

Solution to the hierarchy problem with an infinitely large extra dimension and moduli stabilization

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We construct a class of solutions to Einstein's equations for dimensions greater than or equal to 6. These solutions are characterized by a non-trivial warp factor and possess a non-compact extra dimension. We study in detail a simple model in six dimensions containing two four-branes. One of each brane's four spatial directions is compactified. The hierarchy problem is resolved by the enormous difference between the warp factors at the positions of the two-branes, with the standard model fields residing on the brane with a small warp factor. Both branes can have positive tensions. Their positions, and the size of the compact dimension, are determined in terms of the fundamental parameters of the theory by a combination of two independent and comparable effects—an anisotropic contribution to the stress tensor of each brane from quantum fields residing on it and a contribution to the stress tensor from a bulk scalar field. One overall fine-tuning of the parameters of the theory is required—that for the cosmological constant.

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I. INTRODUCTION

The work of Arkani-Hamed, Dimopoulos and Dvali (ADD) [1] and of Randall and Sundrum (RS) [2] has stimulated interest in explaining the observed weakness of gravity (the “hierarchy problem”) using extra dimensions. The ADD solution requires the new dimensions to have a finite but large volume, which introduces a new hierarchy between the volume of the compact dimension and the fundamental scale of the theory.¹ RS proposed instead five dimensional spacetime with a curvature comparable to the fundamental scale, and showed that a massless graviton can be localized to a 3+1 dimensional hypersurface known as the “Planck brane.” In their setup the extra dimension may be taken infinitely large, and four dimensional general relativity still agrees to a high precision with long distance experimental measurements. The weakness of observed gravity is explained provided the standard model fields are localized to a 3+1 dimensional “TeV brane,” where the graviton wave function is small. Because of the exponential falloff of the graviton wave function away from the Planck brane, the distance between the TeV brane and the Planck brane does not need to be large in units of the fundamental scale. Lykken and Randall (LR) [5] have shown that an infinite fifth dimension is experimentally quite consistent with such a resolution of the hierarchy problem.

In such a picture, it is necessary to introduce dynamics which determines the location of the TeV brane relative to the Planck brane. If this interbrane distance is not fixed, it becomes a massless modulus which leads to unacceptable cosmology [6] and experimental consequences. Goldberger and Wise showed that adding a scalar field which propagates in the bulk and has a source on the branes is sufficient to fix

this distance [7]. Several other suggestions have been made for bulk dynamics to fix the extra dimensional configuration [8].

In this paper we first construct a class of solutions to Einstein's equations for dimensions greater than or equal to 6. These solutions are characterized by a non-trivial warp factor and possess a non-compact extra dimension. We then study in detail a simple model in six dimensions containing two four branes that employs a metric of this form to address the hierarchy problem. One of each brane's four spatial directions is compactified on a circle of small radius. The hierarchy problem is resolved as in the RS and LR models by the enormous difference in the warp factors at the positions of the two branes, with the standard model fields residing on the brane with a small warp factor.

In this model the positions of the branes, and hence the magnitude of the hierarchy, are determined by the combination of two independent effects. The first is an anisotropic contribution to the stress tensor of each brane arising from the quantum effects of fields localized to it. The second is a contribution to the stress tensor from a scalar bulk field as in the model of Goldberger and Wise. These effects are naturally comparable in size and together can yield a sufficiently large value of the brane spacing to solve the hierarchy problem without a fine-tuning of the parameters.

The reason for the first effect is that the theory contains a compact dimension, in addition to the noncompact dimension r . The size of the compact dimension is in general an r dependent function, which is determined from Einstein's equations. We argue that in general the component of the brane tension in the compact dimension will depend on its size, due to the quantum effects of fields localized to the brane. Then a consistent solution to Einstein's equations will fix each brane location at a particular value of r . However, in this model this effect by itself does not give rise to a large hierarchy without fine tuning. Nevertheless, when combined with the effect of a bulk scalar field on the geometry it is possible to realize a large hierarchy without a fine-tuning of the parameters.

Both branes in the theory can have positive tensions and

¹For examples of theories which naturally have a finite but exponentially large volume for the additional dimensions see Ref. [3]. For earlier work on large extra dimensions and/or a low quantum gravity scale see Ref. [4].

the solution is free of singularities where general relativity might break down. One overall fine-tuning of the parameters in the theory is required to adjust the four dimensional cosmological constant to zero.

II. ANSATZ, EQUATIONS OF MOTION AND SOLUTIONS

We begin by looking for solutions to the Einstein equations in D dimensions, where D is greater than or equal to 6, in the presence of a constant background bulk cosmological constant. We assume that all sources other than the bulk cosmological constant are restricted to subspaces of lower dimension. Hence our approach will be to first solve the equations of motion in the bulk to obtain solutions with a number of constants of integration that can then be adjusted to find solutions for various boundary conditions.

The action in the bulk is

$$S = \int d^D x \sqrt{-G} (2M_*^{D-2} R - \Lambda_B). \quad (1)$$

We label a general coordinate by x^M where M runs from 0 to $(D-1)$. We restrict our search to metrics of the simple form

$$ds^2 = f(z) \eta_{\mu\nu} dx^\mu dx^\nu + s(z) dy^2 + dz^2 \quad (2)$$

where μ and ν run from 0 to $D-3$. The remaining two coordinates are labeled by y and z . Here the warp factors f and s are assumed to be functions only of z .

Einstein's equations in the bulk take the form

$$2M_*^{D-2} (R_{MN} - g_{MN} R) = -\frac{1}{2} g_{MN} \Lambda_B. \quad (3)$$

The nontrivial components of this equation are

$$\begin{aligned} & \frac{1}{2} f'' (3-D) - \frac{1}{2} f \frac{s''}{s} + f' \frac{s'}{s} \frac{(3-D)}{4} + \frac{1}{4} f \left(\frac{s'}{s} \right)^2 \\ & - f \left(\frac{f'}{f} \right)^2 \left(\frac{D^2 - 9D + 18}{8} \right) \\ & = -\alpha^2 f \frac{(D-2)^2 + (D-2)}{8} \end{aligned} \quad (4)$$

$$\begin{aligned} & \frac{1}{2} \left[(D-2) \frac{f''}{f} + \frac{(D-2)(D-5)}{4} \left(\frac{f'}{f} \right)^2 \right] \\ & = \alpha^2 \frac{(D-2)^2 + (D-2)}{8} \end{aligned} \quad (5)$$

$$\begin{aligned} & \frac{(D-2)(D-3)}{8} \left(\frac{f'}{f} \right)^2 + \frac{D-2}{4} \frac{f'}{f} \frac{s'}{s} \\ & = \alpha^2 \frac{(D-2)^2 + (D-2)}{8} \end{aligned} \quad (6)$$

where α is defined by

$$\alpha^2 \frac{(D-2)^2 + (D-2)}{8} = -\frac{\Lambda_B}{4M_*^{(D-2)}}. \quad (7)$$

To solve these equations note that we can rewrite Eq. (5) in the form

$$\begin{aligned} & \frac{1}{2} \left\{ (D-2) \left[\left(\frac{f'}{f} \right)' + \left(\frac{f'}{f} \right)^2 \right] + \frac{(D-2)(D-5)}{4} \left(\frac{f'}{f} \right)^2 \right\} \\ & = \alpha^2 \frac{(D-2)^2 + (D-2)}{8}. \end{aligned} \quad (8)$$

This has the form of a first order differential equation for f'/f . This differential equation is straightforward to solve, and we can then obtain f itself by performing a simple integration. We then use the result obtained for f in Eq. (6) and the problem of determining s also then reduces to performing a simple integral. The results are

$$f = f_0 e^{\alpha z} [1 - c e^{-(D-1)/2 \alpha z}]^{4/(D-1)} \quad (9)$$

$$\begin{aligned} s &= s_0 e^{\alpha z} [1 - c e^{-(D-1)/2 \alpha z}]^{-2(D-3)/D-1} \\ & \times [1 + c e^{-(D-1)/2 \alpha z}]^2. \end{aligned} \quad (10)$$

Here f_0 , s_0 and c are constants to be determined by boundary conditions. In the limits of vanishing c and infinite c we recover the usual anti-de Sitter (AdS) metric. It is easy to see that in fact for any values of these constants the warp factors f and s change very rapidly as function of z . In particular there are always values of z where they are changing exponentially quickly. This suggests that these metrics are good candidates for a solution to the hierarchy problem.

It is possible to use the class of metrics above to find solutions to the Einstein equations for various source configurations and geometries.² In the next section we exhibit a potential solution to the hierarchy problem based on these metrics.

III. MODEL

A. Metric

In this section we limit our interest to a solution where the fifth dimension y is compact and corresponds to an angle in the higher dimensional space. We relabel y by ϕ in this section and hereafter to emphasize its angular character. The angle ϕ runs from zero to 2π . We allow the coordinate z to be non-compact and run from zero to infinity. It corresponds to a ‘radius’ in the higher dimensional space. We relabel it by r to emphasize its radial character.

The geometry of our model consists of two four-branes localized in the higher dimensional space at different values of r . Their positions are specified by the equations $r=a$ and $r=b$ where $a < b$. They can therefore be thought of as being similar to the surfaces of two infinitely long ‘concentric cyl-

²Note that these solutions are coordinate transformations of the bulk solutions found in Ref. [9].

inders'' in the higher dimensional space, with the regular four dimensions parallel to the common axis of the cylinders, the fifth dimension going around the surface but perpendicular to the axis, and the sixth dimension being the radius. (This intuitive picture does not account for the fact that the space is curved.) The standard model fields are localized to the brane at $r=a$. The hierarchy problem will be resolved by the enormous difference between the values of the warp factor at $r=a$ and $r=b$.

Because the coordinate ϕ is compact, the four-brane appears as a three-brane at sufficiently long length scales.

The branes divide the space into three distinct sections: $0 < r < a$, $a < r < b$, and $r > b$. In general there is no reason for the bulk cosmological constants in these three sections to be the same since the branes may be separating different phases of the theory. In what follows we will assume that they are different and will associate the three regions with three different values of α : α_1 , α_2 and α_3 respectively.

The solutions of Einstein's equations in the three regions will be of the form of Eqs. (9) and (10) but with different values of the constants f_0 , s_0 and c . We will give these constants an additional subscript i , where i runs from 1 to 3, in order to differentiate them in the three different regions.

Now the constant c_1 is fixed to be -1 by the requirement that the solution be non-singular at the origin. This requirement also fixes the value of s_{01} to be $2^{16/5}/(25\alpha^2)$. To see that this choice does indeed smooth out the singularity at the origin we first examine the behavior of the the functions $f(r)$ and $s(r)$ as r tends to 0:

$$f(r) = \text{const} + O(r^2) \quad (11)$$

$$s(r) = r^2 + O(r^4). \quad (12)$$

We then go to the "Cartesian" coordinate system which is smooth at the origin:

$$x' = r \cos \phi \quad (13)$$

$$y' = r \sin \phi. \quad (14)$$

It is straightforward to verify that in this coordinate system the components of the metric and their first and second derivatives (which go into the Ricci tensor) are smooth at the origin, showing that there is no singularity there.

The requirement that the metric be bounded at infinity determines that the space outside $r=b$ must be AdS. This corresponds to setting c_3 to infinity while keeping the products $c_3 f_{03}$ and $c_3 s_{03}$ finite. We relabel these products by f_3 and s_3 respectively.

We are now in a position to write down the forms of the solutions in the three regions. For $r < a$,

$$f = f_{01} e^{\alpha_1 r} [1 + e^{-(5/2)\alpha_1 r}]^{4/5} \quad (15)$$

$$s = s_{01} e^{\alpha_1 r} [1 + e^{-(5/2)\alpha_1 r}]^{-6/5} [1 - e^{-(5/2)\alpha_1 r}]^2. \quad (16)$$

For $b > r > a$,

$$f = f_{02} e^{\alpha_2 r} [1 - c_2 e^{-(5/2)\alpha_2 r}]^{4/5} \quad (17)$$

$$s = s_{02} e^{\alpha_2 r} [1 - c_2 e^{-(5/2)\alpha_2 r}]^{-6/5} [1 + c_2 e^{-(5/2)\alpha_2 r}]^2. \quad (18)$$

For $r > b$,

$$f = f_3 e^{-\alpha_3 r}, \quad (19)$$

$$s = s_3 e^{-\alpha_3 r}. \quad (20)$$

The constant f_{01} is determined by normalizing the warp factor f to 1 at the position of the visible brane. The constant s_{01} was determined earlier. All the other constants above as well as the positions of the branes a and b must be determined by matching across the branes.

We now write down the action for the branes:

$$S_b = \int dx^M \sqrt{-\bar{G}} \{ \delta(r-a) [L_{M1} - \bar{\Lambda}_1] + \delta(r-b) \} \\ \times [L_{M2} - \bar{\Lambda}_2]. \quad (21)$$

Here \bar{G} is the determinant of the metric tensor in the five dimensional subspace, $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$ are the cosmological constants on the two branes, and L_{M1} and L_{M2} are the Lagrangians of matter fields localized to the branes.

The stress tensor for each brane has the form

$$T_{AB} = T^\Lambda_{AB} + \langle T^M_{AB} \rangle \quad (22)$$

where T^Λ is the contribution from the cosmological constant and $\langle T^M \rangle$ the expectation value of the stress tensor of the matter fields residing on the brane. We will be working in the semi-classical limit, treating gravity completely classically but accounting for the quantum effects of matter localized to the branes.

Assuming the matter on the brane is in its ground state, T_{AB} is constrained by four dimensional Lorentz invariance to be of the form

$$\frac{8\pi}{2M_*^4} T_{AB} = - \begin{pmatrix} -\beta^2 f & 0 & 0 & 0 & 0 \\ 0 & \beta^2 f & 0 & 0 & 0 \\ 0 & 0 & \beta^2 f & 0 & 0 \\ 0 & 0 & 0 & \beta^2 f & 0 \\ 0 & 0 & 0 & 0 & \gamma^2 s \end{pmatrix}, \quad (23)$$

where β^2 and γ^2 are constants associated with each brane that we require to be positive. The form of T^Λ is more constrained since it is proportional to \bar{G}_{AB} , which would imply that $\beta^2 = \gamma^2$ if the only contribution to T came from the cosmological constant on the brane. Thus the deviation of T from the \bar{G}_{AB} form is due entirely to the contribution from the matter Lagrangian. In a subsequent section we will show that the contribution to the stress tensor from the zero point energies of fields residing on the brane leads to $\beta \neq \gamma$. In general β and γ will depend on the geometrical factors a , b

and c_2 due to the quantum contribution to the stress tensor from matter localized on the brane.

Now Einstein's equations for the upper 5 by 5 block of the Ricci tensor get modified in the presence of the branes to

$$2M_*^4(R_{AB} - g_{AB}R) = -\frac{1}{2}g_{AB}\Lambda_B + 8\pi\delta(r-a)T_{AB}^a + 8\pi\delta(r-b)T_{AB}^b. \quad (24)$$

Since we already have the solution in the bulk, we can get the complete solution by matching across the branes. The metric tensor is continuous across the branes but its derivatives are not. The above equation fixes the jump discontinuity in the derivatives across the boundary.

In terms of components the conditions on the derivatives are

$$-\frac{3}{2}\Delta\frac{f'}{f} - \frac{1}{2}\Delta\frac{s'}{s} = \beta^2 \quad (25)$$

$$2\Delta\frac{f'}{f} = -\gamma^2. \quad (26)$$

We wish to apply these conditions to our solution. To simplify matters we first define

$$\bar{\gamma}^2 = 4\beta^2 - 3\gamma^2 \quad (27)$$

$$F(c, r, \alpha) = \frac{2\alpha c}{(e^{(5/2)r\alpha} - c)}. \quad (28)$$

The conditions on the derivatives at the first boundary are

$$\alpha_2 - \alpha_1 + F(c_2, a, \alpha_2) - F(-1, a, \alpha_1) = -\frac{1}{2}\bar{\gamma}_1^2 \quad (29)$$

$$\begin{aligned} \alpha_2 - \alpha_1 - \frac{3}{2}[F(c_2, a, \alpha_2) - F(-1, a, \alpha_1)] \\ + \frac{5}{2}[F(-c_2, a, \alpha_2) - F(1, a, \alpha_1)] = -\frac{1}{2}\bar{\gamma}_1^2. \end{aligned} \quad (30)$$

The conditions at the second boundary are

$$-\alpha_3 - \alpha_2 - F(c_2, b, \alpha_2) = -\frac{1}{2}\bar{\gamma}_2^2 \quad (31)$$

$$-\alpha_3 - \alpha_2 + \frac{3}{2}F(c_2, b, \alpha_2) - \frac{5}{2}F(-c_2, b, \alpha_2) = -\frac{1}{2}\bar{\gamma}_2^2. \quad (32)$$

Looking at the above equations we have three parameters c_2 , a and b which have to satisfy four independent equations. Hence a fine-tuning is necessary. This is the fine-tuning necessary to set the effective four dimensional cosmological constant to zero. Once this fine-tuning has been made, it is straightforward to find solutions to the above

equations without singularities where all parameters are of order 1 in terms of the fundamental scale M_* . However, since we want to generate a hierarchy, we want $\alpha_2 b = O(40) \gg 1$. From now on we will assume that b is the only large parameter in the problem and that it is therefore responsible for generating the hierarchy. Subtracting Eq. (31) from Eq. (32) we see that

$$\frac{5}{2}F(c_2, b, \alpha_2) - \frac{5}{2}F(-c_2, b, \alpha_2) = -\frac{1}{2}\bar{\gamma}_2^2 + \frac{1}{2}\gamma_2^2. \quad (33)$$

Note that from the definition of $\bar{\gamma}$ it is clear that if $\beta = \gamma$, then $\bar{\gamma} = \gamma$. But the difference between β and γ arises from the vacuum energy of quantum fields localized to the brane (the Casimir effect), which vanishes in the limit of large proper radius for the compact dimension. As will be discussed in a subsequent section the Casimir effect is finite and regulator independent in the limit that the cutoff is taken to infinity. Then by dimensional considerations, if the fields on the brane are massless, the right hand side must be of order $[2\pi s(b)]^{-5/2}$ since the proper size of the compact dimension is the only scale in the problem. Then the equation above becomes

$$\begin{aligned} \frac{5}{2}F(c_2, b, \alpha_2) - \frac{5}{2}F(-c_2, b, \alpha_2) \\ = \text{const}[e^{(-5/2)\alpha_2 b}(1 - c_2 e^{-(5/2)\alpha_2 b})^3 \\ \times (1 + c_2 e^{(-5/2)\alpha_2 b})^{-5}], \end{aligned} \quad (34)$$

where the constant above is of order 1 in units of the fundamental scale. While we see that both sides of this equation are the same order of magnitude even for large b we also see that b only appears in the combination $e^{(-5/2)\alpha_2 b}$. It is this combination which is determined in terms of the various tensions. Hence, although the brane spacing is fixed, a large value for $\alpha_2 b$ is only possible through a fine-tuning of parameters which is over and above the fine-tuning necessary to set the four dimensional cosmological constant to zero. In the next section we shall show that by adding a scalar field in the bulk in the manner of Goldberger and Wise [7] the matching conditions at the branes can be made more sensitive to b and the hierarchy can be made natural.³

Equating the various components of the metric across the boundary fixes the values of the coefficients f_{02} , s_{02} , f_3 , and s_3 . Once again there are four equations but there are now four unknowns, so there is no further fine tuning required.

This completes the determination of the metric in the absence of any bulk matter. The warp factor for the fifth dimension is plotted in Fig. 1 for a choice of parameters for

³Note that the necessary fine-tuning of the Planck 4-brane tension might also be natural if the 4-brane is an approximate Bogomol'nyi-Prasad-Sommerfield (BPS) state of a nearly supersymmetric theory. Thus supersymmetry of the bulk action might provide an alternative solution to the hierarchy problem.

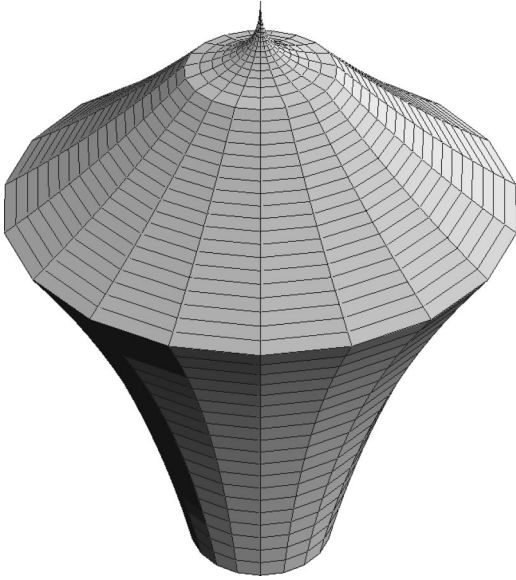


FIG. 1. Plot of warp factor of compact dimension (circumference of circles) versus r (vertical axis), illustrating why we call this the “space needle” metric. The apparent singularity at the top, where $r=0$, is a coordinate singularity only.

which there is a large hierarchy. The warp factor for the usual 3+1 dimensions is qualitatively similar, except near $r=0$ where it goes to a constant. We call this the “space needle” metric for reasons which are obvious from the picture.

B. Model with bulk scalar fields

In this section we show that when the model of the previous section is modified by the inclusion of bulk scalar fields, the hierarchy can be made natural.

The action for the scalar field

$$S_M = \int d^6x \sqrt{-G} \frac{1}{2} (-\partial_M \psi \partial^M \psi - m^2 \psi^2) + \sqrt{-G} F(\psi) \delta(r-a) + \sqrt{-G} H(\psi) \delta(r-b). \quad (35)$$

Here m is the mass of the scalar field in the bulk. We will be interested in the limit $m^2 \ll \alpha^2$, since this is where we naturally obtain a large hierarchy. The scalar field sources $F(\psi)$ and $H(\psi)$ are in general arbitrary functions. For simplicity we will take

$$F(\psi) = \lambda_1 \psi \quad (36)$$

$$H(\psi) = \lambda_2 \psi. \quad (37)$$

The coupled equations for the gravity-matter system are difficult to solve. We shall address the problem by a successive approximation method. We will first solve for the gravitational field in the absence of the scalar field, assuming that the contribution to the stress tensor of this field is small. We will then solve for the scalar field in this geometry, and com-

pute the stress tensor T_ψ of the scalar field. We then can substitute this solution back into Einstein’s equations to determine the correction to the geometry induced by the stress tensor of the scalar field.⁴ If we wish, this procedure, which is essentially an expansion in T_ψ/M_*^6 , can be carried out to higher orders but the lowest order will be sufficient to fix the brane spacing, which we expect will receive only small corrections at higher orders.

For simplicity we will take as a starting metric

$$f_0 = 1, \quad r < a, \quad (38)$$

$$f_0 = e^{-\alpha a} e^{\alpha r}, \quad a < r < b, \quad (39)$$

$$f_0 = e^{-\alpha a} e^{2\alpha b} e^{-\alpha r}, \quad r > b, \quad (40)$$

$$s_0 = r^2, \quad r < a, \quad (41)$$

$$s_0 = a^2 e^{-\alpha a} e^{\alpha r}, \quad a < r < b, \quad (42)$$

$$s_0 = a^2 e^{-\alpha a} e^{2\alpha b} e^{-\alpha r}, \quad r > b. \quad (43)$$

This corresponds to a solution for the special case with no bulk cosmological constant for $r < a$ but the same cosmological constant everywhere outside, and $c_2 = 0$. This metric is flat in the neighborhood of the origin and pure AdS outside a . This is a special case, with α_1 and c_2 set to zero, of the more general class of metrics we have considered in Sec. III A. Non-zero c_2 will be treated perturbatively in what follows and we will also indicate how to include α_1 perturbatively. Hence the only loss of generality arising from this starting metric is that we can only extend our conclusions to the more general class of metrics of the previous section when c_2 and α_1 are sufficiently small that a perturbative approach is valid.

We first solve for the scalar field in this background metric. The equation of motion is

$$-\psi'' - \left(2 \frac{f'}{f} + \frac{1}{2} \frac{s'}{s} \right) \psi' + m^2 \psi = \lambda_1 \delta(r-a) + \lambda_2 \delta(r-b). \quad (44)$$

The solutions in the three regions consistent with smoothness of ψ at the origin and vanishing of ψ at infinity are

$$\psi = A_1 \left(1 + \frac{1}{4} m^2 r^2 + \dots \right), \quad r < a, \quad (45)$$

$$\psi = A_2 e^{\sigma_1 r} + B_2 e^{\sigma_2 r}, \quad a < r < b, \quad (46)$$

$$\psi = A_3 e^{\sigma_3 r}, \quad r > b, \quad (47)$$

where we have neglected higher order terms in $m^2 r^2$ in Eq. (45), and

⁴We neglect quantum corrections to the stress tensor from the scalar field, as we do not expect these to qualitatively alter our conclusions.

$$\sigma_1 = -\frac{5}{4}\alpha - \sqrt{\left(\frac{5}{4}\alpha\right)^2 + m^2} \quad (48)$$

$$\sigma_2 = -\frac{5}{4}\alpha + \sqrt{\left(\frac{5}{4}\alpha\right)^2 + m^2}. \quad (49)$$

For positive m^2 σ_2 is positive and σ_1 negative. Then

$$\sigma_3 = -\sigma_2. \quad (50)$$

The coefficients A_i and B_i are to be determined by matching the solutions for ψ across the boundaries. We require continuity of ψ across the boundaries and the following jump conditions:

$$\Delta\psi'(a) = -F'(\psi) \quad (51)$$

at the first boundary and

$$\Delta\psi'(b) = -H'(\psi) \quad (52)$$

at the second boundary.

Since the expressions for the A_i 's and B_i 's are complicated, we will neglect effects of order e^{-ab} which are very small and further assume $m^2 a^2 \ll 1$ so that such effects can also be neglected. These approximations will not affect our conclusions. Then in this limit the expressions for the A_i 's and B_i 's are

$$A_2 = \frac{\lambda_2 \sigma_2}{(\sigma_3 - \sigma_2) \sigma_1} e^{(\sigma_2 - \sigma_1)a} e^{-\sigma_2 b} - \frac{\lambda_1}{\sigma_1} e^{-\sigma_1 a} \quad (53)$$

$$B_2 = -\frac{\lambda_2}{(\sigma_3 - \sigma_2)} e^{-\sigma_2 b} \quad (54)$$

$$A_3 = -\frac{\lambda_2}{(\sigma_3 - \sigma_2)} e^{-\sigma_3 b}. \quad (55)$$

We now find the contribution to the stress tensor from the field ψ , T_ψ :

$$16\pi T_{\psi 0}^0 = \left\langle -\frac{1}{2}(\psi'^2 + m^2) + F(\psi)\delta(r-a) \right. \\ \left. + H(\psi)\delta(r-b) \right\rangle \quad (56)$$

$$16\pi T_{\psi r}^r = \left\langle \frac{1}{2}(\psi'^2 - m^2) \right\rangle. \quad (57)$$

Next, we substitute the stress tensor back into the Einstein equations to determine the corrections to the geometry. For convenience we define $\bar{T} = 8\pi T_\psi / 2M_*^4$, $\bar{\lambda} = \lambda / 2M_*^4$. Einstein's equations in the region $a < r < b$ take the form

$$-\frac{3}{2}f'' - \frac{1}{2}f\frac{s''}{s} - \frac{3}{4}f'\left(\frac{s'}{s}\right) + \frac{1}{4}f\left(\frac{s'}{s}\right)^2 = -\frac{5}{2}\alpha^2 f + \bar{T}_{00} \quad (58)$$

$$2\frac{f''}{f} + \frac{1}{2}\left(\frac{f'}{f}\right)^2 = \frac{5}{2}\alpha^2 + \bar{T}_3^5 \quad (59)$$

$$\frac{3}{2}\left(\frac{f'}{f}\right)^2 + \frac{f'}{f}\frac{s'}{s} = \frac{5}{2}\alpha^2 + \bar{T}_6^6. \quad (60)$$

The equations inside $r < a$ can be obtained by setting $\alpha = 0$ in the equations above. The equations for $r > b$ are identical to those above. We are interested in the correction to the geometry to linear order in \bar{T}/M_*^2 . To obtain this we expand

$$f = f_0(1 + \epsilon) \quad (61)$$

$$s = s_0(1 + \kappa), \quad (62)$$

and linearize in ϵ and κ . Then for $r < a$ Eqs. (58) and (60) above become

$$\epsilon' = \frac{1}{2}r\bar{T}_6^6 \quad (63)$$

$$-\frac{3}{2}\epsilon'' - \frac{1}{2}\kappa'' - \frac{1}{r}\kappa' - \frac{3}{2r}\epsilon' = -\bar{T}_0^0. \quad (64)$$

These equations yield

$$\epsilon' = \frac{1}{2}r\bar{T}_6^6 \quad (65)$$

$$\kappa' = \frac{1}{2}r\bar{T}_6^6 - \frac{5}{2}\frac{1}{r^2}\int_0^r dr r^2 \bar{T}_6^6, \quad (66)$$

which can be integrated to get ϵ and κ . Here we have used the constraints of smoothness at the origin and the linearized form of the Bianchi identity. The latter is shown below:

$$(r\bar{T}_6^6)' = \bar{T}_5^5. \quad (67)$$

For $a < r < b$ Eqs. (59) and (60) when linearized become

$$2\epsilon'' + 5\alpha\epsilon' = \bar{T}_5^5 \quad (68)$$

$$4\alpha\epsilon' + \alpha\kappa' = \bar{T}_6^6. \quad (69)$$

These yield

$$\epsilon' = D e^{-(5/2)(\alpha r)} + \frac{\bar{T}_6^6}{5\alpha} \quad (70)$$

$$\kappa' = -4D e^{-(5/2)(\alpha r)} + \frac{\bar{T}_6^6}{5\alpha}, \quad (71)$$

which can be integrated to yield ϵ and κ . Here D is a constant of integration which must be determined from matching and once again the linearized form of the Bianchi identity for this region, shown below, has been used:

$$(\overline{T^6_6})' + \frac{5\alpha}{2}\overline{T^6_6} - 2\alpha\overline{T^0_0} - \frac{1}{2}\alpha\overline{T^5_5} = 0. \quad (72)$$

In identical fashion we can get, for the region $r > b$,

$$\epsilon' = \kappa' = -\frac{\overline{T^6_6}}{5\alpha}. \quad (73)$$

Here the requirement that the metric die away at infinity has been imposed. This completes the determination of bulk corrections to the metric. The final step is to match across the branes, thereby determining the brane positions and the undetermined constant D .

The conditions on the continuity of the metric across the brane are straightforward to satisfy because of the additional constants of integration that will be obtained on integrating the expressions for ϵ' and κ' . Once f has been normalized to 1 on the visible brane and the requirement of smoothness has been met at $r=0$, all these additional constants will have been fixed.

We now focus our attention on the jump conditions on the derivatives. At the inner brane we have

$$\begin{aligned} -\frac{3}{2}\Delta\epsilon'(a) - \frac{1}{2}\Delta\kappa'(a) - \frac{1}{2}\left(\alpha - \frac{2}{a}\right) - \frac{3}{2}\alpha \\ = \beta_1^2 - \frac{1}{2}\overline{\lambda}_1\psi(a) \end{aligned} \quad (74)$$

$$2\Delta\epsilon'(a) + 2\alpha = -\gamma_1^2 + \frac{1}{2}\overline{\lambda}_1\psi(a). \quad (75)$$

At the outer brane

$$\begin{aligned} -\frac{3}{2}\Delta\epsilon'(b) - \frac{1}{2}\Delta\kappa'(b) + \frac{1}{2}(2\alpha) + \frac{3}{2}(2\alpha) \\ = \beta_2^2 - \frac{1}{2}\overline{\lambda}_2\psi(b) \end{aligned} \quad (76)$$

$$2\Delta\epsilon'(b) - 4\alpha = -\gamma_2^2 + \frac{1}{2}\overline{\lambda}_2\psi(b). \quad (77)$$

These are four independent equations for the three unknowns a , b and D . Just as in the previous section, the metric and brane locations are completely determined with one overall fine-tuning needed to find a Poincaré invariant solution.

First consider the situation without the scalar field, i.e. $\overline{T} = 0$. This is the same case which was considered in the previous section. In general, as we found that the metric between the branes need not be pure AdS even without a scalar field. The tensions of the branes must be fine-tuned to set D to zero. We are allowing these brane tensions (which are related to β^2 and γ^2 in the above equations) to deviate from those fine-tuned values by small amounts. Here small merely means that perturbation theory is valid. Then it is straightforward to verify that Eqs. (74) to (77) are simply linearized versions of Eqs. (29)–(32) with $\alpha_1 = 0$ and the parameter D is proportional to c_2 . Hence even in the absence of the scalar

field the metric between the branes is not of the AdS form, and our conclusions once the scalar field is introduced will be valid for this more general class of metrics. Setting \overline{T} in the above formulas to a constant non-zero value for $r < a$ corresponds to allowing a non-zero α_1 . Although it is straightforward to accommodate a non-zero α_1 perturbatively in this framework, for simplicity we shall keep $\alpha_1 = 0$ in what follows below, and \overline{T} will be related only to T_ψ and will not include any piece from a cosmological constant.

We now return to the general case with a scalar field. We are interested in a solution with all parameters of order 1 in units of the fundamental scale but we allow m^2/α^2 to be somewhat less than 1 in order to obtain a hierarchy.

Rather than give the exact solution we will give the more informative order of magnitude results. For dimensional considerations any parameter of order 1 in units of the fundamental scale will be denoted by the appropriate power of α .

The matching conditions at the inner brane, Eqs. (74) and (75), determine a and D as functions of b . Provided m^2b/α is less than or of order 1, a and D are of order 1 in units of the fundamental scale. The dependence of D on b is

$$D = O(\alpha) + O(m^2b). \quad (78)$$

This result will turn out to be crucial for the solution of the hierarchy problem.

Both equations at the outer brane are sensitive to b and a only through exponentially small terms. This is because the value of ψ and its derivatives is exponentially insensitive to b and a in this region, which manifests itself in the forms of ϵ' and κ' close to the outer brane. This is similar to the insensitivity to b of Eqs. (31) and (32). One might therefore worry that just as in the previous case an exponentially precise fine-tuning will be necessary to get a hierarchy. However the fine-tuning in the previous case was related to the fact that near the Planck brane the metric was very nearly pure AdS, which is a homogenous space, and the compact dimension was exponentially large, so that the location of the Planck brane had very little effect on the matching conditions at either brane. In the present case with a light bulk scalar the value of the scalar at the TeV brane depends more sensitively on b . To determine b , add Eqs. (76) and (77) to obtain

$$\frac{5}{2}De^{(-5/2)ab} = (\beta_2)^2 - (\gamma_2)^2. \quad (79)$$

This is the analogue of Eq. (33) in the previous section. Recall that $(\beta_2)^2 - (\gamma_2)^2$ arises from the vacuum energy of quantum fields localized on the brane and is of order $\alpha e^{(-5/2)ab}$. Then this equation, together with Eq. (78) which relates D to b , determines b as

$$b = O\left(\frac{\alpha}{m^2}\right). \quad (80)$$

Clearly m need not be much smaller than α to get a sizable hierarchy. The difference of Eqs. (76) and (77), which

we have not yet used, becomes the condition that the effective four dimensional cosmological constant be zero, which is the usual fine-tuning.⁵

One limitation of the above approach is that the inner brane necessarily has negative tension along the non-compact directions. This is a consequence of the choice of a starting metric with zero α_1 and c_2 and the fact that deviations from this metric can only be perturbative. The non-zero α_1 case is not simple but there is a limit in which it is tractable — that in which $(\alpha_1 a)^2$ is perturbatively less than 1, even though α_1 is not small. In this limit, neglecting higher order terms in $(\alpha_1 r)^2$ we find

$$\frac{f'}{f} = \frac{5}{4} \alpha_1^2 r + \dots \quad (81)$$

$$\frac{s'}{s} = \frac{2}{r} - \frac{5}{6} \alpha_1^2 r + \dots \quad (82)$$

We can substitute for this in Eq. (44) and find that to the order shown the solution (45) is unchanged. Now, if $\alpha_1^2 a > \alpha_2$, the inner brane can have positive tension and it is straightforward to verify that all the other conclusions above go through as before.

C. Physical implications

To determine the effective four dimensional Planck scale we concentrate on the higher dimensional Einstein action. When the two extra dimensions are integrated out this will contain the usual four dimensional Einstein action:

$$S_G = 2M_*^4 \int d^6 x^M \sqrt{-GR}^6. \quad (83)$$

Expanding $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and integrating out the two extra dimensions we see that

$$\begin{aligned} S_G &= 2M_*^4 \int d^6 x^M \sqrt{-gR}^4 f \sqrt{s} + \dots \\ &= M_4^2 \int d^4 x^\mu \sqrt{-gR}^4 + \dots, \end{aligned} \quad (84)$$

and consequently

$$2\pi M_*^4 \int dr f \sqrt{s} = M_4^2. \quad (85)$$

Rather than do this integral exactly we will be satisfied with an estimate. The dominant contribution to the integral comes from the region close to the outer brane where the integrand is very large and the forms of f and s are very nearly simple exponentials. For the purposes of the estimate we set $\alpha_3 = \alpha_2 = a^{-1} = M_* = \alpha$:

$$M_4^2 = O(\alpha^2 e^{(3/2)ab}). \quad (86)$$

This large exponent is responsible for the hierarchy between the Planck scale and the weak scale.

Next we analyze the spectrum of linearized tensor fluctuations to see if it is consistent with the results of gravitational experiments. We neglect the effect of the scalar field in what follows since we do not expect it to qualitatively affect our results because its contribution to the energy density is small.

Expanding

$$G_{\mu\nu} = f \eta_{\mu\nu} + h_{\mu\nu} \quad (87)$$

and substituting this into the Lagrangian we get the following equation for the fluctuation h in the bulk:

$$-h'' - \frac{1}{2} \frac{s'}{s} h' - \left(\frac{f'}{f}\right)^2 h + \frac{5}{2} \alpha^2 h = \frac{1}{f} m^2 h, \quad (88)$$

where $m^2 = -\eta_{\mu\nu} p^\mu p^\nu$ and we are restricting our attention to modes that have no momentum in the compact direction. These are expected to be separated by a mass gap from the heavier modes with nonzero momentum in the fifth direction.

The boundary conditions that h has to satisfy are $h'(0) = 0$ at the origin and

$$2\Delta \frac{h'}{h} = -\gamma^2. \quad (89)$$

Clearly $h=f$ is a solution of this equation with $m=0$ by comparison with Eqs. (5) and (6). This is the massless graviton. There is a continuum of other solutions for all positive m^2 , as can be seen from the asymptotic behavior of the equation. However, to extract these solutions is not easy, because of the complicated forms of f and s . However, since we are only interested in order of magnitude estimates, we can simplify the problem by considering instead a simpler problem that retains the physics we are interested in. Consider the metric

$$f=1, \quad r < a, \quad (90)$$

$$f=e^{-\alpha a} e^{\alpha r}, \quad a < r < b, \quad (91)$$

$$f=e^{-\alpha a} e^{2\alpha b} e^{-\alpha r}, \quad r > b, \quad (92)$$

$$s=r^2, r < a, \quad (93)$$

$$s=a^2 e^{-\alpha a} e^{\alpha r}, \quad a < r < b, \quad (94)$$

$$s=a^2 e^{-\alpha a} e^{2\alpha b} e^{-\alpha r}, \quad r > b. \quad (95)$$

This corresponds to a solution for the case with no cosmological constant for $r < a$ but the same cosmological constant everywhere outside. This metric is flat in the neighborhood of the origin, AdS outside a . It clearly has a form very similar to the metric we are interested in for $r \gg a$, although unlike our case, the positions of the branes are not fixed and one has negative tension. In what follows we will assume the

⁵Work is in progress to see whether this fine-tuning can be made more natural though supersymmetry of the bulk action when the Planck brane is approximately BPS.

main features of the excitations we are interested in are common to both metrics and proceed.

With this approximation the equation in the region $r < a$ where α is zero becomes

$$-h'' - \frac{1}{r}h' = m^2h. \quad (96)$$

The solution of this equation to leading order in m^2a^2 is

$$h = N \left[1 - \frac{1}{4}m^2r^2 \right] \quad (97)$$

where N is a normalization constant. Since we are primarily interested in the light modes this will suffice. In the region between the branes the equation for the modes has the form

$$-h'' - \frac{1}{2}\alpha h' + \frac{3}{2}\alpha^2 h = \frac{m^2}{f}h. \quad (98)$$

This equation admits a solution in terms of Bessel functions. The solution is

$$h = Ne^{(-1/4)\alpha r} (A_2 J_{5/2}[mq_2] + B_2 J_{-5/2}[mq_2]) \quad (99)$$

where A_2 and B_2 are constants and q_2 is defined by

$$q_2 = \frac{2}{\alpha} \frac{e^{(-1/2)\alpha r}}{e^{(-1/2)\alpha a}}. \quad (100)$$

The closed form expressions for the relevant Bessel functions are

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\sin x \left[\frac{3}{x^2} - 1 \right] - 3 \frac{\cos x}{x} \right) \quad (101)$$

$$J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\cos x \left[\frac{3}{x^2} - 1 \right] + 3 \frac{\sin x}{x} \right). \quad (102)$$

Since we are interested in values of m such that $mq_1 \ll 1$, we can conveniently approximate the solution between the branes by

$$h = N \left[\bar{A}_2 \left(\frac{2m}{\alpha} \right)^{5/2} e^{(-3/2)\alpha r} + 3\bar{B}_2 \left(\frac{2m}{\alpha} \right)^{-5/2} e^{\alpha r} + \frac{1}{2}\bar{B}_2 \left(\frac{2m}{\alpha} \right)^{-1/2} e^{\alpha a} \right]. \quad (103)$$

Coming to the region $r > b$ we can also obtain a solution in terms of Bessel functions:

$$h = Ne^{(+1/4)\alpha r} (A_3 J_{5/2}[mq_3] + B_3 J_{-5/2}[mq_3]) \quad (104)$$

where q_3 is defined by

$$q_3 = \frac{2}{\alpha} \frac{e^{1/2\alpha r}}{e^{(-1/2)\alpha a + \alpha b}}. \quad (105)$$

The values of the A 's and B 's are to be determined by matching. The boundary conditions to be satisfied are continuity of h across the various boundaries and the following jump conditions on the derivatives at $r = a$ and $r = b$ respectively:

$$\Delta \frac{h'}{h} = \alpha \quad (106)$$

$$\Delta \frac{h'}{h} = -2\alpha. \quad (107)$$

The constant N is to be determined by normalization. Since this is not quite in the form of an eigenvalue problem, we make some simple transformations which render it so.

Defining

$$g = \text{const} \times h \left(\frac{s}{f} \right)^{1/4} \quad (108)$$

$$q = \int dr \frac{1}{\sqrt{f}}, \quad (109)$$

we get an equation for g as a function of q which has the form of an eigenvalue equation for m^2 with unit density function. For the continuum modes it is the outer region which is relevant for normalization. But here s and f are proportional, so we can conveniently choose the constant so that $g = h$ in this region. Also q and q_3 as defined differ at most by an additive constant. Hence for the continuum modes we merely have to normalize the solution for h for $r > b$ with respect to q_3 .

Matching at the inner brane we find that \bar{A}_2 is of order $\sqrt{m/\alpha}$, and \bar{B}_2 is of order $(m/\alpha)^{5/2}$. Then matching at the outer brane we find that

$$A_3 = O \left[\left(\frac{\alpha}{m} \right)^{1/2} e^{\alpha b} \right] \quad (110)$$

$$B_3 = O \left[\left(\frac{\alpha}{m} \right)^{-5/2} e^{-(\alpha/2)b} \right]. \quad (111)$$

In the far asymptotic region the A_3 mode is dominant. Normalizing to a box of size L we find

$$N = O \left[\frac{1}{\sqrt{L}} \frac{m}{\alpha} e^{(-3/2)\alpha b} \right]. \quad (112)$$

The situation is slightly different for the zero mode. We write the normalizable wave function as

$$g_0 = N_0 f^{3/4} s^{1/4}. \quad (113)$$

Now the integral relevant for normalization,

$$\int dq g_0^2 = N_0^2 \int dr \frac{1}{\sqrt{f}} f^{3/2} s^{1/2} = N_0^2 \int dr f s^{1/2}. \quad (114)$$

This is the same integral that appears in determination of the four dimensional Planck scale. By exactly the same methods we obtain, on normalizing to unity,

$$N_0 = O(\alpha e^{(-3/4)ab}). \quad (115)$$

We are now in a position to determine the corrections to gravity from the Kaluza-Klein excitations of the graviton. The change in the potential energy between two masses m_1 and m_2 on our brane is given by

$$\begin{aligned} \Delta V &= O\left[\frac{Gm_1m_2}{r} \int dm \left(\frac{N^2}{N_0^2 a}\right) L e^{-mr}\right] \\ &= O\left[Gm_1m_2 \frac{e^{(-3/2)ab}}{\alpha^3 r^4}\right] = O\left[\frac{Gm_1m_2}{r} \left(\frac{10^{-32}}{r^3(\text{TeV})^3}\right)\right]. \end{aligned} \quad (116)$$

From this it is clear that deviations from Newtonian gravity are highly suppressed at long distances.

We now explain why we expect this model to have the same physical implications as the model we started out with. Essentially for $r \gg a$ the general solution of both models will have a similar form. The only difference will be in the magnitudes of the coefficients \bar{A}_2 and \bar{B}_2 . Although these coefficients are determined by matching in the interior, their order of magnitude follows from simple dimensional considerations. This then implies that \bar{A}_3 and \bar{B}_3 and hence the normalization of the modes can be fixed by dimensional considerations. Hence the two theories will give the same order of magnitude estimate for the the corrections to Newtonian gravity.

Because of the isometry of the compact dimension, this theory contains a massless ‘‘gravi-photon’’—a Kaluza-Klein $U(1)$ gauge boson. However, no light or massless fields will carry non-trivial $U(1)$ charge since they have no momentum in the compact dimension. We expect that other than the modes we have already discussed, the remaining spectrum of gravitational excitations will be massive.

A more comprehensive study of the phenomenological implications of this model is left for future work.

D. Stress energy tensor for a field localized to a brane

In this section we consider the form of the stress energy tensor for a field localized to a brane having the metric $\bar{G}_{AB} = \text{diag}(-1, 1, 1, 1, 1)$ but in which the fifth dimension is compact and has a proper size a . We show that in the ground state the stress tensor does indeed have the form given in Eq. (23). For simplicity we limit ourselves to the case of a free scalar field.

The Lagrangian has the form

$$L = -\frac{1}{2} \partial_A \phi \partial^A \phi - \frac{1}{2} m^2 \phi^2. \quad (117)$$

We are interested in the expectation value of the stress tensor in the ground state:

$$\langle T_A^B \rangle = \langle -\partial_A \phi \partial^B \phi - \bar{G}_A^B L \rangle. \quad (118)$$

Because the ground state possesses translational symmetry,

$$\langle T_A^B \rangle = \frac{1}{V} \int d^4 x \langle T_A^B \rangle \quad (119)$$

where V is the volume of the four space dimensions. Performing a Fourier expansion for the field ϕ and making use of the canonical commutation relations this reduces to

$$\langle T_0^0 \rangle = \sum_{\mathbf{k}} \sum_{k_5} \frac{1}{2V} \sqrt{\mathbf{k}^2 + k_5^2 + m^2} \quad (120)$$

$$\langle T_m^n \rangle = -\sum_{\mathbf{k}} \sum_{k_5} \frac{1}{2V} \frac{k_m k^n}{\sqrt{\mathbf{k}^2 + k_5^2 + m^2}}. \quad (121)$$

Since the usual three space dimensions are infinite, the sums over momenta in these three directions can be replaced by integrals. However, since the fifth dimension ϕ is compact, the momenta in this direction remain discrete, $k_5 = n/a$ where n is an integer:

$$\langle T_0^0 \rangle = \sum_{k_5} \frac{1}{(2\pi)^4 a} \int d^3 \mathbf{k} \sqrt{\mathbf{k}^2 + k_5^2 + m^2} \quad (122)$$

$$\langle T_m^n \rangle = -\sum_{k_5} \frac{1}{(2\pi)^4 a} \int d^3 \mathbf{k} \frac{k_m k^n}{\sqrt{\mathbf{k}^2 + k_5^2 + m^2}}. \quad (123)$$

These integrals are infinite and must be regulated in order to yield sensible physical results. We will use a Pauli-Villars regulator, adding massive fields with appropriate statistics until all the divergences have been removed. (We could get similar results in a theory with spontaneously broken supersymmetry.) We simplify to the special case where the boson field is massless. Then all the divergences can be removed by adding three fields with opposite statistics having masses M , M and $2M$ and two fields with the same statistics which both have mass $\sqrt{3}M$. Here M is assumed to be some kind of cutoff for the theory.

Performing the integrals and adding the contributions from the various fields we get the finite but regulator dependent results

$$\begin{aligned} T_\mu^\nu &= \delta_\mu^\nu \sum_{k_5} \frac{1}{4(2\pi)^3 a} [k_5^4 \ln(k_5^2) + 2(k_5^2 + 3M^2)^2 \ln(k_5^2 + 3M^2) \\ &\quad - 2(k_5^2 + M^2)^2 \ln(k_5^2 + M^2) - (k_5^2 + 4M^2)^2 \ln(k_5^2 + 4M^2)] \end{aligned} \quad (124)$$

$$\begin{aligned} T_5^5 &= -\sum_{k_5} \frac{1}{(2\pi)^3 a} k_5^2 [k_5^2 \ln(k_5^2) + 2(k_5^2 + 3M^2) \ln(k_5^2 + 3M^2) \\ &\quad - 2(k_5^2 + M^2) \ln(k_5^2 + M^2) - (k_5^2 + 4M^2) \ln(k_5^2 + 4M^2)] \end{aligned} \quad (125)$$

$$T_\mu^5 = 0. \quad (126)$$

All other components of the stress energy tensor vanish, and it clearly has the form of Eq. (23). In a supersymmetric theory the scale M will be related to the scale of supersymmetry breaking. We now estimate T_m^n and T_5^5 in various limits.

When $M \ll (1/a)$ we can approximate T_m^n and T_5^5 to leading order in $(1/a)^2$ as $O(M^4/a)$ and $O(M^2/a^3)$ respectively. Clearly in this limit β^2 and γ^2 are not equal.

The relative difference between T_5^5 and T_μ^ν must be finite when the cutoff M is taken to infinity and must vanish as the size of the fifth dimension becomes infinite. We now estimate this difference as a function of a and M when the dimensionless quantity aM is large.

To do this we attempt to replace the sum we are interested in by a sum of integrals. We begin by observing that the integral below can be broken up into a sum of integrals over equal subdomains:

$$\int dk^5 T(k^5) = \sum_{k_5} \int_{k_5}^{k_5+1/a} dp^5 T(p^5). \quad (127)$$

Here T represents an arbitrary function of k_5 . If the function T is smooth it can be Taylor expanded:

$$T(p^5) = T(k_5) + T'(k^5)(p^5 - k^5) + T''(k^5) \frac{(p^5 - k^5)^2}{2} + \dots \quad (128)$$

Then performing the integrals over the subdomains we get

$$\int dk^5 T(k^5) = \sum_{k_5} \left[\frac{1}{a} T(k_5) + \frac{1}{2a^2} T'(k_5) + \frac{1}{6a^3} T''(k_5) + \dots \right]. \quad (129)$$

Now if ΣT is T_m^n or T_5^5 , the first term on the right has the form we are interested in. Also notice that the other terms on the right then involve fewer powers of k_5 in the numerator and hence for the seventh term and beyond the sums for individual fields are finite and straightforward to estimate.

ΣT can also be chosen to be derivatives to arbitrary order of T_m^n or T_5^5 . Since these quantities occur in the expansions for T_m^n or T_5^5 , they can then be substituted back to obtain systematic expansions for T_m^n and T_5^5 in terms of integrals.

A complication that arises for the case of a massless field is that the fourth derivative and higher of T_m^n and T_5^5 are not well defined at $k_5 = 0$. We account for this by separating this point from the sum and approximating the rest of the sum by integrals from $(1/a)$ to infinity and $-(1/a)$ to negative infinity.

To obtain a reasonable estimate we must expand in Eq. (128) to at least seventh order in order to account for all possible divergences as powers or logarithms of M . The cal-

ulation is straightforward but lengthy and the details will not be presented here. The result is that the difference between T_m^n and T_5^5 is finite and of order $(1/a)^5$ in the limit that the cutoff M is large.

This $(1/a)^5$ result for the difference could have been anticipated. Since the only counterterm allowed by general covariance is the cosmological constant which contributes equally to both T_m^n and T_5^5 , the difference between these two must be finite and regulator independent in the limit that the cutoff is taken to infinity. For a massless field, a is the only available dimensionful parameter. This result is just a higher dimensional form of the Casimir force.

IV. CONCLUSIONS

We have constructed a set of solutions to Einstein's equations in six or more dimensions, and exhibited a six dimensional setup, "the space needle," with two concentric positive tension 4-branes, which each have one compact dimension. Gravity is mostly localized to the outer brane while we assume the standard model resides on the inner brane, explaining the apparent weakness of gravity in our world. There are no massless moduli associated with either the size of the compact dimension or the brane locations. This provides an explicit demonstration that the gauge hierarchy problem can be solved in six dimensions, without supersymmetry, and with negligible corrections to gravity at distances longer than an inverse TeV. Gravitational effects do become strong at energies of order 1 TeV. We leave the discussion of gravitational collider phenomenology of new noncompact dimensions for future work.

We do not address the important issue of how to obtain chiral fermions on the standard model brane. Ordinary dimensional reduction by compactifying the fifth dimension on a circle always results in a non-chiral theory. One simple alternate possibility is to have the standard model reside on a 3-brane at the center of space. Then it is only necessary to have one 4-brane—the Planck brane. The metric is simply the $a \rightarrow 0$ limit of the space needle metric. Alternatively, it may be possible to generalize our mechanism to additional dimensions with some compactification which does allow a chiral effective theory on the TeV brane.

We also do not address the cosmological constant problem. The effective four dimensional cosmological constant depends on a complicated function of the bulk and brane parameters, and may be fine-tuned to zero or to a small acceptable value.

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- [1] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, *ibid.* **436**, 257 (1998).
- [2] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); **83**, 4690 (1999).
- [3] A. G. Cohen and D. B. Kaplan, Phys. Lett. B **470**, 52 (1999); N. Arkani-Hamed, L. Hall, D. Smith, and N. Weiner, Phys. Rev. D (to be published), hep-ph/9912453.
- [4] O. Klein, Z. Phys. **37**, 895 (1926); Nature (London) **118**, 516 (1926); K. Akama, in *Proceedings of the International Symposium on Gauge Theory and Gravitation* edited by K. Kikkawa, N. Nakanishi, and H. Nariai (Springer-Verlag, 1983), pp. 267–271; Prog. Theor. Phys. **78**, 184 (1987); **79**, 1299 (1988); M. Visser, Phys. Lett. **159B**, 22 (1985); M. Gell-Mann and B. Zwiebach, Nucl. Phys. **B260**, 569 (1985); V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. **125B**, 136 (1983); **125B**, 139 (1983); I. Antoniadis, Phys. Lett. B **246**, 377 (1990); J. D. Lykken, Phys. Rev. D **54**, 3693 (1996); R. Sundrum, J. High Energy Phys. **07**, 001 (1999).
- [5] J. Lykken and L. Randall, hep-th/9908076. See also N. Arkani-Hamed, S. Dimopoulos, G. Dvali, and N. Kaloper, Phys. Rev. Lett. **84**, 586 (2000).
- [6] C. Csaki, M. Graesser, and J. Terning, Phys. Lett. B **456**, 16 (1999); P. Binetruy, C. Deffayet, and D. Langlois, Nucl. Phys. **B565**, 269 (2000); C. Csaki, M. Graesser, C. Kolda, and J. Terning, Phys. Lett. B **462**, 34 (1999); J. M. Cline, C. Grojean, and G. Servant, Phys. Rev. Lett. **83**, 4245 (1999); D. J. Chung and K. Freese, Phys. Rev. D **61**, 023511 (2000); T. Shiromizu, K. Maeda, and M. Sasaki, *ibid.* **62**, 024012 (2000); C. Csaki, M. Graesser, L. Randall, and J. Terning, *ibid.* **62**, 045015 (2000); W. D. Goldberger and M. B. Wise, Phys. Lett. B **475**, 275 (2000); P. Binetruy, C. Deffayet, U. Ellwanger, and D. Langlois, Phys. Lett. B **477**, 285 (2000); P. Kanti, I. I. Kogan, K. A. Olive, and M. Pospelov, Phys. Rev. D **61**, 106004 (2000).
- [7] W. D. Goldberger and M. B. Wise, Phys. Rev. Lett. **83**, 4922 (1999).
- [8] N. Arkani-Hamed, S. Dimopoulos, and J. March-Russell, Phys. Rev. D (to be published), hep-th/9809124; R. Sundrum, *ibid.* **59**, 085010 (1999); M. A. Luty and R. Sundrum, *ibid.* **62**, 035008 (2000).
- [9] A. Chodos and E. Poppitz, Phys. Lett. B **471**, 119 (1999).