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Chiral symmetry restoration at nonzero temperature in the $SU(3)_r \times SU(3)_l$ linear sigma model

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We study patterns of chiral symmetry breaking at zero temperature and its subsequent restoration at nonzero temperature within the $SU(3)_r \times SU(3)_l$ linear sigma model. Gap equations for the masses of the scalar and pseudoscalar mesons and the non-strange and strange quark condensates are systematically derived in the Hartree approximation via the Cornwall-Jackiw-Tomboulis formalism. In the chiral limit, the chiral symmetry restoring transition is found to be first order, as predicted by universality arguments. Taking the experimental values for the meson masses, however, the transition is crossover. The absence of the $U(1)_A$ anomaly is found to drive this transition closer to being first order. At large temperatures, the mixing angles between octet and singlet states approach ideal flavor mixing.

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I. INTRODUCTION

Chiral symmetry is broken in the vacuum of quantum chromodynamics (QCD). At temperatures of about 150 MeV, lattice QCD calculations indicate that chiral symmetry is restored [1]. The order of the phase transition depends on the mass of the non-strange up and down quarks, $m_u \simeq m_d$, and the mass of the strange quark, m_s [2]. In nature, $m_u \simeq m_d \sim 10$ MeV and $m_s \sim 100$ MeV [3]. At temperatures on the order of 150 MeV, heavier quark flavors do not play an essential role.

For N_f massless quark flavors, the QCD Lagrangian has a $SU(N_f)_r \times SU(N_f)_l \times U(1)_A$ symmetry. In the vacuum, a non-vanishing expectation value of the quark condensate, $\langle \bar{q}_l q_r \rangle \neq 0$, spontaneously breaks this symmetry to the diagonal $SU(N_f)_V$ group of vector transformations, $V = r + l$. For $N_f = 3$, the effective, low-energy degrees of freedom of QCD are the scalar and pseudoscalar mesons. Since mesons are $q\bar{q}$ states, they fall in singlet and octet representations of $SU(3)_V$.

The $SU(N_f)_r \times SU(N_f)_l \times U(1)_A$ symmetry of the QCD Lagrangian is also explicitly broken by nonzero quark masses. For $M \leq N_f$ degenerate quark flavors, a $SU(M)_V$ symmetry is preserved. If $M < N_f$, the mass eigenstates are mixtures of singlet and octet states. For instance, in the pseudoscalar meson sector, this mixing occurs between the η and the η' meson, with the η meson being mostly octet and the η' meson being mostly singlet, with a mixing angle of about $-10°$ to $-20°$ [4].

As shown by 't Hooft [5], instantons also break the $U(1)_A$ symmetry explicitly to $Z(N_f)_A$ [6]. For the low-energy dynamics of QCD, however, this discrete symmetry is irrelevant.

Pisarski and Wilczek [6] discussed the order of the chiral transition using renormalization group arguments in the framework of the linear sigma model. This model is the effective theory for the low-energy degrees of freedom of QCD and incorporates the global $SU(N_f)_r \times SU(N_f)_l \times U(1)_A$ symmetry, but not the local $SU(3)_c$ color symmetry. They found that for $N_f = 2$ flavors of massless quarks, the transition can be of second order, if the $U(1)_A$ symmetry is explicitly broken by instantons. It is driven first order by fluctuations, if the $U(1)_A$ symmetry is restored at T_c. For $N_f = 3$ massless flavors, the transition is always first order. In this case, the term which breaks the $U(1)_A$ symmetry explicitly is a cubic invariant, and consequently drives the transition first order. In the absence of explicit $U(1)_A$ symmetry breaking, the transition is fluctuation-induced of first order.

In nature, the chiral symmetry of QCD is explicitly broken by nonzero quark masses. In this case, one has to resort to numerical calculations to determine the order of the chiral transition. At present, however, lattice QCD data have not unambiguously settled this issue. For physical values of the quark masses, calculations with staggered fermions [2] favor a smooth crossover transition, while calculations with Wilson fermions [7] predict the transition to be first order.

As an alternative to lattice QCD calculations, one can also use the linear sigma model to make predictions on the order of the phase transition in QCD. Furthermore, various symmetry-breaking scenarios can be more easily investigated than on the lattice. Studying the linear sigma model at nonzero temperature, however, requires many-body resummation schemes, because infrared divergences cause naive perturbation theory to break down [8].

For $N_f = 2$, effective chiral models for QCD have been studied extensively, because in this case $SU(2)_r \times SU(2)_l$ is isomorphic to $O(4)$, and $O(N)$ models [9] in general are particularly amenable to many-body approximations at nonzero temperature. For an incomplete list of references, see [10], where some of us (J.T.L. and D.H.R.) studied the $O(N)$

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model in the Hartree and large-N approximations.

For $N_f=3$, many-body resummation schemes become considerably more involved due to the larger number of degrees of freedom. The $SU(3)_r\times SU(3)_l$ linear sigma model [11] was previously studied at nonzero temperature in [12]. The genuine problem of these approaches is that they employ methods related to the standard loop expansion to compute the effective potential. At nonzero temperature, the loop expansion is known to fail in the case of spontaneously broken symmetry, since it generates imaginary masses for the particles. The physical reason for this failure is that contributions from thermal excitations to the masses are neglected. This can be amended by self-consistent resummation schemes like the mean-field or the Hartree approximation. In [13], one of us (J.S.-B.) studied the linear sigma model in the mean-field approximation.

In this work, we study the $SU(3)_r\times SU(3)_l$ linear sigma model in the Hartree approximation. We derive this approximation systematically from the Cornwall-Jackiw-Tomboulis (CJT) formalism [14]. Thus, we extend previous work for $N_f=2$ [10] to $N_f=3$. We study possible patterns of symmetry breaking in the vacuum and its subsequent restoration at nonzero temperature. We focus on both the cases where chiral symmetry is and where it is not explicitly broken by nonzero quark masses. Lattice QCD data indicate that the $U(1)_A$ anomaly becomes small near the chiral transition [15]. Therefore, we also study the influence of the $U(1)_A$ anomaly on symmetry restoration at nonzero temperature.

This paper is organized as follows. In Sec. II, we introduce the $SU(3)_r\times SU(3)_l$ linear sigma model. In Sec. III, the possible patterns of symmetry breaking in the vacuum are discussed. Section IV is devoted to the vacuum properties of the model at tree level. In Sec. V, we derive the gap equations for the condensates and the masses in Hartree approximation via the CJT effective potential. Numerical results are presented in Sec. VI. We conclude this work in Sec. VII with a summary of our results.

We use the imaginary-time formalism to compute quantities at nonzero temperature. Our notation is

$$\int_k f(k)\equiv T\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}\, f(2\pi inT,\mathbf{k}),$$

$$\int_x f(x)\equiv\int_0^{1/T}d\tau\int d^3\mathbf{x}\, f(\tau,\mathbf{x}). \tag{1}$$

We use units $\hbar=c=k_B=1$. The metric tensor is $g^{\mu\nu}=\mathrm{diag}(+,-,-,-)$. Throughout this work, all latin subscripts are adjoint U(3) indices, $a=0,\ldots,8$, and a summation over repeated indices is understood.

II. THE LINEAR SIGMA MODEL FOR THREE FLAVORS

The Lagrangian of the $SU(3)_r\times SU(3)_l$ linear sigma model is given by [11]

$$\begin{aligned}\mathcal{L}(\Phi)=&\mathrm{Tr}(\partial_\mu\Phi^\dagger\partial^\mu\Phi-m^2\Phi^\dagger\Phi)-\lambda_1[\mathrm{Tr}(\Phi^\dagger\Phi)]^2\\&-\lambda_2\mathrm{Tr}(\Phi^\dagger\Phi)^2+c[\mathrm{Det}(\Phi)+\mathrm{Det}(\Phi^\dagger)]\\&+\mathrm{Tr}[H(\Phi+\Phi^\dagger)].\end{aligned}\tag{2}$$

Φ is a complex 3×3 matrix parametrizing the scalar and pseudoscalar meson nonets,

$$\Phi=T_a\phi_a=T_a(\sigma_a+i\pi_a), \tag{3a}$$

where σ_a are the scalar fields and π_a are the pseudoscalar fields. The 3×3 matrix H breaks the symmetry explicitly and is chosen as

$$H=T_ah_a, \tag{3b}$$

where h_a are nine external fields. $T_a=\hat{\lambda}_a/2$ are the generators of U(3), where $\hat{\lambda}_a$ are the Gell-Mann matrices with $\hat{\lambda}_0=\sqrt{\frac{2}{3}}\mathbf{1}$. The T_a are normalized such that $\mathrm{Tr}(T_aT_b)=\delta_{ab}/2$ and obey the U(3) algebra with

$$[T_a,T_b]=if_{abc}T_c, \tag{4a}$$

$$\{T_a,T_b\}=d_{abc}T_c, \tag{4b}$$

where f_{abc} and d_{abc} for $a,b,c=1,\ldots,8$ are the standard antisymmetric and symmetric structure constants of SU(3) and

$$f_{ab0}\equiv0,\quad d_{ab0}\equiv\sqrt{\frac{2}{3}}\delta_{ab}. \tag{4c}$$

In Eq. (2), m^2 is squared the tree-level mass of the fields in the absence of symmetry breaking, λ_1 and λ_2 are the two possible quartic coupling constants, and c is the cubic coupling constant. In four dimensions, the cubic and the two quartic terms are the only relevant $SU(3)_r\times SU(3)_l$ invariant operators.

The terms in the first line of Eq. (2) are actually invariant under the larger group of $U(3)_r\times U(3)_l$ symmetry transformations,

$$\Phi\to U_r\Phi U_l^\dagger,\quad U_{r,l}\equiv\exp(i\omega_{r,l}^aT^a). \tag{5}$$

Introducing $\omega_{V,A}^a\equiv(\omega_r^a\pm\omega_l^a)/2$, the right- and left-handed symmetry transformations can be alternatively written as vector, $V=r+l$, and axial vector, $A=r-l$, transformations. It is then obvious that Φ is a singlet under $U(1)_V$ transformations $\exp(i\omega_V^0T^0)$. This $U(1)_V$ is the U(1) of baryon number conservation and thus always respected. The first three terms of Eq. (2) are therefore invariant under $SU(3)_r\times SU(3)_l\times U(1)_A\cong SU(3)_V\times U(3)_A$.

The determinant terms correspond to the $U(1)_A$ anomaly in the QCD vacuum. As shown by 't Hooft [5], they arise from instantons. These terms are invariant under $SU(3)_r\times SU(3)_l\cong SU(3)_V\times SU(3)_A$ transformations, but break the $U(1)_A$ symmetry of the Lagrangrian explicitly. The last term in Eq. (2) breaks the axial and possibly the $SU(3)_V$ vector symmetries explicitly. The patterns of explicit symmetry breaking will be discussed in detail in Sec. III.

The σ_a fields are members of the scalar ($J^P=0^+$) nonet and the π_a fields are members of the pseudoscalar ($J^P=0^-$) nonet:

$$T_a\sigma_a=\frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}}a_0^0+\frac{1}{\sqrt{6}}\sigma_8+\frac{1}{\sqrt{3}}\sigma_0 & a_0^- & \kappa^- \\ a_0^+ & -\frac{1}{\sqrt{2}}a_0^0+\frac{1}{\sqrt{6}}\sigma_8+\frac{1}{\sqrt{3}}\sigma_0 & \overline{\kappa^0} \\ \kappa^+ & \kappa^0 & -\frac{2}{\sqrt{3}}\sigma_8+\frac{1}{\sqrt{3}}\sigma_0 \end{pmatrix}, \tag{6a}$$

$$T_a\pi_a=\frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0+\frac{1}{\sqrt{6}}\pi_8+\frac{1}{\sqrt{3}}\pi_0 & \pi^- & K^- \\ \pi^+ & -\frac{1}{\sqrt{2}}\pi^0+\frac{1}{\sqrt{6}}\pi_8+\frac{1}{\sqrt{3}}\pi_0 & \overline{K^0} \\ K^+ & K^0 & -\frac{2}{\sqrt{3}}\pi_8+\frac{1}{\sqrt{3}}\pi_0 \end{pmatrix}. \tag{6b}$$

Here, $\pi^{\pm}\equiv(\pi_1\pm i\pi_2)/\sqrt{2}$ and $\pi^0\equiv\pi_3$ are the charged and neutral pions, respectively. $K^{\pm}\equiv(\pi_4\pm i\pi_5)/\sqrt{2}$, $K^0\equiv(\pi_6+i\pi_7)/\sqrt{2}$, and $\overline{K^0}\equiv(\pi_6-i\pi_7)/\sqrt{2}$ are the kaons. In general, because the strange quark is much heavier than the up or down quarks, the π_0 and the π_8 are admixtures of the η and the η' meson.

The situation with the scalar nonet is not as clear. The parity partner of the pion is the $a_0(980)$ meson, i.e., $a_0^{\pm}\equiv(\sigma_1\pm i\sigma_2)/\sqrt{2}$ and $a_0^0\equiv\sigma_3$. We identify the parity partner of the kaon with the κ meson [now referred to as $K_0^*(1430)$ in [4]]. Finally, in general the σ_0 and the σ_8 are admixtures of the σ [now also referred to as $f_0(400-1200)$] and $f_0(1370)$ mesons. [Instead of the $f_0(1370)$ meson, one could have chosen the $f_0(980)$ meson. However, as we shall see in Sec. IV, in the linear sigma model the mass of this state is closer to the $f_0(1370)$.] For details concerning the phenomenological status of the scalar nonet, see, for instance [16].

In principle, there is the possibility that the scalar particles are not diquark states, but formed from two quarks and two antiquarks [17]. Then we would associate the κ with the $\kappa(900)$ discovered in πK scattering [18]. Determining the vacuum properties of the linear sigma model, cf. Sec. IV, the mass of the κ turns out to be between that of the $\kappa(900)$ and the $K_0^*(1430)$, while its width [19] is closer to the observed width of the $\kappa(900)$.

Symmetry breaking gives the Φ field a vacuum expectation value:

$$\langle\Phi\rangle\equiv T_a\overline{\sigma}_a. \tag{7}$$

Shifting the Φ field by this vacuum expectation value, the Lagrangian can be rewritten as [20]

$$\mathcal{L}=\frac{1}{2}[\partial_\mu\sigma_a\partial^\mu\sigma_a+\partial_\mu\pi_a\partial^\mu\pi_a-\sigma_a(m_S^2)_{ab}\sigma_b-\pi_a(m_P^2)_{ab}\pi_b]+\left(\mathcal{G}_{abc}-\frac{4}{3}\mathcal{F}_{abcd}\overline{\sigma}_d\right)\sigma_a\sigma_b\sigma_c$$
$$-3\left(\mathcal{G}_{abc}+\frac{4}{3}\mathcal{H}_{abcd}\overline{\sigma}_d\right)\pi_a\pi_b\sigma_c-2\mathcal{H}_{abcd}\sigma_a\sigma_b\pi_c\pi_d-\frac{1}{3}\mathcal{F}_{abcd}(\sigma_a\sigma_b\sigma_c\sigma_d+\pi_a\pi_b\pi_c\pi_d)-U(\overline{\sigma}), \tag{8}$$

where

$$U(\overline{\sigma})=\frac{m^2}{2}\overline{\sigma}_a^2-\mathcal{G}_{abc}\overline{\sigma}_a\overline{\sigma}_b\overline{\sigma}_c+\frac{1}{3}\mathcal{F}_{abcd}\overline{\sigma}_a\overline{\sigma}_b\overline{\sigma}_c\overline{\sigma}_d-h_a\overline{\sigma}_a \tag{9}$$

is the tree-level potential and $\overline{\sigma}_a$ is determined on the tree level by

$$\frac{\partial U(\overline{\sigma})}{\partial\overline{\sigma}_a}=m^2\overline{\sigma}_a-3\mathcal{G}_{abc}\overline{\sigma}_b\overline{\sigma}_c+\frac{4}{3}\mathcal{F}_{abcd}\overline{\sigma}_b\overline{\sigma}_c\overline{\sigma}_d-h_a=0. \tag{10}$$

The coefficients $\mathcal{G}_{abc}$, $\mathcal{F}_{abcd}$, and $\mathcal{H}_{abcd}$ are given by

$$\mathcal{G}_{abc}=\frac{c}{6}\left[d_{abc}-\frac{3}{2}(\delta_{a0}d_{0bc}+\delta_{b0}d_{a0c}+\delta_{c0}d_{ab0})+\frac{9}{2}d_{000}\delta_{a0}\delta_{b0}\delta_{c0}\right], \tag{11a}$$

$$\mathcal{F}_{abcd}=\frac{\lambda_1}{4}(\delta_{ab}\delta_{cd}+\delta_{ad}\delta_{bc}+\delta_{ac}\delta_{bd})+\frac{\lambda_2}{8}(d_{abn}d_{ncd}+d_{adn}d_{nbc}+d_{acn}d_{nbd}), \tag{11b}$$

$$\mathcal{H}_{abcd}=\frac{\lambda_1}{4}\delta_{ab}\delta_{cd}+\frac{\lambda_2}{8}(d_{abn}d_{ncd}+f_{acn}f_{nbd}+f_{bcn}f_{nad}). \tag{11c}$$

The tree-level masses, $(m_S^2)_{ab}$ and $(m_P^2)_{ab}$ are given by

$$(m_S^2)_{ab}=m^2\delta_{ab}-6\mathcal{G}_{abc}\bar{\sigma}_c+4\mathcal{F}_{abcd}\bar{\sigma}_c\bar{\sigma}_d, \tag{12a}$$

$$(m_P^2)_{ab}=m^2\delta_{ab}+6\mathcal{G}_{abc}\bar{\sigma}_c+4\mathcal{H}_{abcd}\bar{\sigma}_c\bar{\sigma}_d. \tag{12b}$$

In general, these mass matrices are not diagonal. Consequently, the fields (σ_a,π_a) in the standard basis of U(3) generators are not mass eigenstates. Since the mass matrices are symmetric and real, diagonalization is achieved by an orthogonal transformation:

$$\tilde{\sigma}_i=O_{ia}^{(S)}\sigma_a, \tag{13a}$$

$$\tilde{\pi}_i=O_{ia}^{(P)}\pi_a, \tag{13b}$$

$$(\tilde{m}_{S,P}^2)_i=O_{ai}^{(S,P)}(m_{S,P}^2)_{ab}O_{bi}^{(S,P)}. \tag{13c}$$

III. PATTERNS OF SYMMETRY BREAKING

In this section, we discuss possible patterns of symmetry breaking in the vacuum. We begin with the most symmetric case, i.e., with the minimum number of nontrivial couplings, and then successively reduce the symmetry.

(1) $H=0$, $c=0$, $\lambda_2=0$: For $m^2>0$, the symmetry group is O(18), on account of

$$\mathrm{Tr}(\Phi^\dagger\Phi)=\frac{1}{2}(\sigma_a^2+\pi_a^2). \tag{14}$$

The physics of the O(N) model has been studied extensively in the past [9,10] and so we shall restrict ourselves in the following to $\lambda_2\neq 0$. Here, we only mention that for $m^2<0$, the O(18) symmetry is spontaneously broken to O(17) and there are 17 Goldstone bosons.

(2) $H=0$, $c=0$, $\lambda_2\neq 0$: For $m^2>0$, the Lagrangian has a global $SU(3)_V\times U(3)_A$ symmetry. For $m^2<0$, Eq. (9) shows that Φ develops a non-vanishing vacuum expectation value, and the $U(3)_A$ symmetry is *spontaneously* broken. By the Vafa-Witten theorem [21], only the axial symmetries can be spontaneously broken, while the vector symmetries remain intact. One can distinguish two cases [22]:

(a) $\lambda_2>0$. $SU(3)_V\times U(3)_A$ is broken to $SU(3)_V$, with $\langle\Phi\rangle\sim$diag(1,1,1) and the appearance of 9 Goldstone bosons which comprise the entire pseudoscalar nonet, i.e., the pions, the kaons, the η, and the η' meson. The nine massive scalar particles fall into irreducible representations of $SU(3)_V$. Since the mesons consist of a quark (a [**3**] of $SU(3)_V$) and an antiquark (a [$\bar{\mathbf{3}}$]), these representations are a singlet and an octet, because $[\mathbf{3}]\times[\bar{\mathbf{3}}]=[\mathbf{1}]+[\mathbf{8}]$. The mass of the singlet, the σ meson, is in general different from the (degenerate) masses of the octet particles.

(b) $\lambda_2<0$. $SU(3)_V\times U(3)_A$ is broken to $SU(2)_V\times U(2)_A$, with $\langle\Phi\rangle\sim$diag(0,0,1) and 10 Goldstone bosons.

(3) $H=0$, $c\neq 0$, $\lambda_2\neq 0$: The symmetry is $SU(3)_V\times SU(3)_A$. A non-vanishing $\langle\Phi\rangle$ spontaneously breaks this symmetry to $SU(3)_V$, with the appearance of 8 Goldstone bosons, which is the complete pseudoscalar octet. The ninth Goldstone boson of case 2(a), the η' meson, becomes massive and thus is no longer a Goldstone boson, because the $U(1)_A$ symmetry is already explicitly broken. The masses of the scalar particles behave as in case 2(a). Note that from Eq. (9), $m^2<0$ is no longer required for spontaneous symmetry breaking when $c\neq 0$.

(4) $H\neq 0$, $c=0$, $\lambda_2\neq 0$: In QCD, this corresponds to non-vanishing quark masses, but vanishing $U(1)_A$ anomaly. Since $\langle\Phi\rangle$ must carry the quantum numbers of the vacuum, only h_0, h_3, and h_8 can be nonzero. One can distinguish three cases:

(a) $h_0\neq 0$, $h_3=h_8=0$. All quark masses are equal, i.e., $m_u=m_d=m_s$. In this case, the $U(3)_A$ axial symmetry is *explicitly* broken, i.e., the 9 Goldstone bosons of case 2(a) become (mass degenerate) pseudo-Goldstone bosons. The $SU(3)_V$ symmetry remains intact and the scalars follow the classification as discussed in case 2(a).

(b) $h_0\neq 0$, $h_3=0$, $h_8\neq 0$. Only the non-strange flavors are degenerate in mass, i.e., $m_u=m_d\neq m_s$. In addition to the explicitly broken $U(3)_A$ symmetry, $SU(3)_V$ is explicitly broken to $SU(2)_V$. For the scalar particles, the following applies. If there was ideal flavor mixing, i.e., the physical particles are also eigenstates of flavor, one particle is an $s\bar{s}$ state (the f_0 meson), while all others contain at least one non-strange quark or antiquark. The latter then fall into irreducible representations of $SU(2)_V$. For the scalar particles containing no strange quark or antiquark, a quark (a [**2**] of $SU(2)_V$) couples with an antiquark (a [$\bar{\mathbf{2}}$]) to form the σ meson singlet and the a_0 meson triplet, since $[\mathbf{2}]\times[\bar{\mathbf{2}}]=[\mathbf{1}]+[\mathbf{3}]$. The strange scalar particles have only one quark in a [**2**] or a [$\bar{\mathbf{2}}$] representation, and therefore fall into doublets of $SU(2)_V$. The κ^+ and κ^0 mesons form a [**2**], while the κ^- and $\bar{\kappa}^0$ mesons form a [$\bar{\mathbf{2}}$]. Because the masses of quarks and antiquarks are identical, these two doublets are mass degenerate. In nature, however, flavor mixing is not ideal and the f_0 meson has a $q\bar{q}$ admixture, just as the σ meson has an $s\bar{s}$ admixture. For the pseudoscalars, only the four non-strange pseudo-Goldstone bosons, the pions and the η' meson, are degenerate in mass. The kaons and the η

meson both have different masses. Since the pions are non-strange $q\bar{q}$ states, and the η' meson is degenerate in mass with the pions, it follows that it is also a non-strange $q\bar{q}$ state. Then, the η meson is a pure $s\bar{s}$ state, i.e., flavor mixing is ideal in the pseudoscalar sector.

(c) $h_0 \neq 0$, $h_3 \neq 0$, $h_8 \neq 0$. Here, $SU(3)_V$ is completely broken. Even the non-strange pseudo-Goldstone bosons are no longer completely degenerate in mass.

(5) $H \neq 0$, $c \neq 0$, $\lambda_2 \neq 0$: Now, from the $U(3)_A \cong SU(3)_A \times U(1)_A$ symmetry, the $U(1)_A$ is explicitly broken by instantons. Again, there are three cases:

(a) $h_0 \neq 0$, $h_3 = h_8 = 0$. In this case, the remaining $SU(3)_A$ axial symmetry is explicitly broken, i.e., the 8 Goldstone bosons of case 3 become (mass degenerate) pseudo-Goldstone bosons. As above, the $SU(3)_V$ symmetry remains intact. The scalar particles behave as in case 4(a).

(b) $h_0 \neq 0$, $h_3 = 0$, $h_8 \neq 0$. Besides the explicitly broken $SU(3)_A$ symmetry, $SU(3)_V$ is explicitly broken to $SU(2)_V$. The scalar and pseudoscalar particles behave as in case 4(b), except that the η' meson mass is different from the pion mass because of the $U(1)_A$ anomaly. Then, flavor mixing is no longer ideal in the pseudoscalar sector.

(c) $h_0 \neq 0$, $h_3 \neq 0$, $h_8 \neq 0$. This is the case realized in nature, although violation of isospin $SU(2)_V$ is small (the charged and the neutral pions are almost degenerate in mass).

In the following, we shall restrict ourselves to studying cases 2(a), 3, 4(b), and 5(b). The first two cases are interesting because they represent the idealized scenario where the quark masses are zero, i.e., the chiral limit. The last two cases are close to the situation in nature where quark masses break the chiral symmetry explicitly. Lattice QCD data indicate that the $U(1)_A$ anomaly becomes small for large temperatures [15]. This motivates our interest in cases 2(a) and 4(b), see also [13]. Since isospin $SU(2)_V$ violation is rather small in nature, it is sufficient and easier to study case 5(b) instead of 5(c).

IV. CONDENSATES AND MASSES IN THE VACUUM

In this section, we establish the vacuum properties of the $SU(3)_r \times SU(3)_l$ model in the various cases selected above. In contrast to the previous section, it is now easier to begin with the most asymmetric case 5(b). This is the case closest to nature and therefore it is natural to use the experimental values for the meson masses and decay constants as input to determine the coupling constants. The linear sigma model has six parameters, m^2, λ_1, λ_2, c, h_0, and h_8. It therefore requires six experimentally known quantities as input. We choose m_π, m_K, f_π, f_K, the average squared mass of the η and η' mesons, $m_\eta^2 + m_{\eta'}^2$, and m_σ. The other masses, m_{a_0}, m_κ, m_{f_0}, the difference $m_\eta^2 - m_{\eta'}^2$, and the mixing angles are then predicted.

In the other cases, nature does not provide us with information about the masses and the decay constants. In case 4(b), $c=0$, and we need to specify only five input parameters. It turns out that we do not need to fix $m_\eta^2 + m_{\eta'}^2$, because $m_{\eta'} = m_\pi$, and the mass of the η meson is predicted as well.

Cases 2(a) and 3 correspond to the chiral limit which is not realized in nature. Therefore, in principle one should not use experimental values for the masses and decay constants to fix the parameters m^2, λ_1, λ_2, and, in case 3, c. For the sake of definiteness, however, we use the following tentative generalizations of experimental data. We use the pion decay constant, f_π, extrapolated to the chiral limit, the σ meson mass, and, in case 3, the η' meson mass. [In case 2(a), $m_{\eta'} = m_\pi = 0$.] For the final input parameter, since the scalar octet is degenerate in mass when the $SU(3)_r \times SU(3)_l$ symmetry is not explicitly broken, we use an average of the experimental values for the masses of the scalar octet.

A. Explicit chiral symmetry breaking with $U(1)_A$ anomaly

The vacuum expectation value is $\langle \Phi \rangle = T_0 \bar{\sigma}_0 + T_8 \bar{\sigma}_8$. The equations for the condensates $\bar{\sigma}_0$ and $\bar{\sigma}_8$ read

$$h_0 = \left[m^2 - \frac{c}{\sqrt{6}} \bar{\sigma}_0 + \left(\lambda_1 + \frac{\lambda_2}{3} \right) \bar{\sigma}_0^2 \right] \bar{\sigma}_0 + \left[\frac{c}{2\sqrt{6}} + (\lambda_1 + \lambda_2) \bar{\sigma}_0 - \frac{\lambda_2}{3\sqrt{2}} \bar{\sigma}_8 \right] \bar{\sigma}_8^2, \tag{15a}$$

$$h_8 = \left[m^2 + \frac{c}{\sqrt{6}} \bar{\sigma}_0 + \frac{c}{2\sqrt{3}} \bar{\sigma}_8 + (\lambda_1 + \lambda_2) \bar{\sigma}_0^2 - \frac{\lambda_2}{\sqrt{2}} \bar{\sigma}_0 \bar{\sigma}_8 + \left(\lambda_1 + \frac{\lambda_2}{2} \right) \bar{\sigma}_8^2 \right] \bar{\sigma}_8. \tag{15b}$$

The partially conserved axial-vector current (PCAC) relations (see the Appendix) determine the values of the condensates from the pion and kaon decay constants, f_π, f_K,

$$\bar{\sigma}_0 = \frac{f_\pi + 2 f_K}{\sqrt{6}}, \tag{16a}$$

$$\bar{\sigma}_8 = \frac{2}{\sqrt{3}} (f_\pi - f_K). \tag{16b}$$

We use the experimental values $f_\pi = 92.4$ MeV, $f_K = 113$ MeV [4].

The nonzero elements of the scalar mass matrix are

$$(m_S^2)_{00} = m^2 - \sqrt{\frac{2}{3}}\, c \bar{\sigma}_0 + (3\lambda_1 + \lambda_2) \bar{\sigma}_0^2 + (\lambda_1 + \lambda_2) \bar{\sigma}_8^2, \tag{17a}$$

$$(m_S^2)_{11} = (m_S^2)_{22} = (m_S^2)_{33} = m^2 + \frac{c}{\sqrt{6}} \bar{\sigma}_0 - \frac{c}{\sqrt{3}} \bar{\sigma}_8 + (\lambda_1 + \lambda_2) \bar{\sigma}_0^2 + \sqrt{2}\, \lambda_2 \bar{\sigma}_0 \bar{\sigma}_8 + \left(\lambda_1 + \frac{\lambda_2}{2} \right) \bar{\sigma}_8^2, \tag{17b}$$

$$(m_S^2)_{44}=(m_S^2)_{55}=(m_S^2)_{66}=(m_S^2)_{77}$$
$$=m^2+\frac{c}{\sqrt{6}}\,\bar{\sigma}_0+\frac{c}{2\sqrt{3}}\,\bar{\sigma}_8+(\lambda_1+\lambda_2)\bar{\sigma}_0^2$$
$$-\frac{\lambda_2}{\sqrt{2}}\,\bar{\sigma}_0\bar{\sigma}_8+\left(\lambda_1+\frac{\lambda_2}{2}\right)\bar{\sigma}_8^2, \tag{17c}$$

$$(m_S^2)_{88}=m^2+\frac{c}{\sqrt{6}}\,\bar{\sigma}_0+\frac{c}{\sqrt{3}}\,\bar{\sigma}_8+(\lambda_1+\lambda_2)\bar{\sigma}_0^2$$
$$-\sqrt{2}\,\lambda_2\bar{\sigma}_0\bar{\sigma}_8+3\left(\lambda_1+\frac{\lambda_2}{2}\right)\bar{\sigma}_8^2, \tag{17d}$$

$$(m_S^2)_{08}=(m_S^2)_{80}=\left[\frac{c}{\sqrt{6}}+2(\lambda_1+\lambda_2)\bar{\sigma}_0-\frac{\lambda_2}{\sqrt{2}}\,\bar{\sigma}_8\right]\bar{\sigma}_8. \tag{17e}$$

While the masses of the a_0 and the κ mesons are given by the (11) and (44) elements of the mass matrix, $m_{a_0}^2 \equiv (m_S^2)_{11}$, $m_\kappa^2 \equiv (m_S^2)_{44}$, the σ and f_0 meson masses are obtained by diagonalizing the (08) sector of the mass matrix. According to Eq. (13),

$$m_\sigma^2 \equiv (\tilde{m}_S^2)_0=(m_S^2)_{00}\cos^2\theta_S+(m_S^2)_{88}\sin^2\theta_S$$
$$+2(m_S^2)_{08}\cos\theta_S\sin\theta_S, \tag{18a}$$

$$m_{f_0}^2 \equiv (\tilde{m}_S^2)_8=(m_S^2)_{00}\sin^2\theta_S+(m_S^2)_{88}\cos^2\theta_S$$
$$-2(m_S^2)_{08}\cos\theta_S\sin\theta_S, \tag{18b}$$

where the scalar mixing angle θ_S is given by

$$\tan 2\theta_S=\frac{2(m_S^2)_{08}}{(m_S^2)_{00}-(m_S^2)_{88}}. \tag{18c}$$

The pseudoscalar mass matrix is given by

$$(m_P^2)_{00}=m^2+\sqrt{\frac{2}{3}}\,c\bar{\sigma}_0+\left(\lambda_1+\frac{\lambda_2}{3}\right)(\bar{\sigma}_0^2+\bar{\sigma}_8^2), \tag{19a}$$

$$(m_P^2)_{11}=(m_P^2)_{22}=(m_P^2)_{33}$$
$$=m^2-\frac{c}{\sqrt{6}}\,\bar{\sigma}_0+\frac{c}{\sqrt{3}}\,\bar{\sigma}_8+\left(\lambda_1+\frac{\lambda_2}{3}\right)\bar{\sigma}_0^2$$
$$+\frac{\sqrt{2}}{3}\lambda_2\bar{\sigma}_0\bar{\sigma}_8+\left(\lambda_1+\frac{\lambda_2}{6}\right)\bar{\sigma}_8^2, \tag{19b}$$

$$(m_P^2)_{44}=(m_P^2)_{55}=(m_P^2)_{66}=(m_P^2)_{77}$$
$$=m^2-\frac{c}{\sqrt{6}}\,\bar{\sigma}_0-\frac{c}{2\sqrt{3}}\,\bar{\sigma}_8+\left(\lambda_1+\frac{\lambda_2}{3}\right)\bar{\sigma}_0^2$$
$$-\frac{\lambda_2}{3\sqrt{2}}\,\bar{\sigma}_0\bar{\sigma}_8+\left(\lambda_1+\frac{7}{6}\lambda_2\right)\bar{\sigma}_8^2, \tag{19c}$$

$$(m_P^2)_{88}=m^2-\frac{c}{\sqrt{6}}\,\bar{\sigma}_0-\frac{c}{\sqrt{3}}\,\bar{\sigma}_8+\left(\lambda_1+\frac{\lambda_2}{3}\right)\bar{\sigma}_0^2$$
$$-\frac{\sqrt{2}}{3}\lambda_2\bar{\sigma}_0\bar{\sigma}_8+\left(\lambda_1+\frac{\lambda_2}{2}\right)\bar{\sigma}_8^2, \tag{19d}$$

$$(m_P^2)_{08}=(m_P^2)_{80}=\left[-\frac{c}{\sqrt{6}}+\frac{2}{3}\lambda_2\bar{\sigma}_0-\frac{\lambda_2}{3\sqrt{2}}\,\bar{\sigma}_8\right]\bar{\sigma}_8. \tag{19e}$$

While the mass of the pion and the kaon are given by the (11) and (44) elements of the mass matrix, $m_\pi^2 \equiv (m_P^2)_{11}$, $m_K^2 \equiv (m_P^2)_{44}$, the η' and η meson masses are obtained by diagonalizing the (08) sector of the mass matrix. According to Eq. (13),

$$m_{\eta'}^2 \equiv (\tilde{m}_P^2)_0=(m_P^2)_{00}\cos^2\theta_P+(m_P^2)_{88}\sin^2\theta_P$$
$$+2(m_P^2)_{08}\cos\theta_P\sin\theta_P, \tag{20a}$$

$$m_\eta^2 \equiv (\tilde{m}_P^2)_8=(m_P^2)_{00}\sin^2\theta_P+(m_P^2)_{88}\cos^2\theta_P$$
$$-2(m_P^2)_{08}\cos\theta_P\sin\theta_P, \tag{20b}$$

where the pseudoscalar mixing angle θ_P is given by

$$\tan 2\theta_P=\frac{2(m_P^2)_{08}}{(m_P^2)_{00}-(m_P^2)_{88}}. \tag{20c}$$

The explicit symmetry breaking terms, h_0 and h_8, are determined from Eqs. (15), (16), (19b), and (19c) as

$$h_0=\frac{1}{\sqrt{6}}(m_\pi^2 f_\pi+2m_K^2 f_K), \tag{21a}$$

$$h_8=\frac{2}{\sqrt{3}}(m_\pi^2 f_\pi-m_K^2 f_K). \tag{21b}$$

Using the experimental pion and kaon masses, one obtains $h_0=(286.094\text{ MeV})^3$ and $h_8=-(310.960\text{ MeV})^3$.

Comparing Eqs. (15b) and (17c) reveals

$$h_8=m_\kappa^2\,\bar{\sigma}_8, \tag{22}$$

i.e., the mass of the κ meson is predicted to be $m_\kappa=1124.315$ MeV, which is about 21% smaller than the experimental value. The average η and η' meson mass squared determines λ_2

$$\lambda_2 = \frac{3(2f_K - f_\pi)m_K^2 - (2f_K + f_\pi)m_\pi^2 - 2(m_{\eta'}^2 + m_\eta^2)(f_K - f_\pi)}{[3f_\pi^2 + 8f_K(f_K - f_\pi)](f_K - f_\pi)}. \tag{23}$$

For $m_{\eta'}^2 + m_\eta^2 = (1103.625\ \text{MeV})^2$, one obtains $\lambda_2 = 46.484$.

The difference of the pion and kaon masses squared and λ_2 determine c,

$$c = \frac{m_K^2 - m_\pi^2}{f_K - f_\pi} - \lambda_2(2f_K - f_\pi) = 4807.835\ \ \text{MeV}. \tag{24}$$

Now, also the mass of the a_0 meson is fixed,

$$m_{a_0}^2 = m_\kappa^2 + (f_K - f_\pi)[c - \lambda_2(2f_K + f_\pi)] = (1028.707\ \ \text{MeV})^2, \tag{25}$$

which is about 4% larger than the experimental value. The pseudoscalar mixing angle is $\theta_P = -5^{\circ}$. This angle determines the individual η and η' meson masses to be $m_\eta = 539.008$ MeV and $m_{\eta'} = 963.046$ MeV. These values are surprisingly close to the experimental ones; the η meson is about 2% lighter and the η' meson is about 0.6% heavier than in nature.

Finally, λ_1 is determined by fixing either the mass of the σ or the f_0 meson; the other mass, the scalar mixing angle, θ_S, and m^2 are then given by solving a nonlinear equation for the fixed mass. Here, we choose $m_\sigma = 600$ MeV to yield $\lambda_1 = 1.400$ and $m_{f_0} = 1221.113$ MeV, about 11% smaller than the experimental value. The scalar mixing angle is $\theta_S = 19.859°$, and $m^2 = (342.523\ \text{MeV})^2$.

B. Explicit chiral symmetry breaking without $U(1)_A$ anomaly

In the absence of the $U(1)_A$ anomaly, $c = 0$, the condensate equations (15a) and (15b) and the equations for the scalar and pseudoscalar mass matrices (17) and (19) simplify. Equations (16) and (21), however, remain the same and thus yield the same values for $\bar{\sigma}_0$, $\bar{\sigma}_8$, h_0, and h_8 as above. The mass of the κ meson is still given by Eq. (22) and is the same as above.

The differences begin with the mass of the η' meson, which is now identical to the pion mass, since the $U(1)_A$ anomaly is absent. Furthermore, λ_2 is given by the kaon and pion masses and decay constants,

$$\lambda_2 = \frac{m_K^2 - m_\pi^2}{(2f_K - f_\pi)(f_K - f_\pi)} = 82.470. \tag{26}$$

This then determines the mass of the η meson,

$$m_\eta^2 = m_\pi^2 + 2\lambda_2 f_K(f_K - f_\pi) = (634.818\ \ \text{MeV})^2, \tag{27}$$

and the a_0 meson,

$$m_{a_0}^2 = m_\kappa^2 - \lambda_2(2f_K^2 - f_\pi f_K - f_\pi^2) = (850.387\ \ \text{MeV})^2. \tag{28}$$

The pseudoscalar mixing angle is given by $\tan 2\theta_P = 2\sqrt{2}$, or $\theta_P = 35.264^{\circ}$. This corresponds to ideal flavor mixing.

As before, λ_1 is given by solving the equation for the σ meson mass, which in turn yields the f_0 meson mass, the scalar mixing angle, and m^2. For $m_\sigma = 600$ MeV we find $\lambda_1 = -4.550$, $m_{f_0} = 1341.367$ MeV, $\theta_S = 31.326°$, and $m^2 = -(503.551\ \text{MeV}^2)$.

C. Chiral limit with $U(1)_A$ anomaly

In this case, $\langle\Phi\rangle = T_0\bar{\sigma}_0$. Using $h_a = 0$ and the explicit form for the $\mathcal{G}_{abc}$ and $\mathcal{F}_{abcd}$, the equation for the condensate at the tree level (15a) simplifies to

$$0 = \left[m^2 - \frac{c}{\sqrt{6}}\bar{\sigma}_0 + \left(\lambda_1 + \frac{\lambda_2}{3}\right)\bar{\sigma}_0^2\right]\bar{\sigma}_0. \tag{29}$$

From the PCAC relations (see the Appendix), we now obtain $\bar{\sigma}_0 = \sqrt{3/2}\, f_\pi$. Lattice QCD data indicate a linear behavior of f_π with the quark mass, $f_\pi \simeq a m_q + b$ [23]. Extrapolating these data to the chiral limit, $m_q \to 0$ (as appropriate for $H = 0$), one obtains $f_\pi = 90$ MeV.

The scalar and the pseudoscalar mass matrices are diagonal and have the particularly simple form

$$(m_S^2)_{ab} = \left[m^2 + \frac{c}{\sqrt{6}}\bar{\sigma}_0 + (\lambda_1 + \lambda_2)\bar{\sigma}_0^2\right]\delta_{ab} - \left(\sqrt{\frac{3}{2}}\,c - 2\lambda_1\bar{\sigma}_0\right)\bar{\sigma}_0\delta_{a0}\delta_{b0}, \tag{30a}$$

$$(m_P^2)_{ab} = \left[m^2 - \frac{c}{\sqrt{6}}\bar{\sigma}_0 + \left(\lambda_1 + \frac{\lambda_2}{3}\right)\bar{\sigma}_0^2\right]\delta_{ab} + \sqrt{\frac{3}{2}}\,c\,\bar{\sigma}_0\delta_{a0}\delta_{b0}. \tag{30b}$$

Taking into account Eq. (29), the pseudoscalar octet, $a,b = 1, \ldots, 8$, is massless, as expected from group theory, see case (3) in Sec. III. The singlet, the η' meson, is massive, because the $U(1)_A$ symmetry is explicitly broken by the $U(1)_A$ anomaly,

$$m_{\eta'}^2 \equiv (m_P^2)_{00} = \frac{3}{2}c f_\pi. \tag{31}$$

If we use the experimental value of the η' meson mass, $c = 6791.157$ MeV.

The scalar particles fall into a singlet, the σ meson, with mass

$$m_\sigma^2 \equiv (m_S^2)_{00} = m^2 - c f_\pi + \frac{3}{2}(3\lambda_1 + \lambda_2) f_\pi^2, \tag{32}$$

and an octet comprising the a_0, κ, and f_0 mesons,

$$m_{a_0}^2 \equiv (m_S^2)_{11} = m^2 + \frac{c}{2} f_\pi + \frac{3}{2}(\lambda_1 + \lambda_2) f_\pi^2$$
$$\equiv m_\sigma^2 + m_{\eta'}^2 - 3\lambda_1 f_\pi^2 \equiv \lambda_2 f_\pi^2 + \frac{2}{3} m_{\eta'}^2, \tag{33}$$

where the last identity follows from Eq. (29). This equation determines λ_1 and λ_2 for given m_σ, m_{a_0}, and $m_{\eta'}$. As the octet is mass degenerate, we average the experimental values for $m_{a_0}^2$, m_κ^2, and $m_{f_0}^2$, weighted by the respective isospin degeneracy, to obtain an average mass of $m_{a_0} = 1225.795$ MeV. For the σ meson we again take $m_\sigma = 600$ MeV. This results in $\lambda_1 = -9.291$ and $\lambda_2 = 110.046$. Then, m^2 follows from any of the mass equations as $m^2 = -(164.921 \text{ MeV})^2$.

Note that the scalar singlet is lighter than the scalar octet, but the pseudoscalar singlet is heavier than the pseudoscalar octet. This ''inverted mass spectrum'' for the scalar mesons relative to the pseudoscalar mesons [17] is a general feature in the presence of the $U(1)_A$ anomaly. It arises from the relative difference in sign of the terms $\sim \mathcal{G}_{abc}$ in Eqs. (12).

D. Chiral limit without $U(1)_A$ anomaly

In this case, as the $U(1)_A$ symmetry is not explicitly broken, all nine pseudoscalar particles are massless. This is readily derived from Eqs. (29) and (30b) when $c = 0$.

The scalar particles again fall into a singlet, the σ meson, with mass

$$m_\sigma^2 \equiv (m_S^2)_{00} = m^2 + \frac{3}{2}(3\lambda_1 + \lambda_2) f_\pi^2, \tag{34}$$

and an octet comprising the a_0, κ, and f_0 mesons,

$$m_{a_0}^2 \equiv (m_S^2)_{11} = m^2 + \frac{3}{2}(\lambda_1 + \lambda_2) f_\pi^2 \equiv m_\sigma^2 - 3\lambda_1 f_\pi^2 \equiv \lambda_2 f_\pi^2, \tag{35}$$

where the last identity follows from Eq. (29) with $c = 0$. This equation determines λ_1 and λ_2 for given m_σ and m_{a_0}. We again use $m_{a_0} = 1225.795$ MeV and $m_\sigma = 600$ MeV. This results in $\lambda_1 = -47.019$ and $\lambda_2 = 185.503$. Then, m^2 follows from any of the mass equations, $m^2 = -(424.264 \text{ MeV})^2$, which is negative, as required in this case for spontaneous symmetry breaking.

V. THE EFFECTIVE POTENTIAL IN THE CORNWALL-JACKIW-TOMBOULIS FORMALISM

The effective potential of the Cornwall-Jackiw-Tomboulis formalism [14] is

$$V[\bar{\sigma}, \mathcal{S}, \mathcal{P}] = U(\bar{\sigma}) + \frac{1}{2}\int_k \{[\ln \mathcal{S}^{-1}(k)]_{aa} + [\ln \mathcal{P}^{-1}(k)]_{aa}\}$$
$$+ \frac{1}{2}\int_k [S_{ab}^{-1}(k;\bar{\sigma})\mathcal{S}_{ba}(k) + P_{ab}^{-1}(k;\bar{\sigma})\mathcal{P}_{ba}(k)$$
$$- 2\delta_{ab}\delta_{ba}] + V_2[\bar{\sigma}, \mathcal{S}, \mathcal{P}]. \tag{36}$$

Here, $U(\bar{\sigma})$ is the tree-level potential of Eq. (9), and

$$S_{ab}^{-1}(k;\bar{\sigma}) = -k^2 \delta_{ab} + (m_S^2)_{ab}, \tag{37a}$$

$$P_{ab}^{-1}(k;\bar{\sigma}) = -k^2 \delta_{ab} + (m_P^2)_{ab}, \tag{37b}$$

are the tree-level propagators for scalar and pseudoscalar particles, with the respective mass matrices (12).

The expectation values for the scalar fields, $\bar{\sigma}_a$, and the full propagators for scalar, $\mathcal{S}(k)$, and pseudoscalar, $\mathcal{P}(k)$, particles are determined from the stationarity conditions

$$\frac{\delta V[\bar{\sigma}, \mathcal{S}, \mathcal{P}]}{\delta \bar{\sigma}_a} = 0, \tag{38a}$$

$$\frac{\delta V[\bar{\sigma}, \mathcal{S}, \mathcal{P}]}{\delta \mathcal{S}_{ab}(k)} = 0, \tag{38b}$$

$$\frac{\delta V[\bar{\sigma}, \mathcal{S}, \mathcal{P}]}{\delta \mathcal{P}_{ab}(k)} = 0. \tag{38c}$$

With Eq. (36), the latter two can be written in the form

$$\mathcal{S}_{ab}^{-1}(k) = S_{ab}^{-1}(k;\bar{\sigma}) + \Sigma_{ab}(k), \tag{39a}$$

$$\mathcal{P}_{ab}^{-1}(k) = P_{ab}^{-1}(k;\bar{\sigma}) + \Pi_{ab}(k), \tag{39b}$$

where

$$\Sigma_{ab}(k) \equiv 2\frac{\delta V_2[\bar{\sigma}, \mathcal{S}, \mathcal{P}]}{\delta \mathcal{S}_{ba}(k)}, \tag{40a}$$

$$\Pi_{ab}(k) \equiv 2\frac{\delta V_2[\bar{\sigma}, \mathcal{S}, \mathcal{P}]}{\delta \mathcal{P}_{ba}(k)}, \tag{40b}$$

are the self-energies for the scalar and pseudoscalar particles.

In general, $V_2[\bar{\sigma}, \mathcal{S}, \mathcal{P}]$ is the sum of all two-particle irreducible (2PI) diagrams, with all lines representing full propagators. Here, we restrict ourselves to the most simple class of 2PI diagrams, the double-bubble diagrams of Fig. 1, which is equivalent to the Hartree approximation. Explicitly,

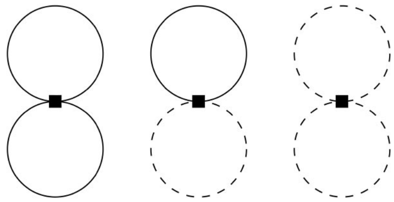

FIG. 1. The double-bubble diagrams. Full lines are scalar particles, dashed lines are pseudoscalar particles.

$$V_2[\bar{\sigma},\mathcal{S},\mathcal{P}]=\mathcal{F}_{abcd}\left[\int_k \mathcal{S}_{ab}(k)\int_p \mathcal{S}_{cd}(p)+\int_k \mathcal{P}_{ab}(k)\int_p \mathcal{P}_{cd}(p)\right]+2\mathcal{H}_{abcd}\int_k \mathcal{S}_{ab}(k)\int_p \mathcal{P}_{cd}(p). \tag{41}$$

Note that, in the Hartree approximation, V_2 is actually independent of $\bar{\sigma}_a$. Therefore, the stationarity conditions for the condensates are

$$h_a=m^2\bar{\sigma}_a-3\mathcal{G}_{abc}\left\{\bar{\sigma}_b\bar{\sigma}_c+\int_k[\mathcal{S}_{cb}(k)-\mathcal{P}_{cb}(k)]\right\}+4\mathcal{F}_{abcd}\left[\frac{1}{3}\bar{\sigma}_b\bar{\sigma}_c+\int_k \mathcal{S}_{cb}(k)\right]\bar{\sigma}_d+4\mathcal{H}_{bcad}\bar{\sigma}_d\int_k \mathcal{P}_{cb}(k). \tag{42a}$$

Since the self-energies (40) are independent of momentum in the Hartree approximation, the Schwinger-Dyson equations (39) for the full propagators assume the simple form

$$\mathcal{S}^{-1}_{ab}(k)=-k^2\delta_{ab}+(M_S^2)_{ab}, \tag{42b}$$

$$\mathcal{P}^{-1}_{ab}(k)=-k^2\delta_{ab}+(M_P^2)_{ab}, \tag{42c}$$

where the scalar and pseudoscalar mass matrices in the Hartree approximation are given by

$$(M_S^2)_{ab}=m^2\delta_{ab}-6\mathcal{G}_{abc}\bar{\sigma}_c+4\mathcal{F}_{abcd}\left[\bar{\sigma}_c\bar{\sigma}_d+\int_k \mathcal{S}_{cd}(k)\right]+4\mathcal{H}_{abcd}\int_k \mathcal{P}_{cd}(k), \tag{43a}$$

$$(M_P^2)_{ab}=m^2\delta_{ab}+6\mathcal{G}_{abc}\bar{\sigma}_c+4\mathcal{H}_{abcd}\left[\bar{\sigma}_c\bar{\sigma}_d+\int_k \mathcal{S}_{cd}(k)\right]+4\mathcal{F}_{abcd}\int_k \mathcal{P}_{cd}(k). \tag{43b}$$

In general, the mass matrices are not diagonal in the standard basis of U(3) generators, see Sec. IV. Consequently, the propagators are also not diagonal in this basis. Physically, however, only mass eigenstates can propagate. Therefore, we have to diagonalize the propagators before we compute the loop integrals in Eqs. (42a) and (43).

In the Hartree approximation, all particles are stable quasiparticles, i.e., the imaginary parts of the self-energies vanish. Therefore, the inverse propagators (42b) and (42c) are real-valued. They are also symmetric in the standard basis of U(3) generators and thus diagonalizable via an orthogonal transformation. This transformation is given by Eq. (13c), with the obvious replacements

$$(m^2_{S,P})_{ab}\to(M^2_{S,P})_{ab},\quad (\tilde{m}^2_{S,P})_i\to(\tilde{M}^2_{S,P})_i. \tag{44}$$

In cases 2(a) and 3 in Sec. III, the mass matrices are diagonal at zero temperature, cf. Sec. IV, and are taken to be diagonal at nonzero temperature T as well. In cases 4(b) and 5(b), they have off-diagonal elements in the (08) sector at zero temperature, and consequently also have off-diagonal contributions in this sector at nonzero temperature. In the latter case, diagonalization proceeds as in Eq. (13c) with the replacements (44).

The propagator matrices are diagonalized by the same orthogonal transformation as their inverse. The loop integrals in Eqs. (42a) and (43) are therefore computed, for example, as

$$\int_k \mathcal{S}_{ab}(k)=O^{(S)}_{ai}\int_k \tilde{\mathcal{S}}_i(k)O^{(S)}_{bi}, \tag{45}$$

where $\tilde{\mathcal{S}}_i(k)$ is the scalar propagator in the mass eigenbasis.

The loop integral in Eq. (45) requires renormalization. Renormalization of many-body approximation schemes is nontrivial [10], but does not change the results qualitatively. We therefore simply omit the vacuum contributions to the loop integrals and set

$$\int_k \tilde{\mathcal{S}}_i(k)=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\epsilon_{\mathbf{k}}[(\tilde{M}_S^2)_i]}\left(\exp\left\{\frac{\epsilon_{\mathbf{k}}[(\tilde{M}_S^2)_i]}{T}\right\}-1\right)^{-1}, \tag{46}$$

and similarly for the pseudoscalar loop integrals. Here, $\epsilon_{\mathbf{k}}[(\tilde{M}_S^2)_i]=\sqrt{\mathbf{k}^2+(\tilde{M}_S^2)_i}$ is the relativistic energy of the ith scalar quasiparticle with momentum $\mathbf{k}$.

VI. RESULTS

In this section, we discuss the numerical results at nonzero temperature for the four cases of interest.

A. Chiral limit

In Fig. 2(a), the masses are shown in the chiral limit with the $U(1)_A$ anomaly, $c\neq 0$. This corresponds to case 3 of Sec. III, for which the zero-temperature properties were discussed in Sec. IV C. Accordingly, there are eight Goldstone bosons, the three pions, the four kaons and the η meson, while the η' meson has a large mass due to the $U(1)_A$ anomaly. The scalar octet, comprising the three a_0 mesons, the four κ mesons, and the f_0 meson, is mass degenerate, while the singlet, the σ meson, has a different mass.

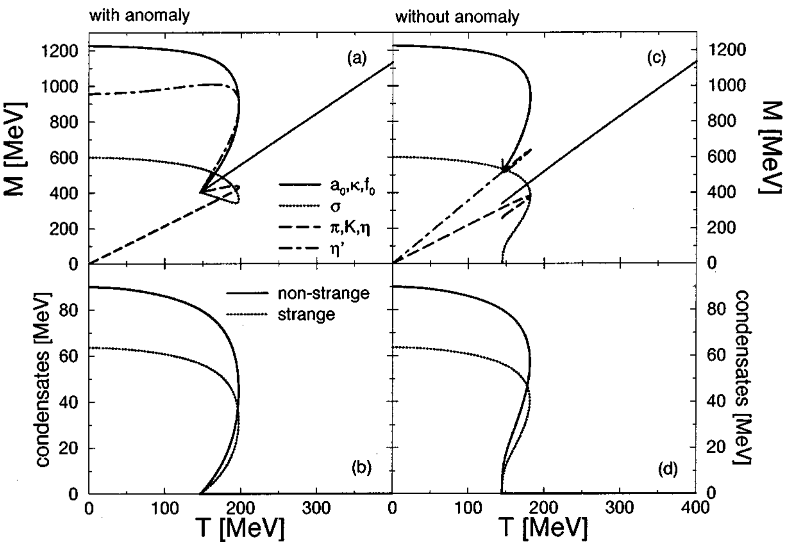

FIG. 2. The meson masses and the condensates as a function of temperature for (a,b) the case with $U(1)_A$ anomaly ($c \neq 0$) and (c,d) the case without $U(1)_A$ anomaly ($c=0$), for $H=0$. In (a,c), the full lines are the masses of the a_0, κ, and f_0 mesons, the dotted lines represent the σ meson mass, the dashed lines are π, K, and η meson masses, and the dot-dashed lines are the η' meson mass. In (b,d), the full lines represent the non-strange condensate, and the dotted lines represent the strange condensate.

As the temperature increases, the scalar masses decrease while the pseudoscalar masses increase. The mass of the Goldstone bosons (the pseudoscalar octet) increases because the Hartree approximation does not respect Goldstone's theorem at nonzero temperature [10]. We do not consider this to be a serious shortcoming: as shown in [24], supplementing the Hartree approximation with the random-phase approximation cures this problem. The analogous treatment within the CJT formalism will be deferred to a forthcoming publication [25].

At some critical temperature, $T_c \sim 170$ MeV, there is a first order phase transition between the low-temperature phase where chiral symmetry is broken and the high-temperature phase where chiral symmetry is restored and all meson masses are equal. The exact numerical value of T_c can be determined by computing the effective potential.

At the point where the condensates vanish, all masses become degenerate. The reason is that at this point the condensate equation enforces the constraint

$$0=\frac{c}{6\sqrt{6}}\left\{2\int_k[\mathcal{S}_{00}(k)-\mathcal{P}_{00}(k)] - \sum_{a=1}^{8}\int_k[\mathcal{S}_{aa}(k)-\mathcal{P}_{aa}(k)]\right\}, \qquad (47)$$

which is fulfilled when all masses are equal.

In Fig. 2(b), the non-strange and strange condensates, $\varphi_{\rm ns}$ and $\varphi_{\rm s}$, respectively, are shown as a function of temperature. In the standard basis of U(3) generators, these are defined as

$$\langle\Phi\rangle=\frac{1}{2}\begin{pmatrix}\varphi_{\rm ns} & 0 & 0\\ 0 & \varphi_{\rm ns} & 0\\ 0 & 0 & \sqrt{2}\varphi_{\rm s}\end{pmatrix}. \qquad (48)$$

When $\langle\Phi\rangle=T_0\bar{\sigma}_0$, $\varphi_{\rm ns}=\sqrt{2/3}\bar{\sigma}_0$, and $\varphi_{\rm s}=\bar{\sigma}_0/\sqrt{3}$, i.e., $\varphi_{\rm s}=\varphi_{\rm ns}/\sqrt{2}$, as borne out by Fig. 2(b).

In Figs. 2(c) and 2(d), the masses and condensates are shown for the case without explicit $U(1)_A$ symmetry breaking, $c=0$. This corresponds to case 2(a) of Sec. III. The zero-temperature properties were discussed in Sec. IV D. Now, there are nine Goldstone bosons, the three pions, the four kaons, the η, and the η' meson. The behavior of the scalar octet is similar to the previous case.

As the temperature increases, the behavior of the scalar and pseudoscalar masses is quite similar to the masses in Fig. 2(a), i.e., the scalar masses decrease and the pseudoscalar masses increase, until they become degenerate in a first order phase transition. The critical temperature for this transition, however, appears to be slightly lower than for $c\neq 0$. A notable difference between Figs. 2(a) and 2(c) is that the masses do not become degenerate continuously as the condensates vanish. The reason is that, for $c=0$, the above constraint equation (47) is absent. However, these phenomena are irrelevant for the thermodynamic properties of the model, as they occur in a region where the solutions are thermodynamically unstable and the stable solution has to be found from Maxwell's construction for first order phase transitions.

Another interesting feature is that the mass of the σ meson is proportional to the condensates. The reason for this is that the condensate equation and the equation for the σ mass can be combined to give

$$M_\sigma^2=\frac{8}{3}\mathcal{F}_{0000}\bar{\sigma}_0^2. \qquad (49)$$

B. Explicit chiral symmetry breaking with $U(1)_A$ anomaly

In Fig. 3, we show the masses for the scalars (a), the pseudoscalars (b), the condensates (c), and the mixing angles (d) for explicit chiral symmetry breaking, including the $U(1)_A$ anomaly. The masses behave according to the discussion of case 5(b) in Sec. III. As the temperature increases, chiral symmetry is restored in a crossover transition and all masses become approximately degenerate. The temperature range of the crossover transition is $\sim$220 MeV, i.e., about 50 MeV higher than in the case where there is no explicit chiral symmetry breaking. A notable feature is that the κ becomes lighter than the a_0 meson, and the η' meson lighter than the kaon, at about 240 MeV.

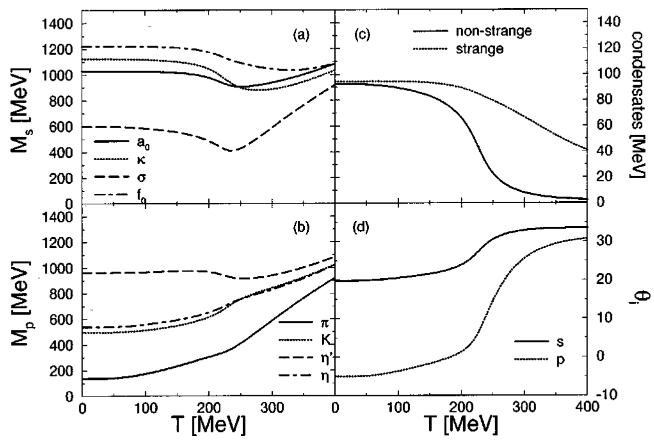

FIG. 3. The meson masses, the condensates, and the mixing angles as a function of temperature for $h_0, h_8 \neq 0$, $h_3 = 0$, and $c \neq 0$. (a) For the scalar mesons, the full line is the a_0, the dotted line is the κ, the dashed line is the σ, and the dot-dashed line is the f_0 meson mass. (b) For the pseudoscalar mesons, the full line is the pion, the dotted line is the kaon, the dashed line is the η', and the dot-dashed line is the η meson mass. (c) The non-strange (full) and strange (dotted) condensates. (d) The scalar (full) and the pseudoscalar (dotted) mixing angles.

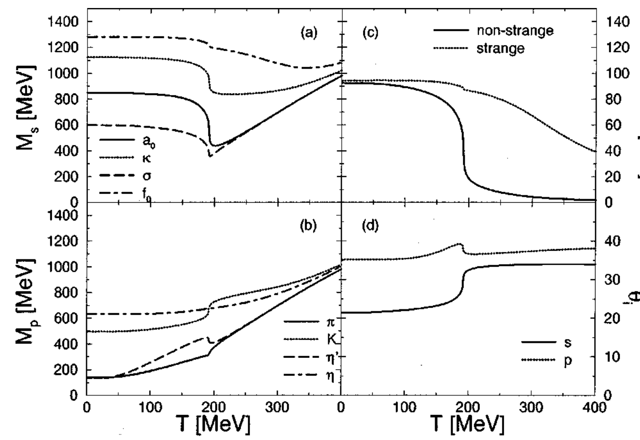

FIG. 4. As in Fig. 3, but for $c = 0$.

The non-strange and the strange condensates, φ_{ns} and φ_s, are shown in Fig. 3(c). Note that the strange condensate decreases much more slowly with temperature than the non-strange condensate. Figure 3(d) shows the scalar and pseudoscalar mixing angles. At large temperatures, these approach $\arcsin(1/\sqrt{3}) \simeq 35.264°$. From Eq. (13),

$$\sigma \equiv \tilde{\sigma}_0 = \sqrt{\frac{2}{3}}\,\sigma_0 + \frac{1}{\sqrt{3}}\sigma_8, \tag{50a}$$

$$f_0 \equiv \tilde{\sigma}_8 = -\frac{1}{\sqrt{3}}\sigma_0 + \sqrt{\frac{2}{3}}\,\sigma_8. \tag{50b}$$

On the other hand,

$$\varphi_{ns} = \sqrt{\frac{2}{3}}\,\bar{\sigma}_0 + \frac{1}{\sqrt{3}}\,\bar{\sigma}_8, \tag{51a}$$

$$\varphi_s = \frac{1}{\sqrt{3}}\,\bar{\sigma}_0 - \sqrt{\frac{2}{3}}\,\bar{\sigma}_8. \tag{51b}$$

This shows that the σ meson becomes an excitation of the non-strange condensate, i.e., a purely non-strange $q\bar{q}$ state. On the other hand, the f_0 meson is an excitation of the strange condensate and a pure $s\bar{s}$ state. Similarly, the η' meson is purely non-strange and the η meson is purely strange. This is what was referred to as ideal flavor mixing earlier.

The transition occurs at temperatures which are not significantly larger than the strange quark mass. Therefore, the explicit $SU(3)_r \times SU(3)_l$ symmetry breaking by the strange quark mass cannot be neglected and, at first, only the (approximate) $SU(2)_r \times SU(2)_l$ symmetry is restored. This means that the pion becomes degenerate with the σ meson, and the a_0 becomes degenerate with the η' meson. (The η' is purely non-strange due to ideal flavor mixing.) However, due to the fact that the strange condensate decreases rather slowly with temperature, the explicit $U(1)_A$ symmetry breaking term $\sim \mathcal{G}_{abc}\bar{\sigma}_c$ in Eqs. (43) is not small. This causes the pion/σ meson mass still to be different from the a_0/η' meson mass. Only when both condensates approach zero, the $U(1)_A$ symmetry is effectively restored and the masses of all non-strange particles become degenerate.

When the temperature becomes significantly larger than m_s, the (approximate) $SU(3)_r \times SU(3)_l$ symmetry is restored. Then, all scalar octet states become degenerate, likewise all pseudoscalar octet states become degenerate. If this happened when the explicit $U(1)_A$ breaking term was still large, then the complete pseudoscalar octet would become degenerate in mass with the scalar singlet, and the scalar octet degenerate in mass with the pseudoscalar singlet. As it turns out, however, the explicit $U(1)_A$ symmetry breaking becomes small around the same point where the (approximate) $SU(3)_r \times SU(3)_l$ symmetry is restored.

C. Explicit chiral symmetry breaking without $U(1)_A$ anomaly

In Fig. 4, we show the masses for the scalars (a), the pseudoscalars (b), the condensates (c), and the mixing angles (d) for explicit chiral symmetry breaking in the absence of the $U(1)_A$ anomaly. The masses behave according to the discussion of case 4(b) in Sec. III. As the temperature increases, the chiral symmetry restoration crossover transition is much more rapid than in the previous case, and occurs at a slightly smaller temperature, ~200 MeV. A notable feature is the inverse mass ordering of the η meson and the kaon. At small temperatures and above the transition, the masses of the pion and the η' meson are the same. In the temperature range from about 50 to 210 MeV, however, they differ. We perceive this to be an artifact of the violation of Goldstone's theorem in the Hartree approximation, cf. Fig. 2(c).

The melting of the condensates, Fig. 4(c), is similar to the previous case, Fig. 3(c). The mixing angles, Fig. 4(d), again approach ideal flavor mixing at large temperatures. The difference here, however, is that the η and η' mesons are also ideally flavor-mixed at zero temperature, cf. Sec. IV B.

Due to the absence of the $U(1)_A$ anomaly, once the (approximate) $SU(2)_r \times SU(2)_l$ symmetry is restored, the pion, the η', the a_0, and the σ mesons simultaneously become degenerate in mass. (The η' meson belongs to this class of non-strange particles due to ideal flavor mixing.) Once the temperature becomes large compared to the strange quark mass, the masses of the strange mesons converge with those of the non-strange mesons.

VII. CONCLUSIONS

In this work, we computed properties of the $SU(3)_r \times SU(3)_l$ linear sigma model in the Hartree approximation at nonzero temperature. We first classified possible patterns of symmetry breaking, with special attention to the cases where the $U(1)_A$ anomaly is either absent or present, and the cases of zero, degenerate or nonzero, non-degenerate quark masses. We then determined the coupling constants from the vacuum values for the masses and the decay constants in the various cases of interest. We systematically derived the Hartree approximation within the CJT formalism. Within this approximation, we computed the masses of scalar and pseudoscalar particles, the non-strange and strange condensates, and the scalar and pseudoscalar mixing angles as a function of temperature. We checked that our results are consistent with the mean-field approximation employed in [13] to compute these quantities.

For the $SU(N_f)_r \times SU(N_f)_l$ model, in the case where the quark masses are zero, universality arguments predict the chiral symmetry restoring transition to be first order for $N_f=3$ and $N_f=2$ in the absence of the $U(1)_A$ anomaly, and second order for $N_f=2$ in the presence of the $U(1)_A$ anomaly [6]. We find that the Hartree approximation correctly gives a first order transition in the case $N_f=3$. This is not necessarily an indication for the validity of this approximation, because earlier work has shown that it incorrectly predicts a first order transition when $N_f=2$ and the $U(1)_A$ anomaly is present [10]. The transition temperature is on the order of 170 MeV.

As expected, when the $U(1)_A$ anomaly is absent, the η' meson becomes a Goldstone boson for zero quark masses. A surprising result is that then the σ meson mass is directly proportional to the condensate.

For nonzero quark masses, $m_u=m_d \neq m_s$, we find the transition to be a crossover transition, but for vanishing $U(1)_A$ anomaly the crossover region is much more narrow than in the presence of the $U(1)_A$ anomaly. In the chirally symmetric phase, the mixing angles approach the situation of ideal flavor mixing, i.e., the σ and η' mesons are pure non-strange $q\bar{q}$ states, while the f_0 and the η mesons are pure $s\bar{s}$ states.

As an outlook, the present framework can be used as an alternative to lattice QCD studies [2] to study the order of the chiral symmetry restoring transition as a function of the strange and non-strange quark masses [26]. Moreover, other meson properties such as the decay widths and the spectral functions can be self-consistently computed at nonzero temperature [19]. These properties can be experimentally investigated in relativistic nuclear collisions, for instance at Brookhaven National Laboratory's Relativistic Heavy-Ion Collider.

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APPENDIX: DERIVATION OF EQS. (16)

The infinitesimal form of the $SU(3)_r \times SU(3)_l \times U(1)_A$ symmetry transformation (5) is

$$T^a \phi^a \to T^a \phi^a - i\omega_V^a [T^a, T^b]\phi^b + i\omega_A^a \{T^a, T^b\}\phi^b. \quad \text{(A1)}$$

For axial-vector transformations, $\omega_V^a \equiv 0$, and the associated (axial-vector) Noether current is

$$\begin{aligned} \mathcal{J}_a^\mu &\equiv \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_b)} i d_{abc}\phi_c + \text{H.c.} \\ &= \frac{i}{2}(\partial^\mu \sigma_b - i\partial^\mu \pi_b) d_{abc}(\sigma_c + i\pi_c) + \text{H.c.} \\ &= d_{abc}(\sigma_b \partial^\mu \pi_c - \pi_b \partial^\mu \sigma_c). \end{aligned} \quad \text{(A2)}$$

Inserting this into the PCAC relation,

$$\langle 0 | \mathcal{J}_a^\mu | \pi_a \rangle \equiv i p^\mu f_a, \quad \text{(A3)}$$

where f_a is the decay constant corresponding to the field π_a, and shifting the scalar fields by their vacuum expectation values, $\sigma_a \to \sigma_a + \bar{\sigma}_a$, one obtains

$$f_a = d_{aab}\bar{\sigma}_b, \quad \text{(A4)}$$

where one sums over the index b but not over a.

In the case that $\bar{\sigma}_0$, $\bar{\sigma}_8 \neq 0$, one obtains for the pion and kaon decay constants

$$f_\pi \equiv f_1 = d_{11a}\bar{\sigma}_a = \sqrt{\frac{2}{3}}\,\bar{\sigma}_0 + \frac{1}{\sqrt{3}}\,\bar{\sigma}_8, \quad \text{(A5a)}$$

$$f_K \equiv f_4 = d_{44a}\bar{\sigma}_a = \sqrt{\frac{2}{3}}\,\bar{\sigma}_0 - \frac{1}{\sqrt{12}}\,\bar{\sigma}_8. \quad \text{(A5b)}$$

In the case that $\bar{\sigma}_8 = 0$, this simplifies to $f_\pi = f_K = \sqrt{2/3}\bar{\sigma}_0$.

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