Bounds on curved domain walls in 5D gravity

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We discuss maximally symmetric curved deformations of the flat domain wall solutions of five-dimensional dilaton gravity that appeared in a recent approach to the cosmological constant problem. By analyzing the bulk field configurations and the boundary conditions at a four-dimensional maximally symmetric curved domain wall, we obtain constraints on such solutions. For a special dilaton coupling to the brane tension that appeared in recent works, we find no curved deformations, confirming and extending slightly a result of Arkani-Hamed *et al.* which was argued using a Z_2 symmetry of the solution. For more general dilaton-dependent brane tension, we find that the curvature is bounded by the Kaluza-Klein scale in the fifth dimension.

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I. INTRODUCTION

There has recently been renewed interest in the old idea that placing our world on a domain wall in a higher-dimensional bulk space can provide a useful new perspective on the cosmological constant problem [1]. Work in this direction appeared in [2,3]. More recently, concrete examples of 4D domain wall universes which have bound state gravitons [4] and a 4D cosmological constant which is insensitive to quantum loops of matter fields localized on the wall appeared in [5,6]. Similar domain wall solutions have been explored [in the context of the AdS conformal field theory (CFT) correspondence] in [7,8].

In this paper, we extend our work [5] in one respect. There, we concentrated for the most part on 5D gravity theories with a bulk scalar dilaton ϕ, and action

$$S=M_*^3\int d^5x\sqrt{-G}\left[R-\frac{4}{3}(\nabla\phi)^2\right]+\int d^4x\sqrt{-g}(-f(\phi)) \tag{1.1}$$

(with vanishing bulk cosmological term). In Eq. (1.1), G is the 5D bulk metric, while g is the induced metric on the domain wall, which is located at $x_5=0$. We demonstrated that one can find flat domain wall solutions for fairly generic thin wall δ function sources $f(\phi)$, i.e, without ''fine-tuning'' the brane tension $f(\phi)$. This is important because quantum loops of brane matter fields will in the most general circumstances correct the form of $f(\phi)$; it demonstrates some insensitivity of the existence of a flat 4D world to brane quantum loops. However, we did not address the issue of curved [de Sitter or anti–de Sitter (AdS)] solutions to the same 5D equations of motion.

Here we find curved solutions with maximal symmetry in four dimensions. More specifically, for both negatively curved and positively curved deformations we find that the largest scale of curvature possible is given by the scale set by the inverse proper length of the fifth dimension. In particular, the curvature can at most reach the mass scale of Kaluza-Klein modes in the fifth dimension. Unfortunately this upper bound is essentially equivalent to a 4D vacuum energy of the order of the scale of brane physics.

The organization of this paper is as follows. In Sec. II we describe the bulk gravity solutions with maximally symmetric curved domain walls and the matching boundary conditions at the domain wall. In Sec. III we explain the bounds on the curvature of curved solutions that result from these solutions. In Sec. IV, we discuss some additional issues of interest in analyzing the physics of the solutions discussed here and in [5,6], including the singularities.

II. CURVED SOLUTIONS AND MATCHING CONDITIONS

We will make the following ansatz for the metric (following [9,10]):

$$ds^2=e^{2A(x_5)}\bar{g}_{\mu\nu}dx^\mu dx^\nu+dx_5^2, \tag{2.1}$$

where

$$\bar{g}_{\mu\nu}=\mathrm{diag}(-1,e^{2\sqrt{\Lambda}x_1},e^{2\sqrt{\Lambda}x_1},e^{2\sqrt{\Lambda}x_1})$$

for de Sitter space and

$$\bar{g}_{\mu\nu}=\mathrm{diag}(-e^{2\sqrt{-\Lambda}x_4},e^{2\sqrt{-\Lambda}x_4},e^{2\sqrt{\Lambda}x_4},1)$$

for anti–de Sitter space.

Plugging this ansatz into the dilaton equations and Einstein's equations gives

$$\frac{8}{3}\phi''+\frac{32}{3}A'\phi'-\frac{\partial f}{\partial\phi}\delta(x_5)=0, \tag{2.2}$$

$$6(A')^2-\frac{2}{3}(\phi')^2-6\bar{\Lambda}e^{-2A}=0, \tag{2.3}$$

$$3A''+\frac{4}{3}(\phi')^2+3\bar{\Lambda}e^{-2A}+\frac{1}{2}f(\phi)\delta(x_5)=0. \tag{2.4}$$

Note here that the zero mode $A(0)$ always appears together with $\bar{\Lambda}$ here; we will take $A(0)=0$ in what follows.

Integrating the first equation in the bulk gives

$$\phi'=\gamma e^{-4A} \tag{2.5}$$

for some integration constant γ. Substituting this into the second equation gives

$$A' = \epsilon\sqrt{\tfrac{1}{9}\gamma^2 e^{-8A} + \bar{\Lambda} e^{-2A}}, \tag{2.6}$$

where $\epsilon = \pm 1$ determines the branch of the square root that we choose in the solution. Note here that this solution only makes sense when the argument of the square root in Eq. (2.6) is positive; for anti–de Sitter slices (negative $\bar{\Lambda}$) this gives a constraint on $\bar{\Lambda}$ which we will discuss in Sec. III.

This equation can be integrated to yield

$$\int^A \frac{\epsilon dA}{\sqrt{\frac{1}{9}\gamma^2 e^{-8A} + \bar{\Lambda} e^{-2A}}} = x_5 + \frac{3}{4}c. \tag{2.7}$$

The left-hand side of Eq. (2.7) is

$$\frac{3}{4}\epsilon\frac{1}{|\gamma|}e^{4A}\,{}_2F_1\left(\frac{1}{2},\frac{2}{3},\frac{5}{3},-\frac{9\bar{\Lambda}}{\gamma^2}e^{6A}\right) = x_5 + \frac{3}{4}c, \tag{2.8}$$

where ${}_2F_1(\frac{1}{2},\frac{2}{3},\frac{5}{3},z) \equiv F(z)$ is a hypergeometric function. It is analytic on $\mathbb{C} - \{[1,\infty) \subset \mathbb{R}\}$ and increases monotonically from $F(-\infty)=0$ through $F(0)=1$ until it attains its maximum at

$$F(1) = F_{\max} = \frac{\Gamma(\frac{5}{3})\Gamma(\frac{1}{2})}{\Gamma(\frac{7}{6})} = 1.725.$$

Because $F \geqslant 0$, the solution (2.8) is valid only on one side of $x_5 = -\frac{3}{4}c$ (determined by the sign ϵ). At $x_5 = -c$ there is a curvature singularity. As in [5,6], we make the assumption that the space can be truncated at this singularity, at least as far as low-energy physics is concerned.

Let us now introduce a domain wall at $x_5 = 0$. We must match the bulk solutions [given implicitly in Eqs. (2.5) and (2.8)] on the two sides of the wall, consistent with the δ-function terms in Eqs. (2.2) and (2.4). Let us denote the integration constants on the left ($x_5 < 0$) side of the wall by c_1, γ_1, d_1 and those on the right ($x_5 > 0$) side by c_2, γ_2, d_2. Here d_i refers to the zero mode $\phi(0)$ of the dilaton field on the ith side of the wall. Imposing continuity of ϕ at the wall fixes d_2.

Defining $\tilde{c}_i = c_i / F|_{x_5=0}$, we find the matching conditions

$$-\frac{8}{3}M_*^3\left(\frac{1}{\tilde{c}_1} + \frac{1}{\tilde{c}_2}\right) = \frac{\partial f}{\partial \phi}(\phi(0)), \tag{2.9}$$

$$M_*^3\left(\sqrt{\left(\frac{1}{\tilde{c}_1}\right)^2 + 9\bar{\Lambda}} + \sqrt{\left(\frac{1}{\tilde{c}_2}\right)^2 + 9\bar{\Lambda}}\right) = \frac{1}{2}f(\phi(0)). \tag{2.10}$$

We here used the fact, which follows from Eq. (2.8) evaluated at $x_5 = 0$ with $A(0)=0$, that $|\gamma_i||\tilde{c}_i| = 1$.

III. BOUNDS ON CURVED DEFORMATIONS

A. Asymmetric solutions (I) [General $f(\phi)$]

We have now gathered the information we need to determine the extent of curvature of these curved-slice deformations of the flat solutions of [5,6]. We will first discuss a bound on $|\bar{\Lambda}|$, which basically constrains it to be less than the inverse proper length of the fifth dimension, that applies to both signs of $\bar{\Lambda}$. We will then discuss a tighter bound that arises in the case of positive $\bar{\Lambda}$.

Consider the Eq. (2.8) at $x_5 = 0$:

$$\epsilon|\tilde{c}|F(-9\Lambda(\tilde{c})^2) = c. \tag{3.1}$$

Defining $y \equiv \sqrt{9|\bar{\Lambda}|}|\tilde{c}|$, this equation implies

$$|yF(-y^2)| = |c|\sqrt{9|\bar{\Lambda}|}. \tag{3.2}$$

Now the quantity $yF(-y^2)$ is bounded. In fact, its maximum value (attained as $y \to \infty$) is 4. We have from Eq. (3.2) that

$$\sqrt{9|\bar{\Lambda}|} < 4\left|\frac{1}{c_i}\right|, \tag{3.3}$$

where we added the index to c_i since this bound applies on either side of the domain wall.

Note from the metric (2.1) that $|c_i|$ is the proper distance to the singularity on the ith side of the wall. So for either sign of $\bar{\Lambda}$, we find that the effective 4D cosmological constant of the curved solutions is bounded to be smaller than the Kaluza-Klein scale in the bulk.

This reflects the same physical point made in [5,6]: there is no contribution from physics localized on the brane to the 4D cosmological constant. A brane-scale cosmological constant would have manifested itself in a contribution to Eq. (3.3) which depends on $f(\phi(0))$, and such terms are absent. These bounds arise from the matching conditions, but note that it is not the case that the singularities recede to ∞ (or come in to the origin) as one saturates the bound.

The largest phenomenologically viable value for the proper distance c is roughly a millimeter [11]. This would give us a bound on $\bar{\Lambda}$ of about 10^{-6} eV^2. This is much larger than the observed value $\bar{\Lambda} \sim 10^{-64}$ eV^2 of the cosmological constant. Note that we here are using "general relativity" conventions for the cosmological constant $\bar{\Lambda}$; the standard "particle physics" cosmological constant is $\Lambda_4 = M_4^2\tilde{\Lambda} \sim \mathrm{mm}^{-4}$. Unfortunately this is within a couple of orders of magnitude of the standard model scale of TeV^4. In a model with supersymmetry spontaneously broken at the TeV scale, this would be the scale of a brane cosmological constant.

For positive $\bar{\Lambda}$ this $1/c$ scale is itself bounded by a further constraint. Consider the matching condition (2.10). It implies that

$$\frac{1}{\tilde{c}_i^2} < \frac{1}{2M_*^3}f(\phi). \tag{3.4}$$

Therefore, since $c = \tilde{c}/F|_0$, we can extend Eq. (3.3) to

$$\sqrt{9\bar{\Lambda}}<4F_{\max}\frac{1}{2M_*^3}f(\phi). \tag{3.5}$$

In fact we can do better than Eq. (3.5). The 4D Planck scale M_4 is given by

$$M_4^2=M_*^3\int dx_5\,e^{2A}=\frac{M_*^3}{9\bar{\Lambda}}\left(\sqrt{\frac{1}{|\tilde{c}_1|^2}+9\bar{\Lambda}}+\sqrt{\frac{1}{|\tilde{c}_2|^2}+9\bar{\Lambda}}-\frac{1}{|\tilde{c}_1|}-\frac{1}{|\tilde{c}_2|}\right). \tag{3.6}$$

Multiplying Eq. (3.6) by $\bar{\Lambda}$, and dividing by M_4^2, we get an equation for $\bar{\Lambda}$. For positive $\bar{\Lambda}$, we can use the matching condition (2.10) to replace the first two terms in the parentheses in Eq. (3.6) with $\frac{1}{2}f(\phi(0))$. We then obtain the inequality

$$\bar{\Lambda}<\frac{1}{18}\frac{f(\phi(0))}{M_4^2} \tag{3.7}$$

(for negative $\bar{\Lambda}$, we would not obtain such an inequality). So for instance if the value of $f(\phi(0))$ is TeV scale, which is natural if we take the standard model (cut off at about a TeV) to live on the brane, then

$$\bar{\Lambda}<10^{-33}\ \ (\mathrm{TeV})^2. \tag{3.8}$$

This is of the same order as the contribution of a brane with supersymmetry spontaneously broken at a TeV.

B. Symmetric solutions (II) $[f(\phi)=e^{\pm(4/3)\phi}]$

When we pick $f(\phi)=e^{\pm(4/3)\phi}$, we find the matching condition (2.9) becomes

$$M_*^3\left(\frac{1}{|\tilde{c}_1|}+\frac{1}{|\tilde{c}_2|}\right)=\pm\frac{1}{2}e^{\pm(4/3)\phi}, \tag{3.9}$$

which agrees with the second condition (2.10) when $\bar{\Lambda}=0$. When $\bar{\Lambda}\neq 0$, the two conditions (3.9) and (2.10) contradict each other, and there are no solutions. This means that the symmetric solutions of [5,6] [solutions (II) in the classification of [5]] do not have any deformations with 4D de Sitter or anti–de Sitter symmetry. This slightly extends the result of Arkani-Hamed *et al.* [6], who observed that such deformations would violate the Z_2 symmetry of the solution, and thus could not appear in a Z_2 orbifold of this solution.

IV. DISCUSSION AND FURTHER ISSUES

A priori there is a question as to whether the space of integration constants is parametrized by vacuum expectation values of fluctuating fields in four dimensions, or whether instead different members of this family arise from different four-dimensional Lagrangians.[1] The existence of anti–de Sitter and de Sitter deformations (bounded though they are) suggests that these deformations constitute parameters in the 4D effective theory. If the effective 4D cosmological constant were parametrized by a field, then in solving its equations of motion one would end up with one consistent possibility for the value of the 4D cosmological constant. The fact that we find a family of solutions suggests that this is not the case here. Indeed, naive calculation of the coefficient of the kinetic term for the mode which moves one from flat to curved 4D metrics does suggest that it is not a dynamical mode (it has infinite kinetic term). However the divergence in the calculation arises at the singularities, so this conclusion depends sensitively on how the singularities are resolved by microphysics.

To a 4D effective field theorist, the choice of which member of the family to start with constitutes a tuning of the 4D cosmological constant. From the point of view of the microscopic 5D theory, this tuning involves a parameter in the solution and not a parameter in the Lagrangian. If this system can be embedded consistently into string theory, where there are no input parameters in the ''Lagrangian,'' the mere existence of Poincare invariant solutions after some quantum corrections have been taken into account would be significant, even if such solutions lie in a family of curved solutions that signal the appearance of fine-tuning at low energies. In any case, our results here indicate that the apparent fine-tuning required to choose a flat slice is independent of standard model physics, though it can arise at the same scale.

Having understood better the situation with respect to this issue of fine-tuning, one is led to consider the main challenge identified in [5,6]: the question of possible microphysical constraints on the (codimension one) singularities in the solutions. The type of analysis we did here might help resolve an issue raised in [12], as we will mention presently, after first discussing the issue in a little more generality.

One possibility is that boundary conditions are required at the singularities, as in the case analyzed in [13]. It is then important to check whether the appropriate boundary conditions, along with the equations of motion and matching conditions, can be solved within the space of curved solutions we have identified [5,6].

There are some singularities in string theory (like conifolds, orbifolds, and brane-orientifold systems) which have a well-understood quantum resolution involving new degrees of freedom at the singularity; in these cases the resolution does not imply any extra boundary conditions in the effective long-wavelength theory.

It has recently been suggested that the singularities that appear in our solutions do not permit a finite-temperature deformation accessible with a long-wavelength general relativistic analysis [12]. This is a criterion that does not appear to contradict the microscopic consistency of orbifolds or conifolds, and the case of orientifolds and their duals must be considered carefully. Because of the large curvatures (and in

[1]We thank S. Dimopoulos and R. Sundrum for discussions on this point.

some cases large couplings) in the backgrounds we consider here, such an analysis is necessarily limited. However, the general question of how finite temperature can be obtained in these backgrounds is an important one.

Within the context of the analysis of [12], it is notable that our solutions lie on the boundary between (conjecturally) allowed and (conjecturally) disallowed singularities. It is important to redo this analysis for solutions which include some bulk corrections. In particular, a nontrivial bulk dilaton potential of the right sign [as in our case (III) [5]] may put us in the allowed region according to the conjectured criterion of [12]. Instead of fine-tuning to obtain 4D Poincare invariant slices as we did in case (III) of [5], one can consider curved solutions of the sort given here. In the context of the type (III) situation where there is a bulk potential for ϕ, this is in fact natural if we do not wish to fine-tune the parameters in the 5D Lagrangian in order to obtain a 4D Poincare invariant solution. It is possible that this bulk correction will induce a sub-TeV correction to the 4D cosmological constant, while satisfying the conjectured constraints coming from the long-wavelength analysis of [12].

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