Nonperturbative mass renormalization in quenched QED from the worldline variational approach

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Following Feynman's successful treatment of the polaron problem we apply the same variational principle to quenched QED in the worldline formulation. New features arise from the description of fermions by Grassmann trajectories, the supersymmetry between bosonic and fermionic variables and the much more singular structure of a renormalizable gauge theory such as QED in $3+1$ dimensions. We take as a trial action a general retarded quadratic action both for the bosonic and fermionic degrees of freedom and derive the variational equations for the corresponding retardation functions. We find a simple analytic, non-perturbative, solution for the anomalous mass dimension $\gamma_m(\alpha)$ in the MS scheme. For small couplings we compare our result with recent four-loop perturbative calculations while at large couplings we find that $\gamma_m(\alpha)$ becomes proportional to $\sqrt{\alpha}$. The anomalous mass dimension shows no obvious sign of the chiral symmetry breaking observed in calculations based on the use of Dyson-Schwinger equations; however, we find that a perturbative expansion of $\gamma_m(\alpha)$ diverges for $\alpha > 0.7934$. Finally, we investigate the behavior of $\gamma_m(\alpha)$ at large orders in perturbation theory.

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I. INTRODUCTION

Variational methods are widely used in many areas of physics but are not very prominent in field theory [1]. This is due to the infinite number of degrees of freedom and the singular short-distance behavior of relativistic field theories. A very successful application of variational methods in a *non-relativistic* field theory is provided by Feynman's treatment of the polaron [2]: after integrating out the phonon degrees of freedom and approximating variationally the remaining effective action by a retarded quadratic trial action one obtains the best approximation scheme which works for both small and large coupling constants. Detailed numerical investigations [3] have shown that Feynman's approximate solution deviates at most 2.2% from the true ground state energy for all coupling constants. It is therefore very attractive to apply similar techniques to problems in relativistic quantum field theory where there is much need for non-perturbative methods. In previous publications we have done that in the context of a scalar, super-renormalizable model theory [4].

In this paper we present the first results obtained by applying polaron variational methods to a realistic theory, namely quantum electrodynamics (QED) in the quenched approximation where electron-positron loops are neglected. While the actual coupling constant between electrons and photons, $\alpha = e^2/(4\pi) \simeq 1/137$, is small enough to apply perturbation theory in most cases, there is enough interest to study the theory at larger coupling: first, the strong coupling behavior of any physical theory is of interest in itself, second, the possibility of chiral symmetry breaking [5] demands an investigation at large α and, finally, bound state problems are inherently non-perturbative and involve powers of $\ln 1/\alpha \simeq 4.92$ in radiative corrections.

The extension of our methods to QED requires a formalism to include fermions and a treatment of the more severe singularities encountered in a renormalizable field theory rather than a super-renormalizable or non-relativistic one. We do this within the worldline technique which has recently experienced a revival [6]. In this formulation, the degrees of freedom describing the electron are its bosonic worldline $x_\mu(t)$, which is the four-dimensional analogue of the polaron trajectory, as well as a Grassmannian path $\zeta_\mu(t)$ needed to describe the electron's spin [7]. Here t is the proper time which parametrizes the paths and runs from 0 to T. The dynamics of the electron in an external vector field $A_\mu(x)$ with field strength $F_{\mu\nu}(x)$ are then described by the following worldline Lagrangian:

$$L = -\frac{\kappa_0}{2}\dot{x}^2 + i\zeta\cdot\dot{\zeta} + \frac{1}{T}\dot{x}\cdot\zeta\,\chi - e\,\dot{x}\cdot A(x) - \frac{ie}{\kappa_0}F_{\mu\nu}(x)\,\zeta^\mu\zeta^\nu. \tag{1}$$

Here κ_0 is an arbitrary parameter which may be used to reparametrize the proper time without changing the physics and χ is a Grassmannian (super-)partner of the proper time T. Note that the above action exhibits a well-known *supersymmetry* between bosonic and fermionic degrees of freedom [8]. For further details about the application of the worldline formalism to QED we refer the reader to Ref. [9].

The photon field A_μ may be integrated out exactly in complete analogy to the phonons in the polaron case, resulting in an effective action for the electron only:

$$S_{\rm eff}=S_0-\frac{e^2}{2}\int_0^T dt_1\,dt_2\int\frac{d^4k}{(2\pi)^4}G^{\mu\nu}(k)\times\left[\dot{x}_\mu(t_1)+\frac{2}{\kappa_0}\zeta_\mu(t_1)\,k\cdot\zeta(t_1)\right]\times\left[\dot{x}_\nu(t_2)-\frac{2}{\kappa_0}\zeta_\nu(t_2)k\cdot\zeta(t_2)\right]e^{-ik\cdot[\,x(t_1)-x(t_2)\,]}. \tag{2}$$

Here S_0 denotes the free action,

$$S_0=\int_0^T dt\left[-\frac{\kappa_0}{2}\dot{x}^2(t)+i\zeta(t)\cdot\dot{\zeta}(t)+\frac{1}{T}\dot{x}(t)\cdot\zeta(t)\,\chi\right], \tag{3}$$

and $G^{\mu\nu}(k)$ the gauge-fixed photon propagator. As described in Ref. [9], the electron propagator is obtained by carrying out a path integral over the degrees of freedom $x_\mu(t)$ and $\zeta_\mu(t)$, as well as a weighted integral over the proper times T and χ, and finally identifying the Grassmannian variable $\Gamma=\zeta(0)+\zeta(T)$ with the Dirac matrix γ:

$$G_2(p)=e^{\gamma\cdot\frac{\partial}{\partial\Gamma}}\int_0^\infty dT\int d\chi\times\exp\left\{\frac{i}{2\kappa_0}[(p^2-M_0^2)T+(p\cdot\Gamma-M_0)\chi]\right\}\times\left.\frac{\int\mathcal{D}\tilde{x}\,\mathcal{D}\zeta\,\exp[ip\cdot x+\zeta(0)\cdot\zeta(T)]\exp(iS_{\rm eff})}{\int\mathcal{D}\tilde{x}\,\mathcal{D}\zeta\,\exp[ip\cdot x+\zeta(0)\cdot\zeta(T)]\exp(iS_0)}\right|_{\Gamma=0}. \tag{4}$$

Here M_0 is the bare mass and $\mathcal{D}\tilde{x}$ contains an integration over the end point $x=x(T)$. Note that we have divided and multiplied by the path integral for the free theory, so the bare propagator may be obtained by just ignoring the last line. For non-zero couplings, of course, the path integrals in the last line cannot be performed; these we shall approximate variationally in the next section.

II. VARIATIONAL APPROACH

Feynman's variational principle has its root in Jensen's inequality for convex functions applied to $\exp(-S_E)$, where S_E is a Euclidean action. In Minkowski space and/or for complex actions the variational principle remains valid; however, it becomes a stationary principle rather than a minimum principle. To be more precise, the path integral over bosonic and fermionic paths obeys

$$\langle\exp[\,i(S-S_t)\,]\rangle_t\overset{\rm stat}{\simeq}\exp[\,i\langle S-S_t\rangle_t\,], \tag{5}$$

where $\langle\cdots\rangle_t$ indicates an average involving the weight function e^{iS_t} in the relevant functional integral and S_t is a suitable trial action. Note that corrections to this variational approximation may be calculated in a systematic way and that, furthermore, to first order in the interaction (i.e. to order α) the relation is in fact an equality if S_t reduces to the free action for small couplings.

For the trial action required in Eq. (5) we choose a general *retarded* quadratic action which is a two-time modification of the free action in Eq. (3)

$$\tilde{S}_t=S_0+i\kappa_0^2\int_0^T dt_1dt_2\left[-g_B(\sigma)\,\dot{x}(t_1)\cdot\dot{x}(t_2)+\frac{2i}{\kappa_0}g_F'(\sigma)\times\zeta(t_1)\cdot\zeta(t_2)-2\frac{\sigma}{\kappa_0T}g_{SO}'(\sigma)\,\dot{x}(t_1)\cdot\zeta(t_2)\chi\right]+\lambda_1\,p\cdot x-i\lambda_2\,\zeta(0)\cdot\zeta(T). \tag{6}$$

Here the variational parameters are contained in the retardation functions $g_i(\sigma)$ for bosonic, fermionic and spin-orbit interactions; these are even functions of $\sigma=t_1-t_2$ and they become identical for a supersymmetric trial action.[1] The variational principle ''adjusts'' these functions in order to compensate for the fact that the true effective action (2) is not quadratic in the variables $x(t)$, $\zeta(t)$. Feynman's polaron result was obtained by taking a specific ansatz for the retardation functions but here we leave their functional form free. This is because one expects that the correct short-time behavior of these functions is much more important for a renormalizable theory like QED than for the polaron problem which does not exhibit any ultraviolet divergences. Indeed one finds that for small σ the ''best'' $g_B(\sigma)$ behaves like $\sqrt{\sigma}$, $\ln\sigma$ and $1/\sigma$ in the polaron, super-renormalizable and QED case, respectively. The ''tilde'' over the trial action indicates that it includes the boundary terms already present in Eq. (4) and that we are using ''momentum averaging'' [4]. These terms, involving the external momentum p and the Grassmann variable Γ, are multiplied by additional variational parameters λ_1 and λ_2, respectively. They provide additional freedom to modify the strength of the boundary terms. We have allowed this freedom because of our experience in scalar relativistic field theory [4], where the variational parameter λ_1 turned out to be essential for describing the instability of the Wick-Cutkosky model.

Since the trial action (6) is at most quadratic in $x(t)$ and $\zeta(t)$, it is possible to evaluate the various averages required

[1]We have explicitly separated out the free action in Eq. (6), which could have alternatively been added into the second term by adding $\delta(\sigma)/(2i\kappa_0)$ to each of the $g_i(\sigma)$'s. This way our retardation functions contain no distributions. Also, in the supersymmetric limit our trial action could be written in the explicitly supersymmetric notation of Ref. [9] as $i\kappa_0^2\int_0^T dt_1dt_2\int d\theta_1d\theta_2 g(T_{12})\,DX_1\cdot DX_2$, with a single retardation function.

in Eq. (5) analytically. A particular simplification occurs if one restricts oneself to $p^2=M^2$, where M is the physical (i.e. pole) mass: as discussed in [4], the divergence of the propagator on its mass shell results from a divergence of the integral over the proper time T. Indeed, the variational approximation (5) results in an electron propagator [see Eq. (4)] which has the form

$$G_2^{\rm var}(p)=e^{\gamma\cdot\partial/\partial\Gamma}\int_0^\infty dT\int d\chi \times\exp\left\{\frac{iT}{2\kappa_0}[-M_0^2+p^2(2\lambda-\lambda^2)]\right\} \times\exp\left\{-\frac{iT}{\kappa_0}(\Omega[A_B]-\Omega[A_F]+V[\mu_B^2,\mu_F^2]) +F(\chi,\Gamma;T;p)\right\}\Big|_{\Gamma=0}, \tag{7}$$

where λ (which is defined below), the Ω's and V are T independent and the function $F(\chi,\Gamma;T;p)$ is subleading in T. The latter therefore contains information relevant for the wave function renormalization of $G_2^{\rm var}(p)$, and not the pole structure. We leave the discussion of this function, which also contains the entire χ and Γ dependence (and hence the spin structure of the propagator), for a future publication as it is not required for our present investigation.

From Eq. (7) we see that the bare and physical masses are related through

$$M_0^2=M^2(2\lambda-\lambda^2)-2(\Omega[A_B]-\Omega[A_F]+V[\mu_B^2,\mu_F^2]). \tag{8}$$

We have labeled this relationship *Mano's equation* as Mano first applied polaron techniques to a scalar relativistic field theory [10]. Note that, on the mass shell, the variational equations resulting from Eq. (5) are equivalent to demanding stationarity of Mano's equation.

The nomenclature in Mano's equation corresponds to that introduced in Ref. [4]: $\Omega[A_B]$ and $\Omega[A_F]$ originate from contributions (bosonic and fermionic, respectively) of the terms in Eq. (5) involving S_0 and S_t only. They are the analogue of the kinetic term in variational quantum mechanical calculations, while the analogue of the contribution from a potential term (explicitly proportional to the strength of the coupling) resides in V.

Similarly to Ref. [4], it is useful to express the retardation functions in terms of the variational ''profile functions'' $A_i(E)$ and the ''pseudotimes'' $\mu_i^2(\sigma), i=B,F$ defined by

$$A_i(E)=1+i\kappa_0\int_0^\infty d\sigma\, g_i(\sigma)\cos(E\sigma) \tag{9}$$

$$\mu_i^2(\sigma)=\frac{4}{\pi}\int_0^\infty dE\,\frac{1}{E^2A_i(E)}\sin^2\left(\frac{E\sigma}{2}\right), \tag{10}$$

respectively.[2] Furthermore, it is convenient to define $\lambda=\lambda_1/A_B(0)$. Indeed it turns out that the averages in Eq. (5) can be directly expressed in terms of these quantities. The kinetic terms in Eqs. (7) and (8) become

$$\Omega[A_i]=\frac{d\kappa_0}{2i\pi}\int_0^\infty dE\left(\log A_i(E)+\frac{1}{A_i(E)}-1\right), \tag{11}$$

where d is the spacetime dimension $d=4-2\epsilon$. This is identical to the result in Ref. [4] if $d=4$ and $\kappa_0=i$ (i.e. the Euclidean formulation) are taken. The specific properties of QED are encoded in the ''interaction'' term V which, with $V=V_1+V_2$, reads

$$V_1[\mu_B^2,\mu_F^2]=-(d-1)\pi\alpha\frac{\nu^{2\epsilon}}{\kappa_0}\int_0^\infty d\sigma\int\frac{d^dk}{(2\pi)^d} \times\{[\dot\mu_F^2(\sigma)]^2-[\dot\mu_B^2(\sigma)]^2\}E(k,\sigma) \tag{12}$$

$$V_2[\mu_B^2]=-\frac{4\pi\alpha\nu^{2\epsilon}\lambda^2}{\kappa_0}\int_0^\infty d\sigma\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2} \times\left[M^2+(d-2)\frac{(k\cdot p)^2}{k^2}\right]E(k,\sigma). \tag{13}$$

Note that by $\dot\mu^2(\sigma)$ we mean $(d/d\sigma)\mu^2(\sigma)$ (and not $[(d/d\sigma)\mu(\sigma)]^2$), the function $E(k,\sigma)$ is defined to be $E(k,\sigma)=\exp\{i[k^2\mu_B^2(\sigma)-2\lambda k\cdot p\sigma]/(2\kappa_0)\}$ and of course $p^2=M^2$. The fermionic contributions, both in the ''kinetic term'' Ω_F as well as in V_1, appear with an opposite sign to the bosonic contributions. The reason for the separation of V into two pieces will become apparent below.

By construction Mano's equation is *stationary* under variation of the parameters. It is important to note that we have not demanded the various retardation functions $g_{B,F}$ (as well as g_{SO}, which only plays a role for the residue) to be identical (before variation). Had we done so, the resulting profile functions A_B and A_F would have also been identical, the pseudotimes $\mu_{B,F}^2$ would have been one and the same and hence $\Omega[A_B]-\Omega[A_F]$ as well as V_1 would have vanished. The absence of a ''kinetic'' contribution would have been fatal to the variational principle as this contribution provides the restoring ''force'' to the potential V. On the other hand, closer examination of V_1 reveals that $\dot\mu_B^2\neq\dot\mu_F^2$ is also dangerous: The contribution of each of these terms is quadratically (UV) divergent if the dimensional regularization is replaced by a momentum cutoff. This may be checked by either directly substituting the small σ limit of $\mu_i^2(\sigma)$ into

[2]It is easy to show from Eq. (10) that, as long as $A_i(0)$ and $A_i(\infty)$ exist, for both asymptotically large and infinitesimally small times σ the functions $\mu_i^2(\sigma)$ become proportional to σ; hence the label ''pseudotime.'' In the free case one has $A_i(E)=1,\mu_i^2(\sigma)=\sigma,\lambda=1,\Omega_i=0$. The other variational function $A_{SO}(E)$ and parameter λ_2 are linked to the spin structure of the propagator and therefore do not show up in Mano's equation.

V_1 or by noting that for scalar QED, where the Grassmannian path integrals are absent, the remaining contribution from $\dot{\mu}_B^2$ gives rise to the quadratically divergent one-loop diagram of that theory. It is *the combination* $(\dot{\mu}_B^2)^2-(\dot{\mu}_F^2)^2$ which displays the usual logarithmic UV divergence of QED. Although at leading order in the coupling we are guaranteed to reproduce the correct perturbative result [see Eq. (5)], at higher orders the cancellation of these quadratic divergences is ensured by the supersymmetry. To summarize, on the one hand the trial action cannot be restricted to contain only supersymmetric terms but on the other hand allowing non-supersymmetric terms may destroy the renormalizability of the theory.

The way out of this predicament is provided by the variational principle itself: although it is unavoidable that the trial action breaks supersymmetry, the actual solutions to the variational equations may in fact be nearly supersymmetric. That this indeed turns out to be the case may be seen by recognizing that V_1 is the most singular part of the interaction whereas V_2, which involves only bosonic contributions and is the only source of supersymmetry breaking, is similar in structure to the scalar super-renormalizable model studied before [11]. Divergent contributions in the limit $\epsilon\to 0$ to the variational equations are solely determined by V_1. Therefore, the divergent contributions to $A_B(E)$ and $A_F(E)$, and hence to $\dot{\mu}_B^2$ and $\dot{\mu}_F^2$, are identical.

In this paper we confine ourselves to studying this divergent structure and so it is sufficient to set $A_B(E)=A_F(E)\equiv A(E)$. The corresponding variational equation becomes, after performing the k integration in Eq. (12),

$$A(E)=1+(1-\epsilon)\,c_\epsilon\,\nu^{2\epsilon}\int_0^\infty d\sigma\,\frac{\sin E\sigma}{E}\,\frac{\dot{\mu}^2(\sigma)}{[\mu^2(\sigma)]^{2-\epsilon}}\times\exp\left[-i\frac{\lambda^2M^2\sigma^2}{2\kappa_0\mu^2(\sigma)}\right]. \tag{14}$$

Note that here we have now also dropped the subscript on the pseudotime as it is no longer relevant and we have defined

$$c_\epsilon=\frac{\alpha}{\pi}\left(\frac{2\pi i}{\kappa_0}\right)^\epsilon\frac{3-2\epsilon}{(1-\epsilon)(2-\epsilon)}\xrightarrow{\epsilon\to 0}\frac{3\alpha}{2\pi}. \tag{15}$$

Since $\mu^2(\sigma)\to\sigma$ for small σ one sees that the σ integral in Eq. (14) would diverge for $\epsilon=0$; this just reflects the $1/\sigma$ behavior of the retardation function in Eq. (9) as was discussed before. The crucial difference between super-renormalizable and renormalizable theories therefore is that for the latter ones the variational equations themselves are UV divergent. In this way the divergent structure of higher-order diagrams is effectively summed up.

We may now simplify V by making use of the above ''asymptotic'' supersymmetry. The only remaining contribution is that of $V_2\equiv\lambda^2M^2W_2$ which becomes, after carrying out the integration over the momentum k,

$$W_2=\frac{(2-\epsilon)(1-\epsilon)}{2}c_\epsilon\,\nu^{2\epsilon}\int_0^\infty\frac{d\sigma}{[\mu^2(\sigma)]^{1-\epsilon}}\int_0^1\frac{du}{u^\epsilon}\times[\epsilon+(1-\epsilon)u]\exp\left(-\frac{i}{2\kappa_0}\frac{\lambda^2M^2\sigma^2}{\mu^2(\sigma)}u\right) \tag{16}$$

where the u integration arises from an exponentiation of the photon propagator in Eq. (13) in a similar way as in Ref. [4]. With this, the variational equation for λ in this asymptotic limit becomes

$$\lambda=1-\frac{\partial}{\partial\lambda}(\lambda^2W_2). \tag{17}$$

III. MASS RENORMALIZATION

Renormalizability of (quenched) QED means that all divergences can be collected in the mass and wave function renormalization constants. In the present investigation we concentrate on the mass renormalization constant in the minimal subtraction (MS) scheme, $Z_M^{\rm MS}$, defined via $M_0=Z_M^{\rm MS}M_\nu$ where M_ν is an intermediate mass scale. In this scheme it has the perturbative expansion

$$Z_M^{\rm MS}=1+\frac{b_{11}}{\epsilon}\frac{\alpha}{\pi}+\left[\frac{b_{22}}{\epsilon^2}+\frac{b_{12}}{\epsilon}\right]\left(\frac{\alpha}{\pi}\right)^2+\cdots, \tag{18}$$

where it is known from perturbation theory [12] that the expansion coefficients b_{ij} are pure, i.e. mass independent, numbers. Furthermore, the renormalization group provides relations between many of these coefficients; at order n in perturbation theory only the coefficient b_{1n} contains new information. This is encapsulated in the solution of the renormalization group equation for $Z_M^{\rm MS}$, namely

$$Z_M^{\rm MS}=\exp\left[-\frac{1}{2\epsilon}\int_0^\alpha dx\,\frac{\gamma_m(x)}{x}\right]=\exp\left[-\frac{1}{2\epsilon}\sum_{n=1}^\infty\frac{\gamma_{n-1}}{n}\left(\frac{\alpha}{\pi}\right)^n\right], \tag{19}$$

where $\gamma_m(\alpha)$ is the anomalous mass dimension of the electron [13]. In perturbation theory, $\gamma_m(\alpha)$ can be extracted from perturbative QCD calculations, which have been performed up to 4-loop order. One obtains $\gamma_0=3/2,\gamma_1=3/16$ [14], $\gamma_2=\frac{129}{64}=2.0156$ and $\gamma_3=-\frac{1}{128}[\frac{1261}{8}+336\,\zeta(3)]=-4.3868$ [15].

As the variational calculation is applicable for arbitrary values of the coupling, comparison to perturbation theory provides a useful guide to its utility. As mentioned before, to first order in the coupling the calculation is guaranteed to be exact as long as one has used a trial action which can reduce to the free action in the limit $\alpha\to 0$. A genuine test of the variational scheme is only obtained by comparing the coefficients in higher order. It should be noted that this test is much more demanding than in the polaron case where one can only compare the numerical value of the second-order coefficient for the energy: here, in addition, one tests the ϵ

dependence of this coefficient and also whether it is mass independent as it should be in the exact theory.

In order to know Ω and V at second order in α one requires variational parameters up to first order in α. These may be obtained by inserting the zeroth order results $\mu^2(\sigma)=\sigma$ and $\lambda=1$ into the variational equations for the profile function, Eq. (14), and λ, Eq. (17). The solutions then need to be substituted back into Ω [Eq. (11)] and V_2 [Eq. (16)]. Having done this, $Z_M^{\rm MS}$ may then be extracted from Mano's equation, yielding $b_{22}^{\rm var}=9/32$, which is correct, and $b_{12}^{\rm var}=0$, which should be compared to the exact value of $b_{12}=-3/64$. As in the Wick-Cutkosky model, the λ variation is of crucial importance: for example, fixing $\lambda=1$ would give a wrong result for b_{22} and a logarithmic mass dependence for b_{12}.

It is possible to develop the perturbative expansion of the variational result further, with the result that no mass dependence in the coefficients appears even at higher order. Indeed, it turns out that it is in fact possible to obtain the *full* analytic expression for the anomalous mass dimension in the worldline variational approximation. We shall sketch the derivation below, leaving the technical details for the Appendix to this paper.

To begin with, we first drop the mass term in the variational equation (14) since it only affects long-distance physics and not the ultraviolet behavior contained in $Z_M^{\rm MS}$. Then we change variables from σ to $y=c_\epsilon(\nu^2\sigma)^\epsilon$, and equivalently for E. This has the effect of making the system of integral equations (10),(14),(16) independent of the coupling. We write these explicitly in the Appendix, where it is shown that if, for small ϵ, the pseudotime has the form

$$\frac{\mu^2(\sigma)}{\sigma}=\exp\left[-\frac{\omega_0(y)}{\epsilon}+O(\epsilon^0)\right], \tag{20}$$

then the anomalous mass dimension may be written in terms of this function $\omega_0(y)$, i.e.

$$\gamma_m^{\rm var}=\frac{v(y_0)}{1-v(y_0)}, \tag{21}$$

where $v(y)=y\omega_0'(y)$ and y_0 is determined by the implicit equation

$$y_0=\frac{3\alpha}{2\pi}e^{\omega_0(y_0)}. \tag{22}$$

On the other hand, it can also be shown (see the Appendix) that the variational equation for the pseudotime translates into an equation for $\omega_0(y)$, i.e.

$$\frac{e^{\omega_0(y)}}{y}=\frac{\pi}{2}[\,1-v(y)\,]\cot\left[\frac{\pi}{2}v(y)\right]. \tag{23}$$

This equation must in general be solved numerically. However, we note that the calculation of the anomalous mass dimension in Eq. (21) only requires knowledge of the function $v(y)$ at $y=y_0$. Furthermore, it is remarkable that, at this value of y, the combination $e^{\omega_0(y)}/y$ is precisely the combination that is fixed in terms of the coupling constant [see Eq. (22)]. Hence, at $y=y_0$, the left hand side (LHS) of Eq. (23) may be written in terms of α while on the RHS we can eliminate $v(y_0)$ completely in terms of $\gamma_m^{\rm var}$ by making use of Eq. (21). One is left with a simple implicit algebraic equation for the anomalous dimension,

$$\frac{3}{4}\alpha=(\,1+\gamma_m^{\rm var}\,)\tan\left(\frac{\pi/2\gamma_m^{\rm var}}{1+\gamma_m^{\rm var}}\right), \tag{24}$$

without ever having actually solved the variational equations themselves. Equation (24) is the main result of this paper.

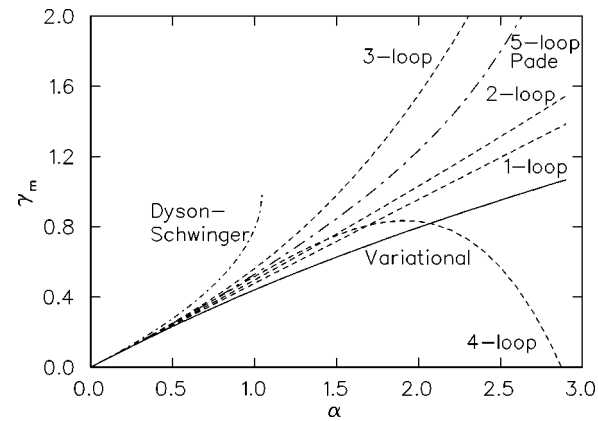

FIG. 1. Anomalous mass dimension γ_m as function of the coupling constant α in quenched QED. The variational result (24) is shown as a solid curve while the solution from the Dyson-Schwinger equations in rainbow approximation is indicated as a dot-dashed curve. The curves labeled ''n-loop'' show the result up to n-loop perturbation theory. Finally, the Padé estimation of the 5-loop result is also shown.

IV. DISCUSSION

When expanded in powers of α, Eq. (24) immediately yields

$$\gamma_m^{\rm var}(\alpha)=\frac{3}{2}\frac{\alpha}{\pi}-\frac{9}{32}\pi^2\left(\frac{\alpha}{\pi}\right)^3+\frac{27}{32}\pi^2\left(\frac{\alpha}{\pi}\right)^4 -\frac{243}{128}\pi^2\left(1-\frac{\pi^2}{20}\right)\left(\frac{\alpha}{\pi}\right)^5+\mathcal{O}(\alpha^6), \tag{25}$$

which may be compared to perturbation theory. Numerically the values of the coefficients are different but of the same order of magnitude as the exact perturbative results. Note, however, that this comparison is not particularly meaningful: the variational result is an approximation which is valid at all α. It need not have the same, or even approximately the same, perturbative expansion in α as the exact result. It should, however, be *numerically* similar. In Fig. 1 we plot the variational result as a function of the coupling and compare it to perturbation theory up to 4-loop order. For $\alpha\gtrsim 1$ the 3- and 4-loop anomalous dimensions start to deviate so much from each other that one cannot trust either of them. Also shown is the result up to 5 loops, where the 5-loop coefficient has been estimated from Padé approximations to the perturbation theory [see Eq. (2.12) of Ref. [16], which

needs to be adapted to QED with $n_f=0$ flavors; one finds $\gamma_4^{\rm Pade}=3.848$]. Clearly this does not significantly extend the numerical validity of the perturbative result. In short, the variational estimate for γ_m is roughly in agreement with (albeit apparently a little below) the perturbative result in the region where the perturbative result can be trusted.

Also shown in Fig. 1 is the only other easily available non-perturbative result for $\gamma_m^{\rm MS}$ based on the use of dimensionally regularized Dyson-Schwinger (DS) equations in ''rainbow approximation'' within the Landau gauge. This may be obtained by adapting the discussion in Ref. [17] to finite M_0, with the result that $\gamma_m^{\rm DS}=1-\sqrt{1-3\alpha/\pi}$ (which is the same as derived by Miransky [18] using a hard momentum cutoff). We see that this result deviates from perturbation theory in a region where, at least numerically, perturbation theory still appears to converge. Above $\alpha=\pi/3=1.047$ the DS result becomes complex, this value of the coupling constant coinciding with the coupling α_{cr} at which the onset of chiral symmetry breaking takes place in those calculations. This is in contrast to the variational result which remains real for all values of the coupling and in fact has the strong coupling limit

$$\gamma_m^{\rm var}(\alpha)\overset{\alpha\to\infty}{\to}\frac{1}{4}\sqrt{6\pi\alpha}-\frac{1}{2}+\mathcal{O}\left(\frac{1}{\sqrt{\alpha}}\right). \tag{26}$$

Further investigations are necessary to clarify the absence of any obvious sign of chiral symmetry breaking in the variational result for $\gamma_m^{\rm MS}(\alpha)$.[3] Indeed, in order to investigate the issue of dynamical chiral symmetry breaking, it would seem to be more straightforward, at least conceptually, to set M_0 on the right hand side of Mano's equation (8) to zero and to see if the variational equations can be satisfied in this case (for a finite physical mass M). This, however, goes considerably beyond the scope of this paper: we have merely calculated $Z_M^{\rm MS}=M_0/M_\nu$ (or, more precisely, $\gamma_m^{\rm MS}$), which means that we could simplify the calculation by (i) restricting ourselves to considering the supersymmetric massless limit of the variational equations in Sec. II and (ii) only taking into account the most divergent contributions (as $\epsilon\to 0$) to the variational equations, as well as to W_2, in Sec. III and the Appendix of this paper. In a full calculation of the RHS of Mano's equation (i.e. the additional calculation of the finite renormalization M_ν/M) these two simplifications should not be made.[4] In other words, even if $Z_M^{\rm MS}\neq 0$, dynamical chiral symmetry breaking can still occur if M_ν/M vanishes for finite M.

[3]The reader should note that the issue of dynamical chiral symmetry breaking in a dimensionally regulated theory is a notoriously subtle problem; see Ref. [17]. In particular, it was shown there that if four dimensional quenched QED breaks chiral symmetry *above* a critical coupling, then the dimensionally regularized theory will break it for *all* couplings at finite ϵ.

[4]This situation is analogous to what is the case in perturbative calculations, where anomalous dimensions of operators are far easier to calculate than finite contributions.

It is interesting to note, however, that there are also some strong similarities in the analytic structure of the variational and DS result. A perturbative inversion of Eq. (24), i.e. the expansion $\gamma_m^{\rm var}(\alpha)=\Sigma_{n=1}^{\infty}c_n\alpha^n$, has a finite radius of convergence due to a branch cut in the complex α plane. The position of this cut, and hence the radius of convergence, can be determined most easily by searching for the value of α at which Eq. (24) has two solutions for $\gamma_m^{\rm var}$ which are infinitesimally close to each other. This amounts to demanding that Eq. (24) be satisfied and at the same time the derivative of its RHS vanish, i.e.

$$0=\cot\left(\frac{\pi/2}{1+\gamma_m^{\rm var}}\right)+\frac{\pi/2}{1+\gamma_m^{\rm var}}\Big/\sin^2\left(\frac{\pi/2}{1+\gamma_m^{\rm var}}\right). \tag{27}$$

One finds that $\alpha_{con}=0.7934$, which is not too different from the radius of convergence of the DS result [19]. It is not clear whether this similarity between α_{cr} and α_{con} is accidental or not.

In connection with this, it is interesting to note that for large n the behavior of the expansion coefficients c_n in both the variational result and the DS result are rather similar:

$$c_n\approx\alpha_{con}^{-n}\frac{e^{-\beta}}{n^{3/2}}\sin\left[\left(a+\frac{5\pi}{7}\right)n-\frac{3\pi}{7}+b\right], \tag{28}$$

where numerically $\beta\approx 1.38$, $a\approx 2.3\times 10^{-3}$ and $b\approx -8.27\times 10^{-2}$. For the DS result one obtains $\beta=\log(2\sqrt{\pi})=1.27$ and the sine function is absent. It is the sine function in the variational result which is responsible for placing the branch point (which, for the DS result, is on the positive real axis) into the complex plane. Furthermore, it is remarkable that the large-α limit of $|\gamma_m(\alpha)|$ obtained in Eq. (26) is almost the same as for the DS result: $|\gamma_m^{\rm var}(\alpha)|\to 1.09\sqrt{\alpha}$ vs $|\gamma_m^{\rm DS}(\alpha)|\to 0.98\sqrt{\alpha}$.

It should be pointed out that a finite radius of convergence of the perturbation expansion is *not* what one generally expects from calculations of large orders of perturbation theory using the methods of Lipatov and others [20]. Rather, the factorial growth of the number of diagrams at nth order in perturbation theory tends to lead to a vanishing radius of convergence. As has been observed elsewhere [4], it can be shown that the variational calculation contains (pieces of) all possible Feynman diagrams at any order in perturbation theory. One concludes, therefore, that at nth order in perturbation theory there are either strong cancellations between diagrams in the variational calculation or that $\mathcal{O}(n!)$ of them give a vanishing contribution.

V. SUMMARY AND OUTLOOK

We have applied polaron variational techniques to quenched QED in $3+1$ dimensions and obtained, within the MS scheme, a remarkably simple expression for the anomalous mass dimension valid for arbitrary couplings. The approach has considerable advantages over other techniques in that it automatically maintains gauge invariance, as well as the requirements of the renormalization group, and corrections can be systematically calculated (as has been done in

the polaron case [21]). Furthermore, we have shown that the numerical results for γ_m are rather reasonable at small coupling and that at large couplings the perturbative expansion of this quantity fails in a way similar to rainbow DS results. It would be interesting to compare to DS calculations which go beyond the ladder approximation, thus decreasing the strong gauge dependence inherent in that approximation. Furthermore, variational calculations with more general trial actions could give an indication whether this analytic structure is robust, thus indicating possible large cancellations between diagrams at high order in the perturbation theory of quenched QED, or whether this structure is just an artifact of the particular trial action used in this paper. Finally, we note that the calculation of physical observables or application to bound state problems also seems feasible within the variational worldline approach developed here.

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APPENDIX

In this appendix we provide some of the technical details which enter into the derivation of the variational approximation to the anomalous dimension. To begin with, we shall scale the trivial σ dependence out of $\mu^2(\sigma)$ and define the reduced pseudotime $s(\sigma)$ as

$$\mu^2(\sigma)=\sigma s(\sigma). \tag{A1}$$

As argued in the main text, mass terms can be dropped for the calculation of the mass anomalous dimension. A perturbative evaluation of the variational equations (14) and (10) for $M=0$ then shows that the profile function and reduced pseudotime have an expansion in powers of $E^{-\epsilon}$ and σ^{ϵ}, respectively:

$$A(E)=1+\sum_{n=1} A_n\left(\frac{\nu^2}{E}\right)^{n\epsilon}, \quad s(\sigma)=1+\sum_{n=1} s_n(\nu^2\sigma)^{n\epsilon}. \tag{A2}$$

One finds

$$A_1=c_\epsilon\Gamma(\epsilon)\cos\left(\frac{\epsilon\pi}{2}\right)\overset{\epsilon\to 0}{\to}\frac{3\alpha}{2\pi}\frac{1}{\epsilon}, s_1=-\frac{c_\epsilon}{\epsilon(1+\epsilon)}\overset{\epsilon\to 0}{\to}-\frac{3\alpha}{2\pi}\frac{1}{\epsilon}$$

$$A_2=\frac{1}{2}\left(\frac{c_\epsilon}{\epsilon}\right)^2\frac{1-\epsilon}{1+\epsilon}\Gamma(1+2\epsilon)\cos(\epsilon\pi)\overset{\epsilon\to 0}{\to}\frac{1}{2}\left(\frac{3\alpha}{2\pi}\frac{1}{\epsilon}\right)^2 \tag{A3}$$

$$s_2=\frac{1}{2}\left(\frac{c_\epsilon}{\epsilon}\right)^2\frac{1}{1+2\epsilon}\left[\left(1+\frac{1}{\cos(\epsilon\pi)}\right)\frac{\Gamma^2(1+\epsilon)}{\Gamma(1+2\epsilon)}-\frac{1-\epsilon}{1+\epsilon}\right]$$
$$\overset{\epsilon\to 0}{\to}\frac{1}{2}\left(\frac{3\alpha}{2\pi}\frac{1}{\epsilon}\right)^2.$$

This suggests that perhaps the leading ϵ behavior of the coefficients is

$$A_n\overset{\epsilon\to 0}{\to}\frac{1}{n!}\left(\frac{3\alpha}{2\pi}\frac{1}{\epsilon}\right)^n, \quad s_n\overset{\epsilon\to 0}{\to}\frac{(-1)^n}{n!}\left(\frac{3\alpha}{2\pi}\frac{1}{\epsilon}\right)^n. \tag{A4}$$

In view of these results we rewrite all equations in terms of the dimensionless quantities

$$y=c_\epsilon(\nu^2\sigma)^\epsilon, \quad z=c_\epsilon\left(\frac{\nu^2}{E}\right)^\epsilon \tag{A5}$$

where we have also rescaled by c_ϵ (which is linear in the coupling) because the mass scale ν always appears in the combination $\alpha\,\nu^{2\epsilon}$ in dimensional regularization. In a similar way, the variational parameter λ almost always appears in the combination λM, so it is convenient to define the dimensionless combination

$$a_\epsilon=c_\epsilon\left(\frac{2\kappa_0\nu^2}{i\lambda^2M^2}\right)^\epsilon. \tag{A6}$$

With these definitions, the massless variational equation (14) for $A(E)$ may be brought into the form

$$A(z)=1+\frac{1}{\epsilon}\int_0^\infty dy\,\frac{\cos(y/z)^{1/\epsilon}}{[s(y)]^{1-\epsilon}}. \tag{A7}$$

The reduced pseudotime [i.e. the rescaled version of Eq. (10)] is now given by

$$s(y)=\frac{2}{\pi}\frac{1}{\epsilon}\int_0^\infty dz\,\frac{1}{z}\left(\frac{z}{y}\right)^{1/\epsilon}\frac{1-\cos(y/z)^{1/\epsilon}}{A(z)}, \tag{A8}$$

while the rescaled potential W_2 becomes a function of a_ϵ alone,

$$W_2(a_\epsilon)=\frac{(2-\epsilon)(1-\epsilon)}{2\epsilon}\int_0^\infty\frac{dy}{[s(y)]^{1-\epsilon}}\int_0^1\frac{du}{u^\epsilon}$$
$$\times[\epsilon+(1-\epsilon)u]\exp\left[-(y/a_\epsilon)^{1/\epsilon}\frac{u}{s(y)}\right], \tag{A9}$$

and hence the variational equation (17) for λ becomes

$$\frac{1}{\lambda}=1+2W_2(a_\epsilon)-2\epsilon a_\epsilon W_2'(a_\epsilon). \tag{A10}$$

The anomalous mass dimension in the MS scheme may be defined [see Eq. (19)] through

$$\gamma_m = -\lim_{\epsilon\to 0}\epsilon\frac{\alpha}{Z_M^2}\frac{\partial}{\partial\alpha}Z_M^2. \tag{A11}$$

Note that this equation is correct independently of whether Z_M has been calculated in the MS scheme or whether it is defined through Mano's equation by $Z_M \equiv M_0/M$. It is because of this fact that we can derive a nonperturbative expression for γ_m in the MS scheme, even though this scheme is usually only used within the context of perturbation theory.

As the α dependence of Z_M now only enters through the variable a_ϵ (and of course implicitly through the variational parameters), it is not surprising that we may use the variational equation (A10) for λ to simplify γ_m. Indeed, differentiating Mano's equation with respect to the coupling gives

$$\frac{\partial}{\partial\alpha}Z_M^2 = \frac{\partial\lambda}{\partial\alpha}\frac{\partial}{\partial\lambda}Z_M^2 - 2\frac{\partial a_\epsilon}{\partial\alpha}\lambda^2 W_2'(a_\epsilon). \tag{A12}$$

The first term is zero because of the variational equation for λ, $\partial a_\epsilon/\partial\alpha$ is just a_ϵ/α and by substituting the variational equation for λ into Mano's equation we find

$$Z_M^2 = \lambda\,[\,1 - 2\epsilon\lambda a_\epsilon W_2'(a_\epsilon)\,]. \tag{A13}$$

Hence the anomalous mass dimension is just given by

$$\gamma_m = \lim_{\epsilon\to 0}\frac{2\epsilon\lambda a_\epsilon W_2'(a_\epsilon)}{1 - 2\epsilon\lambda a_\epsilon W_2'(a_\epsilon)}. \tag{A14}$$

In order to proceed further, we need to evaluate $W_2(a_\epsilon)$. In general one would need to do this numerically; however, fortunately in Eq. (A14) only the small-ϵ limit is required. Let us assume that the reduced pseudotime may be written as

$$s(y) = \exp\left[-\frac{\omega(y,\epsilon)}{\epsilon}\right], \tag{A15}$$

where $\omega_0(y) \equiv \lim_{\epsilon\to 0}\omega(y,\epsilon)$ is finite. This is supported by the perturbative results given in Eqs. (A2) and (A4) and we shall show that this holds in general when we solve the variational equations below. In this case the exponential in Eq. (A9) has the argument

$$-\left(\frac{y}{a_\epsilon}e^{\omega(y,\epsilon)}\right)^{1/\epsilon} u. \tag{A16}$$

If the term in brackets is larger than 1, this argument will become arbitrarily large (and negative) in the limit $\epsilon\to 0$; hence it will lead to a vanishing contribution to the integral. If the term in brackets is smaller than 1, however, the argument goes to zero, the exponential factor in Eq. (A9) may be replaced by unity and the integral over u may be performed, yielding

$$W_2(a_\epsilon) \overset{\epsilon\to 0}{\to} \frac{1}{2\epsilon}\int_0^{y_0} dy \exp\left[\frac{\omega(y,\epsilon)}{\epsilon}(1-\epsilon)\right], \tag{A17}$$

where y_0 is given by the equation

$$y_0\, e^{\omega_0(y_0)} = \lim_{\epsilon\to 0} a_\epsilon. \tag{A18}$$

We have assumed here that $y e^{\omega_0(y)}$ is an increasing function of y, which will turn out to be the case. The leading term in Eq. (A17) may be obtained by integration by parts, with the result

$$W_2(a_\epsilon) \overset{\epsilon\to 0}{\to} \frac{1}{2\omega'(y_0,\epsilon)}\exp\left[\frac{\omega(y_0,\epsilon)}{\epsilon}(1-\epsilon)\right]. \tag{A19}$$

We also require the derivative of this function, which is most easily obtained by direct differentiation of Eq. (A17):

$$W_2'(a_\epsilon) \overset{\epsilon\to 0}{\to} \frac{1}{2\epsilon}\exp\left[\frac{\omega(y_0,\epsilon)}{\epsilon}(1-\epsilon)\right]\frac{e^{-\omega(y_0,\epsilon)}}{1 + y_0\omega'(y_0,\epsilon)}. \tag{A20}$$

Substitution into the variational equation for λ yields

$$\lambda \overset{\epsilon\to 0}{\to} \omega'(y_0,\epsilon)\frac{1 + y_0\,\omega'(y_0,\epsilon)}{\exp\left[\dfrac{\omega(y_0,\epsilon)}{\epsilon}(1-\epsilon)\right]} \tag{A21}$$

and hence

$$2\epsilon\lambda a_\epsilon W_2'(a_\epsilon) \overset{\epsilon\to 0}{\to} y_0\,\omega'(y_0,\epsilon) \tag{A22}$$

so that the anomalous dimension becomes

$$\gamma_m = \frac{y_0\,\omega_0'(y_0)}{1 - y_0\omega_0'(y_0)}. \tag{A23}$$

We stress that only the last equation is exact while the previous ones have correction terms for finite ϵ. In particular, the calculation of $Z_M = M_0/M$ (as opposed to $Z_M^{\rm MS} = M_0/M_\nu$) would require these additional terms and hence the result, unfortunately, does not shed light on whether Z_M could in fact be zero for finite M, which would signal chiral symmetry breaking. Note that the limit $\epsilon\to 0$ in Eq. (A18) needs some care: naively, one would conclude from the definition (A6) that the RHS equals $\lim_{\epsilon\to 0} c_\epsilon = 3\alpha/(2\pi)$ but Eq. (A21) shows that the variational parameter λ vanishes like $\exp(-\omega_0(y_0)/\epsilon)$ and therefore also gives a contribution

$$y_0\, e^{\omega_0(y_0)} = \frac{3\alpha}{2\pi}e^{2\omega_0(y_0)}, \tag{A24}$$

this being the result (22) quoted in the main text.

It now remains to calculate the function $\omega_0(y)$. The arguments used to derive the approximate expression for $W_2(a_\epsilon)$ in Eq. (A17) are more difficult to apply to the variational equation (A7) for $A(z)$ and the definition (A8) of $s(y)$ because of the rapidly oscillating trigonometric functions appearing in their integrands. We shall therefore adopt a more systematic approach at this stage and note that it is possible to write these equations in a differential form. Consider, for example, an integral of the type

$$I_\epsilon(z)=\int_0^\infty dy f(y)\cos\left(\frac{y}{z}\right)^{1/\epsilon}. \tag{A25}$$

By changing integration variable to $y^{1/\epsilon}$ and Taylor expanding the function $f(y)$ we may carry out the integration term by term by making use of the integral

$$\int_0^\infty dy y^{q-1}\cos y=\Gamma(q)\cos\left(q\frac{\pi}{2}\right). \tag{A26}$$

Hence we obtain

$$I_\epsilon(z)=\sum_{n=0}^\infty f^{(n)}(0)\frac{z^{n+1}}{(n+1)!}\Gamma[1+(n+1)\epsilon]\cos\left[(n+1)\epsilon\frac{\pi}{2}\right]. \tag{A27}$$

This expression may be resummed, by defining the dilatation operator $D_z\equiv z\,d/dz$, into the compact form

$$\begin{aligned} I_\epsilon(z)&=\Gamma(1+\epsilon D_z)\cos\left(\frac{\pi}{2}\epsilon D_z\right)\int_0^z dy\, f(y)\\ &=:\gamma_c(\epsilon D_z)\int_0^z dy\, f(y). \end{aligned} \tag{A28}$$

Hence the variational equation for $A(z)$ becomes

$$A(z)=1+\frac{1}{\epsilon}\gamma_c(\epsilon D_z)\int_0^z dy\frac{1}{[s(y)]^{1-\epsilon}} \tag{A29}$$

and in a similar way we can rewrite Eq. (A8) as

$$s(y)=\frac{1}{1+\epsilon D_y}\frac{1}{\gamma_c(\epsilon D_y)}\frac{1}{A(y)}. \tag{A30}$$

Inverting Eq. (A30) and substituting into Eq. (A29) eliminates the profile function $A(z)$:

$$\frac{1}{(1+\epsilon D_y)\gamma_c(\epsilon D_y)\,s(y)}=1+\frac{1}{\epsilon}\gamma_c(\epsilon D_y)\int_0^y dx\frac{1}{[s(x)]^{1-\epsilon}}. \tag{A31}$$

Finally, one can eliminate the integral by operating with D_y on both sides of this equation, so that

$$D_y\frac{1}{(1+\epsilon D_y)\gamma_c(\epsilon D_y)\,s(y)}=\frac{1}{\epsilon}\gamma_c(\epsilon D_y)\frac{y}{[s(y)]^{1-\epsilon}}. \tag{A32}$$

This equation may be solved systematically by defining $s(y)$ in terms of the function $\omega(y,\epsilon)$ [see Eq. (A15)] and by making the ansatz that $\omega(y,\epsilon)$ has a power expansion in ϵ:

$$\omega(y,\epsilon)=\omega_0(y)+\epsilon\omega_1(y)+\cdots. \tag{A33}$$

The crucial observation is that repeated application of the dilatation operator on an exponential of the form of Eq. (A15) results in $(\epsilon D_y)^n\exp(\omega/\epsilon)=[(y\,\omega')^n+\mathcal{O}(\epsilon)]\exp(\omega/\epsilon)$ so that at leading order in ϵ, for any function $F(\epsilon D_y)$ acting on $\exp(\pm\omega/\epsilon)$, we have

$$F(\epsilon D_y)\exp(\pm\omega/\epsilon)\overset{\epsilon\to 0}{\longrightarrow}F(\pm y\,\omega_0')\exp(\pm\omega/\epsilon). \tag{A34}$$

Applying this relation to Eq. (A32) provides the following equation for $\omega_0(y)$:

$$\frac{\omega_0'}{(1-y\,\omega_0')\,\gamma_c(-y\,\omega_0')}=\gamma_c(y\,\omega_0')\,e^{-\omega_0}, \tag{A35}$$

which is ϵ independent, justifying the ansatz (A33) *a posteriori*. This equation may be simplified considerably by making use of the reflection formula $\Gamma(z)\,\Gamma(1-z)=\pi/\sin\pi z$ for Γ functions. By defining $v(y)\equiv y\omega_0'(y)$, we then find

$$\frac{e^{\omega_0(y)}}{y}=\frac{\pi}{2}[1-v(y)]\cot\left[\frac{\pi}{2}v(y)\right], \tag{A36}$$

which is Eq. (23) in the main text. Together with the boundary condition $\omega_0(0)=0$ [i.e. $\mu^2(\sigma)\to\sigma$ for $\sigma\to 0$, as discussed in footnote 2 below Eq. (10)] the first-order nonlinear differential equation (A36) determines the function $\omega_0(y)$. Remarkably, as shown in the main text, it is not actually necessary to solve it in order to obtain the anomalous mass dimension γ_m. It is also interesting to note that due to the reflection formula all Γ functions have disappeared, which has the consequence that in a perturbative expansion of $\gamma_m^{\rm var}$ no Riemann ζ functions, but only powers of π, occur.

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