right)\chi_r(U). \tag{3.8}$$

For the next example we choose $G(x,t)$ to be the generating function of the modified Bessel functions

$$G(x,t)=\exp\left[\frac{x}{2}\left(t+\frac{1}{t}\right)\right]=\sum_{n=-\infty}^{+\infty}I_n(x)t^n. \tag{3.9}$$

Since the index $n$ takes negative as well as positive values we need to use Eq. (2.17) which yields the character expansion

$$\exp\left[\frac{x}{2}\mathrm{Tr}(U+U^\dagger)\right]=\sum_{m_1=0}^{+\infty}\sum_{m_2=0}^{+\infty}\cdots\sum_{m_{N-1}=0}^{+\infty}\sum_{n_N=-\infty}^{+\infty}\det[I_{n_j+i-j}(x)]\times(\det U)^{n_N}\chi_{(l_1,l_2,\ldots,l_N)}(U). \tag{3.10}$$

This expansion was previously obtained by direct integration for $SU(N)$ group ($\det U=1$) [7]. Because of the symmetry in the argument of the exponential, Eq. (3.10) can be equivalently written as

$$\exp\left[\frac{x}{2}\,\mathrm{Tr}(U+U^\dagger)\right] = \sum_{m_1=0}^{+\infty}\sum_{m_2=0}^{+\infty}\cdots\sum_{m_{N-1}=0}^{+\infty}\sum_{n_N=-\infty}^{+\infty}\det[I_{n_j+i-j}(x)] \times(\det U^\dagger)^{n_N}\chi_{(l_1,l_2,\ldots,l_N)}(U^\dagger). \quad (3.11)$$

In using these expressions it is useful to remember that $(\det U)^n$ for $n\geqslant 0$ is the character of the representation where all $n_i=n, i=1,\ldots,N$:

$$(\det U)^n=\chi_{(n,n,\ldots,n)}(U). \quad (3.12)$$

This can be proven rewriting Eq. (2.3) in terms of the elementary symmetric functions and conjugate partitions; for the definitions see Appendix A and for a proof see, e.g., Ref. [8].

## IV. ITZYKSON-ZUBER INTEGRALS

In 1980 Itzykson and Zuber were able to calculate the group integral [9]

$$\int dU\,\exp[\beta\mathrm{Tr}(M_1UM_2U^\dagger)] = \left(\prod_{p=o}^{N-1}p!\right)\beta^{-N(N-1)/2}\left[\frac{\det[\exp(\beta\lambda_i\nu_j)]}{\Delta(\lambda)\Delta(\nu)}\right], \quad (4.1)$$

where $\lambda$ and $\nu$ are eigenvalues of the matrices $M_1$ and $M_2$, respectively. This result, which is a special case of a more general formula by Harish-Chandra [10] was extensively used in the theory of matrix models. Here we present a very simple direct derivation using the character expansions.

To derive the Itzykson-Zuber formula using Eq. (3.1) we expand the integrand

$$\exp[\beta\,\mathrm{Tr}(M_1UM_2U^\dagger)]=\sum_r \alpha_r\chi_r(M_1UM_2U^\dagger). \quad (4.2)$$

In writing Eq. (4.2) we assumed that the constant matrices $M_1$ and $M_2$ belong to the group algebra. The group integration is easily carried out using the formula

$$\int dU\mathcal{U}^{(r)}_{\sigma\beta}\mathcal{U}^{*(r')}_{\gamma\delta}=\frac{1}{d_r}\delta^{rr'}\delta_{\sigma\gamma}\delta_{\beta\delta}, \quad (4.3)$$

where $\mathcal{U}^{(r)}$ is the group matrix element in the representation $r$, $d_r$ is the dimension of the representation, and the Greek indices run from 1 to $d_r$. A proof of Eq. (4.3) is given in standard texts, see, e.g., Ref. [11]. Since the character is given by $\chi_r(U)=\Sigma_\alpha\mathcal{U}^{(r)}_{\alpha\alpha}$ setting $\sigma=\gamma$ and $\beta=\delta$ in Eq. (4.3) gives the orthogonality formula for the characters:

$$\int dU\chi_r(U)\chi_{r'}(U)=\delta^{rr'}. \quad (4.4)$$

Using Eqs. (4.2) and (4.3) one gets

$$\int dU\,\exp[\beta\,\mathrm{Tr}(M_1UM_2U^\dagger)]=\sum_r\frac{\alpha_r}{d_r}\chi_r(M_1)\chi_r(M_2), \quad (4.5)$$

which, using Eq. (3.3) can be written as

$$\int dU\,\exp[\beta\,\mathrm{Tr}(M_1UM_2U^\dagger)] =\sum_r \beta^{n_1+n_2+\cdots+n_N}\left[\prod_{i=1}^{N}\frac{(N-i)!}{(N+n_i-i)!}\right]\chi_r(M_1)\chi_r(M_2). \quad (4.6)$$

Using Weyl's formula, Eq. (2.1), one can rewrite Eq. (4.6) as

$$\int dU\,\exp[\beta\,\mathrm{Tr}(M_1UM_2U^\dagger)] =\sum_{n_1\geqslant n_2\geqslant\cdots\geqslant n_N}\beta^{n_1+n_2+\ldots+n_N}\left[\prod_{i=1}^{N}\frac{(N-i)!}{(N+n_i-i)!}\right] \times\left[\frac{\det(\lambda_i^{n_j+N-j})\det(\nu_i^{n_j+N-j})}{\Delta(\lambda)\Delta(\nu)}\right]. \quad (4.7)$$

Replacing $n_i$ by $k_i$ of Eq. (2.9), the above equation takes the form

$$\int dU\,\exp[\beta\,\mathrm{Tr}(M_1UM_2U^\dagger)] =\sum_{k_1>k_2>\cdots>k_N}\beta^{k_1+k_2+\ldots+k_N-N(N-1)/2}\left(\prod_{i=1}^{N}\frac{(N-i)!}{k_i!}\right) \times\left[\frac{\det(\lambda_i^{k_j})\det(\nu_i^{k_j})}{\Delta(\lambda)\Delta(\nu)}\right]. \quad (4.8)$$

Using the power series expansion of the exponential function and the theorem Eq. (B2) in the Appendix B one can easily rewrite the right-hand side of Eq. (4.8) to yield the result given in Eq. (4.1):

$$\int dU\,\exp[\beta\,\mathrm{Tr}(M_1UM_2U^\dagger)] =\left(\prod_{p=o}^{N-1}p!\right)\beta^{-N(N-1)/2}\left[\frac{\det[\exp(\beta\lambda_i\nu_j)]}{\Delta(\lambda)\Delta(\nu)}\right]. \quad (4.9)$$

In a later work a generalized form of the Itzykson-Zuber integral was calculated [12,13]. The integral in question is

$$I=\int dU\int dV\,\exp[i\,\mathrm{Re\,Tr}(UxV^\dagger y)]. \quad (4.10)$$

(This integral for complex rectangular ordinary matrices was previously given in Ref. [14].) To calculate this integral we write

$$\mathrm{Re}(UxV^\dagger y)=\frac{1}{2}(UxV^\dagger y+yVxU^\dagger) \quad (4.11)$$

and expand the resulting exponentials using Eq. (3.1):

$$\exp[i\,\mathrm{Re}\,\mathrm{Tr}(UxV^\dagger y)] =\left[\sum_r \left(\frac{i}{2}\right)^{n_1+n_2+\cdots+n_N}\alpha_r\chi_r(UxV^\dagger y)\right] \times\left[\sum_{r'} \left(\frac{i}{2}\right)^{n_1'+n_2'+\cdots+n_N'}\alpha_{r'}\chi_{r'}(yVxU^\dagger)\right]. \quad (4.12)$$

Inserting Eq. (4.12) into Eq. (4.10), the group integrations can easily be carried out using Eq. (4.3) to obtain

$$I=\sum_r \left(\frac{i}{2}\right)^{2(n_1+n_2+\cdots+n_N)}\left[\frac{\alpha_r^2}{d_r^2}\right]\chi_r(x^2)\chi_r(y^2). \quad (4.13)$$

Using Eqs. (2.9), (3.3), and Weyl's character formula this equation takes the form

$$I=\sum_r \left(\frac{-1}{4}\right)^{n_1+n_2+\cdots+n_N}\frac{\left[\prod_{i=1}^N (N-i)!\right]^2}{\left[\prod_{i=1}^N k_i!\right]^2} \times\left[\frac{\det(x_i^{2k_j})\det(y_i^{2k_j})}{\Delta(x^2)\Delta(y^2)}\right]. \quad (4.14)$$

Using the power series expansion of the ordinary Bessel function of zeroth order

$$J_0(x)=\sum_{n=0}^\infty \frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}, \quad (4.15)$$

and the theorem in the Appendix B, Eq. (4.14) can be evaluated to be

$$I=(2i)^{N(N+1)}\left[\sum_{i=0}^{N-1} i!\right]^2\frac{\det[J_0(x_iy_j)]}{\Delta(x^2)\Delta(y^2)}, \quad (4.16)$$

which, up to the normalization factors, is the result obtained in Refs. [12,13].

## V. EFFECTIVE QCD PARTITION FUNCTIONS

In this section we show that character expansions can be used to calculate some group integrals which appear in studying effective QCD partition functions. The finite volume effective partition function of QCD in the sector with topological charge $\nu$ can be written as an integral over $U(N_f)$ [15]

$$Z_\nu=\int dU(\det U)^\nu \exp[V\Sigma\,\mathrm{Re}\,\mathrm{Tr}(\mathcal{M}U^\dagger)], \quad (5.1)$$

where $\Sigma$ is a constant related to the value of the quark condensate in the chiral limit and $\mathcal{M}$ is the quark mass matrix. In Ref. [15] it was shown that for equal quark masses and $N_f$ flavors this partition function is a particular determinant with modified Bessel function entries. A similar form for different quark masses was conjectured in Ref. [13], but only proven for $\nu=0$. A proof for different masses was given in the case $\nu\neq 0$ was given in Refs. [16–18].

To calculate the partition function for the case of equal masses we rewrite it as

$$Z_\nu=\int dU(\det U)^\nu \exp\left[\frac{1}{2}V\Sigma m\,\mathrm{Tr}(U+U^\dagger)\right]. \quad (5.2)$$

Using the character expansion in Eq. (3.11) this takes the form

$$Z_\nu=\int dU\sum_{m_1=0}^{+\infty}\sum_{m_2=0}^{+\infty}\cdots\sum_{m_{N-1}=0}^{+\infty}\sum_{n_N=-\infty}^{+\infty} \times\det[I_{n_j+i-j}(V\Sigma m)](\det U)^{\nu-n_N}\chi_{(l_1,l_2,\ldots,l_N)}(U^\dagger), \quad (5.3)$$

where we used $\det U^\dagger=(\det U)^{-1}$. Rewriting $(\det U)^{\nu-n_N} =\chi_{(\nu-n_N,\nu-n_N,\ldots,\nu-n_N)}(U)$, and carrying out the group integration using Eq. (4.3) we get $\nu-n_N=l_i=0$ for all $i$. Equations (2.11) and (2.14) then indicate that the only surviving partition is $(\nu,\nu,\ldots,\nu)$ and we get

$$Z_\nu=\det[I_{\nu+i-j}(V\Sigma m)], \quad (5.4)$$

which is the desired result.

For unequal masses we rewrite the partition function as

$$Z_\nu=\int dU(\det U)^\nu \exp\left[\frac{1}{2}\mathrm{Tr}(M^\dagger U+U^\dagger M)\right]. \quad (5.5)$$

Note that in this case we cannot use the character expansion in Eq. (3.11) since in general $M^\dagger U$ is not a unitary matrix. Instead, using Eq. (3.1) we write

$$\exp(x\,\mathrm{Tr}\,U^\dagger M)=\sum_r \alpha_r(x)\chi_r(U^\dagger M). \quad (5.6)$$

Similarly using Eq. (3.5) we write

$$(\det M^\dagger U)^\nu e^{x\,\mathrm{Tr}\,M^\dagger U} =\sum_r x^{n_1+\cdots+n_N-N\nu}\left(\det\frac{1}{(n_j-\nu+i-j)!}\right) \times\chi_{n_1,\ldots,n_N}(M^\dagger U). \quad (5.7)$$

Inserting Eqs. (5.6) and (5.7) into Eq. (5.5) the group integral can easily be done:

$$Z_\nu=(\det M^\dagger)^{-\nu}\sum_r \left(\frac{1}{2}\right)^{2(n_1+\ldots+n_N)-N\nu} \times\left(\det\frac{1}{(n_j-\nu+i-j)!}\right)\left[\prod_{i=1}^N\frac{(N-i)!}{(N+n_i-i)!}\right]\chi_r(M^\dagger M). \tag{5.8}$$

Changing the labels $n_i$ into $k_i$ of Eq. (2.9) and rewriting $\chi_r(M^\dagger M)$ as a ratio of two determinants [see Eq. (2.1)] we rewrite the partition function as

$$Z_\nu=(\det M^\dagger)^{-\nu}\left(\frac{1}{2}\right)^{-N(N-1)-N\nu} \times\left[\prod_{i=1}^N (N-i)!\right]\frac{1}{\Delta(\mu_1^2,\ldots,\mu_N^2)} \times\sum_{k_1>k_2>\ldots>k_N}\det\left[\frac{1}{k_j!(k_j-N-\nu+i)!}\right] \times\det\left[\left(\frac{\mu_i}{2}\right)^{2k_j}\right], \tag{5.9}$$

where $\mu_i$ are the eigenvalues of the matrix $M$. Noting the power series expansion of the Bessel function

$$\frac{1}{\sqrt{y^\lambda}}I_\lambda(2\sqrt{y})=\sum_{k=0}^\infty\frac{1}{k!(\lambda+k)!}y^k, \tag{5.10}$$

and using Eq. (B4) of the Appendix B, Eq. (5.9) takes the form

$$Z_\nu=\left(\frac{1}{2}\right)^{-N(N-1)/2} \times\left[\prod_{i=1}^N (N-i)!\right]\frac{1}{\Delta(\mu_1^2,\ldots,\mu_N^2)} \times\det[\mu_j^{N-i}I_{-\nu-N+i}(\mu_j)] \tag{5.11}$$

which is the desired result. Using the fact $I_n=I_{-n}$, the determinant in the Eq. (5.11) can be rearranged to yield the often-quoted form

$$Z_\nu=\left(\frac{1}{2}\right)^{-N(N-1)/2}\left[\prod_{i=1}^N (N-i)!\right] \times\frac{1}{\Delta(\mu_1^2,\ldots,\mu_N^2)}\det[\mu_i^{j-1}I_{\nu+j-1}(\mu_i)]. \tag{5.12}$$

## VI. CONCLUSIONS

Character expansion is a powerful group theoretical technique which should have more widespread use than it currently enjoys [19]. One obstacle was the difficulty in calculating the coefficients of the characters in the expansions and the formulas presented in this paper should help in this regard. We tried to illustrate the utility of the method by rederiving a number of results in the literature in a much more direct way. In the future papers we will cover new applications of the character expansion technique. For example, in Ref. [20] some remarkable relations are derived among the effective partition functions relevant for describing microscopic Dirac spectrum. Our method can be used to explore the nature of such relations and derive new ones.

The character expansions derived using our formula can be generalized to the supergroup $U(N/M)$. The characters of the covariant class I representations of this supergroup are given by a formula similar to Eq. (2.3) except that the complete symmetric functions are replaced by the graded homogeneous symmetric functions [6]. The former can be written in terms of the traces of the fundamental representation. The latter are given by similar expressions except that traces are replaced by supertraces [6,21]. Since our character expansion formulas are basically combinatorial in nature they are applicable in principle to the covariant representations of the supergroup $U(N/M)$. More general representations of $U(1/1)$ were considered in Ref. [22]. One should however note that the characters [21] and the invariant integration [23] for the orthosymplectic supergroup $Osp(N/2M)$ are much more complicated than those of $U(N/M)$. However an approach based on Gelfand-Tzetlin coordinates may be gainfully utilized for both $U(N/M)$ and $Osp(N/2M)$ type supergroups [24]. A detailed analysis of the extension of our character expansion formula to supergroups will de deferred to later work.

Another possible application of our formulas is in the random matrix theory. It was shown that random-matrix theories provide common concepts to various aspects of quantum phenomena [25]. Using random-matrix theory concepts it is possible to study various aspects of the QCD Dirac operator [26,16,27]. In this case symmetry considerations lead to not only chiral unitary, but also chiral Gaussian orthogonal and symplectic ensembles, which in turn require generalization of our character expansion formula to orthogonal and symplectic groups.

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## APPENDIX A: SYMMETRIC FUNCTIONS

The complete homogeneous symmetric function, $h_n(x)$, of degree $n$ in the arguments $x_i, i=1,\ldots,N$, is defined as the sum of the products of the variables $x_i$, taking $n$ of them at a time. For three variables $x_1,x_2,x_3$, the first few complete homogeneous symmetric functions are

$$h_1(x)=x_1+x_2+x_3,$$

$$h_2(x)=x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3,$$

$$h_3(x)=\sum_i x_i^3+\sum_{i\neq j} x_i^2x_j+x_1x_2x_3.$$

One can write the generating function for $h_n$ as

$$\frac{1}{\prod_{i=1}^{N}(1-x_iz)}=\sum_n h_n(x)z^n. \quad \text{(A1)}$$

If $x_1,x_2,x_3$ are the eigenvalues of a matrix $B$, the symmetric functions can be written in terms of traces of powers of $B$, e.g.,

$$h_1(x)=\mathrm{Tr}\ B,$$

$$h_2(x)=\frac{1}{2}[\mathrm{Tr}\ B^2+(\mathrm{Tr}\ B)^2],$$

and so on.

The elementary symmetric functions, $a_n(x)$, are defined in a similar way except that no $x_i$ can be repeated in any product. Again for three variables $x_1,x_2,x_3$, the first few elementary symmetric functions are

$$a_1=h_1,$$

$$a_2=x_1x_2+x_1x_3+x_2x_3,$$

$$a_3=x_1x_2x_3.$$

One takes $a_n=0$ if $n>N$ and $a_0=h_0=1$. The generating function for $a_n$ is given by

$$\prod_{i=1}^{N}(1-x_iz)=\sum_n (-1)^n a_n(x)z^n. \quad \text{(A2)}$$

Note that, since the generating functions in Eqs. (A1) and (A2) are inverses of each other one can write $h_k$ in terms of $a_i, i=1,\ldots,k$ and vice versa. If one takes $x_i, i=1,\ldots,N$ to be eigenvalues of an $N\times N$ matrix $A$, then $a_N(x)=\det A$. If $A$ is an element of $U(N)$, then one can consider Eqs. (A1) and (A2) as character expansions since

$$h_n(U)=\chi_{(n,0,\ldots,0)}(U)$$

[see Eq. (2.3)] and

$$a_N(U)=\chi_{(1,1,\ldots,1)}(U).$$

One can associate a Young Tableau with a given partition $(n_1,\ldots,n_N)$ where the number of boxes at the $i$th row of the Young Tableau is $n_i$. A conjugate partition $(m_1,m_2,\ldots,m_N)$ is defined such that $m_i$ is the number of the boxes at the $i$th column. One can either write the Weyl's formula using Eq. (2.3) in terms of $h_n$'s as written [in terms of the partition $(n_1,n_2,\ldots,n_N)$] or in terms of the conjugate partition by replacing $h_n$'s with $a_n$'s (for details see Ref. [8]).

## APPENDIX B: DETERMINANT EXPANSION THEOREMS

Here we state the expansion theorem we use in the text. A proof is given in Ref. [3]. Consider the power series expansion

$$f(z)=a_0+a_1z+a_2z^2+\ldots, \quad \text{(B1)}$$

convergent for $|z|<\rho$. Then for $|x_iy_j|<\rho, i,j=1,\ldots,N$ one can write

$$\det[f(x_iy_j)]=\sum_{k_1>k_2>\ldots>k_N} a_{k_1}a_{k_2}\ldots a_{k_N}\det(x_i^{k_j})\det(y_i^{k_j}). \quad \text{(B2)}$$

A generalization of this result is known as the Binet-Cauchy formula. If the power series

$$f_i(z)=\sum_{k=0}^{\infty} a_k^{(i)}z^k \quad \text{(B3)}$$

is convergent for $|z|<\rho$, then for $|z_i|<\rho, \forall i$ we have

$$\det f_i(z_j)=\sum_{k_1>k_2>\ldots>k_N}(\det a_{k_j}^{(i)})(\det z_i^{k_j}). \quad \text{(B4)}$$

A proof is given again in Ref. [3].


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